# AN EXTENSION PROBLEM FOR THE CR FRACTIONAL LAPLACIAN

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Abstract. We show that the conformally invariant fractional powers of the sub-Laplacian on the Heisenberg group are given in terms of the scattering operator for an extension problem to the Siegel upper halfspace. Remarkably, this extension problem is different from the one studied, among others, by Caffarelli and Silvestre.

## 1. Introduction and statement of results

There has been a lot of recent interest in the study of CR manifolds, on one hand because of their puzzling geometry and, on the other hand, because they serve as abstract models of real submanifolds of complex manifolds. In particular, orientable CR manifolds of hypersurface type which are strictly pseudoconvex have been the subject of many flourishing studies, due also to the many parallels between their geometry and conformal geometry of Riemannian manifolds. In this context, the Heisenberg group plays the same rôle as $\mathbb{R}^n$ in conformal geometry. Indeed, Folland and Stein [24] constructed normal coordinates which show how the Heisenberg group closely approximates the pseudohermitian structure of a general orientable strictly pseudoconvex CR manifold.

The Heisenberg group $\mathbb{H}^n$ arises also in the description of $n$–dimensional quantum mechanical systems. Moreover there is a rich and fruitful interplay between sub-Riemannian geometry on Carnot groups (of which $\mathbb{H}^n$ is one of the most interesting examples) and control theory in engineering, and there are many works devoted to understanding harmonic analysis on Lie groups.

In this paper we take a closer look at the construction of CR covariant operators of fractional order on $\mathbb{H}^n$ and on orientable CR manifolds of hypersurface type which are strictly pseudoconvex, and how they may be constructed as the Dirichlet-to-Neumann operator of a degenerate elliptic equation in the spirit of Caffarelli and Silvestre [10].

Fractional CR covariant operators of order $2\gamma$, $\gamma \in \mathbb{R}$, may be defined from scattering theory on a Kähler-Einstein manifold $\mathcal{X}$ [21, 45, 40, 33], they are pseudodifferential operators whose principal symbol agrees with the pure fractional powers of the CR sub–Laplacian $(-\Delta_b)^\gamma$ on the boundary $\mathcal{M} = \partial\mathcal{X}$. In the particular case of the Heisenberg group $\mathbb{H}^n$, they are simply the intertwining operators on the CR sphere calculated in [8, 51, 9] using classical representation theory tools. It is precisely the article by Branson, Fontana and Morpurgo [8] that underlined the importance and nice PDE properties of these operators.

There is a rich theory of pseudodifferential operators on the Heisenberg group (see, for instance, [62]). In particular, the fractional sub-Laplacian is the infinitesimal generators of a Levy process [2], although with some particularities because of the extra direction.

Using functional analysis tools, one may formulate an extension problem to construct the pure fractional powers of the sub–Laplacian on Carnot groups (see the related work of [25] and [57]). An interesting feature of our point of view is that one needs to use the underlying complex hyperbolic geometry instead of the abstract functional analysis theory for the construction of the CR covariant version of these operators.

We try to make this paper self-contained in order to make it accessible to analysts and do not assume any prerequisites on CR or complex geometry. In this regard, Sections 2 and 3 are a summary of standard concepts that are included here for convenience of the reader. We emphasize, however, that the reader does not need these concepts for our main results, Theorems 1.1 and 1.2, which only concern the case of the Heisenberg group.

The $n$-dimensional Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group law

$$\hat{\xi} \circ \xi = \big(\hat{x} + x, \hat{y} + y, \hat{t} + t + 2(\langle x, \hat{y}\rangle_{\mathbb{R}^n} - \langle \hat{x}, y\rangle_{\mathbb{R}^n})\big),$$

where $\xi = (x, y, t)$, $\hat{\xi} = (\hat{x}, \hat{y}, \hat{t})$ and $\langle \cdot, \cdot\rangle_{\mathbb{R}^n}$ is the standard inner product in $\mathbb{R}^n$. $\mathbb{H}^n$ can be regarded as a smooth sub–Riemannian manifold; an orthonormal frame is given by the Lie left–invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{for } j = 1, \ldots, n, \quad T_0 = \frac{\partial}{\partial t}. \tag{1.1}$$

Given a smooth function on $\mathbb{H}^n$, we define the sub–Laplacian of $u$ as:

$$\Delta_b u = \frac{1}{2} \sum_{j=1}^{n} \left[X_j^2 + Y_j^2\right] u. \tag{1.2}$$

Note that we have replaced the usual factor 1/4 in front by a 1/2, which will be more convenient for our purposes.

Introducing complex coordinates $z = x + iy \in \mathbb{C}^n$, the Heisenberg group may be identified with the boundary of the Siegel domain $\Omega_{n+1} \subset \mathbb{C}^{n+1}$, which is defined as

$$\Omega_{n+1} := \{\zeta = (z_1, \ldots, z_n, z_{n+1}) = (z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \,|\, q(\zeta) > 0\}$$

with

$$q(\zeta) = \operatorname{Im} z_{n+1} - \sum_{j=1}^{n} |z_j|^2, \tag{1.3}$$

through the map $(z, t) \in \mathbb{H}^n \mapsto (z, t + i|z|^2) \in \partial\Omega_{n+1}$. Introducing the coordinates $(z, t, q) \in \mathbb{C}^n \times \mathbb{R} \times (0, \infty)$ on $\Omega_{n+1}$, the Siegel domain is a Kähler-Einstein manifold with Kähler form

$$\omega_+ = -\frac{i}{2} \partial\bar{\partial} \log q = \frac{i}{2} \left( \frac{\frac{1}{4} dq^2 + \theta^2}{q^2} + \frac{\delta_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}}{q} \right), \tag{1.4}$$

where $\theta^\alpha = dz_\alpha$, $\theta^{\bar{\beta}} = d\bar{z}_\beta$ for $\alpha, \beta = 1, \ldots, n$ and where $\theta$ is given by

$$\theta = \tfrac{1}{2}\big[dt + 2 \sum_{\alpha=1}^{n} (x_\alpha dy_\alpha - y_\alpha dx_\alpha)\big] = \tfrac{1}{2}[dt + i \sum_{\alpha=1}^{n} (z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha)]. \tag{1.5}$$

If, instead, one writes the Kähler metric for the defining function $\rho$ with $q = \rho^2/2$, we have

$$g^+ = \frac{1}{2} \left( \frac{d\rho^2}{\rho^2} + \frac{2\delta_{\alpha\bar{\beta}} \theta^\alpha \otimes \theta^{\bar{\beta}}}{\rho^2} + \frac{4\theta^2}{\rho^4} \right). \tag{1.6}$$

In particular, $\Omega_{n+1}$ may be identified with the complex hyperbolic space $\mathcal{H}^m_{\mathbb{C}}$. Here, and for the rest of the paper, we set

$$m+1.$$

The boundary manifold $\mathcal{M} = \partial\Omega_{n+1} = \{q = 0\}$ inherits a natural CR structure from the complex structure of the ambient manifold given by the 1-form (1.5), that is precisely the contact form that characterizes the Heisenberg group $\mathbb{H}^n$ as a sub–Riemannian (CR) manifold, in which the CR structure is given by the bundle $\mathcal{H}(\mathbb{H}^n) = \operatorname{span}_{\mathbb{C}}\langle Z_1, \dots, Z_n\rangle$, with $Z_j = X_j + iY_j$ and $X_j, Y_j$ defined in (1.1) for $j = 1, \dots, n$. Consequently the Levi distribution on $\mathbb{H}^n$ is given by $H(\mathbb{H}^n) = \operatorname{span}_{\mathbb{R}}\langle X_1, \dots X_n, Y_1, \dots, Y_n\rangle$ and the associated characteristic direction is

$$T = 2T_0 = 2\frac{\partial}{\partial t}.$$

In particular, the associated Laplace-Beltrami operator with this choice of $\theta$ is (1.2); this is the reason for our normalization constant $1/2$.

More generally, one could consider $\mathcal{X}^{n+1}$ a complex manifold with strictly pseudoconvex boundary $\mathcal{M}$ carrying an approximate Kähler-Einstein metric. Then $\mathcal{M}$ inherits a CR structure as explained in Section 2. An even more general setting would be to take an asymptotically complex hyperbolic (ACH) manifold $\mathcal{X}^{m+1}$ with boundary $\mathcal{M}$. An ACH manifold is endowed with a metric $g^+$ that behaves asymptotical like (1.6) near $\mathcal{M}$.

In the first setting, scattering theory tells us that, fixed a defining function $q$, for $s \in \mathbb{C}$, $\Re(s) \geq \frac{m}{2}$, and except for a set of exceptional values, given $f$ smooth on $\mathcal{M}$, the eigenvalue equation

$$-\Delta_{g^+}u - s(m-s)u = 0 \qquad \text{in } \mathcal{X},$$

has a solution $u$ with the expansion

$$\begin{cases} u = q^{(m-s)}F + q^s G, & \text{for some} \quad F, G \in \mathcal{C}^\infty(\bar{\mathcal{X}}), \\ F|_{\mathcal{M}} = f. \end{cases}$$

The scattering operator is defined as

$$S(s) : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$$

by

$$S(s)f := G|_{\mathcal{M}}.$$

It can be shown that $S(s)$ is a meromorphic family of pseudodifferential operators in the $\Theta$-calculus of [21] on $\mathcal{M}$ of order $2(2s-m)$, and it is self-adjoint when $s$ is real.

For $\gamma \in (0, m)\backslash\mathbb{N}$, we set $s = \frac{m+\gamma}{2}$. We define the CR fractional sub–Laplacian on $(\mathcal{M}, [\theta])$ by

$$P^\theta_\gamma f = c_\gamma S(s) f, \tag{1.7}$$

for a constant

$$c_\gamma = 2^\gamma \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{1.8}$$

Note that $c_\gamma < 0$ for $0 < \gamma < 1$. It is proven then that $P^\theta_\gamma$ is a pseudodifferential operator of order $2\gamma$, whose principal symbol is given by (1.12) below. The main property of the operator is its CR covariance. Indeed, given another conformal representative $\hat{\theta} = w^{\frac{2}{m-\gamma}}\theta$ which identifies the CR structure of $\mathcal{M}$, the corresponding operator is given by

$$P_\gamma^{\hat{\theta}}(\cdot) = w^{-\frac{m+\gamma}{m-\gamma}} P_\gamma^\theta(w\,\cdot). \tag{1.9}$$

Then one may define the fractional $Q$-curvature as

$$Q_\gamma^\theta = P_\gamma^\theta(1),$$

which has interesting covariant properties.

For integer powers $\gamma \in \mathbb{N}$, one may still define the operators by taking residues of the scattering operator. In particular, for $\gamma = 1$, one obtains the CR Yamabe operator of Jerison-Lee [48]

$$P_1^\theta = -\Delta_b + \frac{n}{2(n+1)} R_\theta,$$

where $\Delta_b$ is the sub–Laplacian on $(\mathcal{M}, \theta)$ and $R_\theta$ the Webster curvature. In the Heisenberg group case, with the 1-form $\theta$ introduced in (1.5), we can write explicitly

$$P_1^\theta = -\Delta_b, \quad P_2^\theta = (\Delta_b)^2 + T^2, \quad P_k^\theta = \prod_{l=1}^{k} (-\Delta_b + i(k+1-2l)T).$$

These are precisely the GJMS [37] operators in the CR case from [33].

Our first theorem is the precise statement of the extension problem on the Heisenberg group $(\mathbb{H}^n, [\theta])$ for $\theta$ given by (1.5). This relation allows to treat the CR fractional sub–Laplacian as a boundary operator and, in particular, one recovers the formula for CR covariant operators on the Heisenberg group from [8].

**Theorem 1.1.** *Let $\gamma \in (0,1)$, $a = 1-2\gamma$. For each $f \in \mathcal{C}_0^\infty(\mathbb{H}^n)$, there exists a unique solution $U := \mathcal{E}_\gamma f$ for the extension problem*

$$\begin{cases} \partial_{\rho\rho} U + a\rho^{-1}\partial_\rho U + \rho^2 \partial_{tt} U + \Delta_b U = 0, & \text{in } \Omega_{n+1} \simeq \mathbb{H}^n \times \mathbb{R}^+, \\ U = f, & \text{on } \partial\Omega_{n+1} = \{\rho = 0\} \simeq \mathbb{H}^n. \end{cases} \tag{1.10}$$

*Moreover,*

$$P_\gamma^\theta f = \frac{c_\gamma}{\gamma 2^{1-\gamma}} \lim_{\rho \to 0} \rho^a \partial_\rho U. \tag{1.11}$$

*As a consequence, one recovers the symbol of the operator*

$$P_\gamma^\theta = (2|T|)^\gamma \, \frac{\Gamma\left(\frac{1+\gamma}{2} + \frac{-\Delta_b}{2|T|}\right)}{\Gamma\left(\frac{1-\gamma}{2} + \frac{-\Delta_b}{2|T|}\right)}. \tag{1.12}$$

Uniqueness is understood in the natural corresponding Sobolev space, see (1.13) below.

The extension problem (1.10) is similar to the extension considered by Caffarelli and Silvestre [10] for the construction of the fractional Laplacian in the Euclidean case, but surprisingly with the additional term in the $t$-direction $\rho^2\partial_{tt}U$ that appears when one considers the CR sub–Laplacian.

Although here we have concentrated on the Heisenberg group as the boundary of the Siegel domain, on more general ACH manifolds we have the same type of results, although lower order terms appear in the extension problem (1.10). This is exactly what happens in the real case (see [14]), and precisely those lower order terms carry a lot of geometric information.

The main idea in the proof of Theorem 1.1 is to use the group Fourier transform on the Heisenberg group in order to follow the arguments used in the real case by [10]. Their idea of reducing the equation to an ODE in $\rho$ still holds, although we need to take care of the extra direction $\partial_t$. Moreover, the Plancherel identity for the Heisenberg Fourier transform allows to prove an energy identity that yields a sharp trace Sobolev embedding.

Before we state our second theorem precisely, we need to introduce the following notions. Set

$$\mathcal{A}_\gamma[U] = \int_{\Omega_{n+1}} \rho^a \left[ |\partial_\rho U|^2 + \rho^2 |\partial_t U|^2 + \tfrac{1}{2} \sum_{j=1}^{n} \left( |X_j U|^2 + |Y_j U|^2 \right) \right] dx\, dy\, dt\, d\rho \tag{1.13}$$

for functions $U$ on $\overline{\Omega}_{n+1} \simeq \mathbb{H}^n \times [0, \infty)$; this is the weighted Dirichlet energy in the extension. We define the space $\dot{H}^{1,\gamma}(\Omega_{n+1})$ as the completion of $\mathcal{C}_0^\infty(\overline{\Omega}_{n+1})$ with respect to $\mathcal{A}_\gamma^{1/2}$.

Similarly, we denote by $\dot{S}^\gamma(\mathbb{H}^n)$ the closure of $\mathcal{C}_0^\infty(\mathbb{H}^n)$ with respect to the quadratic form $a_\gamma$ associated to the symbol (1.12) through Fourier transform, which is defined by

$$a_\gamma[f] = \frac{1}{2^\gamma} \int_{\mathbb{H}^n} f P_\gamma^\theta f \, dxdydt \tag{1.14}$$

see also Section 6. These are the fractional analogues of the Sobolev spaces on $\mathbb{H}^n$ introduced by Folland and Stein [24].

**Theorem 1.2.** *Let $\gamma \in (0,1)$. Then there exists a unique linear bounded operator $\mathcal{T}$ : $\dot{H}^{1,\gamma}(\Omega_{n+1}) \to \dot{S}^\gamma(\mathbb{H}^n)$ such that $\mathcal{T}U(\cdot) = U(\cdot, 0)$ for all $U \in \mathcal{C}_0^\infty(\overline{\Omega_{n+1}})$. Moreover, for any $U \in \dot{H}^{1,\gamma}(\Omega_{n+1})$ one has*

$$\mathcal{A}_\gamma[U] \geq 2^{1-\gamma} \gamma \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \, a_\gamma[\mathcal{T}U] \, .$$

*Equality is attained if and only if $U = \mathcal{E}_\gamma f$ for some $f \in \dot{S}^\gamma(\mathbb{H}^n)$.*

One may use the Cayley transform to translate Theorem 1.2 to the CR sphere $\mathbb{S}^{2m-1}$ as the boundary of the complex Poincaré ball $\mathcal{H}_{\mathbb{C}}^m$. Indeed, the Cayley transform $\Psi_c$ is a biholomorphism between the unit ball in $\mathbb{C}^{n+1}$ and the Siegel domain; when restricted to the respective boundaries it gives a CR equivalence between $\mathbb{S}^{2m-1}$ minus a point and $\mathbb{H}^n \simeq \partial\Omega_{n+1}$ (which is the CR equivalent of the stereographic projection of the unit sphere on the Euclidean space), see also section 3 and [48]. Combining Theorem 1.2 with the fractional Sobolev embedding on the Heisenberg group from [26] we obtain the following energy inequality:

**Corollary 1.3.** *There exists a sharp constant $S(n,\gamma)$ such that for every $U \in \dot{H}^{1,\gamma}(\mathcal{H}_{\mathbb{C}}^m)$, $f = \big(\mathcal{T}(U \circ \Psi_c^{-1})\big) \circ \Psi_c$ defined on $\mathbb{S}^{2m-1}$,*

$$\|f\|^2_{L^{2^*}(\mathbb{S}^{2n+1})} \leq S(n,\gamma) \mathcal{A}_\gamma[U \circ \Psi_c^{-1}] \tag{1.15}$$

*for $2^* = \frac{2m}{m-\gamma}$. We have equality in (1.15) if and only if and $U \circ \Psi_c^{-1} = \mathcal{E}_\gamma(f \circ \Psi_c^{-1})$ and $f \circ \Psi_c^{-1}$ comes from a CR transformation of the CR sphere (3.15), i.e. $f \circ \Psi_c^{-1} : \mathbb{H}^n \to \mathbb{R}$ with*

$$f\big(\Psi_c^{-1}(z,t)\big) = \left( \frac{1}{(1+|z|^2)^2 + t^2} \right)^{\frac{m-\gamma}{2}}$$

*up to left translations, dilations and multiplication by a constant.*

The importance of this Corollary is that it constitutes the first step in the resolution of the fractional CR Yamabe problem (see [32] and Section 7 for a short discussion).

In the last part of this paper (Section 8), we explore the quaternionic setting and show that both Theorems 1.1 and 1.2 can be proven in a similar manner when one considers the quaternionic Heisenberg group $\mathcal{Q}^n$ as the boundary of the quaternionic hyperbolic space $\mathcal{H}^m_{\mathbb{Q}}$. In particular, the extension problem (1.10) is the same one provided one replaces $\partial_{tt}$ by the Laplacian in the extra three $t$-directions. This happens because of the underlying Lie group structure and, in general, this general construction is possible in symmetric spaces of rank one, which are the real, complex, quaternionic hyperbolic spaces $\mathcal{H}^m_{\mathbb{R}}$, $\mathcal{H}^m_{\mathbb{C}}$, $\mathcal{H}^m_{\mathbb{Q}}$, respectively, and the octonionic hyperbolic plane $\mathcal{H}^2_{\mathbb{O}}$.

There are still many open questions in this field. From the PDE point of view, an extremely interesting problem is to establish a good elliptic theory for (1.10), which is fourth order because of the term $\partial_{tt}$. Indeed the vector field $T_0 = \frac{\partial}{\partial t}$ is a second order differential operator in the CR structure of $\mathbb{H}^n$, as it is obtained from first order commutators of the vector fields $\{X_j, Y_j\}_{j=1,\ldots,n}$. See also sections 2 and 3.

From the complex geometry point of view, observe that two defining functions $\varphi$ and $\psi$ generate the same Kähler metric in $\mathcal{X}$ if and only if $\varphi = e^F\psi$ for a pluriharmonic function $F$, i.e. $\partial\bar{\partial}F = 0$. If $\theta_\varphi$ and $\theta_\psi$ are the corresponding pseudohermitian structures on $\mathcal{M} = \partial\mathcal{X}$ then $\theta_\varphi = e^f\theta_\psi$, where $f = F|_{\mathcal{M}}$. In this case, Branson's $Q$-curvature (that corresponds to our case $\gamma = m$, see (4.1)), vanishes identically and thus it is not an interesting object when restricting to pluriharmonics. Recently, [44] (see also [12]) has constructed new GJMS operators $P'_k$ using the original [37] ambient metric construction. It would be interesting to see those from the scattering and extension point of view.

Fractional order CR operators on the Heisenberg group were introduced in R. Graham's thesis [34, 35] as obstructions to the resolution of an eigenvalue problem. His construction works also for more general operators

$$\mathcal{L}_\alpha := \frac{1}{2}\sum_{j=1}^{n}[X_j^2 + Y_j^2] + i\alpha T, \quad \text{for } \alpha \in \mathbb{R},$$

which up to multiplication by a constant are the only linear differential operators on $\mathbb{H}^n$ which are second order with respect to the natural dilation structure of the Heisenberg group, that are invariant with respect to left translations and that are invariant under the action of unitary transformations of $z = x + iy \in \mathbb{C}^n$. One could try to understand what type of new problems appear.

## 2. Preliminaries

Most of this section is taken from the well written introduction in [45], but we recall it here for convenience of the reader. Two good survey references on Heisenberg groups and CR manifolds are [5], [20]. We also point out the book on complex hyperbolic geometry [31].

2.1. **CR geometry.** A CR-manifold is a smooth manifold $\mathcal{M}$ equipped with a distinguished subbundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C}T\mathcal{M} = T\mathcal{M}\otimes\mathbb{C}$ such that $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$, i.e. $\mathcal{H}$ is closed with respect to the Lie bracket and hence is formally integrable, and $\mathcal{H}\cap\bar{\mathcal{H}} = \{0\}$, i.e. $\mathcal{H}$ is almost Lagrangian. The subbundle $\mathcal{H}$ is called a CR structure on the manifold $\mathcal{M}$. An abstract CR manifold will be called of hypersurface type if $\dim_{\mathbb{R}}\mathcal{M} = 2n+1$ and $\dim_{\mathbb{C}}\mathcal{H}$.

The maximal complex (or Levi) distribution on the CR manifold $\mathcal{M}$ is the real subbundle $H(\mathcal{M}) \subset T\mathcal{M}$ given by $H(\mathcal{M}) = \mathrm{Re}\{\mathcal{H} \oplus \bar{\mathcal{H}}\}$; it carries a natural complex structure $J_b : H(\mathcal{M}) \to H(\mathcal{M})$ defined by

$$J_b(V + \bar{V}) = i(V - \bar{V}) \qquad \text{for any } V \in \mathcal{H}.$$

If the CR manifold $\mathcal{M}$ is of hypersurface type and oriented, as we will always assume throughout the paper, it is possible to associate to its CR structure $\mathcal{H}$ a one–form $\theta$ which is globally defined on $\mathcal{M}$ such that

$$\mathrm{Ker}(\theta) = H(\mathcal{M}). \tag{2.1}$$

This form is unique up to multiplication by a non-vanishing function in $\mathcal{C}^\infty(\mathcal{M})$. More precisely, any two globally defined one–forms $\theta$, $\hat{\theta}$ on $\mathcal{M}$ such that $\mathrm{Ker}(\hat{\theta}) = \mathrm{Ker}(\theta) = H(\mathcal{M})$ are related by

$$\theta = f\hat{\theta}, \tag{2.2}$$

for some nowhere zero smooth function $f$ on $\mathcal{M}$. Any one–form $\theta$ satisfying (2.1) is called a pseudohermitian structure on $\mathcal{M}$.

The Levi form $L_\theta$ associated to a pseudohermitian structure $\theta$ on $\mathcal{M}$ is given by

$$L_\theta(V_1, \bar{V}_2) = -i(d\theta)(V_1, \bar{V}_2), \qquad \forall\, V_1,\, V_2 \in \mathcal{H}, \tag{2.3}$$

where the form $d\theta$ is extended to complex vector fields by $\mathbb{C}$–linearity. The Levi form changes under a change of the pseudohermitian structure given by (2.2) as follows

$$L_{\hat{\theta}} = f L_\theta.$$

We say that an orientable CR manifold of hypersurface type endowed with a pseudohermitian structure is strictly pseudoconvex if its corresponding Levi form is strictly positive definite, while we say that it is nondegenerate if the corresponding Levi form is nondegenerate (i.e. if $V_1 \in \mathcal{H}$ satisfies $L_\theta(V_1, \bar{V}_2) = 0$ for all $V_2 \in \mathcal{H}$, then $V_1 = 0$).

Given a nondegenerate pseudohermitian structure $\theta$ on an oriented CR manifold $\mathcal{M}$ of hypersurface type, $\psi = \theta \wedge (d\theta)^n$ is a volume form on $\mathcal{M}$. Moreover there is a unique globally defined nowhere zero vector field $T$ tangent to $\mathcal{M}$ such that

$$\theta(T) = 1, \qquad T \rfloor d\theta = 0. \tag{2.4}$$

$T$ is called characteristic direction of $(\mathcal{M}, \theta)$ and one can easily show that

$$T\mathcal{M} = H(\mathcal{M}) \oplus \mathbb{R}T.$$

Moreover, the relations

$$g_\theta(V_1, V_2) = d\theta(V_1, J_b V_2), \qquad g_\theta(V_1, T) = 0, \qquad g_\theta(T, T) = 1 \tag{2.5}$$

for every $V_1, V_2 \in H(\mathcal{M})$, where $T$ is the characteristic vector field associated to $\theta$, define the Webster metric on $\mathcal{M}$. If the Levi form $L_\theta$ is strictly positive definite, then $g_\theta$ is a

Riemannian metric on $\mathcal{M}$, but $g_\theta$ is not a CR invariant. Next, for any smooth function $u$ on $\mathcal{M}$ we define $\nabla_{g_\theta} u$ through the relation

$$g_\theta(\nabla_{g_\theta} u, V) = du(V) \tag{2.6}$$

for any vector field $V$ on $\mathcal{M}$. If $\pi_b$ is the canonical projection of the real tangent bundle $T\mathcal{M}$ on the Levi distribution $H(\mathcal{M})$, the horizontal gradient of a smooth function $u$ on $\mathcal{M}$ is given by

$$\nabla_b u = \pi_b \nabla_{g_\theta} u. \tag{2.7}$$

Moreover the sub–Laplacian operator is defined by setting

$$\Delta_b u = \operatorname{div}(\nabla_b u) \tag{2.8}$$

for any smooth function $u$ on $\mathcal{M}$, where $\operatorname{div}(V)$ is defined by

$$\mathcal{L}_V \psi = \operatorname{div}(V)\psi, \tag{2.9}$$

or equivalently by

$$d(V \rfloor \psi) = \operatorname{div}(V)\psi,$$

for any vector field $V$ on $\mathcal{M}$, where $\psi = \theta \wedge (d\theta)^n$ and $\mathcal{L}$ denotes the Lie derivative.

In coordinate notation, let $\{W_\alpha\}_{\alpha=1}^n$ be a local frame for $\mathcal{H}$ and let $W_{\bar\alpha} = \overline{W}_\alpha$. Then the vector fields $\{W_\alpha, W_{\bar\alpha}, T\}$ form a local frame for $\mathbb{C}T\mathcal{M}$, whose dual coframe $\{\theta^\alpha, \theta^{\bar\alpha}, \theta\}$ is admissible if $\theta^\alpha(T) = 0$ for all $\alpha$. The integrability condition is equivalent to the condition that $d\theta = d\theta^\alpha \mod \{\theta, \theta^\alpha\}$. Thus the Levi form is written as

$$L_\theta = h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta},$$

for a Hermitian matrix–valued function $h_{\alpha\bar\beta}$. Its inverse will be denoted by $h^{\alpha\bar\beta}$. We will use $h_{\alpha\bar\beta}$ and $h^{\alpha\bar\beta}$ to lower and raise indexes in the usual way. The horizontal gradient and the sub–Laplacian on $\mathcal{M}$ may be calculated as

$$\nabla_b u = u^{\bar\alpha} W_\alpha + u^\beta W_{\bar\beta}, \qquad \Delta_b u = u_\alpha{}^\alpha + u_{\bar\beta}{}^{\bar\beta}, \tag{2.10}$$

respectively, where $u^{\bar\alpha} = h^{\gamma\bar\alpha} W_{\bar\gamma} u$, $u^\beta = h^{\beta\bar\gamma} W_\gamma u$.

We recall that any real hypersurface $\mathcal{M}$ in a complex manifold $\mathcal{X}$ of complex dimension $m+1$ can be seen as a CR manifold of hypersurface type of dimension $2n+1$, with the CR structure naturally induced by the complex structure of the ambient manifold

$$\mathcal{H} = T^{1,0}(\mathcal{X}) \cap \mathbb{C}T\mathcal{M},$$

where $T^{1,0}(\mathcal{X})$ denotes the bundle of holomorphic vector fields on $\mathcal{X}$.

2.2. **Complex manifolds with CR boundary.** Now suppose that $\mathcal{X}$ is a compact complex manifold of dimension $m+1$ with boundary $\partial\mathcal{X} = \mathcal{M}$. As we have just mentioned, the boundary manifold inherits a natural CR-structure from the ambient manifold. We recall the following facts from [38, 55], where they give a precise description of the asymptotic behavior near the boundary.

Let $\varphi$ be a negative smooth function on $\mathcal{X}$. We say that $\varphi$ is a defining function for $\mathcal{M}$ if $\varphi < 0$ in the interior of $\mathcal{X}$, $\varphi = 0$ on $\mathcal{M}$, $|d\varphi(p)| \neq 0$ for all $p \in \mathcal{M}$. We further suppose that $\varphi$ has no critical points in a collar neighborhood $\mathcal{U}$ of $\mathcal{M}$ so that the level sets $\mathcal{M}^\varepsilon = \varphi^{-1}(-\varepsilon)$ are smooth manifolds for all $\varepsilon$ sufficiently small.

Associated to the defining function $\varphi$ we define the Kähler form

$$\omega_+ = -\frac{i}{2}\partial\bar{\partial}\log(-\varphi) = \frac{i}{2}\left(\frac{\partial\bar{\partial}\varphi}{-\varphi} + \frac{\partial\varphi\wedge\bar{\partial}\varphi}{\varphi^2}\right). \tag{2.11}$$

The manifold $\mathcal{M}^\varepsilon$ inherits a natural CR structure from the complex structure of the ambient manifold with $\mathcal{H}^\varepsilon = \mathbb{C}T\mathcal{M}^\varepsilon \cap T^{1,0}\mathcal{U}$. For a defining function $\varphi$, we define a one–form

$$\Theta = \frac{i}{2}(\bar{\partial} - \partial)\varphi \tag{2.12}$$

and using the natural embedding map $i_\varepsilon : \mathcal{M}^\varepsilon \to \mathcal{U}$, set $\theta_\varepsilon = i_\varepsilon^*\Theta$. The contact form $\theta_\varepsilon$ is a pseudohermitian structure for $\mathcal{M}^\varepsilon$. Note that

$$d\Theta = i\partial\bar{\partial}\varphi, \tag{2.13}$$

and the Levi form on $\mathcal{M}^\epsilon$ is given by

$$L_{\theta_\varepsilon} = -id\theta_\varepsilon. \tag{2.14}$$

We will assume that $L_{\theta_\varepsilon}$ is positive definite for all $\varepsilon > 0$ sufficiently small, so that $\mathcal{M}^\varepsilon$ is strictly pseudoconvex. Moreover, in order to simplify the notation we will write $\theta$ for $\theta_\varepsilon$, suppressing the $\varepsilon$.

It is known that there exists a unique $(1,0)$-vector field $\xi$ on $\mathcal{U}$ such that

$$\partial\varphi(\xi) = 1 \quad \text{and} \quad \xi\rfloor\partial\bar{\partial}\varphi = r\bar{\partial}\varphi$$

for some $r \in \mathcal{C}^\infty(\mathcal{U})$. The function $r$ is called the *transverse curvature*. We decompose

$$\xi = \frac{1}{2}(N - iT),$$

where $N, T$ are real vector fields on $\mathcal{U}$. Then

$$d\varphi(N) = 2, \quad \theta(N) = 0; \quad \theta(T) = 1, \quad T\rfloor d\theta = 0.$$

Thus $T$ is the characteristic vector field for each $M^\varepsilon$ and $N$ is normal to $M^\varepsilon$.

Let $\{W_\alpha\}_{\alpha=1}^n$ be a frame for $\mathcal{H}^\varepsilon$. Then, setting $W_{\bar{\alpha}} = \overline{W}_\alpha$, the vector fields $\{W_\alpha, W_{\bar{\alpha}}, T\}_{\alpha=1}^n$ form a local frame for $\mathbb{C}T\mathcal{M}^\varepsilon$, while $\{W_\alpha, W_{\bar{\alpha}}\}_{\alpha=1}^{n+1}$ form a local frame for $\mathbb{C}T\mathcal{U}$ by setting $W_m = \xi, W_{\bar{m}} = \bar{\xi}$. If $\{\theta^\alpha\}_{\alpha=1}^n$ is a dual coframe for $\{W_\alpha\}_{\alpha=1}^n$ then, setting $\theta_{\bar{\alpha}} = \overline{\theta}_\alpha$, the one–forms $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$ form a dual coframe for $\mathbb{C}T\mathcal{M}^\varepsilon$, while $\{\theta^\alpha, \theta^{\bar{\alpha}}\}_{\alpha=1}^{n+1}$ is a dual coframe for $\mathbb{C}T\mathcal{U}$ if we denote $\theta^m = \partial\varphi, \theta^{\bar{m}} = \bar{\partial}\varphi$. The Levi form on each $\mathcal{M}^\varepsilon$ is given by

$$L_\theta = h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$$

for a $n \times n$ Hermitian matrix valued function $h_{\alpha\bar{\beta}}$.

Recalling (2.13), we can divide $d\Theta$ into tangential and transverse components as follows:

$$\partial\bar{\partial}\varphi = h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} + r\partial\varphi \wedge \bar{\partial}\varphi,$$

which gives the following expression for the Kähler form from formula (2.11)

$$\omega_+ = \frac{i}{2}\left(\frac{h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}}{-\varphi} + \frac{(1 - r\varphi)}{\varphi^2}\partial\varphi \wedge \bar{\partial}\varphi\right). \tag{2.15}$$

For a complex manifold with a Hermitian metric $g = h - i\omega$, the Kähler form $\omega$ combines the metric and the complex structure by $h(V_1, V_2) = \omega(V_1, JV_2)$. Here the Hermitian-Bergman metric may be written as

$$g^+ = \frac{1}{-\varphi} h_{\alpha\bar{\beta}} \theta^\alpha \otimes \theta^{\bar{\beta}} + \frac{(1 - r\varphi)}{\varphi^2} \partial\varphi \otimes \bar{\partial}\varphi.$$

It is easy to see that

$$g^+(N, N) = 4\frac{(1 - r\varphi)}{\varphi^2}$$

and the outward unit normal to $\mathcal{M}^\varepsilon$ is

$$\nu = \frac{-\varphi}{2\sqrt{1 - r\varphi}} N.$$

If $\omega = \frac{i}{2} g_{k\bar{l}} \omega^k \wedge \omega^{\bar{l}}$ is the expression of a Kähler form on a complex manifold in terms of a coframe $\{\omega^k\}$ for $T^{1,0}$, we define the trace of a $(1,1)$-form by

$$\operatorname{Tr}\left(i\eta_{k\bar{l}} \omega^k \wedge \omega^{\bar{l}}\right) = g^{k\bar{l}} \eta_{k\bar{l}}.$$

The Laplace operator on the Kähler manifold $(X, \omega_+)$ is calculated as

$$\Delta_+ u = \operatorname{Tr}(i\partial\bar{\partial}u);$$

note that we are using a different normalization constant.

We recall the following formula from [38], that decomposes the Laplacian into tangential and normal components relative to the level sets of $\varphi$:

$$\Delta_+ = \frac{-\varphi}{4}\left[\frac{-\varphi}{1 - r\varphi}(N^2 + T^2 + 2rN + 2X_r) + 2\Delta_b + 2nN\right], \tag{2.16}$$

where $\Delta_b$ is given by (2.10) and $r$ is the transverse curvature. For any real function $f$, we have defined the real vector field $X_f$, analogous to the gradient of $f$ in Riemannian geometry, by

$$X_f = f^\alpha W_\alpha + f^{\bar{\beta}} W_{\bar{\beta}}.$$

The proof in [38] also shows that $1 - r\varphi > 0$.

2.3. **Approximate Kähler-Einstein and asymptotically complex hyperbolic manifolds.** Let $\Omega$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^m$. Fefferman [22] showed the existence of a defining function $\varphi$ for $\partial\Omega$ which is an approximate solution of the complex Monge-Ampère equation

$$\begin{cases} J[\varphi] = 1, \\ \varphi|_{\partial\Omega} = 0, \end{cases}$$

where $J[\varphi]$ is the Monge-Ampère operator

$$J[\varphi] = (-1)^m \det\begin{pmatrix} \varphi & \varphi_{\bar{j}} \\ \varphi_i & \varphi_{i\bar{j}} \end{pmatrix}.$$

The Kähler metric $g^+$ defined from the Kähler form (2.15) associated to such an approximate solution $\varphi$ is an approximate Kähler-Einstein metric on $\Omega$, i.e. $g^+$ obeys

$$\operatorname{Ric}(g^+) = -(m+1)\omega_+ + \mathcal{O}(\varphi^{m-1}), \tag{2.17}$$

where Ric is the Ricci form.

Under certain conditions (see [15, 55, 45]), Fefferman's local approximate solution of the Monge-Ampère equation can be globalized to an approximate solution of the Monge-Ampère equation near the boundary of a complex manifold $\mathcal{X}$ with strictly pseudoconvex boundary $\mathcal{M}$. It follows that $\mathcal{X}$ carries an approximate Kähler-Einstein metric in the sense of (2.17). This $\varphi$ is called a globally defined approximate solution of the Monge-Ampère equation in $\mathcal{X}$.

More precisely, such a solution exists if and only if $\mathcal{M}$ admits a pseudohermitian structure $\theta$ which is volume-normalized with respect to some locally defined, closed $(n+1,0)$-form in a neighborhood of any point $p \in \mathcal{M}$. If $\dim(\mathcal{M}) \geq 5$, this condition is equivalent to the existence of a pseudo-Einstein, pseudohermitian structure $\theta$ on $\mathcal{M}$ (see [54]). Recall that a CR manifold is pseudo-Einstein if there is a pseudohermitian structure $\theta$ for which the Webster-Ricci curvature is a multiple of the Levi form. Note that $\dim(\mathcal{M}) = 3$ is special. Finally, in the particular case that $\mathcal{X}$ is a pseudoconvex domain in $\mathbb{C}^m$, this condition is trivially satisfied.

We describe now a more general class of manifolds than the ones considered so far. In particular, we consider the $\Theta$-structures introduced by Epstein-Melrose-Mendoza [21]. We will only give brief description and we refer the interested reader to the detailed description in [40].

Let $\mathcal{X}$ be a non-compact manifold of (real) dimension $2n+2$ with a Riemannian metric $g^+$ that compactifies into a smooth $\bar{\mathcal{X}}$, with boundary $\partial\mathcal{X}$. We assume that the boundary admits a contact form $\theta$ and an almost complex structure $J : \mathrm{Ker}(\theta) \to \mathrm{Ker}(\theta)$ such that $d\theta(\cdot, J\cdot)$ is symmetric and positive definite on $\mathrm{Ker}(\theta)$. The associated characteristic direction $T$ is characterized by

$$\theta(T) = 1, \quad d\theta(T, JZ) = 0 \ \text{ for any } \ Z \in \mathrm{Ker}(\theta),$$

see (2.4), (2.5).

We say that $(\mathcal{X}, g^+)$ is an asymptotically complex hyperbolic manifold if there exists a diffeomorphism $\phi : [0, \epsilon) \times \partial\bar{\mathcal{X}} \to \mathcal{X}$ such that $\phi(\{0\} \times \partial\bar{\mathcal{X}}) = \partial\bar{\mathcal{X}}$ and such that the metric splits as a product of the form

$$\phi^* g^+ = \frac{4d\rho^2 + d\theta(\cdot, J\cdot)}{\rho^2} + \frac{\theta^2}{\rho^4} + \rho Q_\rho =: \frac{4d\rho^2 + h(\rho)}{\rho^2},$$

for some smooth symmetric tensors $Q_\rho$ on $\partial\bar{\mathcal{X}}$, where $\rho$ is a defining function for $\partial\bar{\mathcal{X}}$.

Note that if $\rho$ is any boundary defining function, then $\rho^4 g^+|_{\partial\bar{\mathcal{X}}} = e^{4w}\theta^2$ for some smooth function $w$ on $\partial\bar{\mathcal{X}}$. Thus it is natural to define the pair $([\theta], J)$ a conformal pseudohermitian structure on $\partial\bar{\mathcal{X}}$.

We say that $g$ is even at order $2k$ if $h^{-1}(\rho)$ has only even powers in its Taylor expansion at $\rho = 0$ at order $2k$, where $h^{-1}(\rho)$ is the metric dual to $h(\rho)$.

## 3. Model case: the Heisenberg group

The real $n$-dimensional Heisenberg group $\mathbb{H}^n$ is defined as the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the group law

$$(\hat{x}, \hat{y}, \hat{t}) \circ (x, y, t) = \big(\hat{x} + x, \hat{y} + y, \hat{t} + t + 2(\langle x, \hat{y}\rangle_{\mathbb{R}^n} - \langle \hat{x}, y\rangle_{\mathbb{R}^n})\big)$$

for $(\hat{x},\hat{y},\hat{t})$, $(x,y,t)\in\mathbb{H}^n$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ is the standard inner product in $\mathbb{R}^n$. Alternatively, we can use complex coordinates $z=x+iy$ to denote elements of $\mathbb{R}^n\times\mathbb{R}^n\simeq\mathbb{C}^n$, so that the group action in $\mathbb{H}^n$ can be written as

$$(\hat{z},\hat{t})\circ(z,t)=(\hat{z}+z,\hat{t}+t+2\mathrm{Im}\langle\hat{z},z\rangle_{\mathbb{C}^n})$$

for $(\hat{z},\hat{t})$, $(z,t)\in\mathbb{H}^n$, where and $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ is the standard inner product in $\mathbb{C}^n$.

For any fixed $(\hat{z},\hat{t})\in\mathbb{H}^n$, we will denote $\tau_{(\hat{z},\hat{t})}:\mathbb{H}^n\to\mathbb{H}^n$ the left translation on $\mathbb{H}^n$ by $(\hat{z},\hat{t})$ defined by

$$\tau_{(\hat{z},\hat{t})}(z,t)\;=\;(\hat{z},\hat{t})\circ(z,t),\qquad\forall\,(z,t)\in\mathbb{H}^n.$$

For any $\lambda>0$ the dilation $\delta_\lambda:\mathbb{H}^n\to\mathbb{H}^n$ is

$$\delta_\lambda(z,t)\;=\;(\lambda z,\lambda^2t)\;=\;(\lambda x,\lambda y,\lambda^2 t),\qquad\forall\,(z,t)=(x,y,t)\in\mathbb{H}^n.$$

Notice that

$$\delta_\lambda\big((\hat{z},\hat{t})\circ(z,t)\big)=\delta_\lambda\big(\hat{z},\hat{t}\big)\circ\delta_\lambda\big(z,t\big)$$

for every $\lambda>0$ and $(\hat{z},\hat{t})$, $(z,t)\in\mathbb{H}^n$.

As we will see, the $n$-dimensional Heisenberg group can be regarded as a smooth sub–Riemannian manifold. An orthonormal frame on the manifold is given by the Lie vector fields

$$X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\quad Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t}\quad\text{for } j=1,\dots,n,\quad T_0=\frac{\partial}{\partial t},\tag{3.1}$$

which form a base of the Lie algebra of vector fields on the Heisenberg group which are left invariant with respect to the group action $\circ$. Notice that

$$[X_j,Y_k]=-4T_0\delta_{jk},\qquad[X_j,X_k]=[Y_j,Y_k]=0\qquad\text{for } j,k=1,\dots,n,$$

where $\delta_{jk}$ is the Kronecker symbol. We also set

$$Z_j\;=\;\frac{\partial}{\partial z_j}+i\bar{z}_j\frac{\partial}{\partial t},\quad\bar{Z}_j\;=\;\frac{\partial}{\partial\bar{z}_j}-iz_j\frac{\partial}{\partial t},\tag{3.2}$$

where

$$\frac{\partial}{\partial z_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}-i\frac{\partial}{\partial y_j}\right),\quad\frac{\partial}{\partial\bar{z}_j}=\frac{1}{2}\left(\frac{\partial}{\partial x_j}+i\frac{\partial}{\partial y_j}\right),\tag{3.3}$$

for $j=1,\dots,n$.

3.1. **The Heisenberg group as a CR manifold.** The CR structure on the Heisenberg group $\mathbb{H}^n$ is given by the $n$-dimensional complex bundle

$$\mathcal{H}(\mathbb{H}^n)=\mathrm{span}_{\mathbb{C}}\langle Z_1,\dots,Z_n\rangle\tag{3.4}$$

with $Z_1,\dots,Z_n$ given by (3.2). We immediately observe that the associated maximal complex distribution $H(\mathbb{H}^n)=\mathrm{Re}\{\mathcal{H}(\mathbb{H}^n)\oplus\overline{\mathcal{H}}(\mathbb{H}^n)\}$ is simply

$$H(\mathbb{H}^n)=\mathrm{span}_{\mathbb{R}}\langle X_1,\dots,X_n,Y_1,\dots,Y_n\rangle,$$

and that it carries the complex structure $J_b:H(\mathbb{H}^n)\to H(\mathbb{H}^n)$ defined by

$$J_bX_j=Y_j,\qquad J_bY_j=-X_j\qquad\text{for } j=1,\dots,n.$$

The associated one–form $\theta_0$ satisfying (2.1), which is globally defined on $\mathbb{H}^n$, is precisely

$$\theta_0 \ = \ dt + i\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j) \ = \ dt + 2\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)\,. \tag{3.5}$$

From here one immediately calculates

$$d\theta_0 \ = \ 2i\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j \ = \ 4\sum_{j=1}^{n} dx_j \wedge dy_j.$$

Then, the characteristic direction defined through (2.4) is simply the vector field $T_0$ previously defined. The real tangent bundle of $\mathbb{H}^n$ satisfies

$$T\mathbb{H}^n = H(\mathbb{H}^n) \oplus \mathbb{R}T_0.$$

The Levi form $L_{\theta_0}$ associated to the pseudohermitian structure $\theta_0$ on $\mathbb{H}^n$ constructed from (2.3) is given by

$$L_{\theta_0}(Z_j, \bar{Z}_k) = 2\delta_{jk} \qquad \text{for } j,k = 1,\dots,n,$$

which is a positive definite matrix. This tells us that $(\mathbb{H}^n, \theta_0)$ is strictly pseudoconvex. Moreover, one can define the Webster metric $g_{\theta_0}$ on $\mathbb{H}^n$ by the relations (2.5); in particular,

$$g_{\theta_0}(X_j, X_k) = g_{\theta_0}(Y_j, Y_k) = 4\delta_{jk}, \qquad g_{\theta_0}(T_0, T_0) = 1,$$
$$g_{\theta_0}(X_j, Y_k) = g_{\theta_0}(X_j, T_0) = g_{\theta_0}(Y_j, T_0) = 0.$$

A volume form on $\mathbb{H}^n$ is given by

$$\begin{aligned}\psi_0 \ = \ \theta_0 \wedge (d\theta_0)^n \ &= \ n!(2i)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n \wedge dt \\ &= \ n! 2^{2n} dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n \wedge dt.\end{aligned}$$

Now, since $X_1, \ldots X_n, Y_1, \ldots, Y_n, T_0$ is a basis of $T\mathbb{H}^n = H(\mathbb{H}^n) \otimes \mathbb{R}T_0$, we have from (2.6) that

$$\nabla_{g_{\theta_0}} u = T_0(u)T_0 + \frac{1}{4}\sum_{j=1}^{n} X_j(u)X_j + Y_j(u)Y_j$$

for any smooth function $u$ on $\mathbb{H}^n$. We observe that

$$du = T_0 u\, \theta_0 + \sum_{j=1}^{n} X_j u\, dx_j + Y_j u\, dy_j,$$

so

$$d_b u = \sum_{j=1}^{n} X_j u\, dx_j + Y_j u\, dy_j.$$

The horizontal gradient is calculated from (2.7)

$$\nabla_b u := \frac{1}{4}\sum_{j=1}^{n} X_j(u)X_j + Y_j(u)Y_j,$$

and the sub–Laplacian associated to $\theta_0$ from (2.8)

$$\tilde{\Delta}_b u := \frac{1}{4}\sum_{j=1}^{n}\left[X_j^2 + Y_j^2\right] u = \frac{1}{2}\sum_{j=1}^{n}\left[Z_j \bar{Z}_j + \bar{Z}_j Z_j\right] u;$$

the differential operator is linear, second order, degenerate elliptic, and it is hypoelliptic being the sum of squares of (smooth) vector fields satisfying the Hörmander condition [46].

Assume that $u, v$ are smooth functions on $\mathbb{H}^n$ and at least one of them is compactly supported, then we see that

$$-\int_{\mathbb{H}^n} v\tilde{\Delta}_b u\,\psi \;=\; \int_{\mathbb{H}^n} g_{\theta_0}(\nabla_b v, \nabla_b u)\,\psi.$$

The sub–Laplacian on the Heisenberg group or on a orientable, strictly pseudoconvex CR manifold of hypersurface type is the Laplace-Beltrami operator on the manifold, which plays an important role in harmonic analysis and partial differential equations.

For the rest of the paper, we will be working with the new contact form $\theta = \theta_0/2$ on the Heisenberg group, since $\theta$ is the CR structure inherited from the complex hyperbolic space (see (3.9)). Therefore, our sub–Laplacian on the Heisenberg group, by definition will be

$$\Delta_b u := \frac{1}{2}\sum_{j=1}^{n}\left[X_j^2 + Y_j^2\right] u \tag{3.6}$$

and the characteristic direction will be $T = 2T_0$. This explains the choice of constants in (1.2).

3.2. **The Heisenberg group as the boundary of the Siegel domain.** The Heisenberg group may be identified with the boundary of a domain in $\mathbb{C}^{n+1}$. Indeed, let $\Omega_{n+1}$ be the Siegel domain, defined by

$$\Omega_{n+1} := \{\zeta = (z_1, \ldots, z_n, z_{n+1}) = (z, z_{n+1}) \in \mathbb{C}^n \times \mathbb{C} \,|\, q(\zeta) > 0\}\,.$$

where

$$q(\zeta) = \operatorname{Im} z_{n+1} - \sum_{j=1}^{n} |z_j|^2.$$

Its boundary $\partial\Omega_{n+1} = \left\{\zeta \in \mathbb{C}^{n+1} \,|\, q(\zeta) = 0\right\}$ is an oriented CR manifold of hypersurface type with the CR structure induced by $\mathbb{C}^{n+1}$. Now define $G : \mathbb{H}^n \to \partial\Omega_{n+1}$,

$$G(z,t) = (z, t + i|z|^2),\ (z,t) \in \mathbb{H}^n, \tag{3.7}$$

$$G^{-1}(\zeta) = (z_1, \ldots, z_n, \operatorname{Re} z_{n+1}),\ \zeta \in \partial\Omega_{n+1}. \tag{3.8}$$

One may check that $G$ is a CR isomorphism between $\mathbb{H}^n$ and $\partial\Omega_{n+1}$, i.e. it preserves the CR structures.

The boundary manifold inherits a natural CR structure from the complex structure of the ambient manifold, using $\varphi = -q$ as a defining function. The pseudohermitian structure on $\mathbb{H}^n$ is then given (via pullback) by the contact form

$$\theta = \frac{i}{2}(\bar{\partial} - \partial)\varphi = \frac{1}{4}(dz_m + d\bar{z}_m) + \frac{i}{2}\sum_{\alpha=1}^{n} z_\alpha d\bar{z}_\alpha - \bar{z}_\alpha dz_\alpha = \theta_0/2 \tag{3.9}$$

where $\theta_0$ was defined in (3.5).

We now set up the orthonormal frame we will be using for the rest of the paper. The subindexes $\alpha, \beta$ will run from 1 to $n$. As in formula (3.2) let

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\bar{z}_\alpha\frac{\partial}{\partial t}, \quad Z_{\bar{\alpha}} = \frac{\partial}{\partial \bar{z}^\alpha} - iz_\alpha\frac{\partial}{\partial t} \qquad \alpha = 1, \ldots, n.$$

In order to complete a basis for $T\Omega_{n+1}$, define the two real vector fields

$$T = 2\frac{\partial}{\partial t} \quad \text{and} \quad N = -2\frac{\partial}{\partial q}.$$

Note that $T$ differs from the $T_0$ from the previous section by a factor of 2 in order to match with the contact form (3.9). Define also

$$\xi_m = \frac{1}{2}(N - iT) \quad \text{and} \quad \bar{\xi}_m = \frac{1}{2}(N + iT).$$

Then a frame in $\Omega_{n+1}$ is given by

$$\{Z_\alpha, Z_{\bar{\alpha}}, \xi_m, \bar{\xi}_m\}. \tag{3.10}$$

The dual coframe in $\mathbb{C}T\mathbb{H}^n$ of $\{Z_\alpha, Z_{\bar{\alpha}}, T\}$ is given by $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta\}$, while the dual coframe of the basis (3.10) in $\Omega_{n+1}$ is simply $\{\theta^\alpha, \theta^{\bar{\alpha}}, \theta, dq\}$, where $dq = \partial q + \bar{\partial} q$. In particular, $T \rfloor \theta = 1$ and $Z_\alpha \rfloor \theta^\alpha = 1$.

Finally, one may calculate the corresponding Levi form from (2.14)

$$L_\theta = \partial\bar{\partial}\varphi(z) = \sum_{\alpha=1}^{n} dz_\alpha \wedge d\bar{z}_\alpha, \tag{3.11}$$

which may be rewritten as

$$L_\theta = h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}} \quad \text{for} \quad h_{\alpha\bar{\beta}} = \frac{\partial^2\varphi}{\partial z_\alpha \partial z_{\bar{\beta}}} = \delta_{\alpha\bar{\beta}}.$$

Now one may calculate the sub–Laplacian with respect to the contact form $\theta$. Indeed,

$$\Delta_b u = \sum_{\alpha,\beta=1}^{n} h^{\alpha\bar{\beta}} \left[u_{\alpha\bar{\beta}} + u_{\bar{\beta}\alpha}\right] = \frac{1}{2}\sum_{j=1}^{n} \left[X_j^2 + Y_j^2\right] u,$$

which differs from the sub–Laplacian associated to $\theta_0$ by a factor of 2.

On the other hand, let us look at the complex structure of $\Omega_{n+1}$. The functions $t = \operatorname{Re} z_m \in \mathbb{R}$, $z_1, \ldots, z_n \in \mathbb{C}^n$ and $\rho = (2q(z))^{1/2} \in (0, \infty)$ give coordinates in the Siegel domain $\Omega_{n+1} \simeq \mathbb{H}^n \times (0, \infty)$. For the defining function $-\varphi$, one can construct a Kähler form in $\Omega_{n+1}$ as

$$\omega_+ = -\frac{i}{2}\partial\bar{\partial}\log(\varphi) \; = \; \frac{i}{2}\left(\frac{\partial\bar{\partial}\varphi}{-\varphi} + \frac{\partial\varphi \wedge \bar{\partial}\varphi}{\varphi^2}\right). \tag{3.12}$$

The first term $\partial\bar{\partial}\varphi$ is calculated in (3.11), while for the second we observe that

$$\partial\varphi \wedge \bar{\partial}\varphi = \frac{1}{4}\left[(\partial\varphi + \bar{\partial}\varphi)^2 - (\partial\varphi - \bar{\partial}\varphi)^2\right] = \frac{1}{4}\left[d\varphi^2 + 4\theta^2\right],$$

where we have used (2.12) in the last step. Then the Kähler form is simply

$$\omega_+ = \frac{i}{2}\left(\frac{h_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}}{-\varphi} + \frac{\frac{1}{4}d\varphi^2 + \theta^2}{\varphi^2}\right), \tag{3.13}$$

After the change $q = \rho^2/2$, the Hermitian-Bergman metric is given by

$$g^+ = \frac{1}{2}\left(\frac{d\rho^2}{\rho^2} + \frac{2\delta_{\alpha\bar{\beta}}\theta^\alpha \otimes \theta^{\bar{\beta}}}{\rho^2} + \frac{4\theta^2}{\rho^4}\right). \tag{3.14}$$

Then $(\Omega_{n+1}, \omega_+)$ is a Kähler manifold with constant holomorphic curvature.

3.3. **Cayley transform.** The Heisenberg group is also CR isomorphic to the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ minus a point. The CR equivalence between the two manifolds is given by the map $\Psi_c : \mathbb{S}^{2n+1} \setminus \{(0,\dots,0,-1)\} \to \mathbb{H}^n$, defined by

$$\Psi_c(\zeta) = \left( \frac{\zeta_1}{1+\zeta_{n+1}}, \dots, \frac{\zeta_n}{1+\zeta_{n+1}}, \mathrm{Re}\Big( i\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}} \Big) \right),$$

for $\zeta \in \mathbb{S}^{2n+1} \setminus \{(0,\dots,0,-1)\} \subset \mathbb{C}^{n+1}$. Notice that

$$\Psi_c^{-1}(z,t) = \left( \frac{2iz}{t+i(1+|z|^2)}, \frac{-t+i(1-|z|^2)}{t+i(1+|z|^2)} \right), \quad (z,t) \in \mathbb{H}^n.$$

The Jacobian determinant of this transformation is

$$J_c(z,t) = \frac{2^{2n+1}}{((1+|z|^2)^2+t^2)^{n+1}} \tag{3.15}$$

so that

$$\int_{\mathbb{S}^{2n+1}} f\, dS = \int_{\mathbb{H}^n} (f \circ \Psi_c^{-1}) |J_c| \, dzdt.$$

The CR unitary sphere $\mathbb{S}^{2n+1}$ is the boundary of the ball model for the complex hyperbolic space of complex dimension $n+1$, which is the unit ball $B^{n+1} = \{\zeta \in \mathbb{C}^{n+1} : |\zeta| < 1\}$ equipped with the Kähler metric $g_0 = -4\partial\bar\partial \log r$, where $r = 1-|\zeta|^2$. The holomorphic curvature is constant (and negative) and this metric is called the Bergman metric.

## 4. Scattering theory and the conformal fractional sub–Laplacian

Now we go back to the general setting described in section 2.3. Let $\mathcal{X}$ be a $m$-dimensional complex manifold with compact, strictly pseudoconvex boundary $\partial\mathcal{X} = \mathcal{M}$. Let $g^+$ be a Kähler metric on $\mathcal{X}$ such that there exists a globally defined approximate solution $\varphi$ of the Monge-Ampère equation that makes $g^+$ an approximate Kähler-Einstein metric in the sense of (2.17) and $\mathcal{X} = \{\varphi < 0\}$. In particular, the metric $g^+$ belongs to the class of $\Theta$-metrics considered by [21].

The spectrum of the Laplacian $-\Delta_+$ in the metric $g^+$ consists of an absolutely continuous part

$$\sigma_{ac}(-\Delta_+) = \left[ \tfrac{m^2}{4}, \infty \right),$$

and the pure point spectrum satisfying

$$\sigma_{pp}(-\Delta_+) \subset \left( 0, \tfrac{m^2}{4} \right),$$

and moreover it consists of a finite set of $L^2$-eigenvalues. The main result in [21] is the study of the modified resolvent

$$R(s) = (-\Delta_+ - s(m-s))^{-1}$$

considered in $L^2(\mathcal{X})$, which allows to define the Poisson map. More precisely, let

$$\Sigma := \{ s \in \mathbb{C} : \Re(s) > m/2,\ s(m-s) \in \sigma_{pp}(-\Delta_+) \};$$

the resolvent operator is meromorphic for $\Re(s) > \frac{m}{2} - \frac{1}{2}$, having at most finitely many, finite-rank poles at $s \in \Sigma$. Moreover, for $s \notin \Sigma$ and $\Re(s) > \frac{m}{2} - \frac{1}{2}$,

$$R(s) : \dot{\mathcal{C}}^\infty(\mathcal{X}) \to q^s \mathcal{C}^\infty(\mathcal{X}),$$

where $\dot{\mathcal{C}}^\infty(\mathcal{X})$ is the set of $\mathcal{C}^\infty$ functions on $\mathcal{X}$ vanishing up to infinite order at $\partial\mathcal{X}$.

The Poisson operator is constructed as follows (see [45]). Let $s \in \mathbb{C}$ such that $\Re(s) \geq \frac{m}{2}$, $s \notin \mathbb{Z}$ and $2s - m \notin \mathbb{Z}$. Let $q = -\varphi$. Then, given $f \in \mathcal{C}^\infty(\partial\mathcal{X})$, there exists a unique solution $u_s$ of the eigenvalue problem

$$-\Delta_+ u_s - s(m-s)u_s = 0,$$

with the expansion

$$\begin{cases} u_s = q^{(m-s)}F + q^s G, \quad \text{for some} \quad F, G \in \mathcal{C}^\infty(\bar{\mathcal{X}}), \\ F|_{\mathcal{M}} = f. \end{cases}$$

Then the Poisson map is defined as $\mathcal{P}_s : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathring{\mathcal{X}})$ by $f \mapsto u_s$, and the scattering operator

$$S(s) : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$$

by

$$S(s)f := G|_{\mathcal{M}}.$$

Note that $S(s)$ is a meromorphic family of pseudodifferential operators in the $\Theta$-calculus of [21] of order $2(2s-m)$, and self-adjoint when $s$ is real.

The scattering operator has infinite-rank poles when $\Re(s) > \frac{m}{2}$ and $2s - m \in \mathbb{Z}$ due to the crossing of indicial roots for the normal operator. At those exceptional points $s_k = \frac{m}{2} + \frac{k}{2}$, $k \in \mathbb{N}$, $s_k \notin \Sigma$, the CR operators may be recovered by calculating the corresponding residue

$$\operatorname*{Res}_{s=s_k} S(s) = p_k,$$

where $p_k$ is a CR-covariant differential operator of order $2k$, with principal symbol

$$p_k = \frac{(-1)^k}{2^k k!(k-1)!} \prod_{l=1}^{k} (-\Delta_b + i(k+1-2l)T) \quad + \quad \text{l.o.t.}$$

These correspond precisely to the GJMS operators [37] as constructed by Gover-Graham in [33].

For $\gamma \in (0,m) \backslash \mathbb{N}$, set $s = \frac{m+\gamma}{2}$. We define the CR fractional sub–Laplacian on $(\mathcal{M}, \theta)$ by

$$P^\theta_\gamma f = c_\gamma S(s) f,$$

for a constant

$$c_\gamma = 2^\gamma \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}.$$

$P^\theta_\gamma$ is a pseudodifferential operator of order $2\gamma$ with principal symbol given by (1.12).

The main property of the operator is its CR covariance. Indeed, if $\hat{\varphi} = v^{\frac{2}{m-\gamma}} \varphi$ is another defining function for $\mathcal{M}$ and $v|_{\mathcal{M}} = w$, which gives a relation between the contact forms as $\hat{\theta} = w^{\frac{2}{m-\gamma}} \theta$, then the corresponding operator is given by

$$P^{\hat{\theta}}_\gamma (\cdot) = w^{-\frac{m+\gamma}{m-\gamma}} P^\theta_\gamma (w\, \cdot).$$

In particular, for $\gamma = 1$, we obtain the CR Yamabe operator of Jerison-Lee [48]

$$P^\theta_1 = -\Delta_b + \frac{n}{2(n+1)} R_\theta,$$

where $\Delta_b$ is the sub–Laplacian on $(\mathcal{M}, \theta)$ and $R_\theta$ is the associated Webster curvature.

The Heisenberg group $(\mathbb{H}, \theta)$, for $\theta$ given by (3.9), is the model case for $\mathcal{M}$, when it is understood as the boundary at infinity of the complex Poincaré ball through the Cayley transform. In particular, $P_\gamma^\theta$ agrees with the intertwining operators on the CR sphere calculated in [9, 36, 8]. Moreover, we can write explicitly

$$P_1^\theta = -\Delta_b, \quad P_2^\theta = (\Delta_b)^2 + T^2, \quad P_k^\theta = \prod_{l=1}^{k}(-\Delta_b + i(k+1-2l)T).$$

In the case that $\mathcal{X}$ is an asymptotically hyperbolic manifold as described in [40], not necessarily coming from an approximate solution of the Monge-Ampère equation, one can still construct the scattering operator and the CR fractional sub–Laplacian, except possibly for additional poles at the values $\gamma = \frac{k}{2}$, $k \in \mathbb{N}$. In [40], a careful study of those additional values is obtained with the assumption that the metric is even up to a high enough order.

One may define also the Branson's fractional CR curvature as

$$Q_\gamma^\theta = P_\gamma^\theta(1), \quad \gamma \in (0, m),$$

with a constant in front that we assume to be 1.

In the critical case $\gamma = m$, the operator $P_m^\theta$ was first introduced in [36] (see also [34], [35]) as a compatibility operator for the Dirichlet problem for the Bergman Laplacian for the ball in $\mathbb{C}^m$. This construction was later generalized to the boundary of a strictly pseudoconvex domain in $\mathbb{C}^m$ in [38]. The CR $Q = Q_m$ curvature may be calculated as

$$c_m Q = \lim_{s \to m} S(s)1. \tag{4.1}$$

It is a conformal invariant in the sense that, for a change of contact form $\hat{\theta} = e^{2w}\theta$,

$$e^{2mw} Q^{\hat{\theta}} = Q^\theta + P_m w,$$

as it was shown in [23]. The $Q$-curvature also appears in the calculation of renormalized volume (see [61]).

## 5. The extension problem for the Heisenberg group

In this section we give the proof of Theorem 1.1. Here $(\mathcal{X}, g^+)$ is the Siegel domain $\Omega_{n+1}$ with the complex hyperbolic metric, and with boundary $\mathcal{M} = \{\varphi = 0\}$ the Heisenberg group. Throughout, we assume that $s > m/2$ and we parametrize $s = (m+\gamma)/2$ with $\gamma > 0$. (At the end we will also assume that $\gamma < 1$, but this is not needed for most of the discussion.)

First recall the formula for the calculation of the Laplacian $\Delta_+$ from (2.16). Denote $q = -\varphi$, $N = -2\partial_q$ and $T = 2\partial_t$. Then we may write

$$\Delta_+ = q\left[q(\partial_{qq} + \partial_{tt}) + \tfrac{1}{2}\Delta_b - n\partial_q\right],$$

with $\Delta_b$ the sub–Laplacian (1.2).

Now consider the scattering equation

$$-\Delta_+ u - s(m-s)u = 0. \tag{5.1}$$

We are looking for solutions which are small (in a certain sense) as $q \to \infty$ and which behave for $q \to 0$ like

$$u = q^{m-s}F + q^s G.$$

We are interested in the map $F|_{q=0} \mapsto G|_{q=0}$. To extract the leading term we substitute $u = q^{m-s}U$ into (5.1) and obtain the new equation

$$\left(q\partial_{qq} + (1-\gamma)\partial_q + q\partial_{tt} + \frac{1}{2}\Delta_b\right)U = 0\,. \tag{5.2}$$

One additional change of variables $q = \rho^2/2$ transforms (5.2) into the more familiar extension problem (1.10). This is the analogue to the Caffarelli-Silvestre extension [10] on the Euclidean space, with the additional term in the $t$-direction that appears in the Heisenberg group case.

In addition, one may recover the scattering operator as in [14] by

$$P_\gamma^\theta f = c_\gamma S\left(\frac{m+\gamma}{2}\right) f = \frac{c_\gamma}{\gamma}\lim_{q\to 0} q^{1-\gamma}\partial_q U = \frac{c_\gamma}{\gamma 2^{1-\gamma}}\lim_{\rho\to 0}\rho^a\partial_\rho U.$$

5.1. **The group Fourier transform.** Let us recall the Fourier transform on the Heisenberg group $\mathbb{H}^n$, defined by using the irreducible representation of $\mathbb{H}^n$ from the Stone–Von Neumann theorem (see [3, 30, 63] for the necessary background). For a holomorphic function $\Psi$ on $\mathbb{C}^n$ let

$$\pi_{z,t}^\lambda \Psi(\xi) = \Psi(\xi - \bar{z})e^{i\lambda t + 2\lambda(\xi\cdot z - |z|^2/2)},\ \lambda > 0,$$
$$\pi_{z,t}^\lambda \Psi(\xi) = \Psi(\xi + z)e^{i\lambda t + 2\lambda(\xi\cdot \bar{z} - |z|^2/2)},\ \lambda < 0,$$

where $\xi \cdot z = \sum_{j=1}^n \xi_j z_j$ for $\xi, z \in \mathbb{C}^n$. It is easy to check that the unitary family $\{\pi_{z,t}^\lambda\}, \lambda \in \mathbb{R}\setminus\{0\}$ satisfies $\pi_{z,t}\pi_{\hat{z},\hat{t}} = \pi_{(z,t)\circ(\hat{z},\hat{t})}$. Moreover, it gives all irreducible representations of $\mathbb{H}^n$ (except those trivial on the center).

Consider also the Bargmann spaces

$$\mathcal{G}_\lambda := \{\Psi \text{ holomorphic in } \mathbb{C}^n, \|\Psi\|_{\mathcal{G}_\lambda} < \infty\}$$

where

$$\|\Psi\|_{\mathcal{G}_\lambda}^2 := \left(\frac{2|\lambda|}{\pi}\right)^n \int_{\mathbb{C}^n} |\Psi(\xi)|^2 e^{-2|\lambda||\xi|^2}\,d\xi.$$

The space $\mathcal{G}_\lambda$ is a Hilbert space with orthogonal basis

$$\Psi_{\alpha,\lambda}(\xi) = \frac{(\sqrt{2|\lambda|}\xi)^\alpha}{\sqrt{\alpha!}} \quad \text{for } \alpha \in \mathbb{N}_0^n.$$

Here we adopt the usual multi-index conventions $\xi^\alpha := \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}$, $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$ for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, and $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$. For $\varphi = \sum_{\alpha\in\mathbb{N}_0^n} b_\alpha \xi^\alpha \in \mathcal{G}_\lambda$ we have that

$$\|\varphi\|_{\mathcal{G}_\lambda} := \sum_{\alpha\in\mathbb{N}_0^n} \alpha!|2\lambda|^{-|\alpha|}|b_\alpha|^2.$$

It is clear that the derivative $\partial_{\xi_j}\varphi$ and the multiplication $\xi_j\varphi$ still belong to $\mathcal{G}_\lambda$.

The Fourier transform of a function $h(z,t)$ in $L^1(\mathbb{H}^n)$ is defined by

$$\mathcal{F}(h)(\lambda) = \int_{\mathbb{H}^n} h(z,t)\pi_{z,t}^\lambda\,dzdt. \tag{5.3}$$

Note that $\mathcal{F}(h)(\lambda)$ takes its values in the space of bounded operators on $\mathcal{G}_\lambda$, for every $\lambda$.

We recall two important properties of the Fourier transform:

$$\tfrac{1}{2}\mathcal{F}(\Delta_b h)(\lambda)\Psi_{\alpha,\lambda} = -(2|\alpha| + n)|\lambda|\mathcal{F}(h)(\lambda)\Psi_{\alpha,\lambda} \tag{5.4}$$

and

$$\mathcal{F}(\partial_t h)(\lambda)\Psi = -i\lambda\mathcal{F}(h)(\lambda)\Psi. \tag{5.5}$$

The Plancherel formula is

$$\|h\|^2_{L^2(\mathbb{H}^n)} = \frac{2^{n-1}}{\pi^{n+1}} \sum_{\alpha\in\mathbb{N}_0^n} \int_{\mathbb{R}} \|\mathcal{F}(h)(\lambda)\Psi_{\alpha,\lambda}\|^2_{\mathcal{G}_\lambda} |\lambda|^n \, d\lambda, \tag{5.6}$$

where Heisenberg group is endowed with a smooth left invariant measure, the Haar measure, which in the coordinate system $(x, y, t)$ is simply the Lebesgue measure $dx\, dy\, dt$. And the inversion formula,

$$h(z,t) = \frac{2^{n-1}}{\pi^{n+1}} \int_{\mathbb{R}} \mathrm{tr}\, (\pi^\lambda_{z,t})^* \mathcal{F}(h)(\lambda)\, |\lambda|^n d\lambda,$$

where $(\pi^\lambda_{z,t})^*$ is the adjoint operator of $\pi^\lambda_{z,t}$.

**Remark 5.1.** *In view of* (5.4) *one may then define the (pure) fractional powers of the sub–Laplacian by*

$$\mathcal{F}((-\Delta_b)^\gamma h)(\lambda)\Psi_{\alpha,\lambda} := (2(2|\alpha| + n)|\lambda|)^\gamma \mathcal{F}(h)(\lambda)\Psi_{\alpha,\lambda}. \tag{5.7}$$

*However, it does not agree with the operator $P^\theta_\gamma$ we are interested in since it does not have the CR covariance property* (1.9)*.*

For simplicity, in the following we will denote

$$\hat{h}_\alpha(\lambda) := \mathcal{F}(h)(\lambda)\Psi_{\alpha,\lambda} \quad \text{and} \quad k := k_\alpha = 2|\alpha| + n,$$

and the dependence on each level $\alpha \in \mathbb{N}_0^n$ will be sometimes made implicit.

5.2. **Solution of the ODE.** We now perform a Fourier transform of (5.2), which, in view of (5.5) and (5.4), amounts to replacing $\partial_{tt}$ by $-\lambda^2$ and $\frac{1}{2}\Delta_b$ by $-|\lambda|k$. The new equation, written in the basis $\Psi_{\alpha,\lambda}$, reduces to

$$\left(q\partial_{qq} + (1-\gamma)\partial_q - \lambda^2 q - |\lambda|k\right)\phi = 0\,.$$

Here $\lambda$ and $k$ are parameters and $\phi$ is a function of the single variable $q$. We are looking for a solution of this equation which satisfies $\phi(q) \to 0$ as $q \to \infty$ and $\phi(0) = 1$. We make the ansatz $\phi(q) = e^{-|\lambda|q} g(2|\lambda|q)$ and find that the equation for $\phi$ is equivalent to the following equation for $g(x)$,

$$xg'' + (1-\gamma-x)g' - \frac{1-\gamma+k}{2} g = 0\,.$$

This is already Kummer's equation, but it is convenient to transform it into another equation of the same type. To do so, we set $g(x) = x^\gamma h(x)$, which leads to

$$xh'' + (1+\gamma-x)h' - \frac{1+\gamma+k}{2} h = 0\,.$$

The boundary conditions become

$$\lim_{x\to 0} x^\gamma h(x) = 1 \quad \text{and} \quad \lim_{x\to\infty} e^{-x/2} x^\gamma h(x) = 0.$$

To proceed, we recall the following facts about special functions.

**Lemma 5.2** (Kummer's equation)**.** *Let $a \geq 0$ and $b > 1$. The equation $xw'' + (b-x)w' - aw = 0$ on $(0,\infty)$ has two linearly independent solutions $M(a,b,\cdot)$ and $V(a,b,\cdot)$. They satisfy, as $x \to \infty$,*

$$M(a,b,x) = \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b} \left(1 + \mathcal{O}(x^{-1})\right)$$

*and*

$$V(a,b,x) = x^{-a} \left(1 + \mathcal{O}(x^{-1})\right)$$

*and, as $x \to 0$,*

$$M(a,b,x) = 1 + \mathcal{O}(x) \qquad \textit{and} \qquad x^{-1+b} V(a,b,x) = \frac{\Gamma(b-1)}{\Gamma(a)} + o(1)\,.$$

*The function $M(a,b,\cdot)$ is real analytic. Moreover, for all $x > 0$*

$$V(a,b,x) = \frac{\pi}{\sin \pi b} \left( \frac{M(a,b,x)}{\Gamma(1+a-b)\Gamma(b)} - x^{1-b} \frac{M(1+a-b, 2-b, x)}{\Gamma(a)\Gamma(2-b)} \right).$$

These facts are contained in [1, Chp. 13]. The assumptions $a \geq 0$ and $b > 1$ can be significantly relaxed, but this is not important for us. Instead of referring to the known results of this lemma, one can directly deduce all the properties that we need in the following from the integral representation

$$V(a,b,x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tx} t^{a-1} (1+t)^{b-a-1}\, dt\,.$$

For this formula, see again [1, Chp. 13].

Let us return to our scattering problem (5.2). Lemma 5.2 implies that

$$h(x) = \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma(\gamma)}\, V\left(\tfrac{1+\gamma+k}{2}, 1+\gamma, x\right).$$

Because of the representation of the $V$-functions in terms of the $M$-functions we learn that for $x \to 0$

$$\begin{aligned} h(x) &= \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma(\gamma)} \frac{\pi}{\sin \pi(1+\gamma)} \left( \frac{M(\frac{1+\gamma+k}{2}, 1+\gamma, x)}{\Gamma\left(\frac{1-\gamma+k}{2}\right)\Gamma(1+\gamma)} - x^{-\gamma} \frac{M(\frac{1-\gamma+k}{2}, 1-\gamma, x)}{\Gamma(\frac{1+\gamma+k}{2})\Gamma(1-\gamma)} \right) \\ &= x^{-\gamma} - \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma\left(\frac{1-\gamma+k}{2}\right)} + \mathcal{O}(x^{1-\gamma})\,. \end{aligned}$$

Undoing the substitutions we made we find that

$$\begin{aligned} \phi(q) = e^{-|\lambda|q} (2|\lambda|q)^\gamma h(2|\lambda|q) &= e^{-|\lambda|q} \left( 1 - \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma\left(\frac{1-\gamma+k}{2}\right)} (2|\lambda|q)^\gamma + \mathcal{O}(q) \right) \\ &= 1 - \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma\left(\frac{1-\gamma+k}{2}\right)} (2|\lambda|q)^\gamma + \mathcal{O}(q) \quad \text{as} \quad q \to 0. \end{aligned} \tag{5.8}$$

This proves that for $0 < \gamma < 1$, the scattering operator which maps $F|_{q=0} \mapsto G|_{q=0}$ is diagonal with respect to the Fourier transform and its symbol is

$$-\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k}{2}\right)}{\Gamma\left(\frac{1-\gamma+k}{2}\right)}(2|\lambda|)^{\gamma} .$$

Thus, recalling that $T = 2\partial_t$, and using again the properties (5.4)-(5.5),

$$S(s) = -|T|^{\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma}{2} + \frac{-\Delta_b}{2|T|}\right)}{\Gamma\left(\frac{1-\gamma}{2} + \frac{-\Delta_b}{2|T|}\right)}, \quad s = \frac{m+\gamma}{2}, \quad \gamma \in (0,1) .$$

Taking into account (1.7) and (1.8), we have shown (1.12).

5.3. **The extension problem.** For any $0 < \gamma < 1$ we may define an extension operator $\mathcal{E}_\gamma$ which maps functions $f$ on the Heisenberg group $\mathbb{H}^n$ to functions $\mathcal{E}_\gamma f$ on the Siegel domain $\Omega_{n+1} \simeq \mathbb{H}^n \times (0,\infty)$. For every $q > 0$ we can consider $\mathcal{E}_\gamma f(\cdot, q)$ as a function on the Heisenberg group, which is defined through the Fourier multiplier $\phi_\alpha(2|\lambda|q)$, where

$$\phi_\alpha(x) = \frac{\Gamma\left(\frac{1+\gamma+k_\alpha}{2}\right)}{\Gamma(\gamma)}\ e^{-x/2}\ x^{\gamma}\ V\left(\tfrac{1+\gamma+k_\alpha}{2}, 1+\gamma, x\right) .$$

(We do not indicate the dependence of $\phi_\alpha$ on $\gamma$ in the notation). In other words,

$$\widehat{\mathcal{E}_\gamma f(\cdot, q)}_\alpha(\lambda) = \phi_\alpha(2|\lambda|q)\ \hat{f}_\alpha(\lambda) \tag{5.9}$$

for every $q > 0$, $\alpha \in \mathbb{N}_0^n$ and $\lambda \in \mathbb{R}$. The fact that $\phi_\alpha(0) = 1$ from the previous section implies that for $q = 0$ one has, indeed,

$$\widehat{\mathcal{E}_\gamma f(\cdot, 0)}_\alpha(\lambda) = \hat{f}_\alpha(\lambda)$$

for every $\alpha \in \mathbb{N}_0^n$ and $\lambda \in \mathbb{R}$, that is $\mathcal{E}_\gamma f(\cdot, 0) = f$, which justifies the name ‘extension’.

Moreover, the ODE facts that we established in the previous section imply that for every $\alpha \in \mathbb{N}_0^n$ and $\lambda \in \mathbb{R}$, the function $q \mapsto \widehat{\mathcal{E}_\gamma f(\cdot, q)}_\alpha(\lambda)$ solves the equation

$$\left(q\partial_{qq} + (1-\gamma)\partial_q - \lambda^2 q - |\lambda| k_\alpha\right) \widehat{\mathcal{E}_\gamma f(\cdot, q)}_\alpha(\lambda) = 0 ,$$

and therefore, the function $\mathcal{E}_\gamma f$ on $\Omega_{n+1}$ satisfies

$$\left(q\partial_{qq} + (1-\gamma)\partial_q + q\partial_{tt} + \frac{1}{2}\Delta_b\right) \mathcal{E}_\gamma f = 0 .$$

This completes the proof of Theorem 1.1.

## 6. Sharp Sobolev trace inequalities

Consider the quadratic form $a_\gamma$ associated to the operator $P_\gamma^\theta$ as defined in (1.14). Using Fourier transform it may be rewritten as

$$a_\gamma[f] = \frac{2^{n-1}}{\pi^{n+1}} \sum_{\alpha \in \mathbb{N}_0^n} \frac{\Gamma\left(\frac{1+\gamma+k_\alpha}{2}\right)}{\Gamma\left(\frac{1-\gamma+k_\alpha}{2}\right)} \int_{\mathbb{R}} (2|\lambda|)^{\gamma} \|\hat{f}_\alpha(\lambda)\|_{\mathcal{G}_\lambda}^2\, |\lambda|^n d\lambda . \tag{6.1}$$

We now consider the energy functional in the extension introduced in (1.13), where we recall that $q = \rho^2/2$,

$$\mathcal{A}_\gamma[U] = 2^{1-\gamma} \int_{\Omega_{n+1}} \left( q^{1-\gamma}|\partial_q U|^2 + q^{1-\gamma}|\partial_t U|^2 + \frac{1}{4} q^{-\gamma} \sum_{j=1}^{n} \left(|X_j U|^2 + |Y_j U|^2\right) \right) d\zeta$$

for functions $U$ on $\Omega_{n+1}$. Here $d\zeta = dx_1 \cdots dx_n dy_1 \cdots dy_n dt dq$. We define the space $\dot{\mathcal{H}}^{1,\gamma}(\Omega_{n+1})$ as the completion of $\mathcal{C}_0^\infty(\overline{\Omega_{n+1}})$ with respect to $\mathcal{A}_\gamma^{\frac{1}{2}}$. Here $\overline{\Omega_{n+1}} = \mathbb{H}^n \times [0,\infty)$, including the boundary. One can show (and it also follows essentially from our arguments below) that this completion is a space of functions.

Similarly, we denote by $\dot{S}^\gamma(\mathbb{H}^n)$ the closure of $\mathcal{C}_0^\infty(\mathbb{H}^n)$ with respect to the quadratic form $a_\gamma$. (These are the fractional analogues of the Sobolev spaces introduced by Folland and Stein [24]). The dual space of $\dot{S}^\gamma(\mathbb{H}^n)$, with respect to the inner product of $L^2(\mathbb{H}^n)$, is the space $\dot{S}^{-\gamma}(\mathbb{H}^n)$, which is defined through $a_{-\gamma}$.

The following result contains an energy equality for the fractional norm $\dot{S}^\gamma(\mathbb{H}^n)$, using the extension problem from Theorem 1.1. This idea of using the extension has been successfully employed in several other settings ([27, 28, 4], for instance).

**Proposition 6.1.** *Let $0 < \gamma < 1$ and $f \in \dot{S}^\gamma(\mathbb{H}^n)$. Then $\mathcal{E}_\gamma f \in \dot{\mathcal{H}}^{1,\gamma}(\Omega_{n+1})$ and*

$$\mathcal{A}_\gamma[\mathcal{E}_\gamma f] = 2^{1-\gamma}\gamma \, \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \, a_\gamma[f] \, .$$

*Proof.* We use the shorthand $\hat{U}_\alpha(\lambda, q)$ for $\widehat{U(\cdot,q)}_\alpha(\lambda)$. In this notation, Plancherel's identity (5.6) gives

$$\mathcal{A}_\gamma[U] =$$
$$2^{1-\gamma}\frac{2^{n-1}}{\pi^{n+1}} \sum_\alpha \int_{\mathbb{R}} \int_0^\infty \Big( q^{1-\gamma} \|\partial_q \hat{U}_\alpha(\lambda,q)\|^2_{\mathcal{G}_\lambda} + \left(q^{1-\gamma}\lambda^2 + q^{-\gamma}|\lambda| k_\alpha\right) \|\hat{U}_\alpha(\lambda,q)\|^2_{\mathcal{G}_\lambda} \Big) \, dq \, |\lambda|^n d\lambda.$$

We apply this to $U = \mathcal{E}_\gamma f$ and plug the explicit expression for $\hat{U}_\alpha(\lambda,q)$ from (5.9) into the above formula. After changing variables $x = 2|\lambda| q$ we arrive at

$$\mathcal{A}_\gamma[\mathcal{E}_\gamma f] = 2^{1-\gamma}\frac{2^{n-1}}{\pi^{n+1}} \sum_\alpha C_\alpha \int_{\mathbb{R}} (2|\lambda|)^\gamma \|\hat{f}_\alpha(\lambda)\|^2_{\mathcal{G}_\lambda} \, |\lambda|^n d\lambda \tag{6.2}$$

with the constant

$$C_\alpha = \int_0^\infty \left( x^{1-\gamma}|\phi_\alpha'|^2 + \left(\frac{1}{4}x^{1-\gamma} + \frac{1}{2}x^{-\gamma}k_\alpha\right) |\phi_\alpha|^2 \right) dx \, .$$

Its precise value will be calculated in the next few lines. According to the previous subsection,

$$\left( x\partial_{xx} + (1-\gamma)\partial_x - \frac{1}{4}x - \frac{1}{2}k_\alpha \right) \phi_\alpha = 0 \quad \text{in } (0,\infty)$$

and $\phi_\alpha(0) = 1$. Moreover, $\phi_\alpha$ decays exponentially. Thus, we can multiply the equation for $\phi_\alpha$ by $x^{-\gamma}\phi_\alpha$ and integrate over the interval $(\epsilon, \infty)$ to get

$$C_\alpha = \lim_{\epsilon\to 0} \int_\epsilon^\infty \left( x^{1-\gamma}|\phi_\alpha'|^2 + \left(\frac{1}{4}x^{1-\gamma} + \frac{1}{2}x^{-\gamma}k_\alpha\right)|\phi_\alpha|^2 \right) dx = -\lim_{\epsilon\to 0} \epsilon^{1-\gamma}\phi_\alpha(\epsilon)\phi_\alpha'(\epsilon) \, .$$

We know from the previous subsection that, as $\epsilon \to 0$,

$$\phi_\alpha(\epsilon) = 1 - \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k_\alpha}{2}\right)}{\Gamma\left(\frac{1-\gamma+k_\alpha}{2}\right)} \epsilon^\gamma + \mathcal{O}(\epsilon)$$

and that this expansion may be differentiated. Thus,

$$C_\alpha = \gamma \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{\Gamma\left(\frac{1+\gamma+k_\alpha}{2}\right)}{\Gamma\left(\frac{1-\gamma+k_\alpha}{2}\right)} .$$

Substituting back into (6.2) and recalling (6.1) we conclude that

$$\begin{aligned} \mathcal{A}_\gamma[\mathcal{E}_\gamma f] &= 2^{1-\gamma}\gamma \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} \frac{2^{n-1}}{\pi^{n+1}} \sum_\alpha \frac{\Gamma\left(\frac{1+\gamma+k_\alpha}{2}\right)}{\Gamma\left(\frac{1-\gamma+k_\alpha}{2}\right)} \int_{\mathbb{R}} (2|\lambda|)^\gamma \|\hat{f}_\alpha(\lambda)\|^2_{\mathcal{G}_\lambda} |\lambda|^n d\lambda \\ &= 2^{1-\gamma}\gamma \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} a_\gamma[f] , \end{aligned}$$

as claimed.

*Proof of Theorem 1.2.* Let $U \in \mathcal{C}_0^\infty(\overline{\Omega_{n+1}})$ and $g \in \dot{S}^{-\gamma}(\mathbb{H}^n)$, the dual of $\dot{S}^\gamma(\mathbb{H}^n)$. Then $h := \left(P_\gamma^\theta\right)^{-1} g = P_{-\gamma}^\theta g \in \dot{S}^\gamma(\mathbb{H}^n)$. As we observed before, its extension $H = \mathcal{E}_\gamma h$ from (5.9) satisfies

$$\left( q\partial_{qq} + (1-\gamma)\partial_q + q\partial_{tt} + \frac{1}{2}\Delta_b \right) H = 0 .$$

Using dominated convergence and (5.8), see also (1.11), one can show that

$$H(\cdot,\epsilon) \to h \text{ in } \dot{S}^\gamma(\mathbb{H}^n) \qquad \text{and} \qquad \epsilon^{1-\gamma}\frac{\partial H}{\partial q}(\cdot,\epsilon) \to -\gamma 2^{-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)} P_\gamma^\theta h \text{ in } \dot{S}^{-\gamma}(\mathbb{H}^n) .$$

Thus,

$$\begin{aligned} 2^{-\gamma} \int_{\mathbb{H}^n} \overline{g(\xi)} U(\xi,0)\, d\xi &= 2^{-\gamma} \int_{\mathbb{H}^n} \overline{P_\gamma^\theta h(\xi)} U(\xi,0)\, d\xi \\ &= -\frac{1}{\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \lim_{\epsilon\to 0} \epsilon^{1-\gamma} \int_{\mathbb{H}^n} \overline{\frac{\partial H}{\partial q}}(\xi,\epsilon) U(\xi,\epsilon)\, d\xi \\ &= \frac{1}{\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \lim_{\epsilon\to 0} \iint_{\mathbb{H}^n\times(\epsilon,\infty)} \Big( q^{1-\gamma} \overline{\partial_q H} \partial_q U + q^{1-\gamma} \overline{\partial_t H} \partial_t U \\ &\qquad + \frac{1}{4} q^{-\gamma} \sum_{j=1}^n \left( \overline{X_j H} X_j U + \overline{Y_j H} Y_j U \right) \Big) d\xi\, dq . \end{aligned}$$

By the Schwarz inequality,

$$2^{-\gamma}\left|\int_{\mathbb{H}^n}\overline{g(\xi)}U(\xi,0)\,d\xi\right| \le \frac{1}{\gamma}\,\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)}$$

$$\times \limsup_{\epsilon\to 0}\left(\iint_{\mathbb{H}^n\times(\epsilon,\infty)}\left(q^{1-\gamma}|\partial_q H|^2+q^{1-\gamma}|\partial_t H|^2+\frac{1}{4}q^{-\gamma}\sum_{j=1}^n\left(|X_jH|^2+|Y_jH|^2\right)\right)d\xi\,dq\right)^{\frac12}$$

$$\times \limsup_{\epsilon\to 0}\left(\iint_{\mathbb{H}^n\times(\epsilon,\infty)}\left(q^{1-\gamma}|\partial_q U|^2+q^{1-\gamma}|\partial_t U|^2+\frac{1}{4}q^{-\gamma}\sum_{j=1}^n\left(|X_jU|^2+|Y_jU|^2\right)\right)d\xi\,dq\right)^{\frac12}.$$

Next, by Proposition 6.1,

$$\begin{aligned}\limsup_{\epsilon\to 0}&\iint_{\mathbb{H}^n\times(\epsilon,\infty)}\left(q^{1-\gamma}|\partial_q H|^2+q^{1-\gamma}|\partial_t H|^2+\frac{1}{4}q^{-\gamma}\sum_{j=1}^n\left(|X_jH|^2+|Y_jH|^2\right)\right)d\xi\,dq\\ &=2^{\gamma-1}\mathcal{A}_\gamma[H]=\gamma\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}\,a_\gamma[h]=\gamma 2^{-\gamma}\frac{\Gamma(1-\gamma)}{\Gamma(1+\gamma)}\,\left(g,P^\theta_{-\gamma}g\right).\end{aligned}$$

Thus, we have shown that

$$\left|\int_{\mathbb{H}^n}\overline{g(\xi)}U(\xi,0)\,d\xi\right| \le \left(2^{-1+\gamma}\frac{\Gamma(1+\gamma)}{\gamma\,\Gamma(1-\gamma)}\right)^{1/2} a_{-\gamma}[g]^{1/2}\mathcal{A}_\gamma[U]^{1/2}.$$

By duality, this means that $U(\cdot,0)\in\dot{S}^\gamma(\mathbb{H}^n)$ with

$$a_\gamma[U(\cdot,0)]\le 2^{-1+\gamma}\frac{\Gamma(1+\gamma)}{\gamma\,\Gamma(1-\gamma)}\mathcal{A}_\gamma[U].$$

This inequality, together with the density of $\mathcal{C}_0^\infty(\overline{\Omega_{n+1}})$ in $\dot{\mathcal{H}}^{1,\gamma}(\Omega_{n+1})$, implies the Theorem.

It was conjectured by [8] that on the CR sphere we have the following conformally invariant sharp Sobolev inequality

$$\|f\|^2_{L^{q^*}(S^{2n+1})}\le C(n,\gamma)\fint_{S^{2n+1}} f\mathcal{P}_\gamma f,\quad \text{for}\quad q^*=\frac{2Q}{Q-2\gamma},\quad Q=2(n+1), \tag{6.3}$$

where $\mathcal{P}_\gamma$ are the CR fractional powers of the Laplacian on the sphere. The fact that this inequality is valid with some constant follows from the work of Folland and Stein [24]. In the remarkable work [49] Jerison and Lee found the optimal constant in the case $\gamma=1$. The problem of determining the sharp constant for general $\gamma$ was solved in [26]. It is also shown that in the equivalent $\mathbb{H}^n$ version of (6.3) all optimizers are translates, dilates or constant multiples of the function

$$H=\left(\frac{1}{(1+|z|^2)^2+t^2}\right)^{\frac{Q-2\gamma}{4}}.$$

Putting together (6.3) and Theorem 1.2 we complete the proof of Corollary 1.3.

## 7. The Yamabe problem for the conformal fractional sub–Laplacian

Let $\mathcal{X}$ be a $m$-dimensional complex manifold with strictly pseudoconvex boundary $\mathcal{M}$. Let $g^+$ be a Kähler metric on $\mathcal{X}$ such that there exists a globally defined approximate solution of the Monge-Ampère equation that makes $g^+$ an approximate Kähler-Einstein metric, with $\theta$ as contact form on $\mathcal{M}$, as described in Section 2.3.

Fix $\gamma \in (0,1)$. The fractional Yamabe problem asks to find a contact form $\hat{\theta} = f^{\frac{2}{m-\gamma}}\theta$ for some $f>0$ on $\mathcal{M}$ such that the fractional CR curvature $Q_\gamma^{\hat{\theta}}$ is constant. In PDE language, we need to find a positive solution $f$ of the nonlocal problem

$$P_\gamma^\theta(f) = cf^{\frac{m+\gamma}{m-\gamma}}, \quad \text{on } \mathcal{M}.$$

From the variational point of view, we are looking for minimizers of the functional

$$I_\gamma[f] = \frac{\int_{\mathcal{M}} f P_\gamma^\theta f\, \theta\wedge d\theta^n}{\left(\int_{\mathcal{M}} |f|^{2^*}\,\theta\wedge d\theta^n\right)^{\frac{2}{2^*}}},$$

for $2^* = \frac{2m}{m-\gamma}$. Motivated by the Riemannian case from [32], one may find instead minimizers of the extension functional

$$\overline{I}_\gamma[u] = \frac{\int_{\mathcal{X}} q^{m-1}|\nabla u|^2_{g^+}\,dvol_{g^+} - s(m-s)\int_{\mathcal{X}} q^{m-1}u^2\,dvol_{g^+}}{\left(\int_{\mathcal{M}} |u|^{2^*}\,\theta\wedge d\theta^n\right)^{2/2^*}}.$$

In particular, in the Heisenberg group case we may rewrite the functional as

$$\overline{I}_\gamma[U] = \frac{\int_{\Omega_m}\left(q^{1-\gamma}|\partial_q U|^2 + q^{1-\gamma}|\partial_t U|^2 + \frac{1}{4}q^{-\gamma}\sum_{j=1}^n\left(|X_jU|^2+|Y_jU|^2\right)\right)\,d\zeta}{\left(\int_{\mathbb{H}^n}|U|^{2^*}\,\theta\wedge d\theta^n\right)^{2/2^*}}.$$

for $u = q^{m-s}U$.

We define the CR $\gamma$-Yamabe constant as

$$\Lambda_\gamma(\mathcal{M},[\theta]) = \inf \overline{I}_\gamma[U].$$

It is easy to show that

$$\Lambda_\gamma(\mathcal{M},[\theta]) \le \Lambda_\gamma(\mathbb{H}^n), \tag{7.1}$$

where the Heisenberg group is understood with its canonical contact form.

We conjecture that the fractional CR Yamabe problem is solvable if we have a strict inequality in (7.1), and that this is so unless we are already at the model case. We hope to return to this problem elsewhere.

## 8. Further studies

After all this discussion on the complex hyperbolic space, it is natural to look now at the the quaternionic hyperbolic space $\mathcal{H}^m_{\mathbb{Q}}$. It can be characterized as a Siegel domain whose boundary is precisely the quaternionic Heisenberg group that, with some abuse of notation, will be denoted by $\mathcal{Q}^n$. It will become clear that both Theorems 1.1 and 1.2 are consequences of the rigid underlying structure of hyperbolic space.

8.1. **The quaternionic case.** A quaternion is an object of the form

$$q = x + yi + zj + wk, \quad x, y, z, w \in \mathbb{R},$$

where the three imaginary units satisfy the multiplication rules

$$\begin{aligned} &i^2 = j^2 = k^2 = -1, \\ &ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \end{aligned}$$

The set of quaternions, denoted by $\mathbb{Q}$, is a division ring. In particular, multiplication is still associative and every nonzero element has a unique inverse. The number $x$ is called the real part of $q$ while the three dimensional vector $yi + zj + wk$ is its imaginary part. Conjugation is the same as for complex numbers, indeed, $\bar{q} = x - yi - zj - wk$, and the modulus of $q$ is calculated as $|q|^2 = q\bar{q} = x^2 + y^2 + z^2 + w^2$.

We define the quaternionic Heisenberg group by $\mathcal{Q}^n := \mathbb{Q}^n \times \mathrm{Im}(\mathbb{Q})$, with the group law

$$(\zeta_1, v_1) \circ (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\,\mathrm{Im} \ll \zeta_1, \zeta_2 \gg),$$

where $\ll \zeta_1, \zeta_2 \gg = \overline{\zeta_2}\zeta_1$ is the standard positive definite Hermitian form on $\mathbb{Q}^n$. If we choose coordinates

$$(\zeta_i = x_i + iy_i + jz_i + kw_i)_{i=1}^n \quad \text{and} \quad v = iv_1 + jv_2 + kv_3$$

for the group $\mathcal{Q}^n$, then the following 1-form is a quaternionic contact form:

$$\eta = \begin{pmatrix} dv_1 + 2\sum(x_i dy_i - y_i dx_i + z_i dw_i - w_i dz_i) \\ dv_2 + 2\sum(x_i dz_i - z_i dx_i - y_i dw_i + w_i dy_i) \\ dv_3 + 2\sum(x_i dw_i - w_i dx_i + y_i dz_i - z_i dy_i) \end{pmatrix}.$$

Note that $d\eta$ is non-degenerate and the vector fields

$$\begin{aligned} X_i &= \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial v_1} + 2z_i \frac{\partial}{\partial v_2} + 2w_i \frac{\partial}{\partial v_3}, \\ Y_i &= \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial v_1} - 2w_i \frac{\partial}{\partial v_2} + 2z_i \frac{\partial}{\partial v_3}, \\ Z_i &= \frac{\partial}{\partial z_i} + 2w_i \frac{\partial}{\partial v_1} - 2x_i \frac{\partial}{\partial v_2} - 2y_i \frac{\partial}{\partial v_3}, \\ W_i &= \frac{\partial}{\partial w_i} - 2z_i \frac{\partial}{\partial v_1} + 2y_i \frac{\partial}{\partial v_2} - 2x_i \frac{\partial}{\partial v_3}. \end{aligned}$$

generate the kernel of $\eta$. Then $\{X_i, Y_i, Z_i, W_i\}_{i=1}^n$ generate a $4n$-dimensional distribution which is a contact structure on $\mathcal{Q}_n$.

In the paper [52] the quaternionic hyperbolic space $\mathcal{H}^m_{\mathbb{Q}}$ of quaternionic dimension $m$ is characterized as a Siegel domain with boundary $\mathcal{Q}^n$; we give here the main details of this construction. Consider $\mathbb{Q}^{m,1}$, the quaternionic vector space of quaternionic dimension $m+1$ (so real dimension $4m + 4$) with the quaternionic Hermitian form given by

$$\langle z, w \rangle = \overline{w}_1 z_{m+1} + \overline{w}_2 z_2 + \ldots + \overline{w}_m z_m + \overline{w}_{m+1} z_1,$$

where $z$ and $w$ are the column vectors in $\mathbb{Q}^{m,1}$ with entries $z_1, \ldots, z_{m+1}$ and $w_1, \ldots, w_{m+1}$, respectively. Consider the subspaces $V_-, V_0, V_+$ of $\mathbb{Q}^{m,1}$ given by

$$\begin{aligned}V_- &= \{z \in \mathbb{Q}^{m,1} \,:\, \langle z, z\rangle < 0\},\\ V_0 &= \{z \in \mathbb{Q}^{m,1}\backslash\{0\} \,:\, \langle z, z\rangle = 0\},\\ V_+ &= \{z \in \mathbb{Q}^{m,1} \,:\, \langle z, z\rangle > 0\},\end{aligned}$$

Define a right projection map $P$ from the subspace of $\mathbb{Q}^{m,1}$ consisting of those $z$ with $z_{m+1} \neq 0$ to $\mathbb{Q}^m$ by

$$P : \begin{pmatrix} z_1 \\ \vdots \\ z_{m+1}\end{pmatrix} \to \begin{pmatrix} z_1 z_{m+1}^{-1} \\ \vdots \\ z_m z_{m+1}^{-1}\end{pmatrix}.$$

The quaternionic hyperbolic $m$ space is defined as $\mathcal{H}^m_{\mathbb{Q}} := PV_- \subset \mathbb{Q}^m$. This is a paraboloid in $\mathbb{Q}^m$, called the Siegel domain. The metric on $\mathcal{H}^m_{\mathbb{Q}}$ is defined by

$$g^+ = \frac{-4}{\langle z, z\rangle^2} \det \begin{pmatrix} \langle z, z\rangle & \langle dz, z\rangle \\ \langle z, dz\rangle & \langle dz, dz\rangle \end{pmatrix}.$$

The boundary of this Siegel domain consists of those points in $PV_0$ (defined for points $z$ in $V_0$ with $z_{m+1} \neq 0$) together with a distinguished point at infinity, which we denote $\infty$. The finite points on the boundary of $\mathcal{H}^m_{\mathbb{Q}}$ naturally carry the structure of the generalized Heisenberg group $\mathcal{Q}^n$.

As in the complex case just studied, we define horospherical coordinates on quaternionic hyperbolic space. To each point $(\zeta, v, u) \in \mathcal{Q}^n \times \mathbb{R}^+$ we associate a point $\psi(\zeta, v, u) \in V_-$. Similarly, $\infty$ and each point $(\zeta, v, 0) \in \mathcal{Q}^n \times \{0\}$ is associated to a point in $V_0$ by $\psi$. The map $\psi$ is given by

$$\psi(\zeta, v, u) = \begin{bmatrix} (-|\zeta|^2 - u + v)/2 \\ \zeta \\ 1 \end{bmatrix} \quad \text{if } z \in \overline{\mathcal{H}}^m_{\mathbb{Q}}\backslash\{\infty\}, \quad \psi(\infty) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

With these coordinates, the metric on $\mathcal{H}^m_{\mathbb{Q}}$ may be written as

$$g^+ = \frac{du^2 + 4u \ll d\zeta, d\zeta \gg + \eta^2}{u^2},$$

which maybe put into the more standard form (8.2) by means of the change $u = \rho^2$ modulo factors of 2 depending on the normalization. Finally, the volume form is

$$dvol_{\mathcal{H}^m_{\mathbb{Q}}} = \frac{1}{u^{2n+2}}\, du\, dvol_{\mathcal{Q}^n}.$$

8.2. **Asymptotically hyperbolic metrics - a general formulation.** Here we follow the notation of the book [6]. It is well known that the rank one symmetric spaces of non-compact type are the real, complex and quaternionic hyperbolic spaces, and the Cayley (octonionic) hyperbolic plane. We denote them by $\mathcal{H}^m_{\mathbb{K}}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$ (the quaternions) or $\mathbb{O}$ (the octonions). As homogeneous spaces, we may write $\mathcal{H}^m_{\mathbb{K}} = G_0/G$, where $G_0$ is a real semisimple Lie group and $G$ a maximal compact subgroup, that is the stabilizer of a particular point $*$; more specifically,

$$\mathcal{H}^m_{\mathbb{R}} = SO_{1,m}/SO_m, \quad \mathcal{H}^m_{\mathbb{C}} = SU_{1,m}/U_m,$$
$$\mathcal{H}^m_{\mathbb{Q}} = Sp_{1,m}/Sp_1Sp_m, \quad \mathcal{H}^2_{\mathbb{O}} = F_4^{-20}/Spin_9.$$

If we denote

$$d = \dim_{\mathbb{R}} \mathbb{K},$$

then their real dimension is

$$\dim_{\mathbb{R}} \mathcal{H}^m_{\mathbb{K}} = md,$$

i.e., $m$, $2m$, $4m$ and 16 for the real, complex, quaternionic and octonionic case, respectively.

Let $r$ be the distance to $*$ and denote by $\mathbb{S}_r$ the sphere of radius $r$ centered at $*$. The metric $\kappa$ on the boundary sphere $\mathbb{S}$ of the hyperbolic space $\mathcal{H}^m_{\mathbb{K}}$ is defined as

$$\kappa = \lim_{r\to\infty} e^{-2r} g_{\mathbb{S}_r}.$$

The metric is infinite except on a distribution $V$ of codimension 1 (complex case), 3 (quaternionic case) or 7 (octonionic case). In the real case it is finite and $V = T\mathbb{S}$. The brackets of the vector fields in $V$ generate the whole tangent bundle $T\mathbb{S}$, making $\kappa$ into a Carnot-Carathéodory metric.

Moreover, there is a contact form $\eta$ with values in $\operatorname{Im} \mathbb{K}$, such that the hyperbolic metric on $\mathcal{H}^m_{\mathbb{K}}$ is exactly

$$g^+ = dr^2 + \sinh^2(r)\kappa + \sinh^2(2r)\eta^2. \tag{8.1}$$

In the real case, the $\eta^2$ term does not appear. To give a sense to the formula in the other three cases, we have to choose a supplementary subspace to the distribution $V \subset T\mathbb{S}$. This is given here by the fibers of the fibration

$$\begin{array}{ccc} \mathbb{S}^{d-1} & \longrightarrow & \mathbb{S} \\ & & \downarrow \\ & & \mathbb{KP}^{m-1} \end{array}$$

All this depends on the choice of the base point $*$, but the conformal class $[\kappa]$ is well defined and will be called the conformal infinity of $g^+$. Note that $g^+$ has sectional curvature pinched between -4 and -1.

Note that if in (8.1) we make the change of variables $\rho = e^{-r}$, then the metric becomes the more familiar

$$g^+ = \frac{d\rho^2 + \kappa}{\rho^2} + \frac{\eta^2}{\rho^4}. \tag{8.2}$$

One could generalize these definitions to give a notion of asymptotically $\mathbb{K}$-hyperbolic manifolds, whose metric behaves asymptotically at infinity as (8.1), but we will not pursue this end further.

8.3. **Scattering theory on $\mathcal{H}^m_{\mathbb{Q}}$.** Scattering theory on asymptotically $\mathbb{K}$-hyperbolic manifolds was developed in [7, 59] (see also [11, 60] for the generalization to differential forms). Here we would like to show that in the case of hyperbolic space $\mathcal{H}^m_{\mathbb{Q}}$, the calculation of the conformal fractional Laplacian $P^\eta_\kappa$ from Theorem 1.1 and the energy identity from Theorem 1.2 are analogous.

Denote $m_0 = 2 + 4m$. We calculate

$$\Delta_+ = \rho^2 \partial_{\rho\rho} - (1 + 4m)\rho\partial\rho + \rho^2\Delta_\kappa + \rho^4\Delta_\eta.$$

It is well known that the bottom of the spectrum for $-\Delta_+$ is $(m_0/2)^2 = (1 + 2m)^2$. The scattering equation

$$-\Delta_+ u - s(m_0 - s)u = 0, \quad \text{in } \mathcal{H}^m_{\mathbb{Q}} \tag{8.3}$$

has two indicial roots $s$ and $m_0 - s$, and one seeks a solution

$$u = F\rho^{m_0-s} + G\rho^s, \quad F|_{\rho=0} = f.$$

Change variables $U = \rho^{m_0-s}u$, and set $s = \frac{m_0}{2} + \gamma$, $a = 1 - 2\gamma$. Then equation (8.3) becomes

$$\begin{cases} \partial_{\rho\rho}U + \dfrac{a}{\rho}\partial_\rho U + \Delta_\kappa U + \rho^2\Delta_\eta U = 0, \\ \hfill U = f, \end{cases}$$

which is precisely (1.10). From here we can easily produce (quaternionic–conformal) fractional powers for the sub–Laplacian $\Delta_\kappa$. We leave the details to the interested reader.

Some final references: on harmonic analysis on semisimple Lie groups and symmetric spaces see [17, 16, 51] and the books by Helgason [43, 42, 41]. On the quaternionic Yamabe problem see [47]. However, the problem of finding extremals for fractional Sobolev embeddings in this setting is still an open question.

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