Positive Markov Jump Linear Systems

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Abstract

This paper presents a comprehensive study of continuous-time Positive Markov Jump Linear Systems (PMJLS). A PMJLS can be seen as a dynamical system that switches within a finite set of linear time-invariant subsystems according to a stochastic switching signal modeled as a Markov chain, and describes the time-evolution of nonnegative variables under nonnegative inputs. Contrary to the well-studied general class of Markov Jump Linear Systems (MJLS), positivity endows the model with peculiar properties. The paper collects some existing results together with original developments on the stability analysis of PMJLS and the study of their input-output properties. In particular, conditions for stability of PMJLS are discussed, mainly based on Linear Programming problems. Similar computational tools are derived to analyze performance measures, such as $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_\infty$ costs and the respective input-output induced gains. The second part of the paper is devoted to the class of Dual switching Positive Markov Jump Linear Systems (D-PMJLS), namely PMJLS affected by an additional switching variable which can be either an unknown disturbance or a control signal available to the designer for stabilization and performance optimization. We discuss several problems, including stability, performance analysis, stabilization via switching control, and optimization. Some application examples are introduced to motivate the interest in PMJLS and D-PMJLS.

1

Introduction and Motivations

Positive Markov Jump Linear Systems (PMJLS) are piecewise positive linear systems affected by a stochastic signal generated by a Markov chain. This signal selects at each time instant which subsystem (mode) is active among a finite set of LTI subsystems. Between jumps the dynamics is described by a linear time-invariant model. Since PMJLS are positive systems, for any sample path of the Markov process, the state variables remain nonnegative whenever initialized in the positive orthant and the system is forced by external nonnegative inputs. Positive systems naturally arise in the description of biological systems, compartmental models, population dynamics, traffic modeling, chemical reactions, queue processes, et cetera. A rich literature on positive linear systems is now available. Relevant general textbooks on deterministic positive systems are Farina and Rinaldi [2000], Kaczorek [2002], Haddad et al. [2010]. More specific contributions dealing with Lyapunov functions and input-output norms can be found in Ait Rami and Shamma [2009], Tanaka and Langbort [2011], Rantzer [2011], Briat [2013].

PMJLS belong to the general class of Markov Jump Linear Systems (MJLS) that have been widely investigated in the last decades for their ability to model unexpected events, random faults and uncontrolled

configuration changes. General references for the theory of MJLS are the books Costa et al. [2005], Boukas [2005], Costa et al. [2013].

All theoretical results on MJLS are also valid for PMJLS, including those concerning mean square stability and performance, that rely on technical tools of quadratic Lyapunov functions and Linear Matrix Inequalities (LMI). However, positive systems enjoy specific properties that simplify the study of stability and performance. For instance, a sensible notion of stability for PMJLS is mean stability, corresponding to the exponential convergence to zero of the expectation of the state variables, that can be checked through linear co-positive stochastic Lyapunov functions, or equivalently in terms of Hurwitz stability of suitable Metzler matrices. In the same vein, specific input-output performance measures, such as $\mathcal{L}_1$- and $\mathcal{L}_\infty$-induced norms, can be studied using Linear Programming (LP) tools. Much part of this tutorial paper is devoted to the analysis of stability and input-output norms of PMJLS through this approach.

Up to now, there are just a few papers dealing with PMJLS. To the best of the authors' knowledge, the first contribution pointing out the usefulness of the LP-approach to the study of PMJLS is Ait Rami and Shamma [2009]. More recently, Bolzern et al. [2014a] studied various notions of stability and their relationships, Zhang et al. [2014] provided results on stochastic stabilization, and Zhu et al. [2014] addressed the $\mathcal{L}_1$ optimization problem via LP and filter design in discrete-time. An application to an epidemiological model can be found in Ait Rami et al. [2014].

In this paper, we also consider PMJLS subject to an additional switching signal. Such a dual switching configuration appears, for instance, in multi-plant networked control with bandwidth limitations, where the stochastic signal models the random behavior of the network, while the control switching signal is the scheduling variable deciding which plants are currently attended in closed-loop. The resulting dual switching system offers intriguing research issues due to the interplay between the two switching signals. In general, the additional variable can be treated as either a disturbance or a control signal. In the former case, we address the stability and worst-case gain calcula-

tion problems whereas, in the latter case, the stabilization problem and the minimization of these gains are tackled. It is worth remarking that, in general, only suboptimal solutions can be provided, however based on simple computational tools of LP and the use of piecewise linear stochastic Lyapunov functions. Some preliminary work on these topics was presented in Bolzern et al. [2014b].

The paper is organized as follows. In this first chapter, after some notational remarks, we present a few examples that serve as a motivation to study PMJLS. These examples include telecommunication networks, compartmental systems, consensus problems in multi-agents systems, epidemiological models, and switching control in HIV shrink therapies. We hope that these drafted examples inspire future applications of the class of systems under investigation.

In Chapter 2, we deal with the class of PMJLS and tackle the stochastic stability problem by referring to the notions of mean square (MS) stability, mean (M) stability and almost sure (AS) stability. Necessary and sufficient conditions are provided, where the available results pertaining the theory of deterministic positive systems emerge as basic tools of analysis. In particular, the stability properties are investigated through the notion of co-positive stochastic Lyapunov functions that are inherited by the solution of properly defined inequalities, either linear or quadratic. Thanks to the stability notions the rest of the chapter is devoted to the definition and computation of input-output performance measures, including $\mathcal{L}_1$ and $\mathcal{L}_\infty$ (via LP) and $\mathcal{L}_2$ (via LMI). Remarks are provided to characterize the worst (nonnegative) disturbances (both stochastic and deterministic), associated with the norms of suitably defined deterministic positive systems. The chapter ends with some observations on the complete parametrization of state-feedback stabilizing controllers, state-feedback controllers with guaranteed $\mathcal{L}_\infty$ performance, and robustness at the face of uncertainty in the parameters of the transition rate matrix associated with the Markov process.

In Chapter 3, the class of dual switching PMJLS (D-PJMLS) is considered. Here an additional switching signal is introduced, enriching the dynamic behavior of the PMJLS studied in the previous chapter. When

the additional signal is regarded as a disturbance, results guaranteeing mean stability and $\mathcal{L}_1$ performance for arbitrary switching are provided via LP tools. Conversely, if the additional signal is regarded as a control variable, design algorithms for stabilization and minimization of upper bounds of the $\mathcal{L}_1$-norm are provided. Different strategies are considered, depending on the information available to the controller. They include open-loop strategies, state-dependent strategies that rely on the knowledge of the Markov process (mode-dependent), and mode-independent strategies where the signal generated by the Markov chain is not available for feedback. The chapter ends with a section, showing the links between scalar D-PMJLS and a class of deterministic positive linear systems, where the switches affects only the diagonal entries of the dynamical matrix. Many results presented in Chapter 3 find a mature development and explanation by considering the theory on deterministic positive switched systems presented in the twin paper Blanchini et al. [2015].

Apart from the few cited contributions already available in the literature, most of the content of Chapters 2 and 3 on the characterization of input-output norms and their computation both for PJLMS and D-PJLMS is new.

The developed theory is illustrated by simple numerical examples. A few more realistic examples are also treated, namely a distributed algorithm for power allocation in telecommunication networks, the storage control of a 4-element compartmental hydraulic model and the stabilization of the disease-free equilibrium in a Susceptible-Infective-Susceptible (SIS) epidemiological system. The Monte Carlo simulations are carried out by using standard algorithmic tools for the generation of Markov processes, see e.g. Gillespie [1992].

A brief section with some concluding remarks and perspectives closes the paper.

1.1   Notation

Throughout the paper, we adhere to the convention of using small letters to indicate scalars, bold small letters for vectors and capital

letters for matrices. The i-th entry of vector $\mathbf{x}$ is denoted by $[\mathbf{x}]_i$ and the (i, j)-th entry of matrix M is denoted by $[M]_{ij}$. The symbol $\mathbf{1}_n$ denotes the n-dimensional column vector with all entries equal to 1. The symbol I_n stands for the identity matrix of order n.

A (column or row) vector $\mathbf{x} \in \mathbb{R}^n$ is said to be *nonnegative* if all its entries $[\mathbf{x}]_i$, $i = 1, 2, \ldots, n$, are nonnegative. In this case, we will say that $\mathbf{x} \geq 0$. The symbol $\mathbb{R}_+^n$ indicates the set of n-dimensional nonnegative vectors. A vector $\mathbf{x}$ is *positive* if nonnegative and at least one entry is positive. In this case, we will say that $\mathbf{x} > 0$. The symbol $\mathcal{P}_n$ will denote the *simplex* of dimension n, i.e. the set of positive vectors with $\mathbf{1}_n^\top \mathbf{x} = 1$. A (column or row) vector $\mathbf{x} \in \mathbb{R}^n$ is said to be *strictly positive* if all its entries are greater than 0, and in this case, we will say that $\mathbf{x} \gg 0$. The expressions $\mathbf{x} \geq \mathbf{y}$, $\mathbf{x} > \mathbf{y}$, $\mathbf{x} \gg \mathbf{y}$ indicate that the difference $\mathbf{x} - \mathbf{y}$ is nonnegative, positive or strictly positive, respectively. A similar notation is used for real matrices.

A square matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler* if its off-diagonal entries $[A]_{ij}$, $i \neq j$, are nonnegative. A dynamical linear system described by the differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where A is a Metzler matrix, is called a *positive system* because it enjoys the property that any trajectory starting in the positive orthant remains indefinitely confined in it, see Farina and Rinaldi [2000]. More in general, the state-space representation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \end{aligned}$$

describes a positive system if A is a Metzler matrix and B, C and D are nonnegative matrices. Starting from a nonnegative initial state with a nonnegative input function $\mathbf{u}(t)$ all the state and output trajectories of this system remain nonnegative at all time instants.

In the paper, we will use Metzler matrices $A \in \mathbb{R}^{n \times n}$ satisfying the unit row-sum constraint $A\mathbf{1}_n = 0$. The set of these matrices will be denoted by the symbol $\mathcal{S}_n$.

Every Metzler matrix A has a real dominant eigenvalue λ_F, named *Frobenius eigenvalue*. Associated with λ_F there is always both a left and a right positive eigenvector, known as (left/right) *Frobenius eigenvector*.

A Metzler matrix A is *reducible* if there exists a permutation matrix P such that

$$P^\top A P = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square matrices, otherwise it is *irreducible*.

A square matrix is *Hurwitz* if all its eigenvalues lie in the open left half plane. A Metzler matrix is Hurwitz if and only if there exists a vector $\mathbf{c} \gg 0$ such that $\mathbf{c}^\top A \ll 0$, or, equivalently if and only if there exists a vector $\mathbf{d} \gg 0$ such that $A\mathbf{d} \ll 0$, see e.g. Farina and Rinaldi [2000].

We will use the symbol $M \succ 0$ $(M \succeq 0)$ to indicate that a symmetric matrix $M \in \mathbb{R}^{n \times n}$ is *positive definite (positive semi-definite)*, i.e. the quadratic form $\mathbf{x}^\top M \mathbf{x}$ is strictly positive (nonnegative) for all $\mathbf{x} \neq 0$. The notation $M \prec 0$ $(M \preceq 0)$ is used for *negative definite (negative semi-definite)* symmetric matrices.

If $\{\mathbf{x}_i \in \mathbb{R}^n, i = 1, 2, \ldots, N\}$ is a set of column vectors, the symbol col$\{\mathbf{x}_i\}$ will denote the vector in $\mathbb{R}^{nN}$ obtained by orderly stacking all the vectors $\mathbf{x}_i$ in a single column. The range of values for the index i will be omitted, if clear from the context.

If $\{M_i \in \mathbb{R}^{n \times m}, i = 1, 2, \ldots, N\}$ is a set of matrices, the symbol col$\{M_i\}$ will denote the matrix in $\mathbb{R}^{nN \times m}$ obtained by orderly stacking all the matrices M_i in a single block column, while the symbol diag$\{M_i\}$ will denote the block diagonal matrix in $\mathbb{R}^{nN \times mN}$ obtained by orderly putting M_i on the diagonal. Again, the range for the index i will be omitted, if clear from the context.

For a matrix $M \in \mathbb{R}^{n \times m}$, the symbol vec$\{M\}$ will denote the vector in $\mathbb{R}^{nm}$ obtained by orderly stacking all the column A in a single column.

For a column vector $\mathbf{x} \in \mathbb{R}^n$, the 1-*norm* is defined as $\|\mathbf{x}\|_1 = \sum_{i=1}^n |[\mathbf{x}]_i|$, whereas the ∞-*norm* is $\|\mathbf{x}\|_\infty = \max_i |[\mathbf{x}]_i|$. Of course, for nonnegative vectors, i.e. $\mathbf{x} \in \mathbb{R}^n_+$, the absolute values can be taken off. The 2-*norm* of a vector is defined as $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n [\mathbf{x}]_i^2}$. The 1-norm, the 2-norm and the ∞-norm for matrices are induced from those of vectors. Given a square matrix $M \in \mathbb{R}^{n \times n}$, we have $\|M\|_1 = \max_j \sum_{i=1}^n |[M]_{ij}|$, $\|M\|_2 = \sqrt{\lambda_{max}(M^\top M)}$, where λ_{max} de-

notes the maximum eigenvalue, and $\|M\|_\infty = \max_i \sum_{j=1}^n |[M]_{ij}|$. When dealing with a Metzler matrix M, the absolute values can be dropped out for the off-diagonal entries in the expressions for $\|M\|_1$ and $\|M\|_\infty$. A general reference for norms of vectors and matrices is Horn and Johnson [2012].

The symbol $\mu_*(A)$ stands for the $*$-*measure* of a square matrix $A \in \mathbb{R}^{n \times n}$. It is defined as

$$\mu_*(A) = \lim_{T \to 0} \frac{\|I_n + AT\|_* - 1}{T} \tag{1.1}$$

where I_n is the identity matrix and the $*$-norm is any matrix norm. It is important to recall that the measure $\mu_*(A)$ is the derivative of the $*$-norm of $\exp(At)$ at $t = 0$. If $\mu_*(A) < 0$, then the dynamical system $\dot{x}(t) = Ax(t)$ is "instantaneously norm-contractive", and, consequently, A is Hurwitz. In addition, it follows that $\frac{d\|\mathbf{x}\|_*}{dt} \leq \mu_*(A)\|\mathbf{x}\|_*$. If the matrix norm used in (1.1) is the usual spectral norm, $\mu_*(A) = \mu_2(A)$ coincides with $\lambda_{max}\left((A + A^\top)/2\right)$. For more details see Desoer and Vidyasagar [1975].

For two matrices $A \in \mathbb{R}^{n \times m}$, $B \in \mathbb{R}^{p \times q}$, the expression $C = A \otimes B$ stands for the usual *Kronecker product*, obtained by orderly collecting the blocks $[A]_{ij}B$ into the matrix $C \in \mathbb{R}^{np \times mq}$. For two square matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times p}$, the *Kronecker sum* is defined as $D = A \oplus B = A \otimes I_p + I_n \otimes B \in \mathbb{R}^{np \times np}$. Properties of Kronecker operators can be found in Graham [1981].

The symbol $\Pr\{\mathcal{A}\}$ will be used for the probability of the event $\mathcal{A}$. The symbol $\Pr\{\mathcal{A}|\mathcal{B}\}$ will denote the conditional probability of the event $\mathcal{A}$ given event $\mathcal{A}$, i.e. $\Pr\{\mathcal{A}|\mathcal{B}\} = \Pr\{\mathcal{A} \cap \mathcal{B}\}/\Pr\{\mathcal{B}\}$, provided that $\Pr\{\mathcal{B}\} \neq 0$. The expectation of a stochastic variable v will be denoted as $E[v]$. The conditional expectation of v given event $\mathcal{A}$ will be indicated by $E[v|\mathcal{A}]$. If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N$ are mutually exclusive events whose union covers the entire event space, then the well-known law of total expectation claims that $E[v] = \sum_{i=1}^N E[v|\mathcal{A}_i]\Pr[\mathcal{A}_i]$.

1.2   Motivating applications

A few significant applications are here described, leading to the class of models we will concentrate on in the rest of the paper.

1.2.1 Power allocation in telecommunication networks

Consider the problem of power allocation in a mobile telecommunication system with n users which transmit signals to a receiving station. Upon denoting by p_i the power of the i-th transmitter, the Signal-to-Interference-and-Noise-Ratio (SINR) at the receiver, can be calculated as follows

$$\gamma_i = \frac{p_i g_{ii}}{\sum_{j \in \mathcal{N}_i} g_{ij} p_j + \nu_i}$$

where $g_{ij} > 0$, for all i and j, are scaling coefficients multiplying the power level p_j. When $j \neq i$ these coefficients contribute to the amount of noise generated by the transmitter j affecting the transmission i. Moreover, ν_i is the natural (thermal) noise at the receiver, $\mathcal{N}_i$ denotes the set of all other nodes different from i that interfere with node i. The QoS (quality of service) can be expressed by defining desired objective SINR values γ_i^o for each active user, assuming that the SINR quantification parameters satisfy, for each detector, the constraints

$$\sum_{j \in \mathcal{N}_i} \gamma_i^o \frac{g_{ij}}{g_{ii}} < 1$$

A centralized algorithm is to impose the values of the powers so as to minimize the total power

$$\min_{\mathbf{p}} \mathbf{1}^\top \mathbf{p}$$

where $\mathbf{p} = \mathrm{col}\{p_i\}$, with the constraints

$$\gamma_i \geq \gamma_i^o$$

The solution to this problem is quite easy and given by

$$\bar{\mathbf{p}} = -F^{-1} \Gamma \boldsymbol{\nu}$$

where $\bar{\mathbf{p}} = \mathrm{col}\{\bar{p}_i\}$, $\Gamma = \mathrm{diag}\{\gamma_i^o\}$, $\boldsymbol{\nu} = \mathrm{col}\{\nu_i\}$, and

$$F = \begin{bmatrix} -g_{11} & \gamma_1^o g_{12} & \gamma_1^o g_{13} & \cdots & \gamma_1^o g_{1n} \\ \gamma_2^o g_{21} & -g_{22} & \gamma_2^o g_{23} & \cdots & \gamma_2^o g_{2n} \\ \gamma_3^o g_{31} & \gamma_3^o g_{32} & -g_{33} & \cdots & \gamma_3^o g_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_n^o g_{n1} & \gamma_n^o g_{n2} & \gamma_n^o g_{n3} & \cdots & -g_{nn} \end{bmatrix}$$

Matrix F above is Metzler and Hurwitz, thanks to the imposed constraints on the SINR quantification parameters.

Recently, the above centralized distribution scheme has been extended via the theory of positive switched dynamical systems in Zappavigna et al. [2012]. To be precise, the model of the controlled power network can be written by considering that each transmitter is required to regulate its transmitted power based on the presence of other transmitters. If other transmitters are active, the interference noise increases and hence the transmitter should increment its power. The adaptive algorithm for the i-th transmitter is as follows

$$\dot{p}_i(t) = \kappa_i \left[-p_i(t) + \gamma_i^o \left(\sum_{j \in \mathcal{N}_i} \frac{g_{ij}}{g_{ii}} p_j(t) + \frac{\nu_i}{g_{ii}} \right) \right]$$

where $\kappa_i > 0$ are constant gains. Notice that the equilibrium point is such that

$$\frac{\bar{p}_i g_{ii}}{\sum_{j \in \mathcal{N}_i} g_{ij} \bar{p}_j + \nu_i} = \gamma_i^o$$

so that $\bar{\mathbf{p}} = \mathrm{col}\{\bar{p}_i\}$ coincides with the above optimal vector $\bar{\mathbf{p}} = -F^{-1}\Gamma\boldsymbol{\nu}$. Of course, the equilibrium point needs to be exponentially stable. The system can be written as the positive system

$$\dot{\mathbf{x}}(t) = K[-\mathbf{x}(t) + C\mathbf{x}(t)] + K\mathbf{r} = A\mathbf{x}(t) + B\boldsymbol{\nu}$$

where $\mathbf{x}$ is the vector collecting the variables p_i, $K = \mathrm{diag}\{\kappa_i\}$, $C = \mathrm{diag}\{g_{ii}^{-1}\}F + I$, and $\mathbf{r} = \mathrm{diag}\{g_{ii}^{-1}\}\Gamma\boldsymbol{\nu}$. Notice that $A = K(-I+C)$ is a Metzler matrix and $B = K\mathrm{diag}\{g_{ii}^{-1}\}\Gamma$ is a nonnegative matrix. The multiplicative coefficients g_{ij} depend on the applied linear detector and coding properties but also incorporate path losses, shadowing and multi-path fading. As such we can assume that they are uncertain parameters that may jump among N different values, according to a Markov chain with transition rate matrix Λ. Therefore, a more realistic model is

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\boldsymbol{\nu}$$

where $\sigma(t) \in \{1, 2, \ldots, N\}$ represents the current mode of the network, and matrices A_σ, B_σ are obtained from A and B above by replacing g_{ij} with $g_{ij,\sigma}$. The resulting system is a PMJLS and its stability and

performance will be discussed in the example section of Chapter 2. Simulation results will also be provided.

1.2.2   Compartmental models

Compartmental models are systems composed of interconnected reservoirs (named *compartments*), exchanging flows of a common resource, see e.g. Haddad et al. [2010]. They are frequently used in hydrology to model network of water reservoirs and in biology to describe storage, transport and drainage of certain substances through different components of a biological organism. Since the amount of resource in each compartment is intrinsically a positive variable, positive models are appropriate to describe such phenomena.

In the simplest case, a compartmental model is composed of a number N_c of different compartments, each one being modeled by the linear first-order dynamical system

$$\begin{aligned} \dot{x}_i(t) &= -\alpha_i x_i(t) + u_i(t) \\ y_i(t) &= \beta_i x_i(t) \end{aligned}$$

where $x_i(t)$ is the amount of resource stored in the i-th compartment, while $u_i(t)$ and $y_i(t)$ are the inflow and the outflow, respectively. The parameter β_i is nonnegative and the parameter α_i is also nonnegative, accounting for the outflow and possible losses in the compartment. Typically, it is assumed that, due to losses, the dc-gain $\mu_i = \beta_i/\alpha_i$ is less than 1. The compartments are cross-coupled. Each compartment is fed by a fraction of the outflow from other compartments and possibly by an external inflow. Precisely, the total inflow $u_i(t)$ to the i-th compartment is described by

$$u_i(t) = \sum_{j=1}^{N_c} \gamma_{ji} y_j(t) + \delta_i w(t)$$

where γ_{ji} and δ_i are positive coefficients with $\sum_{i=1}^{N_c} \delta_i = 1$. Thus, $u_i(t)$ is a weighted sum of the flows from the connected compartments and a fraction of the external inflow $w(t)$. Of course, $0 \le \gamma_{ji} \le 1$, and $\sum_{i=1}^{N_c} \gamma_{ji} \le 1$, with the equality holding only if the whole outflow from

the j-th compartment does not exit the system. By stacking all the state variables $x_i(t)$ in a single vector $\mathbf{x}(t) \in \mathbb{R}^{N_c}$, the system is described by the linear model

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bw(t)$$

where $A \in \mathbb{R}^{N_c \times N_c}$ is a Metzler matrix and $B \in \mathbb{R}^{N_c}$ is a nonnegative vector.

A simple example of a 4-element compartmental system is schematically shown in Figure 1.1. The associated system matrices are

$$A = \begin{bmatrix} -\alpha_1 & \gamma_{21}\beta_2 & 0 & 0 \\ 0 & -\alpha_2 & 0 & \gamma_{42}\beta_4 \\ \gamma_{13}\beta_1 & \gamma_{23}\beta_2 & -\alpha_3 + \gamma_{33}\beta_3 & 0 \\ \gamma_{14}\beta_1 & \gamma_{24}\beta_2 & \gamma_{34}\beta_3 & -\alpha_4 \end{bmatrix}, \quad B = \begin{bmatrix} \delta_1 \\ \delta_2 \\ 0 \\ 0 \end{bmatrix}$$
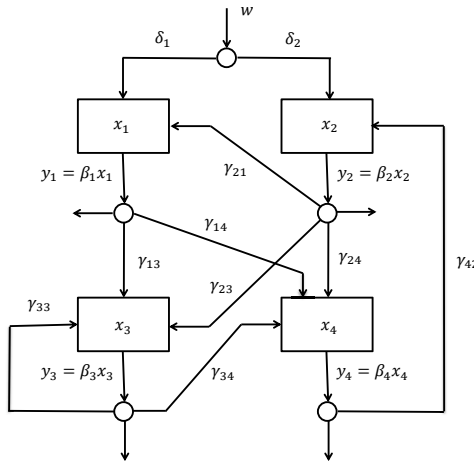

Figure 1.1: Schematic diagram of a 4-element compartmental model. Since the parameter β_2 is subject to stochastic jumps, the system is described as a PMJLS.

Suppose now that the discharge parameter β_2 is subject to sudden fluctuations between two extreme values β_{2min} and β_{2max}, and the transition is governed by a Markov process $\sigma(t)$ taking values in the set $\mathcal{N} = \{1, 2\}$. The value $\sigma = 1$ corresponds to $\beta_2 = \beta_{2max}$ and $\sigma = 2$

corresponds to $\beta_2 = \beta_{2min}$. The compartmental system can now be modeled by the PMJLS

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t) + Bw(t) \tag{1.2}$$

where A_1 and A_2 are obtained by simply putting either $\beta_2 = \beta_{2min}$ or $\beta_2 = \beta_{2max}$ in the expression of matrix A.

If one is interested in the time evolution of total storage in the four compartments, the following output transformation can be introduced

$$\mathbf{z}(t) = C\mathbf{x}(t), \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \tag{1.3}$$

For a given input $w(t)$ and a given realization $\sigma(t)$ of the switching signal, a meaningful cost to be evaluated is $J = \int_0^\infty z(t)dt$. Note that, since $\sigma(t)$ is stochastic, J is actually a random variable. In order to evaluate the system properties, it might be interesting to compute the expected value of J for all possible inputs $w(t)$ in a certain class of functions. We will show in Section 2.5.2 how this problem can be efficiently solved.

Suppose now that some coefficients of the model can be controlled in order to minimize the expected value of J. For instance, the outflow coefficients γ_{33}, γ_{34}, γ_{42} can be switched between two triples of values. In such a case the system is described by a D-PMJLS. The design of a switching strategy guaranteeing an upper bound to the cost will be discussed in Section 3.4.3.

1.2.3 Consensus problems with switching topology

In a consensus problem, a network of agents are required to reach an asymptotic agreement on the value of a common variable through local exchange of information among neighboring agents. A classical solution to this problem is provided by the following protocol, based on Laplacian matrices. Let $x_i(t)$ denote the scalar state of the i-th agent at time t. The time evolution of $x_i(t)$ is governed by the following updating law

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} (x_j(t) - x_i(t))$$

where $\mathcal{N}_i$ denotes the set of the neighbors of agent i. Putting together the states of all the agents in a single state vector $\mathbf{x}(t)$, the dynamics

of the network is modeled by the equation

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t)$$

where L is the so-called Laplacian matrix. The off-diagonal elements $L_{ij}, i \neq j$, of this matrix are either -1 or 0 depending on the fact that agent j is a neighbor to agent i or not. Each diagonal entry L_{ii} represents the degree of agent i, namely the number of its neighbors. Note that $-L$ is a Metzler matrix, with the additional property that the elements of each row sum to 0. If the state starts its evolution in the positive orthant, it stays there forever.

It is quite straightforward to show, that under mild assumptions of network connectivity, the state $\mathbf{x}(t)$ asymptotically tends to a constant vector with identical entries for any initial condition $\mathbf{x}(0)$, i.e. consensus is eventually reached, see Olfati-Saber et al. [2007].

However, if the topology of the network switches with time, the classical methods for LTI systems cannot be used any more to study convergence. In that case the system can be modeled as the switching linear system

$$\dot{\mathbf{x}}(t) = -L_{\sigma(t)}\mathbf{x}(t) \tag{1.4}$$

where the switching signal $\sigma(t)$ takes value in a finite set whose elements are in correspondence with all admissible graph topologies. It may be sensible to describe $\sigma(t)$ as a random process with a Markovian structure. The analysis of consensus is then reduced to the analysis of convergence of the expected value of the state of the PMJLS (1.4) to a constant equilibrium.

1.2.4 Epidemiological models

This example is taken from Ait Rami et al. [2014] (see also Blanchini et al. [2014]). Consider n groups of a population. For each group i, let $I_i(t)$ and $S_i(t)$ denote the number of infectives and suscectives at time t, respectively. For simplicity, assume that the total number $I_i(t)+S_i(t) = N_i$ is constant in time and let $x_i(t) = I_i(t)/N_i$. One can then write, for $i = 1, 2, \ldots, n$, the following balance equations:

$$\dot{x}_i(t) = (1 - x_i(t)) \sum_{j=1}^{N} \frac{\beta_{ij} N_j}{N_i} x_j(t) - (\delta_i + \mu_i)x_i(t) \tag{1.5}$$

where $\beta_{ij} > 0$ is the rate at which susceptibles in group i are infected by infectives in group j, $\delta_i > 0$ is the rate at which an infective individual in group i is cured and $\mu_i > 0$ is the death rate in group i (equal to the birth rate appearing in the equation of $\dot{S}_i = -\dot{I}_i$). Letting $\mathbf{x} = \text{col}\{x_i\}$, system (1.5) is a nonlinear positive systems. Indeed, notice that $\{\mathbf{x} \in \mathbb{R}^n : 0 \le \mathbf{x} \le \mathbf{1}_n\}$ is positively invariant. i.e. if the initial condition $\mathbf{x}(0)$ belongs to this set, then the corresponding state trajectory remains in the set.

If we assume that M alternative therapies can be applied to fight the epidemy, the rate δ_i is not constant but it depends, at every time t, on a certain variable $\gamma(t) \in \{1, 2, \cdots, M\}$, that represents the value at time t of the switching signal that orchestrates among the different therapies. Moreover, we also introduce the simplifying assumption that the change of therapies does not affect the infection and death rates, but that each parameter β_{ij} and μ_i may take values in a finite set $\{1, 2, \ldots, N\}$, depending on the outcomes of a Markov process $\sigma(t)$. Therefore, we replace δ_i in (1.5) with $\delta_{i,\gamma(t)}$, β_{ij} with $\beta_{ij,\sigma(t)}$ and μ_i with $\mu_{i,\sigma(t)}$. The introduction of the therapy scheduling and the stochastic parameters preserves the positive invariance property of the set $\{\mathbf{x} \in \mathbb{R}^n : 0 \le \mathbf{x} \le \mathbf{1}_n\}$. Finally, upon linearizing the system around the *disease free equilibrium point* $\mathbf{x} = 0$, we obtain

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}^{\gamma(t)} \mathbf{x}(t) \tag{1.6}$$

where $\sigma(t) \in \{1, 2, \ldots, N\}$, $\gamma(t) \in \{1, 2, \ldots, M\}$ and $A_i^j = D_i^j + \Lambda_i^\top$, for some Metzler matrix Λ_i and some diagonal matrix D_i^j, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$. Specifically, $[\Lambda_i]_{hk} = \beta_{kh,i} N_h / N_k$, $h \ne k$, $[\Lambda_i]_{hh,i} = \sum_{s \ne h} \beta_{sh,i} N_h / N_k$ and

$$[D_i^j]_{hk} = \begin{cases} -\delta_{h,j} - \mu_{h,i} + \sum_s \beta_{sh,i} N_h / N_s, & \text{if } h = k \\ 0, & \text{if } h \ne k \end{cases}$$

Notice that $\Lambda \mathbf{1}_n = 0$. System (1.6) is a D-PMJLS, object of Chapter 3 of this monograph. A typical performance index is the integral of total fraction of infectives over time, namely

$$J(\mathbf{x}_0) = \int_0^\infty \mathbf{1}_n^\top \mathbf{x}(t) dt \tag{1.7}$$

In Section 3.5, we will provide simulation results on stability and performance of the disease-free equilibrium of the "deterministic" epidemiological model (1.6), i.e. system (1.6) where the coefficients of matrices A_i^j do not depend on the Markov jump signal $\sigma(t)$, that means $A_i^j = D_i^j + \Lambda_i^\top = D^j + \Lambda^\top$, for all $i = 1, 2, \ldots, N$.

1.2.5 Shrink HIV therapy scheduling

Antiretroviral (HAART) therapies have shown to be rather effective in the cure of HIV disease, and a number of control-oriented mathematical models have been developed, see Hernandez Vargas et al. [2014] and references therein. A simple formulation consists in writing the balance equations for the infected ($T^\star$) and uninfected (T) CD4+T cells, infected ($P^\star$) and uninfected (P) macrophages, and viral load V_i, $i = 1, 2, \ldots, n_g$ of the i-th genotype g_i. Therefore, denoting by V_T the total viral load, the following equations can be obtained:

$$
\begin{aligned}
\dot{T} &= s_T + \frac{\rho_T V_T}{C_T + V_T} T - \sum_{i=1}^{n} k_{Ti} T V_i - \delta_T T \\
\dot{T}_i^\star &= k_{Ti} T V_i - \delta_{T^\star} T_i^\star + \mu \sum_{i=1}^{n} m_{ij} V_j T \\
\dot{P} &= s_P + \frac{\rho_P V_T}{C_P + V_T} P - \sum_{i=1}^{n} k_{Pi} P V_i - \delta_P P \\
\dot{P}_i^\star &= k_{Pi} P V_i - \delta_{P^\star} P_i^\star + \mu \sum_{i=1}^{n} m_{ij} V_j P \\
\dot{V}_i &= p_{Ti} T_i^\star + p_{Pi} P_i^\star - \delta_V V_i \\
V_T &= \sum_{i=1}^{N} V_i
\end{aligned}
$$

where s_T and s_P are the generation rates of new T-cells and macrophages, respectively, C_T and C_P are proliferation parameters, ρ_T and ρ_P are the uninfected cell replication rates, k_{Ti} and k_{Pi} represent the infection rates, whereas p_{Ti} and p_{Pi} are the viral proliferation rates. The mutation rate is expressed by μ, and the coefficients $m_{ij} \in [0, 1]$

represent the genetics connection between genotypes. Finally, the parameters δ are the death rates for the relevant species.

Notice that the system is a positive nonlinear system. After the first phase of HAART treatment and before virus rebound, the number of macrophage $P = \bar{P}$ and the number of T-cells $T = \bar{T}$ can be considered almost constant. With this assumption, the equations for $P_i^\star$, $T_i^\star$ and V_i are linear, so that the system is a *positive linear system*. The infection rates are affected by the therapies, usually consisting in M different drugs, so that a parameter k_{Ti} or k_{Pi}, for all $i = 1, 2, \ldots, n_g$, may assume M values each. We can describe this fact by including a switching variable, $\gamma \in \{1, 2, \ldots, M\}$. Finally, almost all other parameters, like the death rates and proliferation rates, are highly uncertain and possible time-varying. This phenomenon can be expressed by saying that they can assume a finite number N of values, described in a stochastic fashion, as the outcome of a Markov chain with a certain transition rate matrix Λ and switching signal $\sigma \in \{1, \ldots, N\}$. An example with 16 genotypes g_i, $i = 1, 2, \ldots, 16$, is sketched in Figure 1.2, which symbolically illustrates the level of efficiency of the therapies on the different genotypes. All in all, defining $\mathbf{x}(t) = \mathrm{col}\{\mathbf{x}_i(t)\}$, with $\mathbf{x}_i(t)^\top = [T_i^\star(t) \ P_i^\star(t) \ V_i(t)]$, the linearized system can be written as

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}^{\gamma(t)} \mathbf{x}(t)$$

where

$$A_i^j = \begin{bmatrix} D_1^j & \cdots & 0 & & \\ 0 & D_2^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n_g}^j \end{bmatrix} + \mu M_u$$

and

$$D_i^j = \begin{bmatrix} -\delta_{T^\star} & 0 & k_{Ti}^j \bar{T} \\ 0 & -\delta_{M^\star} & k_{Pi}^j \\ p_{Ti}^j & p_{Pi}^j \bar{P} & -\delta_V \end{bmatrix}, \quad M_u = [m_{ij}] \otimes \begin{bmatrix} 0 & 0 & \bar{T} \\ 0 & 0 & \bar{P} \\ 0 & 0 & 0 \end{bmatrix}$$

This system is a D-PMJLS studied in Chapter 3, for both analysis (when $\gamma(t)$ is constant), and control performance (when $\gamma(t)$ is a control switching signal).
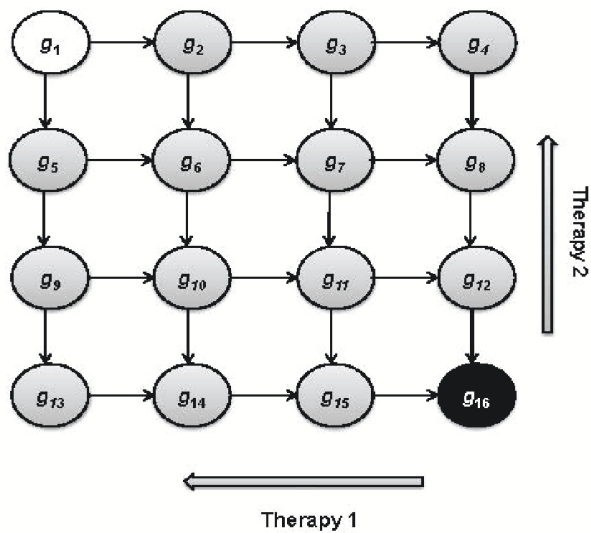
Figure 1.2: The network for 16 genotypes g_i and two drug combinations. The small arrows indicate the admissible mutations. The direction of the big arrows represents the strength of the therapy on the genotypes.

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