FINITE VS INFINITE DECOMPOSITIONS IN CONFORMAL EMBEDDINGS

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ABSTRACT. Building on work of the first and last author, we prove that
an embedding of simple affine vertex algebras $V_{\mathbf{k}}(\mathfrak{g}^0) \subset V_k(\mathfrak{g})$, corre-
sponding to an embedding of a maximal equal rank reductive subal-
gebra $\mathfrak{g}^0$ into a simple Lie algebra $\mathfrak{g}$, is conformal if and only if the
corresponding central charges are equal. We classify the equal rank con-
formal embeddings. Furthermore we describe, in almost all cases, when
$V_k(\mathfrak{g})$ decomposes finitely as a $V_{\mathbf{k}}(\mathfrak{g}^0)$-module.

1. INTRODUCTION

Let V and W be vertex algebras equipped with Virasoro elements ω_V,
ω_W and assume that W is a vertex subalgebra of V.

Definition 1.1. We say that W is conformally embedded in V if $\omega_V = \omega_W$.

In this paper we deal with following problems.

(1) Classify conformal embeddings when V, W are affine vertex algebras
    endowed with ω_V, ω_W given by Sugawara construction.
(2) Decide whether the decomposition of V as a W-module is finite, and
    in such a case find the explicit decomposition.

The general definition of conformal embedding introduced above is a natural
generalization of the following notion, which has been popular in physics
literature in the mid 80's, due to its relevance for string compactifications.

Let $\mathfrak{g}$ be a semisimple finite-dimensional complex Lie algebra and $\mathfrak{g}^0$ a
reductive subalgebra of $\mathfrak{g}$. The embedding $\mathfrak{g}^0 \hookrightarrow \mathfrak{g}$ is called conformal if the
central charge of the Sugawara construction for the affinization $\widehat{\mathfrak{g}}$, acting on
a level 1 integrable module, equals that for the natural embedding of $\widehat{\mathfrak{g}^0}$ in
$\widehat{\mathfrak{g}}$. Such an embedding is called maximal if no reductive subalgebra $\mathfrak{a}$ with
$\mathfrak{g}^0 \subsetneq \mathfrak{a} \subsetneq \mathfrak{g}$ embeds conformally in $\mathfrak{g}$.

Maximal conformal embeddings were classified in [26], [7], and the cor-
responding decompositions are described in [17], [16], [11]. In the vertex
algebra framework the definition can be rephrased as follows: the simple

affine vertex algebras $V_{\mathbf{k}}(\mathfrak{g}^0)$ and $V_1(\mathfrak{g})$ have the same Sugawara conformal vector for some multiindex $\mathbf{k}$ of levels. One may wonder whether the embedding $V_{\mathbf{k}}(\mathfrak{g}^0) \subset V_k(\mathfrak{g})$ is conformal according to Definition 1.1 for some k, not necessarily 1.

Section 3 is devoted to answer question (1). We prove that equality of central charges still detects conformality of maximal equal rank subalgebras: see Theorem 3.1. In Proposition 3.3, we deal with non maximal equal rank embeddings. The proof of Theorem 3.1 is elementary: it is obtained by combining the results of [3] with general results of Panyushev [24] on combinatorics of root systems.

Question (2) has appeared many times in literature. We give a complete answer in Section 4 when $\mathfrak{g}^0$ is semisimple and an almost complete answer in Section 5 when $\mathfrak{g}^0$ has a nonzero center. The missing cases are listed in Remark 5.1. Explicit decompositions are listed in Section 6.

To prove that a conformal pair has finite decomposition, we use an enhancement, given in Theorems 2.3 and 2.4 of results from [3]. Infinite decompositions are settled by exhibiting infinitely many singular vectors: see Proposition 4.3 and Corollary 4.6 for the semisimple case. As a byproduct of our analysis we obtain, in the semisimple case, a criterion for infinite decomposition in terms of the existence of a $\widehat{\mathfrak{g}^0}$-singular vector having conformal weight 2: see Proposition 4.7. Other cases with infinite decomposition when $\mathfrak{g}^0$ has a nonzero center are dealt with in Theorem 5.3. The methods to prove this theorem are a combination of the ideas used in the semisimple case with explicit realization of $V_k(\mathfrak{g})$, more precisely, the Kac-Wakimoto free field realization of $V_{-1}(sl(n+1))$ [20] and Adamović's recent realization of $V_{-3/2}(sl(3))$ in the tensor product of the $N = 4$ superconformal vertex algebra with a suitable lattice vertex algebra [1].

It is worthwhile to note that the methods used in the proof of Theorems 5.1 and 5.3 lead to look for conformal embeddings of simple affine vertex algebras in W-algebras. For instance, it is possible to embed conformally $V_{-\frac{n+1}{2}}(gl(n))$ in the W-algebra $W_{\frac{n-1}{2}}(sl(2|n), f)$ for a suitable nilpotent element in $sl(2|n)$ of even parity. A systematic investigation of conformal embeddings of affine subalgebras in W-algebras has started in [5], [6].

2. Setup and preliminary results

2.1. **Notation.** Let $\mathfrak{g}$ be a simple Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra, Δ the $(\mathfrak{g}, \mathfrak{h})$-root system, Δ^+ a set of positive roots and ρ the corresponding Weyl vector. Let $(\cdot, \cdot)$ denote the bilinear normalized invariant form (i.e., $(\alpha, \alpha) = 2$ for any long root). If $\alpha \in \Delta$, we will denote by X_α a root vector relative to α.

Assume that $\mathfrak{g}^0$ is a reductive equal rank subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}^0$ decomposes as

(2.1) $$\mathfrak{g}^0 = \mathfrak{g}_0^0 \oplus \mathfrak{g}_1^0 \oplus \cdots \oplus \mathfrak{g}_t^0.$$

where $\mathfrak{g}_0^0$ is the (possibly zero) center of $\mathfrak{g}^0$ and $\mathfrak{g}_i^0$ are simple ideals for $i > 0$. Let ρ_0^j be the Weyl vector in $\mathfrak{g}_j^0$ (w.r.t. the set of positive roots induced by Δ^+). Assume that $(\cdot, \cdot)$ is nondegenerate when restricted to $\mathfrak{g}^0$. Let $\mathfrak{p}$ be the $(\cdot, \cdot)$-orthocomplement of $\mathfrak{g}^0$ in $\mathfrak{g}$. If $\mu \in \mathfrak{h}^*$ is a $\mathfrak{g}^0$-dominant integral weight we let $V(\mu)$ be the irreducible finite dimensional $\mathfrak{g}^0$-module with highest weight μ. Clearly we have

$$V(\mu) = \bigotimes_{j=0}^{t} V_{\mathfrak{g}_j^0}(\mu^j),$$

where $V_{\mathfrak{g}_j^0}(\mu^j)$ is a irreducible $\mathfrak{g}_j^0$-module.

We denote by $\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$ the untwisted affinization of $\mathfrak{g}$ (see [15, § 7.2]): d and K denote, respectively, the scaling element and the canonical central element of $\widehat{\mathfrak{g}}$. If $x \in \mathfrak{g}$ we set $x_{(n)} = t^n \otimes x \in \widehat{\mathfrak{g}}$. Let $\widehat{\mathfrak{g}^0}$ denote the subalgebra of $\widehat{\mathfrak{g}}$ generated by $\{x_{(n)} \mid x \in \mathfrak{g}^0, \ n \in \mathbb{Z}\} \cup \{d\}$.

Let $\Lambda_0 \in (\mathfrak{h} + \mathbb{C}K + \mathbb{C}d)^*$ be the weight such that $\Lambda_0(K) = 1$ and $\Lambda_0(\mathfrak{h}) = \Lambda_0(d) = 0$. Fix $k \in \mathbb{C}$. We extend a weight $\mu \in \mathfrak{h}^*$ to $(\mathfrak{h} + \mathbb{C}K + \mathbb{C}d)^*$ by setting $\mu(K) = \mu(d) = 0$ and denote by $L_{\mathfrak{g}}(k, \mu)$ the irreducible highest weight $\widehat{\mathfrak{g}}$-module with highest weight $k\Lambda_0 + \mu$. Let v_μ be a highest weight vector in $L_{\mathfrak{g}}(k, \mu)$. We denote by $\widetilde{L}_{\mathfrak{g}^0}(k, \mu)$ the $\widehat{\mathfrak{g}^0}$-submodule of $L_{\mathfrak{g}}(k, \mu)$ generated by v_μ and by $L_{\mathfrak{g}^0}(k, \mu)$ its irreducible quotient. When there is no chance of confusion we will drop k from the notation denoting $L_{\mathfrak{g}}(k, \mu)$, $\widetilde{L}_{\mathfrak{g}^0}(k, \mu)$, $L_{\mathfrak{g}^0}(k, \mu)$ simply by $L_{\mathfrak{g}}(\mu)$, $\widetilde{L}_{\mathfrak{g}^0}(\mu)$, $L_{\mathfrak{g}^0}(\mu)$, respectively.

We let $V^k(\mathfrak{g})$, $V_k(\mathfrak{g})$ denote, respectively, the universal and the simple affine vertex algebra (see [19, § 4.7 and Example 4.9b]). More generally, if $\mathfrak{a}$ is a reductive Lie algebra that decomposes as $\mathfrak{a} = \mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_s$ with $\mathfrak{a}_0$ abelian and $\mathfrak{a}_i$ simple ideals for $i > 0$ and $\mathbf{k} = (k_1, \ldots, k_s)$ is a multi-index of levels, we let

$$V^{\mathbf{k}}(\mathfrak{a}) = V^{k_0}(\mathfrak{a}_0) \otimes \cdots \otimes V^{k_s}(\mathfrak{a}_s), \quad V_{\mathbf{k}}(\mathfrak{a}) = V_{k_0}(\mathfrak{a}_0) \otimes \cdots \otimes V_{k_s}(\mathfrak{a}_s).$$

We let $\mathbf{1}$ denote the vacuum vector of both $V^{\mathbf{k}}(\mathfrak{a})$ and $V_{\mathbf{k}}(\mathfrak{a})$.

If $j > 0$, let $\{x_i^j\}, \{x_j^i\}$ be dual bases of $\mathfrak{a}_j$ with respect to the normalized invariant form of $\mathfrak{a}_j$ and $h_j^\vee$ its dual Coxeter number. For $\mathfrak{a}_0$, let $\{x_i^j\}, \{x_j^i\}$ be dual bases with respect to any nondegenerate form and set $h_0^\vee = 0$.

Assuming that $k_j + h_j^\vee \neq 0$ for all j, we consider $V^{\mathbf{k}}(\mathfrak{a})$ and all its quotients, including $V_{\mathbf{k}}(\mathfrak{a})$, as conformal vertex algebras with conformal vector $\omega_{\mathfrak{a}}$ given by the Sugawara construction:

$$\omega_{\mathfrak{a}} = \sum_{j=0}^{s} \frac{1}{2(k_j + h_j^\vee)} \sum_{i=1}^{\dim \mathfrak{a}_j} : x_i^j x_j^i : .$$

Let V be a vertex algebra. For $a \in V$, we denote by

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$$

the corresponding field. If V admits a conformal vector ω then we write the corresponding field as

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-2}$$

(so that $\omega_{(n)} = \omega_{n-1}$). Recall that, by definition of conformal vector, ω_0 acts semisimply on V. If x is an eigenvector for ω_0, then the corresponding eigenvalue Δ_x is called the conformal weight of x.

Returning to our pair $(\mathfrak{g}, \mathfrak{g}^0)$, we let $\widetilde{V}_k(\mathfrak{g}^0)$ denote the vertex subalgebra of $V_k(\mathfrak{g})$ generated by $x_{(-1)}\mathbf{1}$, $x \in \mathfrak{g}^0$. We choose $(\cdot, \cdot)_{|\mathfrak{g}_0^0 \times \mathfrak{g}_0^0}$ as nondegenerate form on $\mathfrak{g}_0^0$. Note that there is a uniquely determined multi-index $\mathbf{k}$ such that $\widetilde{V}_k(\mathfrak{g}^0)$ is a quotient of $V^{\mathbf{k}}(\mathfrak{g}^0)$ hence, if $k_j + h_j^\vee \neq 0$ for each j, $\omega_{\mathfrak{g}^0}$ is a conformal vector in $\widetilde{V}_k(\mathfrak{g}^0)$. As an instance of Definition 1.1, we will say that $\widetilde{V}_k(\mathfrak{g}^0)$ is conformally embedded in $V_k(\mathfrak{g})$ if $\omega_{\mathfrak{g}} = \omega_{\mathfrak{g}^0}$.

2.2. Some general results on conformal embeddings.
The basis of our investigation is the following result.

Theorem 2.1. [3, Theorem 1] $\widetilde{V}_k(\mathfrak{g}^0)$ *is conformally embedded in* $V_k(\mathfrak{g})$ *if and only for any* $x \in \mathfrak{p}$ *we have*

$$(2.2) \qquad (\omega_{\mathfrak{g}^0})_0 x_{(-1)} \mathbf{1} = x_{(-1)} \mathbf{1}.$$

The previous theorem has the following useful reformulation. Remark that, in [3], it is assumed that $\mathfrak{g}^0$ is a simple Lie algebra, but the arguments work in the reductive case as well.

Let $\mathfrak{p} = \oplus_i V(\mu_i)$ be the decomposition of $\mathfrak{p}$ as a $\mathfrak{g}^0$-module. Let $(\cdot, \cdot)_0$ denote the normalized invariant bilinear form on $\mathfrak{g}^0$ (i.e. $(\cdot, \cdot)_0$ is the normalized invariant form when restricted to the simple ideals of $\mathfrak{g}^0$ and, on $\mathfrak{g}_0^0$, $(\cdot, \cdot)_0 = (\cdot, \cdot)_{|\mathfrak{g}_0^0 \times \mathfrak{g}_0^0}$).

Corollary 2.2. $\widetilde{V}_k(\mathfrak{g}^0)$ *is conformally embedded in* $V_k(\mathfrak{g})$ *if and only if*

$$(2.3) \qquad \sum_{j=0}^{t} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_0}{2(k_j + h_j^\vee)} = 1$$

for any i.

We now discuss the special case when $\mathfrak{g}^0$ is semisimple and it is the fixed point subalgebra of an automorphism σ of $\mathfrak{g}$ of order m. Let ξ be a primitive m-th root of unity. Since $\mathfrak{g}^0$ is semisimple, the eigenspace associated to the eigenvalue ξ^i (for $i = 1, \ldots, m-1$) is an irreducible $\mathfrak{g}^0$-module $V(\mu_i)$ and $\mathfrak{p} = \oplus_{i=1}^{m-1} V(\mu_i)$. Set also $\mu_0 = 0$.

The automorphism σ can be extended to a finite order automorphism of the simple vertex algebra $V_k(\mathfrak{g})$ which admits the following decomposition

$$V_k(\mathfrak{g}) = V_k(\mathfrak{g})^0 \oplus V_k(\mathfrak{g})^1 \oplus \cdots \oplus V_k(\mathfrak{g})^{m-1},$$

where

$$V_k(\mathfrak{g})^i = \{v \in V_k(\mathfrak{g}) \mid \sigma(v) = \xi^i v\}.$$

Clearly $V_k(\mathfrak{g})^i$ is a $\widehat{\mathfrak{g}^0}$–module.

Theorem 2.3. [3, Theorem 3] *Assume that*

$$(2.4) \qquad V(\mu_i) \otimes V(\mu_j) = V(\mu_l) \oplus \bigoplus_{r=1}^{m_{i,j}} V(\nu_{r,i,j});$$

$$(2.5) \qquad V_k(\mathfrak{g})^l \ \text{does not contain } \widehat{\mathfrak{g}^0} - \text{primitive vectors of weight } \nu_{r,i,j},$$
$$(where \ l = i + j \quad \mod m \ ),$$

for all $i, j \in \{1, \ldots, m-1\}$ *and* $r = 1, \ldots, m_{i,j}$.
    Then

$$V_k(\mathfrak{g}) = L_{\mathfrak{g}^0}(0) \oplus L_{\mathfrak{g}^0}(\mu_1) \oplus \cdots \oplus L_{\mathfrak{g}^0}(\mu_{m-1}).$$

The next result is a version of Theorem 2.3 suitable for the case when $\mathfrak{g}^0$ is a reductive equal rank non-semisimple subalgebra of $\mathfrak{g}$. We also assume here that the embedding $\mathfrak{g}^0 \subset \mathfrak{g}$ is maximal. In this case $\dim \mathfrak{g}_0^0 = 1$, and $\mathfrak{g}^0$ is the fixed point set of an automorphism of $\mathfrak{g}$ of order 2. We can also choose Δ^+ so that the set of simple roots of $\mathfrak{g}^0$ is obtained from the set of simple roots of $\mathfrak{g}$ by dropping one simple root α_p having coefficient 1 in the simple roots expansion of the highest root θ of $\mathfrak{g}$.

Moreover, as a $\mathfrak{g}^0$-module, $\mathfrak{p} = V(\mu_1) \oplus V(\mu_2)$ with $\mu_1 = \theta$ and $\mu_2 = -\alpha_p$. In particular, $\mu_1^0 = \theta_{|\mathfrak{g}_0^0}$, and $\mu_2^0 = -(\alpha_p)_{|\mathfrak{g}_0^0}$. Since $(\cdot, \cdot)$ is nondegenerate on $\mathfrak{p}$ and it is also $\mathfrak{g}^0$-invariant, we have that $V(-\alpha_p) = V(\theta)^*$. In particular the trivial representation occurs in $V(\theta) \otimes V(-\alpha_p)$. Thus we can write

$$(2.6) \qquad V(\theta) \otimes V(-\alpha_p) = \mathbb{C} \oplus \bigoplus_{r=1}^{s} V(\nu_r).$$

with $\nu_i \neq 0$. Let ϖ be the element of $\mathfrak{h}$ such that $\alpha_i(\varpi) = \delta_{ip}$. Then $\mathfrak{g}_0^0 = \mathbb{C}\varpi$. Let $\zeta \in (\mathfrak{g}_0^0)^*$ be defined by setting $\zeta(\varpi) = 1$, so that

$$(2.7) \qquad \mu_1^0 = \zeta, \quad \mu_2^0 = -\zeta.$$

If $q \in \mathbb{Z}$, let $V_k(\mathfrak{g})^{(q)}$ be the eigenspace for the action of $\varpi_{(0)}$ on $V_k(\mathfrak{g})$ relative to the eigenvalue q.

If A, B are subspaces of a vertex algebra V, we set

$$(2.8) \qquad A \cdot B = span\{a_{(n)}b \mid a \in A, b \in B, n \in \mathbb{Z}\}.$$

Theorem 2.4. *Assume $k \neq 0$ and that*

$$(2.9) \qquad V(\theta) \otimes V(-\alpha_p) = \mathbb{C} \oplus \bigoplus_{r=1}^{s} V(\nu_r);$$

$$(2.10) \qquad V_k(\mathfrak{g})^{(0)} \text{ does not contain } \widehat{\mathfrak{g}^0} - \text{primitive vectors of weight } \nu_r,$$
$$\text{(where } r = 1, \ldots, s),$$

Then $\widetilde{V}_k(\mathfrak{g}^0) \cong V_{\mathbf{k}}(\mathfrak{g}^0)$ and as a $\widehat{\mathfrak{g}^0}$-module,

$$(2.11) \qquad V_k(\mathfrak{g})^{(q)} = \begin{cases} L_{\mathfrak{g}^0}(q\theta) & \text{if } q \geq 0, \\ L_{\mathfrak{g}^0}(q\alpha_p) & \text{if } q \leq 0. \end{cases}$$

In particular,

$$(2.12) \qquad V_k(\mathfrak{g}) = L_{\mathfrak{g}^0}(0) \oplus (\sum_{q>0} L_{\mathfrak{g}^0}(q\theta)) \oplus (\sum_{q>0} L_{\mathfrak{g}^0}(-q\alpha_p)).$$

Proof. Let A^+, A^- be the $\widehat{\mathfrak{g}^0}$-submodules of $V_k(\mathfrak{g})$ generated by $V(\theta)_{(-1)}\mathbf{1}$, $V(-\alpha_p)_{(-1)}\mathbf{1}$ respectively.

Then a fusion rules argument shows that a primitive vector in $A^+ \cdot A^-$ must have weight 0 or ν_r for some r. By our hypothesis it has weight 0 so, since the embedding is conformal, it has conformal weight 0. Since the only vector of conformal weight 0 in $V_k(\mathfrak{g})$ is $\mathbf{1}$, we see that

$$(2.13) \qquad A^+ \cdot A^- \subset \widetilde{V}_k(\mathfrak{g}^0).$$

It is clear that $V_k(\mathfrak{g})^{(0)}$ is contained in the sum of all products of type $A_1 \cdot (A_2 \cdot (\cdots A_r)) \cdots)$ with $A_i \in \{A^+, A^-, \widetilde{V}_k(\mathfrak{g}^0)\}$ such that

$$\sharp\{i \mid A_i = A^+\} = \sharp\{i \mid A_i = A^-\}.$$

By the associativity of the $\cdot$ product (2.8) (see [9, Remark 7.6]) we see that (2.13) implies that $A_1 \cdot (A_2 \cdot (\cdots A_r)) \cdots) \subset \widetilde{V}_k(\mathfrak{g}^0)$, so $V_k(\mathfrak{g})^{(0)} = \widetilde{V}_k(\mathfrak{g}^0)$. It follows that $\widetilde{V}_k(\mathfrak{g}^0)$ is simple, hence isomorphic to $V_{\mathbf{k}}(\mathfrak{g}^0)$, and $V_k(\mathfrak{g})^{(q)}$ is a simple $V_{\mathbf{k}}(\mathfrak{g}^0)$-module for all q.

It remains to prove relation (2.11). To do this, we check that $(X_{\theta(-1)})^q\mathbf{1}$ and $(X_{-\alpha_p(-1)})^q\mathbf{1}$ are nonzero singular vectors in $V_k(\mathfrak{g})$ for all $q \in \mathbb{N}$. We first verify that they are singular for $\widehat{\mathfrak{g}^0}$ and then that they are nonzero. It is easy to see that

$$(2.14) \qquad \varpi_{(i)}(X_{\theta(-1)})^q\mathbf{1} = 0, \quad \varpi_{(i)}(X_{-\alpha_p(-1)})^q\mathbf{1} = 0$$

for all $i > 0$. E.g., using relation

$$\varpi_{(i)}(X_{\theta(-1)})^q\mathbf{1} = X_{\theta(i-1)}(X_{\theta(-1)})^{q-1}\mathbf{1} + X_{\theta(-1)}\varpi_{(i)}(X_{\theta(-1)})^{q-1}\mathbf{1}$$
$$= X_{\theta(-1)}\varpi_{(i)}(X_{\theta(-1)})^{q-1}\mathbf{1},$$

an obvious induction gives the leftmost formula in (2.14).

It is also clear that, if $\alpha_i \neq \alpha_p$, then

$$X_{\alpha_i(0)}(X_{\theta(-1)})^q\mathbf{1} = X_{\alpha_i(0)}(X_{-\alpha_p(-1)})^q\mathbf{1} = 0.$$

If β is the highest root of a simple ideal of $\mathfrak{g}^0$ and $\theta - \beta$ is not a root then $X_{-\beta(1)}(X_{\theta(-1)})^q\mathbf{1} = 0$, while, if $\gamma = \theta - \beta \in \Delta^+$ then $X_{-\beta(1)}(X_{\theta(-1)})^q\mathbf{1} = (X_{\gamma(0)})(X_{\theta(-1)})^{q-1}\mathbf{1} = 0$.

Recall that α_p has coefficient 1 in the expansion of θ in terms of simple roots. Hence $\beta + 2\alpha_p \notin \Delta$ and the same argument as above shows that $X_{-\beta(1)}(X_{-\alpha_p(-1)})^q\mathbf{1} = 0$.

We now prove by induction on q that $(X_{\theta(-1)})^q\mathbf{1}$ and $(X_{-\alpha_p(-1)})^q\mathbf{1}$ are nonzero, the base $q = 0$ being obvious.

Assume by induction that $(X_{\theta(-1)})^{q-1}\mathbf{1} \neq 0$. Then, since

$$X_{-\theta(1)}(X_{\theta(-1)})^q\mathbf{1} = -h_{\theta(0)}(X_{\theta(-1)})^{q-1}\mathbf{1} + k(X_{\theta(-1)})^{q-1}\mathbf{1}$$
$$= (-2(q-1) + k)(X_{\theta(-1)})^{q-1}\mathbf{1},$$

we have that $(X_{\theta(-1)})^q\mathbf{1}$ can be 0 only if $k = 2(q-1)$. An embedding in an integrable module can be conformal only when $k = 1$ (see e.g. [7]), thus $(X_{\theta(-1)})^q\mathbf{1} \neq 0$. Computing $x_{\alpha_p(1)}(X_{-\alpha_p(-1)})^q\mathbf{1}$, we see likewise that $(X_{-\alpha_p(-1)})^q\mathbf{1} \neq 0$. $\qquad\square$

Remark 2.1. Condition (2.5) holds whenever

$$(2.15) \qquad \sum_{u=0}^{t} \frac{(\nu_{r,i,j}^u, \nu_{r,i,j}^u + 2\rho_0^u)_0}{2(k_u + h_u^\vee)} \notin \mathbb{Z}_+,$$

for any i, j, while condition (2.10) holds whenever

$$(2.16) \qquad \sum_{u=0}^{t} \frac{(\nu_r^u, \nu_r^u + 2\rho_0^u)_0}{2(k_u + h_u^\vee)} \notin \mathbb{Z}_+$$

for any r.

2.3. **Dynkin indices and combinatorial formulas.** We now review some results by Panyushev [24], which will be used in the proof of Theorem 3.1. Recall that $\mathfrak{g}$ is simple and $\mathfrak{g}^0$ is a reductive equal rank subalgebra of $\mathfrak{g}$ with decomposition as in (2.1). We denote by $\kappa(\cdot, \cdot)$ the Killing form of $\mathfrak{g}$. Recall that $\kappa(\cdot, \cdot) = 2h^\vee(\cdot, \cdot)$, hence, denoting with same symbol the bilinear forms induced on $\mathfrak{h}^*$, we have

$$(2.17) \qquad \kappa(\lambda, \mu) = \frac{1}{2h^\vee}(\lambda, \mu) \quad \forall \lambda, \mu \in \mathfrak{h}^*.$$

Set $d_0 = 1$ and for $j > 0$

$$(2.18) \qquad d_j = \frac{2}{(\theta_j, \theta_j)},$$

where θ_j is the highest root of $\mathfrak{g}_j^0$.

If V is a finite-dimensional $\mathfrak{g}$-module and $x \in \mathfrak{g}$, let $\pi(x) \in End(V)$ denote the action of x on V. The trace form $(x, y)_V := tr(\pi(x)\pi(y))$ is an invariant form on $\mathfrak{g}$, hence there is $d_V \in \mathbb{C}$ such that $d_V(\cdot, \cdot)_V = \kappa(\cdot, \cdot)$. The number d_V is called the Dynkin index of V and denoted by $ind_{\mathfrak{g}}(V)$. The Dynkin index is clearly additive with respect to direct sums and, if V_λ is the irreducible finite-dimensional $\mathfrak{g}$-module with highest weight λ, then

$$(2.19) \qquad ind_{\mathfrak{g}}(V_\lambda) = \frac{\dim V_\lambda}{\dim \mathfrak{g}} \frac{(\lambda, \lambda + 2\rho)}{(\theta, \theta + 2\rho)}.$$

We write (2.19) in terms of the normalized invariant form, but the formula is clearly independent from the choice of the form. Let $C_{\mathfrak{g}}$ denote the Casimir element of κ and $C_{\mathfrak{g}_j^0}$ the Casimir element of $\kappa_{|\mathfrak{g}_j^0 \times \mathfrak{g}_j^0}$.

Proposition 2.5. [24, Proposition 2.2]

(1) *The eigenvalue of $C_{\mathfrak{g}_j^0}$ on $\mathfrak{g}_j^0$ is $\frac{h_j^\vee}{d_j h^\vee}$. (Recall that $h_0^\vee = 0$.)*

(2) *$ind_{\mathfrak{g}_j^0}(\mathfrak{p}) = \frac{d_j h^\vee}{h_j^\vee} - 1, \; j > 0$.*

Proposition 2.6. [24, Corollary 2.7] *Assume that $\mathfrak{g}^0$ is semisimple and that it is the fixed-point set of an automorphism of $\mathfrak{g}$ of prime order m. Then $C_{\mathfrak{g}^0}$ acts scalarly on $\mathfrak{p}$ with eigenvalue $1/m$. Also, if $m = 2$, the above statement holds with $\mathfrak{g}^0$ reductive.*

3. A criterion for conformality

Let $k \neq -h^\vee$. For a simple or abelian Lie algebra $\mathfrak{g}$ set

$$(3.1) \qquad c_{\mathfrak{g}}(k) = \frac{k \dim \mathfrak{g}}{k + h^\vee}.$$

If $\mathfrak{a}$ is a reductive Lie algebra, which decomposes as $\mathfrak{a} = \mathfrak{a}_0 \oplus \cdots \oplus \mathfrak{a}_s$ then we set, for a multiindex $\mathbf{k} = (k_0, \ldots, k_s)$,

$$(3.2) \qquad c_{\mathfrak{a}}(\mathbf{k}) = \sum_{j=0}^{s} c_{\mathfrak{a}_j}(k_j).$$

Theorem 3.1. *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{g}^0$ a maximal equal rank reductive subalgebra. Then $\widetilde{V}_k(\mathfrak{g}^0)$ is a conformal subalgebra of $V_k(\mathfrak{g})$ if and only if*

$$(3.3) \qquad c_{\mathfrak{g}}(k) = c_{\mathfrak{g}^0}(\mathbf{k}).$$

Proof. The statement is trivially verified when $k = 0$, so we can assume $k \neq 0$. Recall that $c_{\mathfrak{g}}(k)$ is the central charge of the conformal vector of $V_k(\mathfrak{g})$, and that $\sum_{j=0}^{t} c_{\mathfrak{g}_j^0}(k_j)$ is the central charge of the conformal vector of $\widetilde{V}_k(\mathfrak{g}^0)$. Hence, if $\widetilde{V}_k(\mathfrak{g}^0)$ is a conformal subalgebra of $V_k(\mathfrak{g})$, equality (3.3) holds.

To prove the converse, by the classification of finite order automorphisms of simple Lie algebras [15] and Borel–de Siebenthal Theorem [10], $\mathfrak{g}^0$ is the fixed point subalgebra of an automorphism of $\mathfrak{g}$ of finite order m, which is a prime number by maximality (indeed $m = 2, 3, 5$). Recall that if $\mathfrak{g}_0^0 \neq \{0\}$, then $\dim \mathfrak{g}_0^0 = 1$ and $m = 2$.

Recall that, as a $\mathfrak{g}^0$-module, $\mathfrak{p} = \oplus_{i=1}^s V(\mu_i)$ with $\mu_i = \sum_{j=0}^t \mu_i^j$. To prove our claim, we verify relation (2.3). We have to estimate $(\mu_i^j, \mu_i^j + 2\rho_0^j)_0$.

We start discussing the cases when m is either 2 or 3. By the additivity of the index, combining (2.19) with part (2) of Proposition 2.5 (and using the conversion formulas (2.17), (2.18)) we get, if $j > 0$,

$$\frac{d_j h^\vee}{h_j^\vee} - 1 = \sum_{i=1}^s \frac{\dim V(\mu_i)}{\dim \mathfrak{g}_j^0} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_0}{(\theta_j, \theta_j + 2\rho_0^j)_0}.$$

If $m = 2, 3$, then $s = 1$ or $s = 2$ and $V(\mu_2) = V(\mu_1)^*$. It follows that, in both cases, $\dim V(\mu_i)$ as well as $(\mu_i^j, \mu_i^j + 2\rho_0^j)_0$ are independent of i, so $\dim V(\mu_i) = \frac{\dim \mathfrak{p}}{s}$ and

$$\frac{d_j h^\vee}{h_j^\vee} - 1 = \frac{\dim \mathfrak{p}}{\dim \mathfrak{g}_j^0} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_0}{(\theta_j, \theta_j + 2\rho_0^j)_0}.$$

Since $(\theta_j, \theta_j + 2\rho_0^j)_0 = 2h_j^\vee$, we have

$$(3.4) \qquad (\mu_i^j, \mu_i^j + 2\rho_0^j)_0 = \frac{2 \dim \mathfrak{g}_j^0}{\dim \mathfrak{p}} (d_j h^\vee - h_j^\vee).$$

Let us discuss the case $j = 0$. Assume that $\mathfrak{g}_0^0 \neq \{0\}$. We need to compute (μ_i^0, μ_i^0) i.e. (ζ, ζ) (cf. (2.7)). Since the roots of $\mathfrak{p}$ are precisely the roots α such that $|\alpha(\varpi)| = 1$, we have $\kappa(\varpi, \varpi) = \dim \mathfrak{p}$, hence $h_\zeta = \frac{1}{\dim \mathfrak{p}} \varpi$ is the unique element of $\mathfrak{h}$ such that $\kappa(h_\zeta, h) = \zeta(h)$. Hence $\kappa(\zeta, \zeta) = \kappa(h_\zeta, h_\zeta) = \frac{1}{\dim \mathfrak{p}}$. It follows form the relation $\kappa(\cdot, \cdot) = \frac{1}{2h^\vee}(\cdot, \cdot)$ that $(\zeta, \zeta) = \frac{2h^\vee}{\dim \mathfrak{p}}$, so (3.4) holds also in this case.

Now we proceed to evaluate (3.1). Setting $g = \dim \mathfrak{g}$, $g_i = \dim \mathfrak{g}_i^0$, $i = 0, \ldots, t$, we may write (3.1) as

$$(3.5) \qquad \frac{gk}{k + h^\vee} - \sum_{j=0}^t \frac{d_j g_j k}{d_j k + h_j^\vee} = 0.$$

Multiplying (3.5) by $\frac{k+h^\vee}{k}$ we find that

$$0 = g - \sum_{j=0}^t \frac{d_j g_j (k + h^\vee)}{d_j k + h_j^\vee} = g - \sum_{j=0}^t g_j + \sum_{j=0}^t g_j - \sum_{j=0}^t \frac{d_j g_j (k + h^\vee)}{d_j k + h_j^\vee}$$

$$= \dim \mathfrak{p} + \sum_{j=0}^t \left(g_j - \frac{d_j g_j (k + h^\vee)}{d_j k + h_j^\vee} \right) = \dim \mathfrak{p} - \sum_{j=0}^t \frac{g_j (d_j h^\vee - h_j^\vee)}{d_j k + h_j^\vee}.$$

It follows that

$$(3.6) \qquad 1 = \sum_{j=0}^{t} \frac{2g_j(d_j h^\vee - h_j^\vee)}{2 \dim \mathfrak{p}(d_j k + h_j^\vee)}$$

and, using (3.4), we find

$$(3.7) \qquad \sum_{j=0}^{t} \frac{(\mu_i^j, \mu_i^j + 2\rho_0^j)_0}{2(d_j k + h_j^\vee)} = 1.$$

In the remaining $m = 5$ case we have that $\mathfrak{g}$ is of type E_8 and $\mathfrak{g}^0$ is of type $A_4 \times A_4$. In this case the equality of central charges reads

$$\frac{248}{k+30} = \frac{48}{k+5}.$$

so that $k = 1$. Formula (2.3) reduces to

$$\frac{1}{12} \sum_{j=1}^{2} (\mu_i^j, \mu_i^j + 2\rho_0^j)_0 = 1.$$

This is readily checked since, by Proposition 2.6, we have $\sum_{j=1}^{2} \kappa(\mu_i^j, \mu_i^j + 2\rho_0^j) = \frac{1}{m}$ hence $\sum_{j=1}^{2} (\mu_i^j, \mu_i^j + 2\rho_0^j)_0 = \frac{2h^\vee}{m} = 12$.

We now discuss the extension of Theorem 3.1 to the embeddings of an equal rank subalgebra $\mathfrak{g}^0$ in $\mathfrak{g}$ which is not maximal. We start with the following

Lemma 3.2. *Let $\mathfrak{g}^0$ be a maximal conformal subalgebra of $\mathfrak{g}$ (i.e., maximal among conformal subalgebras of $\mathfrak{g}$). Then $\mathfrak{g}^0$ is a maximal reductive subalgebra of $\mathfrak{g}$.*

Proof. Suppose by contradiction that there exists a reductive subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ with $\mathfrak{g}^0 \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}$. Then, since the form $(\cdot, \cdot)$ is nondegenerate when restricted to $\mathfrak{k}$, the orthocomplement $\mathfrak{p}$ of $\mathfrak{g}^0$ in $\mathfrak{g}$ can be written as $\mathfrak{p} = \mathfrak{p} \cap \mathfrak{k} \oplus V$ with V the orthocomplement of $\mathfrak{k}$ in $\mathfrak{g}$. If the embedding of $\mathfrak{g}^0$ in $\mathfrak{g}$ is conformal, then, by Theorem 2.1, $(\omega_{\mathfrak{g}^0})_0 x_{(-1)} \mathbf{1} = x_{(-1)} \mathbf{1}$ for all $x \in \mathfrak{p} \cap \mathfrak{k}$. It follows that the embedding of $\mathfrak{g}^0$ in $\mathfrak{k}$ is conformal, hence, clearly, also the embedding of $\mathfrak{k}$ in $\mathfrak{g}$ is conformal. $\qquad \square$

This observation leads to the following criterion.

Proposition 3.3. *Let $\mathfrak{g}^0 = \mathfrak{k}_1 \subset \mathfrak{k}_2 \subset \cdots \subset \mathfrak{k}_t = \mathfrak{g}$ be a sequence of equal rank subalgebras with $\mathfrak{k}_i$ maximal in $\mathfrak{k}_{i+1}$. Let $\mathbf{k}_i$ be the multi-index such that the vertex subalgebra spanned in $V_k(\mathfrak{g})$ by $\{x_{(-1)} \mathbf{1} \mid x \in \mathfrak{k}_i\}$ is a quotient of $V^{\mathbf{k}_i}(\mathfrak{k}_i)$. Then $\mathfrak{g}^0 \subset \mathfrak{g}$ is a conformal embedding if and only if*

$$(3.8) \qquad c_{\mathfrak{k}_i}(\mathbf{k}_i) = c_{\mathfrak{k}_{i-1}}(\mathbf{k}_{i-1}), \ \ i = 1, \ldots, t.$$

Proof. If $\mathfrak{g}^0 \subset \mathfrak{g}$ is a conformal embedding, then, by the above lemma, $\mathfrak{k}_{i-1} \subset \mathfrak{k}_i$ is a conformal embedding. Since the two subalgebras contain a Cartan subalgebra of $\mathfrak{g}$, we have that there are ideals $\bar{\mathfrak{k}}_i, \mathfrak{k}'_i$ of $\mathfrak{k}_i$ with $\bar{\mathfrak{k}}_i$ simple or abelian, and an ideal $\bar{\mathfrak{k}}_{i-1}$ of $\mathfrak{k}_{i-1}$ such that $\mathfrak{k}_i = \bar{\mathfrak{k}}_i \oplus \mathfrak{k}'_i$, $\mathfrak{k}_{i-1} = \bar{\mathfrak{k}}_{i-1} \oplus \mathfrak{k}'_i$ and $\bar{\mathfrak{k}}_{i-1} \subset \bar{\mathfrak{k}}_i$ maximal embedding. Applying Theorem 3.1 to the embedding $\bar{\mathfrak{k}}_{i-1} \subset \bar{\mathfrak{k}}_i$, we obtain $c_{\mathfrak{k}_i}(\mathbf{k}_i) = c_{\mathfrak{k}_{i-1}}(\mathbf{k}_{i-1})$.

If condition (3.8) holds, then, by Theorem 3.1, the embedding $\bar{\mathfrak{k}}_{i-1} \subset \bar{\mathfrak{k}}_i$ is conformal, hence the embedding $\mathfrak{k}_{i-1} \subset \mathfrak{k}_i$ is conformal for $i = 1, \ldots, t$, so the embedding $\mathfrak{g}^0 \subset \mathfrak{g}$ is conformal. $\square$

Definition 3.1. We say that a level k of $V_k(\mathfrak{g})$ for which equality (3.3) holds is a *conformal level* of the pair $(\mathfrak{g}, \mathfrak{g}^0)$.

In the following tables we list the conformal levels for all reductive maximal equal rank subalgebras of simple Lie algebras, thus classifying all possible maximal conformal equal rank embeddings. The cases where $\mathfrak{g}^0$ is not semisimple are denoted by $X \times Z$ where X is the type of $[\mathfrak{g}^0, \mathfrak{g}^0]$ and Z denotes the one-dimensional center of $\mathfrak{g}^0$.

Type A_n, $n \geq 1$

$\mathfrak{g}^0$	conformal level
$A_h \times A_{n-h-1} \times Z$, $h \geq 1$, $n - h \geq 2$	$1; -1; -\frac{n+1}{2}$ (if $h \neq (n-1)/2$)
$A_{n-1} \times Z$, $n \geq 1$	$1; -\frac{n+1}{2}$ (if $n > 1$)

Type D_n, $n \geq 4$

$\mathfrak{g}^0$	conformal levels
$D_h \times D_{n-h}$, $h \geq 2$, $n - h \geq 2$	$1; 2 - n$ (if $h \neq n/2$)
$D_{n-1} \times Z$	$1; 2 - n$
$A_{n-1} \times Z$	$1; -2$

Type C_n, $n \geq 2$

$\mathfrak{g}^0$	conformal levels
$C_h \times C_{n-h}$, $h \geq 1$, $n - h \geq 1$	$-\frac{1}{2}; -1 - \frac{n}{2}$ (if $h \neq n/2$)
$A_{n-1} \times Z$	$1; -\frac{1}{2}$

Type B_n, $n \geq 3$

$\mathfrak{g}^0$	conformal levels
$D_h \times B_{n-h}$, $h \geq 1$, $n - h \geq 1$	$1; \frac{3}{2} - n$
D_n	$\frac{3}{2} - n$
$B_{n-1} \times Z$, $n \geq 4$	$1; \frac{3}{2} - n$

Remark 3.2. Assume $\mathfrak{g}^0$ semisimple and let $\mathfrak{g}^0 = \mathfrak{g}^0_1 \oplus \ldots \oplus \mathfrak{g}^0_t$ be its decomposition into simple ideals. Recall that a set of simple roots for $\mathfrak{g}^0$

can be obtained from $\Pi \cup \{-\theta\}$ by removing a simple root of Π. Call this root α_p, and set

$$\xi = \begin{cases} -\frac{1}{2} & \text{if } \alpha_p \text{ is short,} \\ 1 & \text{otherwise.} \end{cases}$$

Then the previous tables show that equality of central charges occurs at the following levels:

- $k = -\frac{h^\vee}{2} + 1$ if $t = 1$;
- $k \in \{\xi, -\frac{h^\vee}{2} + \xi\}$ if $t > 1$ and there are at least two non-isomorphic simple ideals or all simple ideals are isomorphic but two of them have different Dynkin index;
- $k = \xi$ otherwise, i.e., all simple ideals are isomorphic and have the same Dynkin index.

Type E_6

$\mathfrak{g}^0$	conformal levels
$A_1 \times A_5$	$1; -3$
$A_2 \times A_2 \times A_2$	1
$D_5 \times Z$	$1; -3$

Type E_7

$\mathfrak{g}^0$	conformal levels
$A_1 \times D_6$	$1; -4$
$A_2 \times A_5$	$1; -4$
A_7	1
$E_6 \times Z$	$1; -4$

Type E_8

$\mathfrak{g}^0$	conformal levels
$A_1 \times E_7$	$1; -6$
$A_2 \times E_6$	$1; -6$
$A_4 \times A_4$	1
D_8	1
A_8	1

Type F_4

$\mathfrak{g}^0$	conformal levels
$A_1 \times C_3$	$1; -\frac{5}{2}$
$A_2 \times A_2$	$1; -\frac{5}{2}$
B_4	$-\frac{5}{2}$

Type G_2

$\mathfrak{g}^0$	conformal levels
$A_1 \times A_1$	$1; -\frac{5}{3}$
A_2	$-\frac{5}{3}$

Remark 3.3. Recall the Deligne exceptional series [12], [23]

$$A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8.$$

Set $\mathfrak{a} = Span_{\mathbb{C}}\{X_\beta \mid (\beta, \theta) = 0\} + h_\theta^\perp$. We remark that if $\mathfrak{g} \not\cong A_1, D_4$ is in the Deligne series, then $\mathfrak{a}$ is simple or abelian. Moreover, there is always a maximal equal rank conformal subalgebra of the form $sl(2) \times \mathfrak{a}$ where $sl(2) = \mathbb{C}X_\theta \oplus \mathbb{C}h_\theta \oplus \mathbb{C}X_{-\theta}$. For this maximal subalgebra, the conformal levels are 1 and $-h^\vee/6 - 1$. This numerological coincidence can be explained as follows. Let $W_{min}(k)$ be the minimal simple W-algebra of level k for $\mathfrak{g}$. By [8, Theorem 7.2], $W_{min}(-h^\vee/6 - 1)$ is 1-dimensional. On the other hand, by [21, Theorem 5.1(d)], $W_{min}(k)$ has a subalgebra of currents $\widehat{\mathfrak{a}}$ of level $k + (h^\vee - h_\mathfrak{a}^\vee)/2$, where $h^\vee$ is Coxeter number of $\mathfrak{g}$ and $h_\mathfrak{a}^\vee$ that of $\mathfrak{a}$. Hence this level should be 0 when $k = -h^\vee/6 - 1$, therefore

(3.9)
$$-\frac{h^\vee}{6} - 1 = -\frac{h^\vee - h_\mathfrak{a}^\vee}{2}.$$

Now recall that, for $\mathfrak{g}$ in the Deligne series, we have $\dim \mathfrak{g} = \frac{2(h^\vee+1)(5h^\vee-6)}{h^\vee+6}$ [12]. Since the number of roots not orthogonal to θ is $4h^\vee - 6$ [27], one easily verifies that the degree 3 equation $c_k(\mathfrak{g}) = c_k(\mathfrak{a}) + c_k(sl(2))$ in the variable k has exactly 1 and $-h^\vee/6 - 1$ as nonzero roots.

4. Finite decomposition for maximal equal rank semisimple embeddings

In this section we determine precisely, for all pairs $(\mathfrak{g}, \mathfrak{g}^0)$ with $\mathfrak{g}^0$ semisimple, the conformal levels such that the decomposition is finite. The main result is the following theorem, which will be proved along the section.

Theorem 4.1. If $\mathfrak{g}^0$ is semisimple, the conformal levels different from 1 for which $V_k(\mathfrak{g})$ decomposes as a finite sum of $\widehat{\mathfrak{g}^0}$-irreducibles are the following:

conformal level	$\mathfrak{g}$	$\mathfrak{g}^0$
$-1/2$	C_n	$C_h \times C_{n-h}$
$-n + 3/2$	B_n	D_n
$-5/3$	G_2	A_2
$-5/2$	F_4	B_4

In type C_n, h ranges from 1 to $n - 1$; by C_1 we mean A_1. In type B_3 we set $D_3 = A_3$.

We start the proof of Theorem 4.1 with a simple computation.

Lemma 4.2. Let η be a highest weight of $\mathfrak{p}$. Set $v_m = (X_\eta)_{(-1)}^m \mathbf{1}$. Then

$$(X_{-\eta})_{(1)} v_{m+1} = ((m+1)k - \|\eta\|^2 \binom{m+1}{2}) v_m.$$

Proof. We prove the formula by induction on m. Assume that $(X_\eta, X_{-\eta}) = 1$. If $m = 0$,

$$(X_{-\eta})_{(1)}(X_\eta)_{(-1)}\mathbf{1} = -(h_\eta)_{(0)}\mathbf{1} + k\mathbf{1} = kv_0.$$

If $m > 0$,

$$(X_{-\eta})_{(1)}v_{m+1} = -(h_\eta)_{(0)}v_m + kv_m + (X_\eta)_{(-1)}(X_{-\eta})_{(1)}v_m.$$

hence

$$(X_{-\eta})_{(1)}v_{m+1} = -m\|\eta\|^2 v_m + kv_m + (mk - \|\eta\|^2\binom{m}{2})v_m$$

$$= ((m+1)k - \|\eta\|^2\binom{m+1}{2})v_m.$$

Let $\mathfrak{g}^0$ be semisimple. Let α_p be as in Remark 3.2.

Proposition 4.3. *If α_p is long then the decomposition is finite if and only if $k = 1$.*

Proof. By [15], the decomposition is finite if $k = 1$ and α_p is long. Assume now that the decomposition is finite. Since α_p is a weight of $\mathfrak{p}$, there is μ_i in the decomposition $\mathfrak{p} = \oplus_j V(\mu_j)$ such that α_p occurs as a weight of $V(\mu_i)$. Set $\eta = \mu_i$. Since α_p is a long root, the same holds for η. Let θ_j be the highest root of $\mathfrak{g}^0_j$. Since η is the highest weight of $\mathfrak{p}$, $\eta + \theta_j \notin \Delta$. Since η is long $|(\theta_j, \eta^\vee)| \le 1$. This implies that $-\theta_j + 2\eta \notin \Delta$. Since $-\theta_j - \eta$ is not a root, the η-root string through $-\theta_j$ is $-\theta_j, \ldots, -\theta_j + q\eta$. Since $-q = -(\theta_j, \eta^\vee)$, we see that $q \le 1$.

We now prove by induction on m, that the vectors v_m of Lemma 4.2 are $\widehat{\mathfrak{g}^0}$-singular. If $m = 0$ this is obvious. Let Δ_0 be the $(\mathfrak{g}^0, \mathfrak{h})$-root system and let $\Delta_0^+ = \Delta^+ \cap \Delta_0$ be a set of positive roots. It is clear that if $\alpha \in \Delta_0^+$ then $\eta + \alpha$ is not a root (recall that η is a highest weight in $\mathfrak{p}$), so $[(X_\alpha)_{(0)}, (X_\eta)_{(-1)}] = 0$. This implies that, if $m > 0$, $(X_\alpha)_{(0)}v_m = 0$, so we need only to check that $(X_{-\theta_j})_{(1)}v_m = 0$ for all j. Clearly

$$(X_{-\theta_j})_{(1)}v_m = [X_{-\theta_j}, X_\eta]_{(0)}v_{m-1} + (X_\eta)_{(-1)}(X_{-\theta_j})_{(1)}v_{m-1}$$

$$= [X_{-\theta_j}, X_\eta]_{(0)}v_{m-1}.$$

The last equality follows from the induction hypothesis. Since $-\theta_j + 2\eta$ is not a root we see that $[[X_{-\theta_j}, X_\eta], X_\eta] = 0$. This implies that

$$[X_{-\theta_j}, X_\eta]_{(0)}v_{m-1} = (X_\eta)_{(1)}^{m-1}[X_{-\theta_j}, X_\eta]_{(0)}\mathbf{1} = 0$$

as desired. Since the decomposition is finite, the vectors v_m must span a finite dimensional space. Since they have different weights, they are independent if nonzero, so they are almost all zero. Let M be such that $v_M \ne 0$ and $v_m = 0$ for $m > M$. By Lemma 4.2, since $\|\eta\|^2 = 2$,

$$0 = (X_{-\eta})_{(1)}v_{M+1} = (M+1)(k-M)v_M.$$

so $k = M$. By [22], finite decomposition at positive integral levels can happen only if $k = 1$. □

It remains to deal with the cases with α_p short. Looking at tables in the previous section these are $C_h \times C_{n-h} \hookrightarrow C_n$, $D_n \hookrightarrow B_n$, $A_2 \hookrightarrow G_2$, and $B_4 \hookrightarrow F_4$. The latter three cases are dealt with in the work of Adamović and Perše (see [22, 3.5] for a thorough discussion and precise attributions). The first case is analyzed in the next subsection.

4.1. $C_h \times C_{n-h}$. Consider the level $-\frac{1}{2}$ case. Then Theorem 2.3 applies and the decomposition is finite.

Let us deal with level $-1 - \frac{n}{2}$. Realize as usual the root system of type C_n in terms of the standard basis in $\mathbb{R}^n$ as $\pm\{\epsilon_i \pm \epsilon_j, \mid i \neq j\} \cup \pm\{2\epsilon_i\}$. We can choose root vectors in such a way that the following relations hold:

$$[X_{-2\epsilon_1}, X_{\epsilon_1+\epsilon_{h+1}}] = -X_{-\epsilon_1+\epsilon_{h+1}}, \qquad [X_{-\epsilon_1+\epsilon_{h+1}}, X_{\epsilon_1+\epsilon_{h+1}}] = 2X_{2\epsilon_{h+1}},$$
$$[X_{-2\epsilon_{h+1}}, X_{\epsilon_1+\epsilon_{h+1}}] = -X_{\epsilon_1-\epsilon_{h+1}}, \qquad [X_{\epsilon_1-\epsilon_{h+1}}, X_{\epsilon_1+\epsilon_{h+1}}] = 2X_{2\epsilon_1},$$
$$[X_{-\epsilon_1+\epsilon_{h+1}}, X_{2\epsilon_1}] = X_{\epsilon_1+\epsilon_{h+1}}, \qquad [X_{\epsilon_1-\epsilon_{h+1}}, X_{2\epsilon_{h+1}}] = X_{\epsilon_1+\epsilon_{h+1}}.$$

Moreover
$$(X_{2\epsilon_i}, X_{-2\epsilon_i}) = 1, \ i = 1, \dots, n.$$

With this choice of root vectors set
$$v_{i,j} = (X_{\epsilon_1+\epsilon_{h+1}})^i_{(-1)}(X_{2\epsilon_1})^j_{(-1)}(X_{2\epsilon_{h+1}})^j_{(-1)}\mathbf{1}.$$

Lemma 4.4.

(4.1)
$$(X_{-2\epsilon_1})_{(1)}v_{i,j} = (k-i-j+1)j(X_{2\epsilon_{h+1}})_{(-1)}v_{i,j-1} - i(i-1)(X_{2\epsilon_{h+1}})_{(-1)}v_{i-2,j}.$$

(4.2)
$$(X_{-2\epsilon_{h+1}})_{(1)}v_{i,j} = (k-i-j+1)j(X_{2\epsilon_1})_{(-1)}v_{i,j-1} - i(i-1)(X_{2\epsilon_1})_{(-1)}v_{i-2,j}.$$

(4.3) $\qquad (X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_{i,j} = i(2k - (i-1) - 4j)v_{i-1,j} - j^2 v_{i+1,j-1}.$

Proof. Direct computation, by induction on $i + j$.

Lemma 4.5. *Consider $v_{i,j}$ as an element of $V^k(\mathfrak{g})$ and let S_m be the linear span of $\{v_{i,j} \mid i + 2j = m\}$. Set*

$$v_m = \begin{cases} \sum\limits_{s=0}^{t} \dfrac{\binom{k-t+1}{s}\binom{t}{s}}{\binom{2s}{s}} v_{2s,t-s}, & \text{if } m = 2t, \\ \sum\limits_{s=0}^{t} \dfrac{\binom{k-t}{s}\binom{t}{s}}{(s+1)\binom{2s+1}{s}} v_{2s+1,t-s} & \text{if } m = 2t+1. \end{cases}$$

Then

    (1) *$\mathbb{C}v_m$ is the space of $\widehat{\mathfrak{g}^0}$-singular vector in S_m.*
    (2) *$(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = c_m v_{m-1}$ with*

$$c_m = \begin{cases} \frac{m}{2}(m^2 + (-4-3k)m + 2k^2 + 5k + 3) & \text{if } m \text{ is even,} \\ 2k - 2m + 2 & \text{if } m \text{ is odd.} \end{cases}$$

Proof. Assume first $m = 2t$ even. In order to prove that v_m is $\widehat{\mathfrak{g}}^0$-singular we need only to check that

$$(X_{-2\epsilon_1})_{(1)}v_m = (X_{-2\epsilon_{h+1}})_{(1)}v_m = 0.$$

Set $a_{s,t} = \frac{\binom{k-t+1}{s}\binom{t}{s}}{\binom{2s}{s}}$. Then, by Lemma 4.4,

$$(X_{-2\epsilon_1})_{(1)}v_m = \sum_{s=0}^{t-1} a_{s,t}(k-s-t+1)(t-s)(X_{2\epsilon_{h+1}})_{(-1)}v_{2s,t-s-1}$$

$$-2\sum_{s=1}^{t} a_{s,t}s(2s-1)(X_{2\epsilon_{h+1}})_{(-1)}v_{2s-2,t-s} =$$

(4.4)

$$(X_{2\epsilon_{h+1}})_{(-1)}\sum_{s=0}^{t-1}(a_{s,t}(k-s-t+1)(t-s) - a_{s+1,t}2(s+1)(2s+1))v_{2s,t-s-1}.$$

Since $\frac{(k-s-t+1)(t-s)}{2(s+1)(2s+1)}a_{s,t} = a_{s+1,t}$, (4.4) vanishes, as required. The same computation shows that $(X_{-2\epsilon_{h+1}})_{(1)}v_m = 0$.

Let $v = \sum_{s=0}^{t} c_{s,t}v_{2s,t-s}$ be $\widehat{\mathfrak{g}}^0$-singular. Then the computation above and the fact that the $v_{i,j}$ are linearly independent show that

$$(k-s-t+1)(t-s)c_{s,t} = 2(s+1)(2s+1)c_{s+1,t},$$

so v is determined by the choice of $c_{0,t}$, thus the space of $\widehat{\mathfrak{g}}^0$-singular vectors in S_m is one-dimensional. This proves (1).

Next observe that $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m$ is $\widehat{\mathfrak{g}}^0$-singular. It suffices to prove that

$$(X_{\epsilon_1-\epsilon_2})_{(0)}(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = (X_{\epsilon_{h+1}-\epsilon_{h+2}})_{(0)}(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = 0.$$

Let us check only that $(X_{\epsilon_1-\epsilon_2})_{(0)}(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = 0$, since the other equality is obtained similarly. The latter follows at once from the fact that

$$(X_{\epsilon_1-\epsilon_2})_{(0)}(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = (X_{-\epsilon_2-\epsilon_{h+1}})_{(1)}v_m$$

and that

$$(X_{-\epsilon_2-\epsilon_{h+1}})_{(1)}v_{i,j} = i(X_{\epsilon_1+\epsilon_{h+1}})_{(-1)}^{i-1}(X_{2\epsilon_1})_{(-1)}^{j}(X_{2\epsilon_{h+1}})_{(-1)}^{j}(X_{\epsilon_1-\epsilon_2})_{(0)}\mathbf{1}$$

$$+ j(X_{\epsilon_1+\epsilon_{h+1}})_{(-1)}^{i}(X_{2\epsilon_1})_{(-1)}^{j}(X_{2\epsilon_{h+1}})_{(-1)}^{j-1}(X_{-\epsilon_2+\epsilon_{h+1}})_{(0)}\mathbf{1} = 0.$$

It follows from (4.3) that $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m \in S_{m-1}$, so, since the space of $\widehat{\mathfrak{g}}^0$-singular vectors in S_{m-1} is one-dimensional, $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m = c_m v_{m-1}$. To compute the coefficient c_m we need only to compute the coefficient of $v_{1,m/2-1}$ in $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m$ if m is even and the coefficient of $v_{0,(m-1)/2}$ in $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)}v_m$ if m is odd. By (4.3) this coefficient is $-(m/2)^2 + (2k - 2m + 3)(k - m/2 + 1)m/2$ if m is even and it is equal to $(2k - 2(m-1))$ if m is odd. $\qquad\square$

Corollary 4.6. *In type C_n with $n \geq 2$, if $k = -1 - \frac{n}{2}$, then, for each $m \geq 0$, v_m projects to a nonzero $\widehat{\mathfrak{g}^0}$-singular vector in $V_k(\mathfrak{g})$. In particular, the decomposition of $V_k(\mathfrak{g})$ as $\widehat{\mathfrak{g}^0}$-module cannot be finite.*

Proof. For the first statement, we need only to check that $c_m \neq 0$ if $m \geq 1$. In fact, since $(X_{-\epsilon_1-\epsilon_{h+1}})_{(1)} v_m = c_m v_{m-1}$, the result follows by an obvious induction.

If $k = -1 - \frac{n}{2}$ and m is odd, then $c_m = -n - 2m < 0$. If m is even, then we need to check that

(4.5) $$m^2 + (-1 + \tfrac{3}{2}n)m + \tfrac{n(n-1)}{2} \neq 0.$$

Solving for m the equation $m^2 + (-1 + \frac{3}{2}n)m + \frac{n(n-1)}{2} = 0$, we find $m = 1 - n$ or $m = -\frac{n}{2}$, hence (4.5) holds.

Since the vectors v_m have different weights, they are linearly independent, thus the second statement follows. $\qquad \square$

Proposition 4.7. *If $\mathfrak{g}^0$ is a maximal semisimple equal rank subalgebra of $\mathfrak{g}$ and k is a conformal level, then there are infinitely many $\widehat{\mathfrak{g}^0}$-singular vectors in $V_k(\mathfrak{g})$ if and only if there is a $\widehat{\mathfrak{g}^0}$-singular vector in $V_k(\mathfrak{g})$ having conformal weight 2.*

Proof. Recall that $\mathfrak{g}^0$ is the fixed-points subalgebra of an automorphism σ of $\mathfrak{g}$ of finite order. If the decomposition is finite then the singular vectors are $\mathbf{1}$ and the vectors $x_{(-1)}\mathbf{1}$ with x an highest weight vector for the components $V(\mu_i)$ of $\mathfrak{p}$. In fact, if $k = 1$, then we know from [15] that there are a_p summands (a_p being the label of α_p) in the $\widehat{\mathfrak{g}^0}$-decomposition of $V_k(\mathfrak{g})$. If x is an highest weight vector for $V(\mu_i)$, then $x_{(-1)}\mathbf{1}$ is a singular vector for $\widehat{\mathfrak{g}^0}$. Since a_p coincides with the order of σ, these singular vectors give the whole decomposition. If $k \neq 1$, we apply Theorem 2.3, to obtain the same result.

If the decomposition is not finite, then it is easy to check that at least one of the infinitely many singular vectors that we constructed has conformal weight 2. $\qquad \square$

Remark 4.1. In Appendix 6 we give the explicit decompositions for all the cases where finite decomposition occurs.

5. FINITE DECOMPOSITION FOR MAXIMAL EQUAL RANK REDUCTIVE EMBEDDINGS

Assume that $\mathfrak{g}^0$ is a maximal equal rank subalgebra of $\mathfrak{g}$ such that the center $\mathfrak{g}_0^0$ of $\mathfrak{g}^0$ is nonzero. Let ϖ and ζ be as in Section 2.2. As shown in [11], at level $k = 1$ the decomposition is not finite, but the eigenspaces of the action of $\varpi_{(0)}$ admit finite decomposition. In this section we discuss finite decomposition of $\varpi_{(0)}$-eigenspaces for conformal levels different from 1. In the following we write weights of the simple ideals $\mathfrak{g}_j^0$ of $\mathfrak{g}^0$ as linear combinations of the fundamental weights ω_i of $\mathfrak{g}_j^0$ (to avoid cumbersome notation, we will not make explicit the depencence on j unless it is necessary).

In Theorem 5.1 we list the cases where condition (2.16) holds, hence Theorem 2.4 applies. Case (5) already appears in [4]; case (4) can be derived at once from [4] using results from [13]; the methods of [4] actually cover also case (4) with $n = 3$, where (2.16) does not hold: see Theorem 5.2 below, where more instances of finite decomposition will be given.

Theorem 5.1. *Assume we are in one of the following cases.*

(1) *Type $A_{h-1} \times A_{n-h} \times \mathbb{C}\varpi$ in A_n with $n > 5$, $h > 2$ and $n - h > 1$, conformal level $k = -1$.*
(2) *Type $A_{n-1} \times \mathbb{C}\varpi$ in A_n with $n > 3$, conformal level $k = -\frac{n+1}{2}$.*
(3) *Type $A_{n-1} \times \mathbb{C}\varpi$ in D_n with $n > 4$, conformal level $k = -2$.*
(4) *Type $A_{n-1} \times \mathbb{C}\varpi$ in C_n with $n > 3$, conformal level $k = -\frac{1}{2}$.*
(5) *Type $D_5 \times \mathbb{C}\varpi$ in E_6, conformal level $k = -3$.*
(6) *Type $E_6 \times \mathbb{C}\varpi$ in E_7, conformal level $k = -4$.*

Then $\widetilde{V}_k(\mathfrak{g}^0) \cong V_{\mathbf{k}}(\mathfrak{g}^0)$, the $\varpi_{(0)}$-eigenspaces in $V_k(\mathfrak{g})$ are irreducible $\widehat{\mathfrak{g}^0}$-modules and the decomposition of $V_k(\mathfrak{g})$ as a $\widehat{\mathfrak{g}^0}$-module is given by formula (2.12).

Proof. As observed in Remark 2.1, it is enough to check in each case that condition (2.16) holds:

(1) In this case $V(\theta) = V_{A_{h-1}}(\omega_1) \otimes V_{A_{n-h}}(\omega_{n-h}) \otimes V_{\mathbb{C}\varpi}(\zeta)$ and $V(-\alpha_h) = V_{A_{h-1}}(\omega_{h-1}) \otimes V_{A_{n-h}}(\omega_1) \otimes V_{\mathbb{C}\varpi}(-\zeta)$, thus $V(\theta) \otimes V(-\alpha_h)$ decomposes as

$$V_{A_{h-1}}(0) \otimes V_{A_{n-h}}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{A_{h-1}}(\omega_1 + \omega_{h-1}) \otimes V_{A_{n-h}}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{A_{h-1}}(0) \otimes V_{A_{n-h}}(\omega_1 + \omega_{n-h}) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{A_{h-1}}(\omega_1 + \omega_{h-1}) \otimes V_{A_{n-h}}(\omega_1 + \omega_{n-h}) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summands above $1 + \frac{1}{h-1}$, $1 + \frac{1}{n-h}$, $2 + \frac{1}{h-1} + \frac{1}{n-h}$ respectively.

(2) We have $V(\theta) = V_{A_{n-1}}(\omega_{n-1}) \otimes V_{\mathbb{C}\varpi}(\zeta)$ and $V(-\alpha_1) = V_{A_{n-1}}(\omega_1) \otimes V_{\mathbb{C}\varpi}(-\zeta)$, thus $V(\theta) \otimes V(-\alpha_1)$ decomposes as

$$V_{A_{n-1}}(0) \oplus V(0) \otimes V_{A_{n-1}}(\omega_1 + \omega_{n-1}) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summand above $2 + \frac{2}{n-1}$.

(3) We have $V(\theta) = V_{A_{n-1}}(\omega_2) \otimes V_{\mathbb{C}\varpi}(\zeta)$ and $V(-\alpha_n) = V_{A_{n-1}}(\omega_{n-2}) \otimes V_{\mathbb{C}\varpi}(-\zeta)$, thus $V(\theta) \otimes V(-\alpha_n)$ decomposes as

$$V_{A_{n-1}}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{A_{n-1}}(\omega_1 + \omega_{n-1}) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{A_{n-1}}(\omega_2 + \omega_{n-2}) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summands above $1 + \frac{2}{n-2}$, $2 + \frac{2}{n-2}$, respectively.

(4) We have $V(\theta) = V_{A_{n-1}}(2\omega_1) \otimes V_{\mathbb{C}\varpi}(\zeta)$ and $V(-\alpha_n) = V_{A_{n-1}}(2\omega_{n-1})$, thus $V(\theta) \otimes V(-\alpha_n) \otimes V_{\mathbb{C}\varpi}(-\zeta)$ decomposes as

$$V_{A_{n-1}}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$V_{A_{n-1}}(\omega_1 + \omega_{n-1}) \otimes V_{\mathbb{C}\varpi}(0)$$
$$V_{A_{n-1}}(2\omega_1 + 2\omega_{n-1}) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summands above $1 + \frac{1}{n-1}$, $2 + \frac{2}{n-1}$, respectively.

(5) We have $V(\theta) = V_{D_5}(\omega_4) \otimes V_{\mathbb{C}\varpi}(\zeta)$, $V(-\alpha_1) = V_{D_5}(\omega_5) \otimes V_{\mathbb{C}\varpi}(-\zeta)$, thus $V(\theta) \otimes V(-\alpha_1)$ decomposes as

$$V_{D_5}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$V_{D_5}(\omega_2) \otimes V_{\mathbb{C}\varpi}(0)$$
$$V_{D_5}(\omega_4 + \omega_5) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summands above $\frac{8}{5}$, $\frac{12}{5}$, respectively.

(6) We have $V(\theta) = V_{E_6}(\omega_1) \otimes V_{\mathbb{C}\varpi}(\zeta)$, $V(-\alpha_7) = V_{E_6}(\omega_6) \otimes V_{\mathbb{C}\varpi}(-\zeta)$, thus $V(\theta) \otimes V(-\alpha_7)$ decomposes as

$$V_{E_6}(0) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{E_6}(\omega_2) \otimes V_{\mathbb{C}\varpi}(0)$$
$$\oplus V_{E_6}(\omega_1 + \omega_6) \otimes V_{\mathbb{C}\varpi}(0),$$

so the corresponding formula in (2.16) yields for the nontrivial summands above $\frac{3}{2}$, $\frac{9}{4}$, respectively.

The next result discusses two special cases where we still have finite decomposition of the eigenspaces of $\varpi_{(0)}$, even though the criterion in (2.16) does not apply.

Theorem 5.2. *Consider the embeddings*

(1) $A_2 \times A_2 \times \mathbb{C}\varpi \subset A_5$ *at conformal level* $k = -1$;

(2) $A_2 \times \mathbb{C}\varpi$ *in* C_3 *with conformal level* $k = -1/2$.

Then $\widetilde{V}_k(\mathfrak{g}^0) \cong V_{\mathbf{k}}(\mathfrak{g}^0)$ *and as a* $V_{\mathbf{k}}(\mathfrak{g}^0)$*-module,*

$$(5.1) \qquad V_k(\mathfrak{g})^{(q)} = \begin{cases} L_{\mathfrak{g}^0}(q\theta) & \text{if } q \geq 0, \\ L_{\mathfrak{g}^0}(q\alpha_p) & \text{if } q \leq 0. \end{cases}$$

Proof. Case (1). Recall that $V_{(-1,-1)}(gl(3)) = V_{-1}(A_2) \otimes V_{-1}(\mathbb{C}I_3)$ and let $\zeta_{I_3} \in (\mathbb{C}I_3)^*$ be defined by $\zeta_{I_3}(I_3) = 1$ (I_n being the identity matrix in $gl(n)$). If $x \in gl(3)$ we set $\dot{x} = (x, 0) \in gl(3) \times gl(3)$ and $\ddot{x} = (0, x) \in gl(3) \times gl(3)$. Likewise we write $\lambda \in (gl(3) \times gl(3))^*$ as $\lambda = \dot{\lambda} + \ddot{\lambda}$ with $\dot{\lambda}(\ddot{x}) = \ddot{\lambda}(\dot{x}) = 0$ for all $x \in gl(3)$. Set also $\mathfrak{c} = span(\dot{I}_3, \ddot{I}_3)$ be the center of $gl(3) \times gl(3)$.

We use the Kac-Wakimoto free field realization of $V_{(-1,-1)}(gl(n+1))$ (see [20]). As shown in [4], $V_{(-1,-1)}(gl(n+1))$ can be realized for $n > 1$ as the

subspace F^0 of zero total charge of the universal vertex algebra F generated by fields $a_i^{\pm}$ $(i = 1, \ldots, n+1)$ with λ-products

$$[(a_i^+)_\lambda(a_j^-)] = \delta_{ij}, \quad [(a_i^-)_\lambda(a_j^+)] = -\delta_{ij}, \quad [(a_i^+)_\lambda(a_j^+)] = [(a_i^-)_\lambda(a_j^-)] = 0.$$

Restrict now to the $n = 5$ case. Let F_1 be the vertex subalgebra generated by $a_i^{\pm}$ with $i \leq 3$ and F_2 the subalgebra generated by $a_i^{\pm}$ with $i > 3$. Then, by the explicit realization of F given in [4], we have that, as a $V_{(-1,-1)}(gl(3)) \otimes V_{(-1,-1)}(gl(3))$-module,

$$F = F_1 \otimes F_2,$$

hence

$$F^0 = \sum_{q \in \mathbb{Z}} F_1^q \otimes F_2^{-q},$$

where we define F_j^q to be the subspace of total charge q.

We can now apply Theorem 3.2 of [4] to F_j^q and find that

$$F_j^q = \begin{cases} L_{A_2}(q\omega_1) \otimes L_{\mathbb{C}I_3}(q\zeta_{I_3}) & \text{if } q \geq 0, \\ L_{A_2}(-q\omega_2) \otimes L_{\mathbb{C}I_3}(q\zeta_{I_3}) & \text{if } q < 0. \end{cases}$$

as a $V_{(-1,-1)}(gl(3))$-module.

Thus, as a $V_{(-1,-1)}(A_2 \times A_2) \otimes V_{(-1,-1)}(\mathfrak{c})$-module,

$$F_1^q \otimes F_2^{-q} = \begin{cases} L_{A_2 \times A_2}(q(\dot{\omega}_1 + \ddot{\omega}_2)) \otimes L_{\mathfrak{c}}(q(\dot{\zeta}_{\dot{I}_3} - \ddot{\zeta}_{\ddot{I}_3})) & \text{if } q \geq 0, \\ L_{A_2 \times A_2}(-q(\dot{\omega}_2 + \ddot{\omega}_1)) \otimes L_{\mathfrak{c}}(q(\dot{\zeta}_{\dot{I}_3} - \ddot{\zeta}_{\ddot{I}_3})) & \text{if } q < 0. \end{cases}$$

To recover the action of $V_{-1}(A_2) \otimes V_{-1}(A_2) \otimes V_{-1}(\mathbb{C}\varpi)$ on $F_1^q \otimes F_2^{-q}$ we observe that both $\{\dot{I}_3, \ddot{I}_3\}$ and $\{I_6, \varpi\}$ are orthogonal bases of the center of $gl(3) \times gl(3)$. Since $I_6 = \dot{I}_3 + \ddot{I}_3$ while $\varpi = \frac{1}{2}\dot{I}_3 - \frac{1}{2}\ddot{I}_3$, it follows that $\dot{\zeta}_{\dot{I}_3} = \frac{1}{2}\zeta + \zeta_{I_6}$ while $\ddot{\zeta}_{\ddot{I}_3} = -\frac{1}{2}\zeta + \zeta_{I_6}$. Thus, as a $V_{(-1,-1)}(\mathfrak{c}) = V_{-1}(\mathbb{C}\varpi) \otimes V_{-1}(\mathbb{C}I_6)$-module,

$$L_{\mathfrak{c}}(q(\dot{\zeta}_{\dot{I}_3} - \ddot{\zeta}_{\ddot{I}_3})) = L_{\mathbb{C}\varpi}(q\zeta) \otimes L_{\mathbb{C}I_6}(0).$$

Let $(F^0)^+ = \{v \in F^0 \mid (I_6)_{(r)}v = 0 \text{ for all } r > 0\}$. By Theorem 3.2 of [4], $(F^0)^+ = V_{-1}(A_5)$, so we obtain that, if $q \geq 0$, then

$$V_{-1}(A_5)^{(q)} = (L_{A_2 \times A_2}(q(\dot{\omega}_1 + \ddot{\omega}_2)) \otimes L(q\zeta) \otimes L(0))^+$$
$$= L_{A_2 \times A_2}(q(\dot{\omega}_1 + \ddot{\omega}_2)) \otimes L(q\zeta),$$

and, if $q \leq 0$,

$$V_{-1}(A_5)^{(q)} = (L_{A_2 \times A_2}(-q(\dot{\omega}_2 + \ddot{\omega}_1)) \otimes L(q\zeta) \otimes L(0))^+$$
$$= L_{A_2 \times A_2}(-q(\dot{\omega}_2 + \ddot{\omega}_1)) \otimes L(q\zeta).$$

This in particular shows that $V_{-1}(A_5)^{(0)}$ is simple as a $\widehat{\mathfrak{g}}^0$-module and, since it contains $\widetilde{V}_{-1}(\mathfrak{g}^0)$ as a submodule, we have that $\widetilde{V}_{-1}(\mathfrak{g}^0)$ is simple.

To finish the proof it is enough to observe that

$$\dot{\omega}_1 + \ddot{\omega}_2 + \zeta = \theta, \quad -\dot{\omega}_2 - \ddot{\omega}_1 + \zeta = \alpha_p.$$

Case (2). Again by [4, Theorem 3.2] we have

$$(5.2) \qquad F^q = \begin{cases} L_{A_2}(q\omega_1) \otimes L_{\mathbb{C}I_3}(q\zeta_{I_3}) & \text{if } q \geq 0 \\ L_{A_2}(-q\omega_2) \otimes L_{\mathbb{C}I_3}(q\zeta_{I_3}) & \text{if } q < 0 \end{cases}$$

as a $V_{(-1,-1)}(gl(3))$-module. But by [13, Table XII], the even part of F is the simple vertex algebra $V_{-1/2}(C_n)$. This immediately gives the decomposition

$$(5.3) \qquad V_{-1/2}(C_n) = \bigoplus_{q \in \mathbb{Z}} F^{2q}.$$

Since $2\zeta_{I_3} + 2\omega_1 = \zeta + 2\omega_1 = \theta$ and $2\zeta_{I_3} - 2\omega_2 = \zeta - 2\omega_2 = \alpha_p$, combining (5.2) and (5.3), formula (5.1) follows in this case too. $\qquad\square$

In the next result we collect a few cases where finite decomposition does not occur.

Theorem 5.3. *In the following cases the decomposition of the q-eigenspace for $\varpi_{(0)}$ in $V_k(\mathfrak{g})$ as $\widehat{\mathfrak{g}^0}$-module is not finite for any $q \in \mathbb{Z}$:*

   (1) *Type $A_{h-1} \times A_{n-h} \times \mathbb{C}\varpi$ in A_n with $h \geq 2$ and $n - h \geq 1$, conformal level $k = -\frac{n+1}{2}$.*

   (2) *Type $A_{h-1} \times A_{n-h} \times \mathbb{C}\varpi$ in A_n with either $h = 2$ or $n - h = 1$, conformal level $k = -1$.*

   (3) *Type $A_1 \times \mathbb{C}\varpi$ in A_2, conformal level $k = -\frac{3}{2}$.*

   (4) *Type $A_1 \times \mathbb{C}\varpi$ in $B_2 = C_2$, conformal level $k = -1/2$.*

The proof of the above result employs very different techniques in each of the four cases. We will discuss these cases in the following subsections.

Remark 5.1. The remaining open cases are the following (we set $D_3 = A_3$).

   (1) Type $A_2 \times \mathbb{C}\varpi$ in A_3, conformal level $k = -2$.

   (2) Type $D_{n-1} \times \mathbb{C}\varpi$ in D_n, conformal level $k = 2 - n$, $n \geq 4$.

   (3) Type $B_{n-1} \times \mathbb{C}\varpi$ in B_n, conformal level $k = \frac{3}{2} - n$, $n \geq 3$.

5.1. **Proof of Theorem 5.3 (1).** We can choose the root vectors in such a way that the following relations hold:

$$[X_{\epsilon_i - \epsilon_j}, X_{\epsilon_r - \epsilon_s}] = \delta_{j,r} X_{\epsilon_i - \epsilon_s} - \delta_{s,i} X_{\epsilon_r - \epsilon_j}.$$

Moreover $(X_\alpha, X_{-\alpha}) = 1$ for all roots α. With this choice of roots vectors set

$$v_{i,j,r,s} = (X_{-\epsilon_h + \epsilon_{h+1}})^i_{(-1)} (X_{\epsilon_1 - \epsilon_{n+1}})^j_{(-1)} (X_{\epsilon_1 - \epsilon_h})^r_{(-1)} (X_{\epsilon_{h+1} - \epsilon_{n+1}})^s_{(-1)} \mathbf{1}.$$

Note that $\varpi_{(0)} v_{i,j,r,s} = (j - i) v_{i,j,r,s}$ and that if α is a positive root for $\mathfrak{g}^0$ then $(X_\alpha)_{(0)} v_{i,j,r,s} = 0$.

Lemma 5.4.

$$(5.4) \quad (X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{i,j,r,s} = r(k - i - j - r + 1) v_{i,j,r-1,s} - ij\, v_{i-1,j-1,r,s+1}.$$

$$(5.5) \quad (X_{-\epsilon_{h+1} + \epsilon_{n+1}})_{(1)} v_{i,j,r,s} = s(k - i - j - s + 1) v_{i,j,r,s-1} - ij\, v_{i-1,j-1,r+1,s}.$$

(5.6) $(X_{\epsilon_h - \epsilon_{h+1}})_{(1)} v_{i,j,r,s} = i(k - i - r - s + 1)v_{i-1,j,r,s} - rs\, v_{i,j+1,r-1,s-1}.$

(5.7) $(X_{-\epsilon_1 + \epsilon_{n+1}})_{(1)} v_{i,j,r,s} = j(k - j - r - s + 1)v_{i,j-1,r,s} - rs\, v_{i+1,j,r-1,s-1}.$

(5.8) $$(\varpi)_{(t)} v_{i,j,r,s} = 0, \ t > 0.$$

Proof. The induction is on $i + j + r + s$ with the base case $i = j = r = s = 0$ being clear. We give the details only for (5.4). If $i = j = r = 0$ it is clear that $(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,0,0,s} = 0$; if $i = j = 0, r > 0$, then

$$
\begin{aligned}
&(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,0,r,s} \\
&= (-(h_{-\epsilon_1 + \epsilon_h})_{(0)} + k)v_{0,0,r-1,s} + (X_{\epsilon_1 - \epsilon_h})_{(-1)}(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,0,r-1,s} \\
&= (-2(r - 1) + k)v_{0,0,r-1,s} + (r - 1)(k - r + 2)v_{0,0,r-1,s} \\
&= r(k - r + 1)v_{0,0,r-1,s}.
\end{aligned}
$$

If $i = 0, j > 0$

$$
\begin{aligned}
&(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,j,r,s} = \\
&(X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,j-1,r,s} + (X_{\epsilon_1 - \epsilon_{n+1}})_{(-1)}(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,j-1,r,s} =
\end{aligned}
$$

(5.9) $\quad (X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,j-1,r,s} + r(k - j - r + 2)v_{0,j,r-1,s}.$

We claim that

$$(X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,j,r,s} = -r v_{0,j+1,r-1,s}.$$

Since $[(X_{\epsilon_h - \epsilon_{n+1}})_{(0)}, (X_{\epsilon_1 - \epsilon_{n+1}})_{(-1)}] = 0$, we need only to prove that

$$(X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,0,r,s} = -r v_{0,1,r-1,s}.$$

We use induction on r, the base $r = 0$ being clear. If $r > 0$, then

$$
\begin{aligned}
&(X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,0,r,s} \\
&= -(X_{\epsilon_1 - \epsilon_{n+1}})_{(-1)} v_{0,0,r-1,s} + (X_{\epsilon_1 - \epsilon_h})_{(-1)}(X_{\epsilon_h - \epsilon_{n+1}})_{(0)} v_{0,0,r-1,s} \\
&= -v_{0,1,r-1,s} - (r - 1)v_{0,1,r-1,s} = -r v_{0,1,r-1,s}
\end{aligned}
$$

Substituting in (5.9) we find

$$
\begin{aligned}
(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{0,j,r,s} &= -r v_{0,j,r-1,s} + r(k - j - r + 2)v_{0,j,r-1,s} \\
&= r(k - j - r + 1)v_{0,j,r-1,s}.
\end{aligned}
$$

Finally, if $i > 0$,

$$
\begin{aligned}
&(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{i,j,r,s} \\
&= -(X_{-\epsilon_1 + \epsilon_{h+1}})_{(0)} v_{i-1,j,r,s} + (X_{-\epsilon_h + \epsilon_{h+1}})_{(-1)}(X_{-\epsilon_1 + \epsilon_h})_{(1)} v_{i-1,j,r,s} \\
&= -(X_{-\epsilon_1 + \epsilon_{h+1}})_{(0)} v_{i-1,j,r,s}
\end{aligned}
$$

(5.10) $\quad + r(k - i - j - r + 2)v_{i,j,r-1,s} - (i - 1)j v_{i-1,j-1,r,s+1}.$

We claim that

$$(X_{-\epsilon_1 + \epsilon_{h+1}})_{(0)} v_{i,j,r,s} = r v_{i+1,j,r-1,s} + j v_{i,j-1,r,s+1}.$$

Since $[(X_{-\epsilon_1+\epsilon_{h+1}})_{(0)}, (X_{-\epsilon_h+\epsilon_{h+1}})_{(-1)}] = 0$, we need only to prove that $(X_{-\epsilon_1+\epsilon_{h+1}})_{(0)} v_{0,j,r,s} = r v_{1,j,r-1,s} + j v_{0,j-1,r,s+1}$. The induction is on $j + r$. The base $j = r = 0$ is clear. If $j = 0$ and $r > 0$,

$$
\begin{aligned}
(X_{-\epsilon_1+\epsilon_{h+1}})_{(0)} & v_{0,0,r,s} \\
&= (X_{-\epsilon_h+\epsilon_{h+1}})_{(-1)} v_{0,0,r-1,s} + (X_{\epsilon_1-\epsilon_h})_{(-1)} (X_{-\epsilon_1+\epsilon_{h+1}})_{(0)} v_{0,0,r-1,s} \\
&= v_{1,0,r-1,s} + (r-1) v_{1,0,r-1,s} = r v_{1,0,r-1,s}.
\end{aligned}
$$

If $j > 0$,

$$
\begin{aligned}
(X_{-\epsilon_1+\epsilon_{h+1}})_{(0)} & v_{0,j,r,s} \\
&= (X_{\epsilon_{h+1}-\epsilon_{n+1}})_{(-1)} v_{0,j-1,r,s} + (X_{\epsilon_1-\epsilon_{n+1}})_{(-1)} (X_{-\epsilon_1+\epsilon_{h+1}})_{(0)} v_{0,j-1,r,s} \\
&= v_{0,j-1,r,s+1} + r v_{1,j,r-1,s} + (j-1) v_{0,j-1,r,s+1} = r v_{1,j,r-1,s} + j v_{0,j-1,r,s+1}.
\end{aligned}
$$

Substituting in (5.10), we find

$$
\begin{aligned}
(X_{-\epsilon_1+\epsilon_h})_{(1)} v_{i,j,r,s} &= -r v_{i,j,r-1,s} - j v_{i-1,j-1,r,s+1} \\
&\quad + r(k-i-j-r+2) v_{i,j,r-1,s} - (i-1) j v_{i-1,j-1,r,s+1} \\
&= r(k-i-j-r+1) v_{i,j,r-1,s} - ij v_{i-1,j-1,r,s+1},
\end{aligned}
$$

which is (5.4).

Lemma 5.5. *Consider $v_{i,j,r,s}$ as an element of $V^k(\mathfrak{g})$. Fix $m \geq 0$. Set, for $i = 0, \ldots, m$,*

$$
w_{i,q} = \begin{cases} v_{i,i+q,m-i,m-i} & \text{if } q \geq 0, \\ v_{i-q,i,m-i,m-i} & \text{if } q \leq 0. \end{cases}
$$

Set

$$
S_{m,q} = span(w_{i,q} \mid i = 0, \ldots, m)
$$

and

$$
v_{m,q} = \sum_{i=0}^{m} \frac{\binom{k-m-|q|+1}{i} \binom{m}{i}}{\binom{i+|q|}{|q|}} w_{i,q}.
$$

Then

(1) $\mathbb{C} v_{m,q}$ is the space of $\widehat{\mathfrak{g}^0}$-singular vectors in $S_{m,q}$.
(2) If $q \geq 0$ and $m > 0$ then $(X_{\epsilon_h-\epsilon_{h+1}})_{(1)} v_{m,q} = c_{m,q} v_{m-1,q+1}$ with

$$
(5.11) \qquad c_{m,q} = \frac{m}{q+1}(2+k-m)(1+k-2m-q).
$$

(3) If $q \leq 0$ and $m > 0$ then $(X_{-\epsilon_1+\epsilon_{n+1}})_{(1)} v_{m,q} = d_{m,q} v_{m-1,q-1}$ with

$$
(5.12) \qquad d_{m,q} = \frac{m}{-q+1}(2+k-m)(1+k-2m+q).
$$

Proof. By (5.8), in order to prove that $v_{m,q}$ is $\widehat{\mathfrak{g}^0}$-singular, we need only to check that

$$
(X_{-\epsilon_1+\epsilon_h})_{(1)} v_{m,q} = (X_{-\epsilon_{h+1}+\epsilon_{n+1}})_{(1)} v_{m,q} = 0.
$$

Set $a_i = \frac{\binom{k-m-|q|+1}{i}\binom{m}{i}}{\binom{i+|q|}{|q|}}$ and

$$w'_{i,q} = \begin{cases} v_{i,i+q,m-i-1,m-i} & \text{if } q \geq 0, \\ v_{i-q,i,m-i-1,m-i} & \text{if } q \leq 0. \end{cases}$$

By Lemma 5.4,

$$(X_{-\epsilon_1+\epsilon_h})_{(1)}v_{m,q} = \sum_{i=0}^{m-1} a_i(k-i-m-|q|+1)(m-i)w'_{i,q}-$$

$$\sum_{i=1}^{m} a_i i(i+|q|)w'_{i-1,q}) =$$

$$\sum_{i=0}^{m-1}(a_i(k-i-m-|q|+1)(m-i) - a_{i+1}(i+1)(i+|q|+1))w'_{i,q} = 0.$$

The last equality follows from the following relation

$$\frac{(k-i-m-|q|+1)(m-i)}{(i+1)(i+|q|+1)}a_i = a_{i+1}.$$

The same computation shows that $(X_{-\epsilon_{h+1}+\epsilon_{n+1}})_{(1)}v_{m,q} = 0$.

Let $v = \sum_{s=0}^{m} c_i w_i$ be $\widehat{\mathfrak{g}}^0$-singular. Then the computation above and the fact that the $v_{i,j,r,s}$ are linearly independent show that

$$(k-i-m-|q|+1)(m-i)c_i = (i+1)(i+|q|+1)c_{i+1},$$

so v is determined by the choice of c_0, thus the set of $\widehat{\mathfrak{g}}^0$-singular vectors in $S_{m,q}$ is one-dimensional.

We now prove (2). Set $u = (X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{m,q}$. We first observe that u is $\widehat{\mathfrak{g}}^0$-singular. Indeed, this check reduces to showing that

$$(X_{\epsilon_{h-1}-\epsilon_h})_{(0)}u = (X_{\epsilon_{h+1}-\epsilon_{h+2}})_{(0)}u = 0.$$

Let us check only that $(X_{\epsilon_{h-1}-\epsilon_h})_{(0)}u = 0$; the other equality is obtained similarly. Now

$$(X_{\epsilon_{h-1}-\epsilon_h})_{(0)}(X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{m,q} = (X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}v_{m,q}.$$

We claim that $(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}v_{i,j,r,s} = 0$ for all i,j,r,s. Indeed, since

$$[(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}, (X_{\epsilon_1-\epsilon_{n+1}})_{(-1)}] = [(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}, (X_{\epsilon_1-\epsilon_h})_{(-1)}] = 0$$

we need only to check that

$$(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}v_{i,0,0,s} = 0.$$

We prove this by induction on $i+s$. If $i=0$ then

$$(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}v_{0,0,0,s} = (X_{\epsilon_{h-1}-\epsilon_{n+1}})_{(0)}v_{0,0,0,s-1} = 0$$

If $i>0$, then

$$(X_{\epsilon_{h-1}-\epsilon_{h+1}})_{(1)}v_{i,0,0,s} = (X_{\epsilon_{h-1}-\epsilon_h})_{(0)}v_{i-1,0,0,s} = 0.$$

It follows from (5.6) that $(X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{m,q} \in S_{m-1,q+1}$, so, since the space of $\widehat{\mathfrak{g}}^0$-singular vectors in $S_{m-1,q+1}$ is one-dimensional, $(X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{m,q} = c_{m,q}v_{m-1,q+1}$. To compute the coefficient $c_{m,q}$ we need only to compute the coefficient of $v_{0,q+1,m-1,m-1}$ in $(X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{m,q}$. By (5.6) this coefficient is

$$c_{m,q} = -m^2 + \tfrac{m}{q+1}(k-2m+2)(k-m-q+1)) = \tfrac{m}{q+1}(2+k-m)(1+k-2m-q).$$

We now prove (3). Set $u = (X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{m,q}$. By the same argument used in the proof of (2) one can check that u is $\widehat{\mathfrak{g}}^0$-singular. It follows from (5.7) that $(X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{m,q} \in S_{m-1,q-1}$, so, since the space of $\widehat{\mathfrak{g}}^0$-singular vectors in $S_{m-1,q-1}$ is one-dimensional, $(X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{m,q} = d_{m,q}v_{m-1,q-1}$. To compute the coefficient $d_{m,q}$ we need only to compute the coefficient of $v_{-q+1,0,m-1,m-1}$ in $(X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{m,q}$. By (5.7) this coefficient is

$$d_{m,q} = \tfrac{m}{|q|+1}(2+k-m)(1+k-2m-|q|).$$

Corollary 5.6. *Consider the embedding $A_{h-1} \times A_{n-h} \times \mathbb{C}\varpi \subset A_n$ with $h \geq 2$ and $n-h \geq 1$. If $k = -\frac{n+1}{2}$, then, for each $m \geq 0$ and $q \in \mathbb{Z}$, $v_{m,q}$ projects to a nonzero $\widehat{\mathfrak{g}}^0$-singular vector in $V_k(\mathfrak{g})$. In particular, the decomposition of the q-eigenspace for $\varpi_{(0)}$ in $V_k(\mathfrak{g})$ as $\widehat{\mathfrak{g}}^0$-module cannot be finite.*

Proof. Since $k = -\frac{n+1}{2}$ and $n \geq 3$, it is clear from (5.11) and (5.12) that $c_{m,q} \neq 0$ for all $m \geq 1$. An obvious induction using Lemma 5.5 (2) shows that, if $q \geq 0$ and $v_{0,q+m} \neq 0$, then $v_{m,q} \neq 0$. Likewise, if $q \leq 0$ and $v_{0,q-m} \neq 0$ then $v_{m,q} \neq 0$. We need only to prove that $v_{0,q} \neq 0$ for all q.
  If $q \geq 0$, from (5.7) we deduce that,

$$
\begin{aligned}
(X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{0,q} &= (X_{-\epsilon_1+\epsilon_{n+1}})_{(1)}v_{0,q,0,0} \\
&= q(k-q+1)v_{0,q-1,0,0} = q(k-q+1)v_{0,q-1}.
\end{aligned}
$$

Since $q(k-q+1) \neq 0$ for all $q \geq 1$ an obvious induction shows that $v_{0,q} \neq 0$.
  Similarly, if $q \leq 0$, from (5.6) we deduce that,

$$
\begin{aligned}
(X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{0,q} &= (X_{\epsilon_h-\epsilon_{h+1}})_{(1)}v_{-q,0,0,0} \\
&= -q(k+q+1)v_{-q-1,0,0,0} = -q(k+q+1)v_{0,q+1}.
\end{aligned}
$$

Since $-q(k+q+1) \neq 0$ for all $q \leq -1$ an obvious induction shows that $v_{0,q} \neq 0$.
  The set $\{v_{m,q} \mid m \geq 1\}$ is linearly independent since the vectors $v_{m,q}$ have different weights, thus the second statement follows. $\qquad\square$

5.2. **Proof of Theorem 5.3 (2).** We now discuss the embedding of type $A_{h-1} \times A_{n-h} \times \mathbb{C}\varpi$ in A_n with either $h = 2$ or $n - h = 1$, conformal level $k = -1$. Without loss of generality we may assume $h = 2$.
  We use the Kac-Wakimoto free field realization of $V_{-1}(gl(n+1))$ described in the proof of Theorem 5.2. With the notation used there, we have, as

$V_{-1}(gl(2)) \otimes V_{-1}(gl(n-1))$-modules,

$$V_{(-1,-1)}(gl(n+1)) = \sum_{q \in \mathbb{Z}} F_1^q \otimes F_2^{-q}.$$

We may deduce from [20, Remark 3.3] that

(5.13) $$F_1^q = \sum_{j=0}^{\infty} L_{A_1}((2j+|q|)\omega_1) \otimes L_{\mathbb{C}I_2}(q\zeta_{I_2})$$

as $V_{-1}(A_1) \otimes V_{-1}(\mathbb{C}I_2)$-module.

Arguing as in the proof of Theorem 5.2 we see that, if $n > 3$,

$$V_{-1}(A_n)^{(q)} = \sum_{j \geq 0} L_{A_1}((2j+q)\omega_1) \otimes L_{A_{n-2}}(q\omega_{n-2}) \otimes L_{\mathbb{C}\varpi}(q\zeta)$$

when $q \geq 0$, while, for $q \leq 0$,

$$V_{-1}(A_n)^{(q)} = \sum_{j \geq 0} L_{A_1}((2j-q)\omega_1) \otimes L_{A_{n-2}}(q\omega_1) \otimes L_{\mathbb{C}\varpi}(q\zeta).$$

If $n = 3$,

$$V_{-1}(A_3)^{(q)} = \sum_{j,j' \geq 0} L_{A_1}((2j+|q|)\omega_1) \otimes L_{A_1}((2j'+|q|)\omega_1) \otimes L_{\mathbb{C}\varpi}(q\zeta).$$

In both cases the q-eigenspace of $\varpi_{(0)}$ decomposes with infinitely many factors.

5.3. **Proof of Theorem 5.3 (3).** We now discuss the embedding of type $A_1 \times \mathbb{C}\varpi$ in A_2, conformal level $k = -\frac{3}{2}$.

We consider the lattice vertex algebra V_L associated to the lattice $L = \mathbb{Z}\alpha + \mathbb{Z}\beta + \mathbb{Z}\delta$ such that

$$\langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = \langle \delta, \delta \rangle = 1$$

(other products of basis vectors are zero). Let F_{-1} be the lattice vertex algebra associated to the lattice $\mathbb{Z}\phi$ with $\langle \phi, \phi \rangle = -1$. The operator $\phi_{(0)}$ acts semisimply on F_{-1} and it defines a $\mathbb{Z}$- gradation

$$F_{-1} = \oplus_{\ell \in \mathbb{Z}} F_{-1}^{\ell}, \quad \phi_{(0)|F_{-1}^{\ell}} = -\ell Id.$$

In [1] the following facts are shown:

(1) The simple $N = 4$ superconformal vertex algebra $V = L_c^{N=4}$ is realized as a subalgebra of V_L.

(2) The operator $\delta_{(0)}$ acts semisimply on V and defines a $\mathbb{Z}$-gradation

$$V = \oplus_{\ell \in \mathbb{Z}} V^{\ell}, \quad \delta_{(0)|V^{\ell}} = \ell Id.$$

(3) $V_{-3/2}(sl(2))$ embeds in V^0. This turns V_L into a $\widehat{sl(2)}$-module and $e^{t\delta}$ is a singular vector for this action for all $t \in \mathbb{Z}$.

(4) Let $Q = e_{(0)}^{\alpha+\beta-2\delta}$. Then Q commutes with the action of $\widehat{sl(2)}$.

(5) V is the maximal $sl(2)$-integrable part of the Clifford-Weyl vertex algebra $M \otimes F \subset V_L$.

(6) The subalgebra $\oplus_{\ell \in \mathbb{Z}} V^\ell \otimes F_{-1}^\ell$ of $V \otimes F_{-1}$ is isomorphic to the simple affine vertex algebra $V_{-3/2}(sl(3))$.

As a consequence of (1)–(6) we have

Theorem 5.7. *For each $q \in \mathbb{Z}$ there are infinitely many $\widehat{sl(2)}$-singular vectors in $V_{-3/2}(sl(3))^{(q)}$. In particular $V_{-3/2}(sl(3))^{(q)}$ does not decompose finitely as a $\widehat{sl(2)}$-module.*

Proof. The embedding $\widetilde{V}_{-3/2}(sl(2) \oplus \mathbb{C}\varpi) \subset V_{-3/2}(sl(3))$ corresponds to the pair of embeddings

$$V_{-3/2}(sl(2)) \subset V^0 \otimes \mathbb{C} \subset V^0 \otimes F_{-1}^0, \quad \mathbb{C}\varpi \subset \mathbb{C} \otimes F_{-1}^0,$$

the rightmost one mapping ϖ to $\mathbf{1} \otimes \phi_{(0)} \mathbf{1}$.

Since $e^{t\delta}$ is singular for $\widehat{sl(2)}$ and Q commutes with the action of $\widehat{sl(2)}$, we have that $Q^m e^{t\delta}$ is singular for all $t \in \mathbb{Z}$, $m \in \mathbb{N}$. By [2], $Q^j e^{t\delta} \neq 0$ iff $0 \leq j \leq t$. It is easy to check that $Q^j e^{t\delta}$ is $sl(2)$-integral, so $Q^j e^{t\delta} \in V$. Note also that $Q^j e^{t\delta} \in V^{t-2j}$, hence $Q^j e^{t\delta} \otimes e^{(t-2j)\phi} \in V^{t-2j} \otimes F_{-1}^{t-2j} \subset V_k(sl(3))$ and it is clearly a $\widehat{sl(2) \oplus \mathbb{C}\varpi}$-singular vector.

It follows that, if $\ell \geq 0$, the vectors $v_{\ell,j} = Q^j e^{(\ell+2j)\delta} \otimes e^{\ell\phi}$ are nonzero $\widehat{\mathfrak{g}^0}$-singular vectors for all $j \geq 0$. Since the $sl(2) \oplus \mathbb{C}\varpi$-weight of $v_{\ell,j}$ is $(\ell + 2j)\omega_1 + \ell\zeta$, we see that the set $\{v_{\ell,j} \mid j \geq 0\}$ provides an infinite family of linearly independent singular vectors in the ℓ-eigenspace of $\varpi_{(0)}$.

If $\ell \leq 0$, then the vectors $v_{\ell,j} = Q^{j-\ell} e^{(-\ell+2j)\delta} \otimes e^{\ell\phi}$ are nonzero $\widehat{\mathfrak{g}^0}$-singular vectors for all $j \geq 0$. Since the $sl(2) \oplus \mathbb{C}\varpi$-weight of $v_{\ell,j}$ is $(-\ell + 2j)\omega_1 + \ell\zeta$, we see that the set $\{v_{\ell,j} \mid j \geq 0\}$ provides an infinite family of linearly independent singular vectors in the ℓ-eigenspace of $\varpi_{(0)}$. □

5.4. **Proof of Theorem 5.3 (4).** Now we discuss embedding of $A_1 \times \mathbb{C}\varpi$ in $B_2 = C_2$ at conformal level $k = -1/2$. We know that $V_{-1/2}(B_2)$ is an even subalgebra of the Weyl vertex algebra F generated by $a_i^\pm$, $i = 1, 2$, and Kac-Wakimoto free field realization gives that $V_{-1}(sl_2)$ is realized as the subalgebra of F generated by

$$(5.14) \qquad e = (a_1^+)_{-1} a_2^-, \ f = (a_2^+)_{-1} a_1^-, \ h = -(a_1^+)_{-1} a_1^- + (a_2^+)_{-1} a_2^-.$$

One checks that $\varpi = (a_1^+)_{-1} a_1^- + (a_2^+)_{-1} a_2^-$. By using again [20, Remark 3.3] we have the following decomposition

$$V_{-1/2}(B_2) = \bigoplus_{q \in \mathbb{Z}} F^{2q},$$

and

$$(5.15) \qquad F^{2q} = \bigoplus_{j=0}^{\infty} L_{A_1}((2j + 2|q|)\omega_1) \otimes L_{\mathbb{C}\varpi}(2q\varpi).$$

Remark 5.2. Denote by $v_{j,q}$ the singular vector in $V_{-1/2}(B_2)$ such that

$$V_{-1}(gl_2) \cdot v_{j,q} = L_{A_1}((2j + 2|q|)\omega_1) \otimes L_{\mathbb{C}\varpi}(2q\varpi)$$

(cf. (5.15)).

We shall now provide a sketchy derivation of explicit formulas for these singular vectors. Consider the lattice vertex algebra V_L associated to the lattice $L = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2$ such that

$$\langle \alpha_i, \alpha_j \rangle = -\langle \beta_i, \beta_j \rangle = \delta_{i,j}, \quad \langle \alpha_i, \beta_j \rangle = 0, \qquad i, j \in \{1, 2\}.$$

Then the Weyl vertex algebra F introduced above is isomorphic to the subalgebra of V_L generated by

$$a_1^+ = e^{\alpha_1+\beta_1}, \ a_1^- = -\alpha_1(-1)e^{-\alpha_1-\beta_1}, \ a_1^+ = \alpha_2(-1)e^{-\alpha_1-\beta_1}, \ a_2^- = e^{\alpha_2+\beta_2}.$$

The simple affine vertex algebra $V_{-1}(sl_2)$ is realized as a subalgebra of F generated by e, f, h given by (5.14). Note that $h = -(\beta_1 + \beta_2)$ and that $\varpi = \beta_1 - \beta_2$. We have the following screening operators

$$Q^+ = e_0^{\alpha_1-\alpha_2} = \mathrm{Res}_z Y(e^{\alpha_1-\alpha_2}, z), \quad Q^- = e_0^{\alpha_2-\alpha_1} = \mathrm{Res}_z Y(e^{\alpha_2-\alpha_1}, z).$$

Set $\delta = 2\alpha_2 + \beta_1 + \beta_2$, $\varphi = \beta_2 - \beta_1$. Then the singular vectors are given by the following formulas

$$v_{j,n} = (Q^+)^j e^{(j+n)\delta+n\varphi}, \quad v_{j,-n} = (Q^+)^{2n+j} e^{(j+n)\delta-n\varphi} \qquad (n \geq 0).$$

The proof that $v_{j,n}$ are elements of $V_{-1/2}(B_2) \subset F$ uses description of the Weyl vertex algebra as kernel of certain screening operators, and it is omitted.

6. Explicit decompositions in the semisimple case

As explained in the proof of Proposition 4.7, we have a complete description of the singular vectors occurring in finite decompositions, thus we can compute explicitly the summands of the decomposition. The list of all decompositions is given below. For $k = 1$ all decompositions give simple current extensions (see [22] for definitions). In the non-semisimple cases with finite decomposition we deal with, the explicit decomposition is given by formula (2.12).

6.1. **Type D (level 1).**

$\mathfrak{g}^0$	Decomposition
$D_h \times D_{n-h}$, $h \geq 4$, $n - h \geq 4$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1)$
$A_1 \times A_1 \times D_{n-2}$, $n \geq 6$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_1)$
$A_1 \times A_1 \times A_3$, $n = 5$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_2)$
$A_1 \times A_1 \times A_1 \times A_1$, $n = 4$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0, \ddddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_1, \ddddot{\Lambda}_1)$
$A_3 \times D_{n-3}$, $n \geq 7$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_2, \ddot{\Lambda}_1)$
$A_3 \times A_3$, $n = 6$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_2, \ddot{\Lambda}_2)$

6.2. Type B.

$\mathfrak{g}^0$	level	Decomposition
$D_h \times B_{n-h}$, $h \geq 4$, $n - h \geq 3$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1)$
$A_3 \times B_{n-3}$, $n \geq 6$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_2, \ddot{\Lambda}_1)$
$A_1 \times A_1 \times B_{n-2}$, $n \geq 5$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_1)$
$D_{n-2} \times C_2$, $n \geq 6$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_2)$
$A_1 \times A_1 \times C_2$, $n = 4$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_2)$
$A_3 \times C_2$, $n = 5$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) \oplus (\dot{\Lambda}_2, \ddot{\Lambda}_2)$
$D_{n-1} \times A_1$, $n \geq 5$	1	$(\dot{\Lambda}_0, 2\ddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, 2\ddot{\Lambda}_1)$
$A_1 \times A_1 \times A_1$, $n = 3$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0.2\dddot{\Lambda}_0) \oplus (\dot{\Lambda}_1, \ddot{\Lambda}_1, 2\dddot{\Lambda}_1)$
$A_3 \times A_1$, $n = 4$	1	$(\dot{\Lambda}_0, 2\ddot{\Lambda}_0) \oplus (\dot{\Lambda}_2, 2\ddot{\Lambda}_1)$
D_n, $n \geq 4$	$\frac{3}{2} - n$	$((\frac{3}{2} - n)\dot{\Lambda}_0) \oplus ((\frac{1}{2} - n)\dot{\Lambda}_0 + \dot{\Lambda}_1)$
A_3, $n = 3$	$-\frac{3}{2}$	$(-\frac{3}{2}\dot{\Lambda}_0) \oplus (-\frac{5}{2}\dot{\Lambda}_0 + \dot{\Lambda}_1)$

6.3. Type C (level $-\frac{1}{2}$).

$\mathfrak{g}^0$	Decomposition
$C_h \times C_{n-h}$, $h \geq 2$, $n - h \geq 2$	$(-\frac{1}{2}\dot{\Lambda}_0, -\frac{1}{2}\ddot{\Lambda}_0) \oplus (-\frac{3}{2}\dot{\Lambda}_0 + \dot{\Lambda}_1, -\frac{3}{2}\ddot{\Lambda}_0 + \ddot{\Lambda}_1)$
$A_1 \times C_{n-1}$, $n \geq 3$	$(-\frac{1}{2}\dot{\Lambda}_0, -\frac{1}{2}\ddot{\Lambda}_0) \oplus (-\frac{3}{2}\dot{\Lambda}_0 + \dot{\Lambda}_1, -\frac{3}{2}\ddot{\Lambda}_0 + \ddot{\Lambda}_1)$
$A_1 \times A_1$, $n = 2$	$(-\frac{1}{2}\dot{\Lambda}_0, -\frac{1}{2}\ddot{\Lambda}_0) \oplus (-\frac{3}{2}\dot{\Lambda}_0 + \dot{\Lambda}_1, -\frac{3}{2}\ddot{\Lambda}_0 + \ddot{\Lambda}_1)$

6.4. Type E_6 (level 1).

$\mathfrak{g}^0$	Decomposition
$A_1 \times A_5$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_3)$
$A_2 \times A_2 \times A_2$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0, \dddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_1, \dddot{\Lambda}_1) + (\dot{\Lambda}_2, \ddot{\Lambda}_2, \dddot{\Lambda}_2)$

6.5. Type E_7 (level 1).

$\mathfrak{g}^0$	Decomposition
$A_1 \times D_6$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_6)$
$A_2 \times A_5$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_4) + (\dot{\Lambda}_2, \ddot{\Lambda}_2)$
A_7	$\dot{\Lambda}_0 + \dot{\Lambda}_4$

6.6. Type E_8 (level 1).

$\mathfrak{g}^0$	Decomposition
$A_1 \times E_7$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_7)$
$A_2 \times E_6$	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_2, \ddot{\Lambda}_1) + (\dot{\Lambda}_1, \ddot{\Lambda}_6)$
$A_4 \times A_4$	$(\dot{\Lambda}_0, \dot{\Lambda}_0) + (\dot{\Lambda}_4, \dot{\Lambda}_2) + (\dot{\Lambda}_1, \dot{\Lambda}_3) + (\dot{\Lambda}_2, \dot{\Lambda}_4) + (\dot{\Lambda}_3, \dot{\Lambda}_1)$
D_8	$\dot{\Lambda}_0 + \dot{\Lambda}_7$
A_8	$\dot{\Lambda}_0 + \dot{\Lambda}_6 + \dot{\Lambda}_3$

6.7. **Type F_4.**

$\mathfrak{g}^0$	level	Decomposition
$A_1 \times C_3$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, \ddot{\Lambda}_3)$
$A_2 \times A_2$	1	$(\dot{\Lambda}_0, 2\ddot{\Lambda}_0) + (\dot{\Lambda}_2, 2\ddot{\Lambda}_1) + (\dot{\Lambda}_1, 2\ddot{\Lambda}_2)$
B_4	$-\frac{5}{2}$	$(-\frac{5}{2}\dot{\Lambda}_0) + (-\frac{7}{2}\dot{\Lambda}_0 + \dot{\Lambda}_4).$

6.8. **Type G_2.**

$\mathfrak{g}^0$	level	Decomposition
$A_1 \times A_1$	1	$(\dot{\Lambda}_0, \ddot{\Lambda}_0) + (\dot{\Lambda}_1, 3\ddot{\Lambda}_1)$
A_2	$-\frac{5}{3}$	$(-\frac{5}{3}\dot{\Lambda}_0) + (-\frac{8}{3}\dot{\Lambda}_0 + \dot{\Lambda}_1) + (-\frac{8}{3}\dot{\Lambda}_0 + \dot{\Lambda}_2).$

REFERENCES

[1] D. Adamović, *A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$*, to appear in Transformation Groups, arXiv:1407.1527.

[2] D. Adamović, A. Milas , *On the triplet vertex algebra $W(p)$*, Adv. Math. **217** (2008), 2664–2699.

[3] D. Adamović, O. Perše, *Some General Results on Conformal Embeddings of Affine Vertex Operator Algebras*, Algebr. Represent. Theory **16** (2013), no. 1, 51–64.

[4] D. Adamović, O. Perše, *Fusion Rules and Complete Reducibility of Certain Modules for Affine Lie Algebras*, J. Algebra Appl. **13** no.1. 1350062 (2014), 18pp.

[5] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, *Conformal embeddings of affine vertex algebras I: structural results*, arXiv:1602.04687.

[6] D. Adamović, V. G. Kac, P. Möseneder Frajria, P. Papi, O. Perše, *Conformal embeddings of affine vertex algebras I: decompositions*, arXiv:1604.00893.

[7] R. C. Arcuri, J. F. Gomez, D. I. Olive, *Conformal subalgebras and symmetric spaces*, Nuclear Phys. B **285** (2) (1987) 327–339.

[8] T. Arakawa, A. Moreau, *Joseph ideals and lisse minimal W-algebras*, to appear in J. Inst. Math. Jussieu, arXiv:1506.00710

[9] B. Bakalov, V. G. Kac, *Field algebras.* Int. Math. Res. Not. 2003, no. 3, 123–159.

[10] A Borel, J. de Siebental, *Les sous-groupes fermes de rang maximum de groupes de Lie clos*, Comment. Math. Helv. **23** (1949) 200–221.

[11] P. Cellini, V. G. Kac, P. Möseneder Frajria, P. Papi, *Decomposition rules for conformal pairs associated to symmetric spaces and abelian subalgebras of $\mathbb{Z}_2$-graded Lie algebras*, Adv. Math. **207** (2006), 156–204.

[12] P. Deligne *La serie exceptionalle de groupes de Lie*, C. R. Acad. Sci. Paris Ser. I Math. **322** (1996) 321–326.

[13] A.J. Feingold, I.B. Frenkel *Classical affine algebras*, Adv. Math. **56** (1985), 117–172.

[14] K. Ito, J. O. Madsen, J. L. Petersen, Free Field Representations of Extended Superconformal Algebras, Nucl.Phys. B **398** (1993) 425–458.

[15] V. G. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press, 1990.

[16] V. G. Kac, M. Sanielevici, *Decomposition of representations of exceptional affine algebras with respect to conformal subalgebras*, Physical Review D, **37**, No. 8 (1988), 2231–2237.

[17] V. G. Kac and M. Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Adv. in Math. **70** (1988), 156–236.

[18] V. G. Kac and M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras.* Proc. Nat. Acad. Sci. U.S.A. 85 (1988), no. 14, 4956–4960.

[19] V. G. Kac, *Vertex algebras for beginners*, second ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998.

[20] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell's function.* Comm. Math. Phys. **215** (2001), no. 3, 631–682.

[21] V. G. Kac, M. Wakimoto Quantum reduction and representation theory of superconformal algebras, Adv. Math., **185** (2004), pp. 400–458

[22] V. G. Kac, P. Möseneder Frajria, P. Papi, F. Xu, *Conformal embeddings and simple current extensions*, IMRN, (2015), no. 14, 5229–5288

[23] K. Kawasetsu, *W-algebras with non-admissible levels and the Deligne exceptional series*, arXiv:1505.06985

[24] D. I. Panyushev,*Isotropy representations, eigenvalues of a Casimir element, and commutative Lie subalgebras*, J. London Math. Soc., **64** no. 1 (2001), 61–80.

[25] O. Perše, *Vertex operator algebra analogue of embedding of B_4 into F_4.* J. Pure Appl. Algebra **211** (2007), no. 3, 702–720.

[26] A.N. Schellekens and N.P. Warner, *Conformal subalgebras of Kac-Moody algebras*, Phys. Rev. D (3) **34** (1986), no. 10, 3092–3096.

[27] R. Suter, *Coxeter and dual Coxeter numbers.* Commun. Algebra **26**, (1998) 147–153.

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