

# THE GAP BETWEEN LINEAR ELASTICITY AND VARIATIONAL LIMIT OF FINITE ELASTICITY IN PURE TRACTION PROBLEMS

FRANCESCO MADDALENA, DANILO PERCIVALE, FRANCO TOMARELLI

**ABSTRACT.** A limit elastic energy for pure traction problem is derived from re-scaled non-linear energies of a hyperelastic material body subject to an equilibrated force field. We prove that the strains of minimizing sequences associated to re-scaled nonlinear energies weakly converge, up to subsequences, to the strains of minimizers of a limit energy, provided an additional compatibility condition is fulfilled by the force field. The limit energy is different from classical energy of linear elasticity; nevertheless the compatibility condition entails the coincidence of related minima and minimizers. A strong violation of this condition provides a limit energy which is unbounded from below, while a mild violation may produce unboundedness of strains and a limit energy which has infinitely many extra minimizers which are not minimizers of standard linear elastic energy. A consequence of this analysis is that a rigorous validation of linear elasticity fails for compressive force fields that infringe such compatibility condition.

## CONTENTS

1. Introduction	1
2. Notation and main result	6
3. Proofs	12
4. Limit problem and Linear Elasticity	20
References	25

**AMS Classification Numbers (2010):** 49J45, 74K30, 74K35, 74R10.

**Key Words:** Calculus of Variations, Pure Traction problems, Linear Elasticity, Nonlinear Elasticity, Finite Elasticity, Critical points, Gamma-convergence, Asymptotic analysis, nonlinear Neumann problems.

## 1. INTRODUCTION

*Linear Theory of Elasticity* ([19]) has a prominent role among Mathematical-Physics theories for clearness, rigorous mathematical status and persistence. It was a great achievement of last centuries, which inspired many other theories of Continuum Mechanics and led to the formulation of a more general theory named *Nonlinear Elasticity* ([22],[32]), also known as *Finite Elasticity* to underline that no smallness assumptions is required.

There was always agreement in scholars community that the relation between linear and nonlinear theory amounts to the linearization of strain measure under the assumption of small displacement gradients: this is the precondition advocated in almost all the texts on elasticity.

---

*Date:* May 31, 2019      Research partially supported by C.N.R. INDAM Project 2018:  
*G.N.A.M.P.A. - Problemi asintotici ed evolutivi con applicazioni a metamateriali e reti.*

Nevertheless only at the beginning of present century, when appropriate tools of mathematical analysis were suitably tuned, the problem of a rigorous deduction of any particular theory based on some *approximation hypotheses* from a more general *exact* theory, became a scientific issue related to the general problem of *validation* of a theory, as explained in [31].

In this conceptual framework, G. Dal Maso, M. Negri and D. Percivale in [13] proved that problems ruled by linear elastic energies can be rigorously deduced from problems ruled by non-linear energies in the case of *Dirichlet and mixed boundary conditions*, by exploiting De Giorgi  $\Gamma$ -convergence theory ([12],[14]). This result clarified the mathematical consistency of the linear boundary value problems, under displacements and forces prescribed on the boundary of a three-dimensional material body, via a rigorous deduction from the nonlinear elasticity theory. We mention several papers facing issues in elasticity which are connected with the context of our paper: [1], [2], [3], [4], [5] [6], [7], [8], [9], [21] [23], [24], [25], [27], [28], [29], [30]. The present paper aims to the same general question which was studied in [13], but here we deal with the *pure traction problem*, i.e. the case where the elastic body is subject to a system of equilibrated forces and no Dirichlet condition is imposed on the boundary.

We consider a bounded open set  $\Omega \subset \mathbb{R}^N$ ,  $N = 2, 3$ , as the reference configuration of a hyperelastic material body, hence the stored energy due to a deformation  $\mathbf{y}$  can be expressed as a functional of the deformation gradient  $\nabla \mathbf{y}$

$$\int_{\Omega} \mathcal{W}(\mathbf{x}, \nabla \mathbf{y}) \, d\mathbf{x}$$

where  $\mathcal{W} : \Omega \times \mathcal{M}^{N \times N} \rightarrow [0, +\infty]$  is a frame indifferent function,  $\mathcal{M}^{N \times N}$  is the set of real  $N \times N$  matrices and  $\mathcal{W}(\mathbf{x}, \mathbf{F}) < +\infty$  if and only if  $\det \mathbf{F} > 0$ .

Then due to frame indifference there exists a function  $\mathcal{V}$  such that

$$\mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I})), \quad \forall \mathbf{F} \in \mathcal{M}^{N \times N}, \quad \text{a.e. } \mathbf{x} \in \Omega.$$

We set  $\mathbf{F} = \mathbf{I} + h\mathbf{B}$ , where  $h > 0$  is an adimensional small parameter and

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) := h^{-2} \mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}).$$

We assume that the reference configuration has zero energy and is stress free, i.e.

$$\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \quad D\mathcal{W}(\mathbf{x}, \mathbf{I}) = \mathbf{0} \quad \text{for a.e. } \mathbf{x} \in \Omega,$$

and that  $\mathcal{W}$  is regular enough in the second variable, then Taylor's formula entails

$$\mathcal{V}_h(\mathbf{x}, \mathbf{B}) = \mathcal{V}_0(\mathbf{x}, \text{sym } \mathbf{B}) + o(1) \quad \text{as } h \rightarrow 0_+$$

where  $\text{sym } \mathbf{B} := \frac{1}{2}(\mathbf{B}^T + \mathbf{B})$  and

$$\mathcal{V}_0(\mathbf{x}, \text{sym } \mathbf{B}) := \frac{1}{2} \text{sym } \mathbf{B} \, D^2 \mathcal{V}(\mathbf{x}, \mathbf{0}) \, \text{sym } \mathbf{B}.$$

If the deformation  $\mathbf{y}$  is close to the identity up to a small displacement, say  $\mathbf{y}(\mathbf{x}) = \mathbf{x} + h\mathbf{v}(\mathbf{x})$  with bounded  $\nabla \mathbf{v}$  then, by setting  $\mathbb{E}(\mathbf{v}) := \frac{1}{2}(\nabla \mathbf{v}^T + \nabla \mathbf{v})$ , one plainly obtains

$$(1.1) \quad \lim_{h \rightarrow 0} \int_{\Omega} \mathcal{V}_h(\mathbf{x}, \nabla \mathbf{y}) \, d\mathbf{x} = \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v})) \, d\mathbf{x}.$$

Historically relationship (1.1) was considered as the main justification of linearized elasticity, but such pointwise convergence does not even entail that minimizers fulfilling a given fixed Dirichlet boundary condition actually converge to the minimizers of the corresponding limit boundary value problem: this phenomenon is made explicit by the Example 3.5 in [13] which exhibits a lack of compactness when  $\mathcal{V}$  has several minima.

To set the Dirichlet problem in a variational perspective, referring to a prescribed vector field  $\mathbf{v}_0 \in W^{1,\infty}(\Omega, \mathbb{R}^N)$  as the boundary condition on a given closed subset  $\Sigma$  of  $\partial\Omega$  with  $\mathcal{H}^{N-1}(\Sigma) > 0$  and to a given load  $\mathbf{g} \in L^2(\Omega, \mathbb{R}^N)$ , one has to study the asymptotic behavior of the sequence of functionals  $\mathcal{I}_h$ , which is defined as

$$\mathcal{I}_h(\mathbf{v}) = \begin{cases} \int_{\Omega} \mathcal{V}_h(\mathbf{x}, \nabla \mathbf{v}) d\mathbf{x} - \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\mathbf{x} & \text{if } \mathbf{v} \in H_{\mathbf{v}_0, \Sigma}^1 \\ +\infty & \text{else in } H^1(\Omega, \mathbb{R}^N), \end{cases}$$

where  $H_{\mathbf{v}_0, \Sigma}^1$  denotes the closure in  $H^1(\Omega, \mathbb{R}^N)$  of the space of displacements  $\mathbf{v} \in W^{1,\infty}(\Omega, \mathbb{R}^N)$  such that  $\mathbf{v} = \mathbf{v}_0$  on  $\Sigma$ : it was proved in [13] that (under natural growth conditions and suitable regularity hypotheses on  $\mathcal{W}$ ) every sequence  $\mathbf{v}_h$  fulfilling

$$\mathcal{I}_h(\mathbf{v}_h) = \inf \mathcal{I}_h + o(1)$$

has a subsequence converging weakly in  $H^1(\Omega, \mathbb{R}^N)$  to the (unique) minimizer  $\mathbf{v}_*$  of the functional  $\mathcal{I}$  representing the total energy in linear elasticity, that is

$$\mathcal{I}(\mathbf{v}) = \begin{cases} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v})) d\mathbf{x} - \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{v} d\mathcal{H}^{n-1}(\mathbf{x}) & \text{if } \mathbf{v} \in H_{\mathbf{v}_0, \Sigma}^1 \\ +\infty & \text{else in } H^1(\Omega; \mathbb{R}^N), \end{cases}$$

and that the re-scaled energies converge, namely

$$\lim_{h \rightarrow 0} \mathcal{I}_h(\mathbf{v}_h) = \mathcal{I}(\mathbf{v}_*) = \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_*)) d\mathbf{x} - \int_{\Omega} \mathbf{g} \cdot \mathbf{v}_* d\mathbf{x}.$$

Such result is a complete variational justification of linearized elasticity, at least as far as Dirichlet and mixed boundary value problems are concerned. So it is natural to ask whether a similar result holds true also for pure traction problems whose variational formulation is described in the sequel.

In the present paper we focus our analysis on Neumann boundary conditions, say the pure traction problem in elasticity. Precisely we assume that  $\mathbf{f} \in L^2(\partial\Omega; \mathbb{R}^N)$ ,  $\mathbf{g} \in L^2(\Omega; \mathbb{R}^N)$  are respectively the prescribed boundary and body force fields such that the whole system of forces is equilibrated, namely the condition of equilibrated load

$$(1.2) \quad \mathcal{L}(\mathbf{z}) = 0 \quad \forall \mathbf{z} : \mathbb{E}(\mathbf{z}) \equiv \mathbf{0},$$

(which is a standard necessary condition for pure traction in linear elasticity) is assumed with

$$\mathcal{L}(\mathbf{v}) := \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\mathbf{x}.$$

We consider the sequence of energy functionals

$$(1.3) \quad \mathcal{F}_h(\mathbf{v}) = \int_{\Omega} \mathcal{V}_h(\mathbf{x}, \nabla \mathbf{v}) d\mathbf{x} - \mathcal{L}(\mathbf{v})$$

and we inquire whether the asymptotic relationship  $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$  as  $h \rightarrow 0_+$  implies, up to subsequences, some kind of weak convergence of  $\mathbf{v}_h$  to a minimizer  $\mathbf{v}_0$  of a suitable limit functional in  $H^1(\Omega; \mathbb{R}^N)$ .

First we emphasize that in the case of Neumann condition on the whole boundary things are

not so plain. Indeed even by choosing  $\Omega$  Lipschitz and assuming the simplest dependance of stored energy density  $\mathcal{W}$  on the deformation gradient  $\mathbf{F}$ , say (see [10])

$$(1.4) \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) = \begin{cases} |\mathbf{F}^T \mathbf{F} - \mathbf{I}|^2 & \text{if } \det \mathbf{F} > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

if  $\mathbf{g} \equiv 0$ ,  $\mathbf{f} = f\mathbf{n}$ ,  $f < 0$  and  $\mathbf{n}$  denotes the outward unit normal to  $\partial\Omega$  (so that the global condition (1.2) holds true), then by the same techniques of [13] one can exhibit the Gamma limit of  $\mathcal{F}_h$  with respect to weak  $H^1$  topology:

$$(1.5) \quad \Gamma(w H^1) \lim_{h \rightarrow 0} \mathcal{F}_h(\mathbf{v}) = \mathcal{E}(\mathbf{v}),$$

where

$$(1.6) \quad \mathcal{E}(\mathbf{v}) = 4 \int_{\Omega} |\mathbb{E}(\mathbf{v})|^2 d\mathbf{x} - f \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} d\mathcal{H}^{N-1}(\mathbf{x}),$$

i.e. the classical linear elasticity formulation which achieves a finite minimum over  $H^1(\Omega; \mathbb{R}^N)$  since the condition of equilibrated loads is fulfilled. Nevertheless with exactly the same choices there is a sequence  $\mathbf{w}_h$  in  $H^1(\Omega; \mathbb{R}^N)$  such that  $\mathcal{F}_h(\mathbf{w}_h) \rightarrow -\infty$  as  $h \rightarrow 0^+$  (see Remark 2.7): although minimizers of  $\mathcal{E}$  over  $H^1(\Omega; \mathbb{R}^N)$  exist, functionals  $\mathcal{F}_h$  are not uniformly bounded from below and these facts seem to suggest that, in presence of compressive forces acting on the boundary, minimizing sequences of  $\mathcal{F}_h$  do not converge to minimizers of  $\mathcal{E}$ .

Moreover it is worth noting that if  $\mathcal{W}$  fulfils (1.4) and  $\mathbf{g} \equiv \mathbf{f} \equiv \mathbf{0}$ , hence  $\inf \mathcal{F}_h = 0$  for every  $h > 0$ , then by choosing a fixed nontrivial  $N \times N$  skew-symmetric matrix  $\mathbf{W}$ , a real number  $0 < 2\alpha < 1$  and setting

$$(1.7) \quad \mathbf{z}_h := h^{-\alpha} \mathbf{W} \mathbf{x},$$

we get  $\mathcal{F}_h(\mathbf{z}_h) = \inf \mathcal{F}_h + o(1)$ , nevertheless  $\mathbf{z}_h$  has no subsequence weakly converging in  $H^1(\Omega; \mathbb{R}^N)$ , see Remark 2.4.

Therefore here, in contrast to [13], we cannot expect weak  $H^1(\Omega; \mathbb{R}^N)$  compactness of minimizing sequences, not even in the simplest case of null external forces: although this fact is common to pure traction problems in linear elasticity, we emphasize that in general nonlinear elasticity setting this difficulty cannot be easily circumvented by plain translations since  $\mathcal{F}_h(\mathbf{v}_h) \neq \mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h)$ , with  $\mathbb{P}$  projection on infinitesimal rigid displacements.

We deal with this issue in the paper [26], showing nonetheless that at least for some special  $\mathcal{W}$ , if  $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$  then up to subsequences  $\mathcal{F}_h(\mathbf{v}_h - \mathbb{P}\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$ .

In order to achieve some kind of precompactness for the sequences  $\mathbf{v}_h$  fulfilling  $\mathcal{F}_h(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$ , we work with a very weak notion of convergence: the weak  $L^2(\Omega; \mathbb{R}^N)$  convergence of linear strains. Therefore our approach requires the analysis of variational limit of  $\mathcal{F}_h$  with respect to this convergence. Since weak  $L^2$  convergence of linear strains does not imply an analogous convergence of the skew symmetric part of the displacement gradients, it can be expected that the  $\Gamma$  limit functional is different from the classical linearized elasticity functional which is the pointwise limit of  $\mathcal{F}_h$  (for a reformulation of classical linearized elasticity with linear strain tensor as the “primary” unknown instead of displacement, see [11]).

Indeed under some natural assumptions on  $\mathcal{W}$ , a careful application of the Rigidity Lemma of [18] shows that if  $\mathbb{E}(\mathbf{v}_h)$  are bounded in  $L^2$  then, up to subsequences,  $\sqrt{h}\nabla\mathbf{v}_h$  converges

strongly in  $L^2$  to a constant skew symmetric matrix and the variational limit of the sequence  $\mathcal{F}_h$ , with respect to the w- $L^2$  convergence of linear strains, turns out to be

$$(1.8) \quad \mathcal{F}(\mathbf{v}) := \min_{\mathbf{W}} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2}\mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

where the minimum is evaluated over skew symmetric  $N \times N$  matrices  $\mathbf{W}$  and

$$(1.9) \quad \mathcal{V}_0(\mathbf{x}, \mathbf{B}) := \tfrac{1}{2} \mathbf{B}^T D^2 \mathcal{V}(\mathbf{x}, \mathbf{0}) \mathbf{B} \quad \forall \mathbf{B} \in \mathcal{M}_{sym}^{N \times N}.$$

We emphasize that the functional  $\mathcal{F}$  in (1.8) is different from the functional  $\mathcal{E}$  of linearized elasticity defined as

$$\mathcal{E}(\mathbf{v}) := \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v})) d\mathbf{x} - \mathcal{L}(\mathbf{v})$$

since if  $\mathbf{v}(\mathbf{x}) = \tfrac{1}{2}\mathbf{W}^2\mathbf{x}$  with  $\mathbf{W} \neq \mathbf{0}$  skew symmetric matrix, then  $\mathcal{F}(\mathbf{v}) = -\mathcal{L}(\mathbf{v}) < \mathcal{E}(\mathbf{v})$ . Nevertheless if  $N = 2$  then (see Remark 2.6)

$$\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v}) - \frac{1}{4} \left( \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbf{I}) d\mathbf{x} \right)^{-1} \left[ \left( \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) d\mathbf{x} \right)^- \right]^2,$$

hence  $\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v})$  if

$$N = 2, \quad \text{and} \quad \int_{\Omega} D\mathcal{V}_0(x, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) dx \geq 0.$$

In particular if  $N = 2$  and  $\mathcal{W}$  is the *Saint Venant–Kirchhoff energy density* (1.4) then the previous inequality reduces to

$$\int_{\Omega} \operatorname{div} \mathbf{v} d\mathbf{x} \geq 0$$

which means, roughly speaking, that the area of  $\Omega$  is less than the area of the related deformed configuration  $\mathbf{y}(\Omega)$ , where  $\mathbf{y}(\mathbf{x}) = \mathbf{x} + h\mathbf{v}(\mathbf{x})$  and  $h > 0$ .

The main results of present paper are stated in Theorems 2.2 and 4.1: they show that under a suitable compatibility condition on the forces (subsequent formula (1.10)) the pure traction problem in linear elasticity is deduced via  $\Gamma$ -convergence from pure traction problem in non-linear elasticity, referring to weak  $L^2$  convergence of the linear strains.

Precisely Theorem 2.2 states that, if the loads  $\mathbf{f}$ ,  $\mathbf{g}$  fulfil (1.2) together with the next compatibility condition

$$(1.10) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{W}^2 \mathbf{x} d\mathbf{x} < 0 \quad \forall \text{ skew symmetric matrix } \mathbf{W} \neq \mathbf{0},$$

then every sequence  $\mathbf{v}_h$  with  $\mathcal{F}(\mathbf{v}_h) = \inf \mathcal{F}_h + o(1)$  has a subsequence such that the corresponding linear strains converge weakly in  $L^2$  to the linear strain of a minimizer of  $\mathcal{F}$ , together with convergence (without relabeling) of energies  $\mathcal{F}_h(\mathbf{v}_h)$  to  $\min \mathcal{F}$ . Under the same assumptions Theorem 4.1 states that minimizers of  $\mathcal{F}$  coincide with the ones of linearized elasticity functional  $\mathcal{E}$ , thus providing a full justification of pure traction problems in linear elasticity at least if (1.10) is satisfied. In particular, as it is shown in Remark 2.8, this is true when  $\mathbf{g} \equiv 0$ ,  $\mathbf{f} = f\mathbf{n}$  with  $f > 0$  and  $\mathbf{n}$  is the outer unit normal vector to  $\partial\Omega$ , that is when we are in presence of tension-like surface forces.

About physical interpretation and motivations of compatibility condition (1.10) we refer to subsequent Remarks 2.7 and 2.8.

Moreover, if there exists an  $N \times N$  skew symmetric matrix such that the strict inequality is

reversed in (1.10), then functional  $\mathcal{F}$  is unbounded from below: see Remark 4.5 and Example 4.6. On the other hand if inequality in (1.10) is satisfied in a weak sense by every skew symmetric matrix, then  $\operatorname{argmin} \mathcal{F}$  contains  $\operatorname{argmin} \mathcal{E}$ ,  $\min \mathcal{F} = \min \mathcal{E}$  but  $\mathcal{F}$  may have infinitely many minimizing critical points which are not minimizers of  $\mathcal{E}$  (see Proposition 4.3).

Summarizing, only two cases are allowed: either  $\min \mathcal{F} = \min \mathcal{E}$  or  $\inf \mathcal{F} = -\infty$ ; actually the second case arises in presence of compressive surface load.

By oversimplifying we could say that  $\mathcal{F}$  somehow preserves memory of instabilities which are typical of finite elasticity, while they disappear in the linearized model described by  $\mathcal{E}$ .

In the light of Theorem 2.2 and of remarks and examples of Section 4, it seems reasonable that, as far as it concerns pure traction problems, the range of validity of linear elasticity should be restricted to a certain class of external loads, explicitly those verifying (1.10), a remarkable example in such class is a uniform normal tension load at the boundary as in Remark (2.8); while in the other cases equilibria of a linearly elastic body could be better described through critical points of  $\mathcal{F}$ , whose existence in general seems to be an interesting and open problem. Concerning the structure of the new functional, we emphasize that actually  $\mathcal{F}$  is different from the classical linear elasticity energy functional  $\mathcal{E}$ , though there are many relations between their minimizers (see Theorem 4.1). Further and more detailed information about functional  $\mathcal{F}$  (a suitable property of weak lower semicontinuity, lack of subadditivity, convexity in 2D, nonconvexity in 3D) are described and proved in the paper [26].

## 2. NOTATION AND MAIN RESULT

Assume that the reference configuration of an elastic body is a

$$(2.1) \quad \text{bounded, connected open set } \Omega \subset \mathbb{R}^N \text{ with Lipschitz boundary, } N = 2, 3.$$

The generic point  $\mathbf{x} \in \Omega$  has components  $x_j$  referring to the standard basis vectors  $\mathbf{e}_j$  in  $\mathbb{R}^N$ ;  $\mathcal{L}^N$  and  $\mathcal{B}^N$  denote respectively the  $\sigma$ -algebras of Lebesgue measurable and Borel measurable subsets of  $\mathbb{R}^N$ . For every  $\alpha \in \mathbb{R}$  we set  $\alpha^+ = \alpha \vee 0$ ,  $\alpha^- = -\alpha \vee 0$ .

The notation for vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$  and  $N \times N$  real matrices  $\mathbf{A}, \mathbf{B}, \mathbf{F}$  are as follows:  $\mathbf{a} \cdot \mathbf{b} = \sum_j \mathbf{a}_j \mathbf{b}_j$ ;  $\mathbf{A} \cdot \mathbf{B} = \sum_{i,j} \mathbf{A}_{i,j} \mathbf{B}_{i,j}$ ;  $[\mathbf{AB}]_{i,j} = \sum_k \mathbf{A}_{i,k} \mathbf{B}_{k,j}$ ;  $|\mathbf{F}|^2 = \operatorname{Tr}(\mathbf{F}^T \mathbf{F}) = \sum_{i,j} F_{i,j}^2$  denotes the squared Euclidean norm of  $\mathbf{F}$  in the space  $\mathcal{M}^{N \times N}$  of  $N \times N$  real matrices;  $\mathbf{I} \in \mathcal{M}^{N \times N}$  denotes the identity matrix,  $SO(N)$  denotes the group of rotation matrices,  $\mathcal{M}_{sym}^{N \times N}$  and  $\mathcal{M}_{skew}^{N \times N}$  denote respectively the sets of symmetric and skew-symmetric matrices. For every  $\mathbf{B} \in \mathcal{M}^{N \times N}$  we define  $\operatorname{sym} \mathbf{B} := \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$  and  $\operatorname{skew} \mathbf{B} := \frac{1}{2}(\mathbf{B} - \mathbf{B}^T)$ .

It is well known that matrix exponential maps  $\mathcal{M}_{skew}^{N \times N}$  to  $SO(N)$  and is surjective on  $SO(N)$  (see [20]). Therefore for every  $\mathbf{R} \in SO(N)$  there exist  $\vartheta \in \mathbb{R}$  and  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ ,  $|\mathbf{W}|^2 = 2$  such that  $\exp(\vartheta \mathbf{W}) = \mathbf{R}$ . By taking into account that  $\mathbf{W}^3 = -\mathbf{W}$  if  $N = 2, 3$ , the Taylor's series expansion of  $\vartheta \rightarrow \exp(\vartheta \mathbf{W}) = \sum_{k=0}^{\infty} \vartheta^k \mathbf{W}^k / k!$  yields the *Euler-Rodrigues formula*:

$$(2.2) \quad \exp(\vartheta \mathbf{W}) = \mathbf{R} = \mathbf{I} + \sin \vartheta \mathbf{W} + (1 - \cos \vartheta) \mathbf{W}^2 \quad N = 2, 3.$$

We recall an elementary issue which proves useful in our analysis:

$$(2.3) \quad \text{if } \mathbf{W} \in \mathcal{M}_{skew}^{N \times N}, \quad |\mathbf{W}|^2 = 2, \quad N = 2, 3, \quad \text{then} \quad |\mathbf{W}^2|^2 = 2,$$

and we set

$$(2.4) \quad \mathbb{K} := \{\tau(\mathbf{R} - \mathbf{I}) : \tau \geq 0, \mathbf{R} \in SO(N)\}.$$

For every  $\mathcal{U} : \Omega \times \mathcal{M}^{N \times N} \rightarrow \mathbb{R}$ , with  $\mathcal{U}(\mathbf{x}, \cdot) \in C^2$  a.e.  $\mathbf{x} \in \Omega$ , we denote by  $D\mathcal{U}(\mathbf{x}, \cdot)$  and  $D^2\mathcal{U}(\mathbf{x}, \cdot)$  respectively the gradient and the hessian of  $\mathcal{U}$  with respect to the second variable. For every displacements field  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ ,  $\mathbb{E}(\mathbf{v}) := \text{sym } \nabla \mathbf{v}$  denotes the infinitesimal strain tensor field,  $\mathcal{R} := \{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N) : \mathbb{E}(\mathbf{v}) = \mathbf{0}\}$  denotes the space spanned by the set of the infinitesimal rigid displacements and  $\mathbb{P}\mathbf{v}$  is the orthogonal projection of  $\mathbf{v}$  onto  $\mathcal{R}$ .

We set  $\int_{\Omega} \mathbf{v} d\mathbf{x} = |\Omega|^{-1} \int_{\Omega} \mathbf{v} d\mathbf{x}$ .

We consider a body made of a hyperelastic material, say there exists a  $\mathcal{L}^N \times \mathcal{B}^{N^2}$  measurable  $\mathcal{W} : \Omega \times \mathcal{M}^{N \times N} \rightarrow [0, +\infty]$  such that, for a.e.  $\mathbf{x} \in \Omega$ ,  $\mathcal{W}(\mathbf{x}, \nabla \mathbf{y}(\mathbf{x}))$  represents the stored energy density, when  $\mathbf{y}(x)$  is the deformation and  $\nabla \mathbf{y}(\mathbf{x})$  is the deformation gradient.

Moreover we assume that for a.e.  $\mathbf{x} \in \Omega$

$$(2.5) \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) = +\infty \quad \text{if } \det \mathbf{F} \leq 0 \quad (\text{orientation preserving condition}),$$

$$(2.6) \quad \mathcal{W}(\mathbf{x}, \mathbf{R}\mathbf{F}) = \mathcal{W}(\mathbf{x}, \mathbf{F}) \quad \forall \mathbf{R} \in SO(N) \quad \forall \mathbf{F} \in \mathcal{M}^{N \times N} \quad (\text{frame indifference}),$$

$$(2.7) \quad \exists \text{ a neighborhood } \mathcal{A} \text{ of } SO(N) \text{ s.t. } \mathcal{W}(\mathbf{x}, \cdot) \in C^2(\mathcal{A}),$$

$$(2.8) \quad \exists C > 0 \text{ independent of } \mathbf{x} : \mathcal{W}(\mathbf{x}, \mathbf{F}) \geq C |\mathbf{F}^T \mathbf{F} - \mathbf{I}|^2 \quad \forall \mathbf{F} \in \mathcal{M}^{N \times N} \quad (\text{coerciveness}),$$

$$(2.9) \quad \mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \quad D\mathcal{W}(\mathbf{x}, \mathbf{I}) = 0, \quad \text{for a.e. } \mathbf{x} \in \Omega,$$

that is the reference configuration has zero energy and is stress free, so by (2.6) we get also

$$\mathcal{W}(\mathbf{x}, \mathbf{R}) = 0, \quad D\mathcal{W}(\mathbf{x}, \mathbf{R}) = 0 \quad \forall \mathbf{R} \in SO(N).$$

By frame indifference there exists a  $\mathcal{L}^N \times \mathcal{B}^N$ -measurable  $\mathcal{V} : \Omega \times \mathcal{M}^{N \times N} \rightarrow [0, +\infty]$  such that for every  $\mathbf{F} \in \mathcal{M}^{N \times N}$

$$(2.10) \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) = \mathcal{V}(\mathbf{x}, \tfrac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}))$$

and by (2.7)

$$(2.11) \quad \exists \text{ a neighborhood } \mathcal{O} \text{ of } \mathbf{0} \text{ such that } \mathcal{V}(\mathbf{x}, \cdot) \in C^2(\mathcal{O}), \text{ a.e. } \mathbf{x} \in \Omega.$$

In addition we assume that there exists  $\gamma > 0$  independent of  $\mathbf{x}$  such that

$$(2.12) \quad |\mathbf{B}^T D^2 \mathcal{V}(\mathbf{x}, \mathbf{D}) \mathbf{B}| \leq 2\gamma |\mathbf{B}|^2 \quad \forall \mathbf{D} \in \mathcal{O}, \quad \forall \mathbf{B} \in \mathcal{M}^{N \times N}.$$

By (2.9) and Taylor expansion with Lagrange reminder we get, for a.e.  $\mathbf{x} \in \Omega$  and suitable  $t \in (0, 1)$  depending on  $\mathbf{x}$  and on  $\mathbf{B}$ :

$$(2.13) \quad \mathcal{V}(\mathbf{x}, \mathbf{B}) = \tfrac{1}{2} \mathbf{B}^T D^2 \mathcal{V}(\mathbf{x}, t\mathbf{B}) \mathbf{B}.$$

Hence by (2.12)

$$(2.14) \quad \mathcal{V}(\mathbf{x}, \mathbf{B}) \leq \gamma |\mathbf{B}|^2 \quad \forall \mathbf{B} \in \mathcal{M}^{N \times N} \cap \mathcal{O}.$$

According to (2.10) for a.e.  $\mathbf{x} \in \Omega$ ,  $h > 0$  and every  $\mathbf{B} \in \mathcal{M}^{N \times N}$  we set

$$(2.15) \quad \mathcal{V}_h(\mathbf{x}, \mathbf{B}) := \tfrac{1}{h^2} \mathcal{W}(\mathbf{x}, \mathbf{I} + h\mathbf{B}) = \tfrac{1}{h^2} \mathcal{V}(\mathbf{x}, h \text{sym } \mathbf{B} + \tfrac{1}{2} h^2 \mathbf{B}^T \mathbf{B}).$$

Taylor's formula with (2.9),(2.15) entails  $\mathcal{V}_h(\mathbf{x}, \mathbf{B}) = \frac{1}{2} (\text{sym } \mathbf{B}) D^2 \mathcal{V}(\mathbf{x}, \mathbf{0}) (\text{sym } \mathbf{B}) + o(1)$ , so

$$(2.16) \quad \mathcal{V}_h(\mathbf{x}, \mathbf{B}) \rightarrow \mathcal{V}_0(\mathbf{x}, \text{sym } \mathbf{B}) \text{ as } h \rightarrow 0_+,$$

where the pointwise limit of integrands is the quadratic form  $\mathcal{V}_0$  defined by

$$(2.17) \quad \mathcal{V}_0(\mathbf{x}, \mathbf{B}) := \frac{1}{2} \mathbf{B}^T D^2 \mathcal{V}(\mathbf{x}, \mathbf{0}) \mathbf{B} \quad \text{a.e. } \mathbf{x} \in \Omega, \mathbf{B} \in \mathcal{M}^{N \times N}.$$

The symmetric fourth order tensor  $D^2 \mathcal{V}(\mathbf{x}, \mathbf{0})$  in (2.17) plays the role of classical linear elasticity tensor.

By (2.8) we get

$$(2.18) \quad \mathcal{V}_h(x, \mathbf{B}) = \frac{1}{h^2} \mathcal{W}(x, \mathbf{I} + h\mathbf{B}) \geq C |2 \text{sym } \mathbf{B} + h \mathbf{B}^T \mathbf{B}|^2$$

so that (2.17) and (2.18) imply the ellipticity of  $\mathcal{V}_0$ :

$$(2.19) \quad \mathcal{V}_0(\mathbf{x}, \text{sym } \mathbf{B}) \geq 4C |\text{sym } \mathbf{B}|^2 \quad \text{a.e. } \mathbf{x} \in \Omega, \mathbf{B} \in \mathcal{M}^{N \times N}.$$

Let  $\mathbf{f} \in L^2(\partial\Omega; \mathbb{R}^N)$  and  $\mathbf{g} \in L^2(\Omega; \mathbb{R}^N)$  be respectively the surface and body force field.

For a suitable choice of the adimensional parameter  $h > 0$ , the functional representing the total energy is labeled by  $\mathcal{F}_h : H^1(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R} \cup \{+\infty\}$  and defined as follows

$$(2.20) \quad \mathcal{F}_h(\mathbf{v}) := \int_{\Omega} \mathcal{V}_h(\mathbf{x}, \nabla \mathbf{v}) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

where

$$(2.21) \quad \mathcal{L}(\mathbf{v}) := \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{v} d\mathcal{H}^{n-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\mathbf{x}.$$

In this paper we are interested in the asymptotic behavior as  $h \downarrow 0_+$  of functionals  $\mathcal{F}_h$  and to this aim we introduce the limit energy functional  $\mathcal{F} : H^1(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$  defined by

$$(2.22) \quad \mathcal{F}(\mathbf{v}) = \min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}).$$

We emphasize that the minimum in right-hand side of definition (2.22) exists: precisely the finite dimensional minimization problem has exactly two solutions which differs only by a sign, since by (2.19),

$$(2.23) \quad \lim_{|\mathbf{W}| \rightarrow +\infty, \mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} = +\infty$$

and  $\mathcal{V}_0(\mathbf{x}, \cdot)$  is strictly convex by (2.17), (2.19).

All along this paper we assume (2.1) together with the *standard structural conditions* (2.5)-(2.9),(2.12) as it is usually done in scientific literature concerning elasticity theory and we refer to the notations (2.10),(2.15),(2.17),(2.20)-(2.22).

The pair  $\mathbf{f}, \mathbf{g}$  describing the load is said to be *equilibrated* if

$$(2.24) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{z} d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{z} d\mathbf{x} = 0 \quad \forall \mathbf{z} \in \mathcal{R},$$

and it is said to be *compatible* if

$$(2.25) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} + \int_{\Omega} \mathbf{g} \cdot \mathbf{W}^2 \mathbf{x} d\mathbf{x} < 0 \quad \forall \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \text{ s.t. } \mathbf{W} \neq \mathbf{0}.$$



**Definition 2.1.** We say that  $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$  is a minimizing sequence of the sequence of functionals  $\mathcal{F}_{h_j}$ , if  $(\mathcal{F}_{h_j}(\mathbf{v}_j) - \inf \mathcal{F}_{h_j}) \rightarrow 0$  as  $h_j \rightarrow 0_+$ .

We will show (see Lemma 3.1) that, if compatibility (2.25) holds true then  $\inf_h \inf \mathcal{F}_h > -\infty$ , hence for every infinitesimal sequence  $h_j$  a minimizing sequences of the sequence of functionals  $\mathcal{F}_{h_j}$  do exist.

Now we can state the main result, whose proof is postponed.

**Theorem 2.2.** Assume that the standard structural conditions and (2.24), (2.25) hold true. Then for every sequence of strictly positive real numbers  $h_j \rightarrow 0_+$  there exist minimizing sequences of the sequence of functionals  $\mathcal{F}_{h_j}$ .

Moreover for every minimizing sequence  $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$  of  $\mathcal{F}_{h_j}$  there exist: a subsequence, a displacement  $\mathbf{v}_0 \in H^1(\Omega; \mathbb{R}^N)$  and a constant matrix  $\mathbf{W}_0 \in \mathcal{M}_{skew}^{N \times N}$  such that, without relabeling,

$$(2.26) \quad \mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v}_0) \quad \text{weakly in } L^2(\Omega; \mathcal{M}^{N \times N}),$$

$$(2.27) \quad \sqrt{h_j} \nabla \mathbf{v}_j \rightarrow \mathbf{W}_0 \quad \text{strongly in } L^2(\Omega; \mathcal{M}^{N \times N}).$$

$$(2.28) \quad \lim_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) = \mathcal{F}(\mathbf{v}_0) = \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(\mathbf{v}),$$

$$(2.29) \quad \mathcal{F}(\mathbf{v}_0) = \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \tfrac{1}{2} \mathbf{W}_0^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}_0).$$

**Remark 2.3.** In the sequel (see Corollary 4.2) we prove that structural assumptions together with (2.24) and (2.25) entail even further refinement in previous statement: explicitly  $\mathbf{W}_0 = \mathbf{0}$ .

**Remark 2.4.** It is worth to underline that in contrast to the case of Dirichlet problem faced in [13], here in pure traction problem we cannot expect even weak  $H^1(\Omega; \mathbb{R}^N)$  convergence of minimizing sequences.

Indeed choose:  $\mathbf{f} = \mathbf{g} \equiv 0$  and

$$(2.30) \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) = \begin{cases} |\mathbf{F}^T \mathbf{F} - \mathbf{I}|^2 & \text{if } \det \mathbf{F} > 0, \\ +\infty & \text{otherwise,} \end{cases}$$

$$(2.31) \quad \mathbf{v}_j := h_j^{-\alpha} \mathbf{W} \mathbf{x} \quad \text{with } \mathbf{W} \in \mathcal{M}_{skew}^{N \times N}, \quad 0 < 2\alpha < 1, \quad h_j \rightarrow 0_+.$$

Then  $\mathcal{F}_{h_j}(\mathbf{v}_j) = o(1)$  and, due to  $\inf \mathcal{F}_{h_j} = 0$ , the sequence  $\mathbf{v}_j$  is a minimizing sequence which has no subsequence weakly converging in  $H^1(\Omega; \mathbb{R}^N)$ . It is well known that such phenomenon takes place for pure traction problems in linear elasticity too, but in nonlinear elasticity this difficulty cannot be easily circumvented in general, since the fact that  $\mathbf{v}_j$  is a minimizing sequence does not entail that also  $\mathbf{v}_j - \mathbb{P} \mathbf{v}_j$  is minimizing sequence. In [26] we show that for some special integrand  $\mathcal{W}$ , as in the case of *Saint Venant–Kirchhoff energy density*, if  $\mathbf{v}_j$  is a minimizing sequence then  $\mathbf{w}_j := \mathbf{v}_j - \mathbb{P} \mathbf{v}_j$  is a minimizing sequence too and there exist a (not relabeled) subsequence of functionals  $\mathcal{F}_{h_j}$  such that the related *a minimizing subsequence*  $\mathbf{w}_j$  converges weakly in  $H^1(\Omega; \mathbb{R}^N)$  to a minimizer  $\mathbf{v}_0$  of  $\mathcal{F}$ , provided (2.24) and (2.25) hold true.

**Remark 2.5.** A careful inspection of the proof (see also [13]) shows that Theorem 2.2 remains true if coercivity condition (2.8) on  $\mathcal{W}$  is weakened by assuming either

$$(2.32) \quad \exists C > 0 \text{ independent of } \mathbf{x} : \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) \geq C \operatorname{dist}^2(\mathbf{F}, SO(N)) \quad \forall \mathbf{F} \in \mathcal{M}^{N \times N},$$

or by assuming these three conditions on  $\mathcal{V}$

$$(2.33) \quad \inf_{|\mathbf{B}| \geq \rho} \inf_{x \in \Omega} \mathcal{V}(x, \mathbf{B}) > 0 \quad \forall \rho > 0,$$

$$(2.34) \quad \exists \alpha > 0, \rho > 0 \text{ such that } \inf_{x \in \Omega} \mathcal{V}(x, \mathbf{B}) \geq \alpha |\mathbf{B}|^2 \quad \forall |\mathbf{B}| \leq \rho,$$

$$(2.35) \quad \liminf_{|\mathbf{B}| \rightarrow +\infty} \frac{1}{|\mathbf{B}|} \inf_{x \in \Omega} \mathcal{V}(x, \mathbf{B}) > 0.$$

This can be shown by exploiting Lemma 3.1 in [13]; notice that under such weaker assumption the constant appearing on the right-hand side of inequality (3.4) must be modified accordingly in Lemma 3.1.

It is worth noting that (2.8) implies (2.32), (2.33), (2.34) and (2.35).

In this Section we show some properties of the limit functional  $\mathcal{F}$  and several preliminary results to be used in the proof of Theorem 2.2.

**Remark 2.6.** If  $N = 2$ , then for every  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  there is  $a \in \mathbb{R}$  such that  $\mathbf{W}^2 = -a^2 \mathbf{I}$ , hence (2.22) reads

$$(2.36) \quad \mathcal{F}(\mathbf{v}) = \min_{a \in \mathbb{R}} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) + \frac{a^2}{2} \mathbf{I}) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

therefore, a minimizer  $a_*(\mathbf{v})$  of functional (2.36) (for  $a \in \mathbb{R}$  with fixed  $\mathbf{v}$ ) fulfils

$$a_*^3(\mathbf{v}) \int_{\Omega} \mathcal{V}_0(x, \mathbf{I}) d\mathbf{x} + a_*(\mathbf{v}) \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) d\mathbf{x} = 0$$

that is

$$(2.37) \quad a_*^2(\mathbf{v}) = \left( \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbf{I}) d\mathbf{x} \right)^{-1} \left( \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) d\mathbf{x} \right)^{-}$$

and

$$(2.38) \quad \mathcal{F}(\mathbf{v}) = \int_{\Omega} \mathcal{V}_0 \left( \mathbf{x}, \mathbb{E}(\mathbf{v}) + \frac{a_*^2(\mathbf{v})}{2} \mathbf{I} \right) d\mathbf{x} - \mathcal{L}(\mathbf{v}).$$

Hence, by taking into account that  $\mathcal{V}_0$  is a quadratic form it is readily seen that for  $N = 2$

$$(2.39) \quad \begin{aligned} \mathcal{F}(\mathbf{v}) &= \\ &= \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v})) d\mathbf{x} - \frac{1}{4} \left( \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbf{I}) d\mathbf{x} \right)^{-1} \left[ \left( \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) d\mathbf{x} \right)^{-} \right]^2 - \mathcal{L}(\mathbf{v}) \\ &= \mathcal{E}(\mathbf{v}) - \frac{1}{4} \left( \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbf{I}) d\mathbf{x} \right)^{-1} \left[ \left( \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbf{I}) \cdot \mathbb{E}(\mathbf{v}) d\mathbf{x} \right)^{-} \right]^2. \end{aligned}$$

Even more explicit, if  $N = 2$ ,  $\lambda, \mu > 0$  and

$$(2.40) \quad \mathcal{W}(\mathbf{x}, \mathbf{F}) = \begin{cases} \mu |\mathbf{F}^T \mathbf{F} - \mathbf{I}|^2 + \frac{\lambda}{2} |\text{Tr}(\mathbf{F}^T \mathbf{F} - \mathbf{I})|^2 & \text{if } \det \mathbf{F} > 0 \\ +\infty & \text{otherwise,} \end{cases}$$

then  $\mathcal{V}_0(\mathbf{x}, \mathbf{B}) = 4\mu |\mathbf{B}|^2 + 2\lambda |\text{Tr} \mathbf{B}|^2$  and we get

$$(2.41) \quad a_*^2(\mathbf{v}) = |\Omega|^{-1} \left( \int_{\Omega} \text{div } \mathbf{v} d\mathbf{x} \right)^{-}.$$

This conclusion could be approximately rephrased as follows: in 2D the global energy  $\mathcal{F}(\mathbf{v})$  of a displacement  $\mathbf{v}$  is the same of linearized elasticity if the area of the associated deformed configuration  $\mathbf{y}(\Omega) = (\mathbf{I} + \mathbf{v})(\Omega)$  is not less than the area of  $\Omega$ .

**Remark 2.7.** The compatibility condition (2.25) cannot be dropped in Theorem 2.2 even if the (necessary) condition (2.24) holds true. Moreover plain substitution of strong with weak inequality in (2.25) leads to a lack of compactness for minimizing sequences.

Indeed, if  $\mathbf{n}$  denotes the outer unit normal vector to  $\partial\Omega$  and we choose  $\mathbf{f} = f\mathbf{n}$  with  $f < 0$ ,  $\mathbf{g} \equiv 0$  then, by Divergence Theorem,

$$(2.42) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} \, d\mathcal{H}^{N-1} = f (\text{Tr } \mathbf{W}^2) |\Omega| > 0 \quad \forall \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \setminus \{\mathbf{0}\},$$

so that the strict inequality in (2.25) is reversed in a strong sense by any  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \setminus \{\mathbf{0}\}$ ; fix a sequence of positive real numbers such that  $h_j \rightarrow 0_+$   $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ ,  $\mathbf{W} \neq \mathbf{0}$ , and set  $\mathbf{v}_j = h_j^{-1}(\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W})\mathbf{x}$ , then  $\mathbf{I} + h\mathbf{v}_j = \mathbf{I} + (\frac{1}{2}\mathbf{W}^2 + \frac{\sqrt{3}}{2}\mathbf{W}) \in SO(N)$ , due to representation (2.2) with  $\vartheta = \pi/3$ . Hence, by frame indifference,

$$(2.43) \quad \mathcal{F}_{h_j}(\mathbf{v}_j) = \mathcal{L}(\mathbf{v}_j) = -\frac{f}{2h_j} \int_{\partial\Omega} \mathbf{W}^2 \mathbf{x} \cdot \mathbf{n} \, d\mathcal{H}^{n-1} = -\frac{f}{2h_j} (\text{Tr } \mathbf{W}^2) |\Omega| \rightarrow -\infty.$$

On the other hand, assume  $N = 2, 3$ ,  $\Omega \subset \mathbb{R}^N$  a bounded open set with Lipschitz boundary and  $\mathcal{W}$  as in (2.30) and  $\mathbf{f} = \mathbf{g} \equiv \mathbf{0}$ , so that the compatibility inequality is substituted by the weak inequality; if  $\mathbf{v}_j$  are still defined as above then, hence by frame indifference,

$$(2.44) \quad \mathcal{F}_{h_j}(\mathbf{v}_j) = 0 = \inf \mathcal{F}_{h_j},$$

namely,  $\mathbf{v}_j$  is a minimizing sequence for  $\mathcal{F}_{h_j}$ , but  $\mathbb{E}(\mathbf{v}_j)$  has no weakly convergent subsequences in  $L^2(\Omega; \mathcal{M}^{N \times N})$ .

**Remark 2.8.** It is worth noticing that the compatibility condition (2.25) holds true when  $\mathbf{g} \equiv 0$ ,  $\mathbf{f} = f\mathbf{n}$  with  $f > 0$  and  $\mathbf{n}$  the outer unit normal vector to  $\partial\Omega$ .

Indeed let  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ ,  $\mathbf{W} \neq \mathbf{0}$ : hence by (2.24) and the Divergence Theorem we get

$$(2.45) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} \, d\mathcal{H}^{N-1} = f (\text{Tr } \mathbf{W}^2) |\Omega| < 0$$

thus proving (2.25) in this case. This means that in case of uniform tension-like force field at the boundary and null body force field the compatibility condition holds true.

**Remark 2.9.** It is possible to observe some analogy between the energy functional (2.22) and the results in [15],[16], where the approximate theory of small strain together with *moderate rotations* is discussed under suitable kinematical assumptions. More precisely, if  $\mathbf{F} = \mathbf{I} + h\nabla\mathbf{v}$  is the deformation gradient,  $\mathbf{F} = \mathbf{R}\mathbf{U}$  the polar decomposition, [15] shows that the assumptions

$$(2.46) \quad \mathbf{R} = \mathbf{I} + O(\sqrt{h}), \quad \mathbf{U} = \mathbf{I} + O(h) \quad \text{as } h \rightarrow 0_+$$

in the sense of pointwise convergence are equivalent to

$$(2.47) \quad \mathbb{E}(\mathbf{v}) = O(1), \quad h(\text{skew} \nabla \mathbf{v}) = O(\sqrt{h}) \quad \text{as } h \rightarrow 0_+$$

still in the sense of pointwise convergence. Therefore

$$(2.48) \quad \mathbf{U} = \mathbf{I} + h(\mathbb{E}(\mathbf{v}) - \frac{1}{2}(\text{skew} \nabla \mathbf{v})^2) + o(h)$$

and the pointwise limit of  $\mathcal{F}_h$  ( not the  $\Gamma$ -limit !) becomes

$$\int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2}(\text{skew} \nabla \mathbf{v})^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

which is quite similar to (2.22).

We highlight the fact that (2.46) cannot be understood in the sense of  $L^2(\Omega, \mathcal{M}^{N \times N})$  whenever  $\mathbf{v} \equiv \mathbf{v}_*$  on a closed subset  $\Sigma$  of  $\partial\Omega$  with  $\mathcal{H}^{n-1}(\Sigma) > 0$ , since by Korn and Poincaré inequalities we get

$$\int_{\Omega} |\nabla \mathbf{v}|^2 d\mathbf{x} \leq C \left( \int_{\Omega} |\mathbb{E}(\mathbf{v})|^2 d\mathbf{x} + \int_{\Sigma} |\mathbf{v}_*|^2 d\mathcal{H}^{N-1} \right),$$

therefore if  $\mathbb{E}(\mathbf{v}) = O(1)$  then  $h\nabla \mathbf{v} = O(h)$ , thus contradicting the second of (2.46). On the other hand a careful application of the rigidity Lemma of [18] show that if  $\mathbb{E}(\mathbf{v}) = O(1)$  and  $\mathbf{U} = \mathbf{I} + O(h)$  in the sense of  $L^2(\Omega, \mathcal{M}^{N \times N})$ , then there exists a constant skew symmetric matrix  $\mathbf{W}$  such that  $h\nabla \mathbf{v}^T \nabla \mathbf{v} = -\mathbf{W}^2 + o(1)$  in the sense of  $L^1(\Omega, \mathcal{M}^{N \times N})$  (see the proof of Lemma 3.4 below). Therefore

$$(2.49) \quad \mathbf{U} = \mathbf{I} + h(\mathbb{E}(\mathbf{v}) - \mathbf{W}^2/2) + o(h)$$

where equality is understood in the sense of  $L^1(\Omega, \mathcal{M}^{N \times N})$  and  $\mathbf{W}$  a constant skew symmetric matrix.

### 3. PROOFS

We recall some basic inequalities exploited in the sequel, for both reader's convenience and labeling the related constants.

**Poincaré Inequality.** There exists a constant  $C_P = C_P(\Omega)$  such that

$$(3.1) \quad \|\mathbf{v} - f_{\Omega} \mathbf{v}\|_{L^2(\Omega; \mathbb{R}^N)} + \|\mathbf{v} - f_{\Omega} \mathbf{v}\|_{L^2(\partial\Omega; \mathbb{R}^N)} \leq C_P \|\nabla \mathbf{v}\|_{L^2(\Omega; \mathcal{M}^{N \times N})} \quad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^N).$$

**Korn Inequality.** There exists a constant  $C_K = C_K(\Omega)$  such that

$$(3.2) \quad \|\mathbf{v} - \mathbb{P} \mathbf{v}\|_{L^2(\Omega; \mathbb{R}^N)} + \|\mathbf{v} - \mathbb{P} \mathbf{v}\|_{L^2(\partial\Omega; \mathbb{R}^N)} \leq C_K \|\mathbb{E}(\mathbf{v})\|_{L^2(\Omega; \mathcal{M}^{N \times N})} \quad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^N).$$

**Geometric Rigidity Inequality** ([18]). There exists a constant  $C_G = C_G(\Omega)$  such that for every  $\mathbf{y} \in H^1(\Omega; \mathbb{R}^N)$  there is an associated rotation  $\mathbf{R} \in SO(N)$  such that we have

$$(3.3) \quad \int_{\Omega} |\nabla \mathbf{y} - \mathbf{R}|^2 d\mathbf{x} \leq C_G \int_{\Omega} \text{dist}^2(\nabla \mathbf{y}; SO(N)) d\mathbf{x}.$$

The first step in our analysis is the next lemma showing that if (2.24), (2.25) hold true then the functionals  $\mathcal{F}_h$  are bounded from below uniformly with respect to  $h > 0$ : this implies the existence of minimizing sequences of the sequence of functionals  $\mathcal{F}_{h_j}$  (see Definition 2.1).

**Lemma 3.1.** *Assume (2.24) and (2.25). Then*

$$(3.4) \quad \inf_{h>0} \inf_{\mathbf{v} \in H^1} \mathcal{F}_h(\mathbf{v}) \geq -\frac{C_P^2 C_G}{C} (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2),$$

where  $C$  is the coercivity constant in (2.18) and  $C_P, C_G$  are the constants related to the basic inequalities above.

Actually the claim holds true even if strict inequality is replaced by weak inequality in (2.25).

*Proof.* Let  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  and  $\mathbf{y} = \mathbf{x} + h\mathbf{v}$ . Since  $\det \nabla \mathbf{y} > 0$  a.e., by polar decomposition for a.e.  $\mathbf{x}$  there exist a rotation  $\mathbf{R}_h(\mathbf{x})$  and a symmetric positive definite matrix  $\mathbf{U}_h(\mathbf{x})$  such that  $\nabla \mathbf{y}(\mathbf{x}) = \mathbf{R}_h(\mathbf{x})\mathbf{U}_h(\mathbf{x})$ , hence  $\nabla \mathbf{y}^T \nabla \mathbf{y} = \mathbf{U}_h^2$ , so that for a.e.  $\mathbf{x}$

$$(3.5) \quad \begin{aligned} |\nabla \mathbf{y}^T \nabla \mathbf{y} - \mathbf{I}|^2 &= |\mathbf{U}_h^2 - \mathbf{I}|^2 = |(\mathbf{U}_h - \mathbf{I})(\mathbf{U}_h + \mathbf{I})|^2 \geq |\mathbf{U}_h - \mathbf{I}|^2 = \\ &= |(\nabla \mathbf{y} - \mathbf{R}_h)|^2 \geq \text{dist}^2(\nabla \mathbf{y}, SO(N)). \end{aligned}$$

By (2.8), (3.5) and the Geometric Rigidity Inequality (3.3) there exists a constant rotation  $\mathbf{R}$  such that

$$(3.6) \quad \begin{aligned} \mathcal{F}_h(\mathbf{v}) &\geq Ch^{-2} \int_{\Omega} |\nabla \mathbf{y}^T \nabla \mathbf{y} - \mathbf{I}|^2 d\mathbf{x} - h^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}) \geq \\ &\geq \frac{C}{C_G} h^{-2} \int_{\Omega} |\nabla \mathbf{y} - \mathbf{R}|^2 d\mathbf{x} - h^{-1} \mathcal{L}(\mathbf{y} - \mathbf{x}). \end{aligned}$$

If

$$\mathbf{c} := |\Omega|^{-1} \int_{\Omega} (\mathbf{y} - \mathbf{R}\mathbf{x}) d\mathbf{x},$$

then by Poincaré inequality (3.1)

$$\|\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}\|_{L^2(\Omega)} + \|\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}\|_{L^2(\partial\Omega)} \leq C_P \|\nabla(\mathbf{y} - \mathbf{R}\mathbf{x})\|_{L^2} = C_P \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2},$$

and by (2.24) and Young inequality we get, for every  $\alpha > 0$ ,

$$\begin{aligned} \mathcal{L}(\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}) &\leq C_P \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2} (\|\mathbf{f}\|_{L^2(\partial\Omega)} + \|\mathbf{g}\|_{L^2(\Omega)}) \leq \\ &\leq \alpha^{-1} \frac{C_P}{2} \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha \frac{C_P}{2} (\|\mathbf{f}\|_{L^2} + \|\mathbf{g}\|_{L^2})^2 \\ &\leq \alpha^{-1} \frac{C_P}{2} \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha C_P (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2). \end{aligned}$$

By choosing  $\alpha = h C_P C_G / C$

$$(3.7) \quad \begin{aligned} \mathcal{L}(\mathbf{y} - \mathbf{x}) &= \mathcal{L}(\mathbf{y} - \mathbf{R}\mathbf{x} - \mathbf{c}) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}) \leq \\ &\leq \alpha^{-1} \frac{C_P}{2} \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \alpha C_P (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}) = \\ &= h^{-1} \frac{C/C_G}{2} \|\nabla \mathbf{y} - \mathbf{R}\|_{L^2}^2 + \frac{C_P^2}{C/C_G} h (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2) + \mathcal{L}(\mathbf{R}\mathbf{x} - \mathbf{x}). \end{aligned}$$

Exploiting the standard representation (2.2) of the rotation  $\mathbf{R} = \mathbf{I} + \mathbf{W} \sin \vartheta + (1 - \cos \vartheta) \mathbf{W}^2$  for suitable  $\vartheta \in \mathbb{R}$  and  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  with  $|\mathbf{W}|^2 = 2$ , by (2.24), (2.25) we get

$$(3.8) \quad \mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) = (\sin \vartheta) \mathcal{L}(\mathbf{W}\mathbf{x}) + (1 - \cos \vartheta) \mathcal{L}(\mathbf{W}^2) < 0,$$

hence, by (3.6), (3.7) and (3.8) we conclude

$$(3.9) \quad \begin{aligned} \mathcal{F}_h(\mathbf{v}) &\geq \frac{C/C_G}{2} h^{-2} \int_{\Omega} |\nabla \mathbf{y} - \mathbf{R}|^2 d\mathbf{x} - \frac{C_P^2}{C/C_G} (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2) - h^{-1} \mathcal{L}((\mathbf{R} - \mathbf{I})\mathbf{x}) > \\ &> - \frac{C_P^2 C_G}{C} (\|\mathbf{f}\|_{L^2}^2 + \|\mathbf{g}\|_{L^2}^2) \quad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^N), \quad \forall h > 0. \quad \blacksquare \end{aligned}$$

**Lemma 3.2.** *Let  $\mathbf{v}_n \in H^1(\Omega; \mathbb{R}^N)$  be a sequence such that  $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbf{T}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$ . Then there exists  $\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)$  such that  $\mathbf{T} = \mathbb{E}(\mathbf{w})$ . If in addition  $\nabla \mathbf{v}_n \rightharpoonup \mathbf{G}$  in  $L^2(\Omega; \mathbb{R}^N)$  then there exists a constant matrix  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  such that  $\nabla \mathbf{w} = \mathbf{G} - \mathbf{W}$ .*

*Proof.* Since  $\mathbb{E}(\mathbf{v}_n) \rightharpoonup \mathbf{T}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$ , we have  $\mathbb{E}(\mathbf{v}_n - \mathbb{P}\mathbf{v}_n) = \mathbb{E}(\mathbf{v}_n)$  are equibounded in  $L^2(\Omega; \mathcal{M}^{N \times N})$ , then by Korn inequality  $\mathbf{v}_n - \mathbb{P}\mathbf{v}_n$  are equibounded in  $H^1(\Omega; \mathbb{R}^N)$ , where  $\mathbb{P}$  the projection on the set  $\mathcal{R}$  of infinitesimal rigid displacements. Therefore, up to subsequences, we can assume that  $\mathbf{w}_n := \mathbf{v}_n - \mathbb{P}\mathbf{v}_n \rightharpoonup \mathbf{w}$  in  $H^1(\Omega; \mathbb{R}^N)$  and we get

$$(3.10) \quad \nabla \mathbf{w}_n = \mathbb{E}(\mathbf{w}_n) + \text{skew} \nabla \mathbf{w}_n = \mathbb{E}(\mathbf{v}_n) + \text{skew} \nabla \mathbf{w}_n.$$

Hence there exists  $\mathbf{S} \in L^2(\Omega; \mathcal{M}_{skew}^{N \times N})$  such that  $\text{skew} \nabla \mathbf{w}_n \rightharpoonup \mathbf{S}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  and by letting  $n \rightarrow +\infty$  into (3.10) we have  $\mathbf{T} + \mathbf{S} = \nabla \mathbf{w}$ . Since  $\mathbf{S} \in L^2(\Omega; \mathcal{M}_{skew}^{N \times N})$  it is readily seen that  $\mathbb{E}(\mathbf{w}) = \mathbf{T}$  and if in addition  $\nabla \mathbf{v}_n \rightharpoonup \mathbf{G}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  then there exists a constant  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  such that  $\nabla \mathbb{P}\mathbf{v}_n \rightharpoonup \mathbf{W}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$ , actually converging in the finite dimensional space of constant skew symmetric matrices, thus proving the Lemma. ■

**Remark 3.3.** It is worth noting that if  $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbf{T}$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  then by Lemma 3.2 there exists  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  such that  $\mathbf{T} = \mathbb{E}(\mathbf{v})$  and if  $\mathbf{T} = \mathbb{E}(\mathbf{w})$  for some  $w \in H^1(\Omega; \mathbb{R}^N)$ , then  $\mathbf{v} - \mathbf{w}$  is an infinitesimal rigid displacement in  $\Omega$ , i.e.  $\mathbb{E}(\mathbf{v} - \mathbf{w}) = \mathbf{0}$ .

Next we show a preliminary convergence property: we compute a kind of Gamma limit of the sequence of functional  $\mathcal{F}_h$  with respect to weak  $L^2$  convergence of linearized strains.

**Lemma 3.4. (energy convergence)** *Assume that (2.24) holds true and let  $h_j \rightarrow 0$  be a decreasing sequence. Then*

i) *For every  $\mathbf{v}_j, \mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  such that  $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  we have*

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \mathcal{F}(\mathbf{v}).$$

ii) *For every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  there exists a sequence  $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$  such that  $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  and*

$$\limsup_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \leq \mathcal{F}(\mathbf{v}).$$

*Proof.* First we prove i). We set  $\mathbf{y}_j = \mathbf{x} + h_j \mathbf{v}_j$  and denote various positive constants by  $C', C'', \dots, L', L''$ . We may assume without restriction that  $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq C'$ ; by taking into account (2.8) we get

$$Ch_j^{-2} \int_{\Omega} |\nabla \mathbf{y}_j^T \nabla \mathbf{y}_j - \mathbf{I}|^2 d\mathbf{x} - \mathcal{L}(\mathbf{v}_j) \leq \mathcal{F}_{h_j}(\mathbf{v}_j)$$

and by (2.24)

$$h_j^{-2} \int_{\Omega} |\nabla \mathbf{y}_j^T \nabla \mathbf{y}_j - \mathbf{I}|^2 d\mathbf{x} \leq C' + \mathcal{L}(\mathbf{v}_j) = C' + \mathcal{L}(\mathbf{v}_j - \mathbb{P}\mathbf{v}_j),$$

where  $\mathbb{P}\mathbf{v}_j$  is the projection of  $\mathbf{v}_j$  onto the set of infinitesimal rigid displacements. Hence by Korn inequality we have

$$(3.11) \quad h_j^{-2} \int_{\Omega} |\nabla \mathbf{y}_j^T \nabla \mathbf{y}_j - \mathbf{I}|^2 d\mathbf{x} \leq C' + C'' \left( \int_{\Omega} |\mathbb{E}(\mathbf{v}_j)|^2 d\mathbf{x} \right)^{\frac{1}{2}} \leq C'''.$$

Inequality (3.11) together with the Rigidity Lemma of [18] and (3.5) imply that for every  $h_j$  there exists a constant rotation  $\mathbf{R}_j \in SO(N)$  and a constant  $C'''$ , dependent only on  $\Omega$ , such that

$$\int_{\Omega} |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 d\mathbf{x} \leq C'''' h_j^2$$

that is

$$(3.12) \quad \int_{\Omega} |\mathbf{I} + h_j \nabla \mathbf{v}_j - \mathbf{R}_j|^2 d\mathbf{x} \leq C'''' h_j^2.$$

Due to the representation (2.2) of rotations, for every  $j \in \mathbb{N}$  there exist  $\vartheta_j \in (-\pi, \pi]$  and  $\mathbf{W}_j \in \mathcal{M}_{skew}^{N \times N}$ ,  $|\mathbf{W}_j|^2 = 2$  such that  $\mathbf{R}_j = \exp(\vartheta_j \mathbf{W}_j)$  and

$$(3.13) \quad \mathbf{R}_j = \exp(\vartheta_j \mathbf{W}_j) = \mathbf{I} + \sin \vartheta_j \mathbf{W}_j + (1 - \cos \vartheta_j) \mathbf{W}_j^2$$

hence by (3.12)

$$(3.14) \quad \int_{\Omega} |h_j \nabla \mathbf{v}_j - \sin \vartheta_j \mathbf{W}_j - (1 - \cos \vartheta_j) \mathbf{W}_j^2|^2 d\mathbf{x} \leq C'''' h_j^2.$$

Since

$$\text{sym} \left( h_j \nabla \mathbf{v}_j - \sin \vartheta_j \mathbf{W}_j - (1 - \cos \vartheta_j) \mathbf{W}_j^2 \right) = h_j \mathbb{E}(\mathbf{v}_j) - (1 - \cos \vartheta_j) \mathbf{W}_j^2$$

we get

$$\int_{\Omega} |\mathbb{E}(\mathbf{v}_j) - (1 - \cos \vartheta_j) h_j^{-1} \mathbf{W}_j^2|^2 d\mathbf{x} \leq C''''.$$

By recalling that  $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  and  $|\mathbf{W}_j^2|^2 = 2$  due to (2.3), we deduce for suitable  $L > 0$

$$(3.15) \quad |1 - \cos \vartheta_j| = \frac{1}{\sqrt{2}} |(1 - \cos \vartheta_j) \mathbf{W}_j^2| \leq L h_j$$

hence

$$(3.16) \quad |\sin \vartheta_j| \leq \sqrt{2L h_j}.$$

By (3.14), (3.15) we have

$$(3.17) \quad \int_{\Omega} |\sqrt{h_j} \nabla \mathbf{v}_j - h_j^{-1/2} \sin \vartheta_j \mathbf{W}_j|^2 d\mathbf{x} \leq (C''' + 2L|\Omega|) h_j,$$

hence, by compactness of the sequence  $h^{-1/2} \sin \vartheta_j \mathbf{W}_j$  in  $\mathcal{M}_{skew}^{N \times N}$ , there exists a constant matrix  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  such that, up to subsequences,

$$(3.18) \quad \sqrt{h_j} \nabla \mathbf{v}_j \rightarrow \mathbf{W} \quad \text{strongly in } L^2(\Omega; \mathcal{M}^{N \times N})$$

and therefore

$$(3.19) \quad h_j \nabla \mathbf{v}_j^T \nabla \mathbf{v}_j \rightarrow \mathbf{W}^T \mathbf{W} = -\mathbf{W}^2 \quad \text{strongly in } L^1(\Omega; \mathcal{M}^{N \times N}).$$

By Lemma 4.2. of [13] for every  $k \in \mathbb{N}$  there exist an increasing sequence of Caratheodory functions  $\mathcal{V}_j^k : \Omega \times \mathcal{M}_{sym}^{N \times N} \rightarrow [0, +\infty)$  and a measurable function  $\mu^k : \Omega \rightarrow (0, +\infty)$  such that  $\mathcal{V}_j^k(\mathbf{x}, \cdot)$  is convex for a.e.  $\mathbf{x} \in \Omega$  and satisfies

$$(3.20) \quad \mathcal{V}_j^k(\mathbf{x}, \mathbf{D}) \leq \mathcal{V}(x, h_j \mathbf{D}) / h_j^2 \quad \forall \mathbf{D} \in \mathcal{M}_{sym}^{N \times N},$$

$$(3.21) \quad \mathcal{V}_j^k(\mathbf{x}, \mathbf{D}) = \left(1 - \frac{1}{k}\right) \mathcal{V}_0(\mathbf{x}, \mathbf{D}) \quad \text{for } \mathcal{V}_0(\mathbf{x}, \mathbf{D}) \leq \mu^k(x) / h_j^2.$$

By setting  $\mathbf{D}_j := \mathbb{E}(\mathbf{v}_j) + \frac{1}{2}h_j \nabla \mathbf{v}_j^T \nabla \mathbf{v}_j$ , properties (3.19), (3.20) and (3.21) entail

$$(3.22) \quad \int_{\Omega} \mathcal{V}_{h_j}(x, \nabla \mathbf{v}_j) d\mathbf{x} \geq \int_{\Omega} \mathcal{V}_j^k(x, \mathbf{D}_j) d\mathbf{x}$$

and

$$(3.23) \quad \lim_{j \rightarrow +\infty} \mathcal{V}_j^k(\mathbf{x}, \mathbf{D}) = \left(1 - \frac{1}{k}\right) \mathcal{V}_0(\mathbf{x}, \mathbf{D}) \quad \text{a.e. } \mathbf{x} \in \Omega, \forall \mathbf{D} \in \mathcal{M}_{sym}^{N \times N}$$

respectively. Then by taking into account that

$$\mathbf{D}_j \rightharpoonup \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2 \quad \text{in } L^1(\Omega; \mathcal{M}^{N \times N}),$$

(3.22) and Lemma 4.3 of [13] yield

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \int_{\Omega} \mathcal{V}_{h_j}(x, \nabla \mathbf{v}_j) d\mathbf{x} &\geq \liminf_{j \rightarrow +\infty} \int_{\Omega} \mathcal{V}_j^k(x, \mathbf{D}_j) d\mathbf{x} \\ &\geq \int_{\Omega} (1 - 1/k) \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} \quad \forall k \in \mathbb{N}. \end{aligned}$$

Up to subsequences  $\mathbf{v}_j - \mathbb{P}\mathbf{v}_j \rightharpoonup \mathbf{w}$  in  $H^1(\Omega; \mathbb{R}^N)$ , moreover  $\mathbb{E}(\mathbf{v}) = \mathbb{E}(\mathbf{w})$ . Then by (2.24) for every  $k \in \mathbb{N}$  we obtain

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) &\geq \int_{\Omega} \left(1 - \frac{1}{k}\right) \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{w}) = \\ &= \int_{\Omega} \left(1 - \frac{1}{k}\right) \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) \end{aligned}$$

Taking the supremum as  $k \rightarrow \infty$  we deduce

$$(3.24) \quad \liminf_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) \geq \mathcal{F}(\mathbf{v})$$

which proves i).

We are left to prove claim ii). To this aim, we set for every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ :

$$(3.25) \quad \mathbf{W}_{\mathbf{v}} \in \operatorname{argmin} \left\{ \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} : \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \right\}.$$

Without relabeling,  $\mathbf{v}$  denotes also a fixed compactly supported extension in  $H^1(\mathbb{R}^N; \mathbb{R}^N)$  of the given  $\mathbf{v}$  (such extension exists since  $\Omega$  is Lipschitz due to (2.1)).

We may define a recovery sequence  $\mathbf{w}_j \in C^1(\bar{\Omega}; \mathbb{R}^N)$  for every  $j$ , as follows. Set

$$(3.26) \quad \mathbf{w}_j = h_j^{-1/2} \mathbf{W}_{\mathbf{v}} \mathbf{x} + \mathbf{v} \star \varphi_j$$

where  $\varphi_j(\mathbf{x}) = \varepsilon_j^{-N} \varphi(\mathbf{x}/\varepsilon_j)$  is a mollifier supported in  $B_{\varepsilon_j}(\mathbf{0})$  and the sequence  $\varepsilon_j$  is chosen in such a way that  $h_j \varepsilon_j^{-3} \rightarrow 0$  holds true. Sobolev embedding entails  $\mathbf{v} \in L^6(\mathbb{R}^N; \mathbb{R}^N)$ , since  $\mathbf{v} \in H^1(\mathbb{R}^N; \mathbb{R}^N)$  and  $N = 2, 3$ ; then by Young Theorem and since  $0 < \varepsilon \leq 1$  we have

$$(3.27) \quad \|\nabla(\mathbf{v} \star \varphi_j)\|_{L^\infty} \leq \|v\|_{L^6} \|\nabla \varphi_j\|_{L^{6/5}} \leq \varepsilon_j^{-N/6-1} \|\nabla \varphi\|_{L^{6/5}} \|v\|_{L^6} \leq \varepsilon_j^{-3/2} \|\nabla \varphi\|_{L^{6/5}} \|v\|_{L^6}.$$

By  $\nabla \mathbf{w}_j = h_j^{-1/2} \mathbf{W}_{\mathbf{v}} + \nabla(\mathbf{v} \star \varphi_j)$  and  $\mathbf{W}_{\mathbf{v}}^T = -\mathbf{W}_{\mathbf{v}}$  we get

$$\mathbb{E}(\mathbf{w}_j) = \mathbb{E}(\mathbf{v}) \star \varphi_j,$$

$$h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j = -\mathbf{W}_{\mathbf{v}}^2 + h_j \nabla(\mathbf{v} \star \varphi_j)^T \nabla(\mathbf{v} \star \varphi_j) + h_j^{1/2} (\nabla(\mathbf{v} \star \varphi_j)^T \mathbf{W}_{\mathbf{v}} - \mathbf{W}_{\mathbf{v}} \nabla(\mathbf{v} \star \varphi_j)),$$



hence, by taking into account (3.27) and  $h_j \varepsilon_j^{-3} \rightarrow 0$ , we get

$$(3.28) \quad \mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j \rightarrow \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2 \quad \text{in } L^2(\Omega, \mathbb{R}^N),$$

$$(3.29) \quad h_j \left( \mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j \right) \rightarrow \mathbf{0} \quad \text{in } L^\infty(\Omega, \mathbb{R}^N).$$

Therefore Taylor's expansion of  $\mathcal{V}$  entails

$$(3.30) \quad \lim_{j \rightarrow +\infty} \mathcal{V}_{h_j}(\mathbf{x}, \mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j) = \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2) \quad \text{for a.e. } \mathbf{x} \in \Omega,$$

and taking into account (2.14), (2.15), (2.16), (3.29) we have

$$(3.31) \quad \mathcal{V}_{h_j}(\mathbf{x}, \mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j) \leq \gamma |\mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2,$$

hence the Lebesgue dominated convergence theorem yields

$$\mathcal{F}_{h_j}(\mathbf{w}_j) \rightarrow \min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}),$$

thus proving ii). ■

**Remark 3.5.** If  $\mathcal{W}$  is a convex function of  $\mathbf{F}^T \mathbf{F} - \mathbf{I}$  then (3.24) is a straightforward consequence of weak  $L^1(\Omega; \mathcal{M}^{N \times N})$  convergence of  $\mathbf{B}_j$  and the construction of  $\mathcal{V}_j^k$  can be avoided in the proof. Hence the restriction to decreasing sequences  $h_j$  (needed in order to apply Lemmas 4.2 and 4.3 of [13]) can be removed in the assumptions of Lemma 3.4 if  $\mathcal{W}$  is convex.

Lemma 3.1 entails existence of minimizing sequences for the sequence of functionals  $\mathcal{F}_h$ . Next technical lemma 3.6 shows a (very weak) relative compactness property of these sequences: if  $\mathbf{v}_j$  is a minimizing sequence then  $E(\mathbf{v}_j)$  is equibounded in  $L^2$ . The key idea of the proof consists in showing that  $\|E(\mathbf{v}_j)\|_{L^2} \rightarrow +\infty$  entails the contradiction  $E(\mathbf{v}_j)/\|E(\mathbf{v}_j)\|_{L^2} \rightarrow 0$  strongly in  $L^2$ , by a careful analysis of all possible cases related to different balance of involved parameters

**Lemma 3.6. (Compactness of minimizing sequences)** *Assume that (2.24), (2.25) hold true,  $h_j \rightarrow 0_+$  is a sequence of strictly positive real numbers and the sequence of displacements  $\mathbf{v}_j \in H^1(\Omega; \mathbb{R}^N)$  fulfil  $(\mathcal{F}_{h_j}(\mathbf{v}_j) - \inf \mathcal{F}_{h_j}) \rightarrow 0$ , namely  $\mathbf{v}_j$  is a minimizing sequence for  $\mathcal{F}_{h_j}$ . Then there exists  $M > 0$  such that  $\|\mathbb{E}(\mathbf{v}_j)\|_{L^2} \leq M$ .*

*Proof.* By Lemma 3.1 there exists  $c$  such that

$$(3.32) \quad -\infty < c \leq \inf \mathcal{F}_{h_j} \leq \mathcal{F}_{h_j}(\mathbf{0}) = 0.$$

Assume by contradiction that  $t_j := \|\mathbb{E}(\mathbf{v}_j)\|_{L^2} \rightarrow +\infty$  and set  $\mathbf{w}_j = t_j^{-1} \mathbf{v}_j$ . By Lemma 3.2 there exist  $\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)$  and a subsequence such that without relabeling  $\mathbb{E}(\mathbf{w}_j) \rightharpoonup \mathbb{E}(\mathbf{w})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$ . By (3.32) we can assume up to subsequences that  $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq 1$ .

By setting  $\mathbf{y}_j = \mathbf{x} + h_j \mathbf{v}_j = \mathbf{x} + h_j t_j \mathbf{w}_j$ , arguing as at the beginning of Lemma 3.4 proof and exploiting Korn inequality (3.2), we obtain that for every  $j \in \mathbb{N}$  there exists a constant rotation  $\mathbf{R}_j \in SO(N)$  such that

$$\int_{\Omega} |\nabla \mathbf{y}_j - \mathbf{R}_j|^2 d\mathbf{x} \leq h_j^2 + C_K (\|f\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\Omega)}) t_j h_j^2,$$

that is, by setting  $C' = C_K(\|f\|_{L^2(\partial\Omega)} + \|g\|_{L^2(\Omega)})$ ,

$$(3.33) \quad \int_{\Omega} |\mathbf{I} + h_j t_j \nabla \mathbf{w}_j - \mathbf{R}_j|^2 d\mathbf{x} \leq h_j^2 (1 + C' t_j).$$

Possibly up to further subsequence extraction, one among these three alternatives take place:

$$a) \ h_j t_j \rightarrow \lambda > 0, \quad b) \ h_j t_j \rightarrow 0, \quad c) \ h_j t_j \rightarrow +\infty.$$

If condition  $a)$  holds true we have

$$\int_{\Omega} \left| \nabla \mathbf{w}_j - \frac{\mathbf{R}_j - \mathbf{I}}{h_j t_j} \right|^2 d\mathbf{x} \leq \frac{1}{t_j^2} + \frac{C'}{t_j},$$

hence, up to subsequences,

$$(3.34) \quad \nabla \mathbf{w}_j \rightarrow \frac{\mathbf{R} - \mathbf{I}}{\lambda}$$

strongly in  $L^2(\Omega; \mathcal{M}^{N \times N})$  for a suitable constant matrix  $\mathbf{R} \in SO(N)$  and by Lemma 3.2 we get  $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}_{skew}^{N \times N}$ .

If condition  $b)$  holds true, then by using formulæ (3.13) and (3.33) there exist  $\vartheta_{h_j} \in (-\pi, \pi]$  and a constant matrix  $\mathbf{W}_j \in \mathcal{M}_{skew}^{N \times N}$  with  $|\mathbf{W}_j|^2 = |\mathbf{W}_j^2|^2 = 2$  such that

$$(3.35) \quad \int_{\Omega} |h_j t_j \nabla \mathbf{w}_j - \sin \vartheta_{h_j} \mathbf{W}_j - (1 - \cos \vartheta_{h_j}) \mathbf{W}_j^2|^2 d\mathbf{x} \leq h_j^2 (1 + C' t_j).$$

Since  $\mathbf{W}_j$  and  $\mathbf{W}_j^2$  are respectively skew-symmetric and symmetric, (3.35) yields

$$(3.36) \quad \int_{\Omega} \left| \mathbb{E}(\mathbf{w}_j) - \frac{(1 - \cos \vartheta_j)}{h_j t_j} \mathbf{W}_j^2 \right|^2 d\mathbf{x} \leq t_j^{-2} + C' t_j^{-1}$$

and bearing in mind that  $\int_{\Omega} |\mathbb{E}(\mathbf{w}_j)|^2 d\mathbf{x} = 1$  we get

$$(3.37) \quad \left| \frac{(1 - \cos \vartheta_j)}{h_j t_j} \right| = \frac{1}{\sqrt{2}} \left| \frac{(1 - \cos \vartheta_j)}{h_j t_j} \mathbf{W}_j^2 \right| \leq C'',$$

hence

$$(3.38) \quad |\sin \vartheta_{h_j}| \leq \sqrt{2(1 - \cos \vartheta_{h_j})} \leq \sqrt{2C'' h_j t_j}$$

Estimate (3.35) together with  $t_j \rightarrow +\infty$  yields

$$(3.39) \quad \int_{\Omega} \left| \sqrt{h_j t_j} \nabla \mathbf{w}_j - \frac{\sin \vartheta_j}{\sqrt{h_j t_j}} \mathbf{W}_j - \frac{(1 - \cos \vartheta_j)}{\sqrt{h_j t_j}} \mathbf{W}_j^2 \right|^2 d\mathbf{x} \leq C''' h_j.$$

By (3.37) we know that  $1 - \cos \vartheta_j = o(\sqrt{h_j t_j})$ , hence (3.38), (3.39) entail the existence of a constant matrix  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  such that, up to subsequences,  $\sqrt{h_j t_j} \nabla \mathbf{w}_j \rightarrow \mathbf{W}$  strongly in  $L^2(\Omega; \mathcal{M}^{N \times N})$ . Moreover by (2.8) and Korn inequality

$$\begin{aligned} t_j^2 \int_{\Omega} |\mathbb{E}(\mathbf{w}_j) + \tfrac{1}{2} h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 d\mathbf{x} &\leq C^{IV} + \mathcal{L}(\mathbf{w}_j) \\ &= C^{IV} + \mathcal{L}(\mathbf{w}_j - \mathbb{P} \mathbf{w}_j) \\ &\leq C^V \left( \int_{\Omega} |\mathbb{E}(\mathbf{w}_j)|^2 d\mathbf{x} \right)^{\frac{1}{2}}, \end{aligned}$$

hence

$$\int_{\Omega} |\mathbb{E}(\mathbf{w}_j) + \frac{1}{2} h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 d\mathbf{x} \rightarrow 0.$$

On the other hand, by Fatou Lemma,

$$\liminf_{j \rightarrow +\infty} \int_{\Omega} |2\mathbb{E}(\mathbf{w}_j) + h_j t_j \nabla \mathbf{w}_j^T \nabla \mathbf{w}_j|^2 d\mathbf{x} \geq \int_{\Omega} |2\mathbb{E}(\mathbf{w}) - \mathbf{W}^2| d\mathbf{x},$$

we get  $2\mathbb{E}(\mathbf{w}) = \mathbf{W}^2$  which implies  $\nabla \mathbf{w} = \text{skew} \nabla \mathbf{w} + \frac{1}{2} \mathbf{W}^2$ , hence  $\text{skew}(\nabla \mathbf{w})$  is a gradient field, then there is a constant skew-symmetric matrix  $\mathbf{Z}$  such that  $\text{skew}(\nabla \mathbf{w}) = \mathbf{Z}$ . By setting

$$\mathbf{R} := \mathbf{I} + \frac{1}{2} \mathbf{W}^2 + \frac{\sqrt{3}}{2} \mathbf{W}$$

and by applying formula (3.13) we get  $\mathbf{R} \in SO(N)$  that is  $\nabla \mathbf{w} - (\mathbf{R} - \mathbf{I}) \in \mathcal{M}_{skew}^{N \times N}$  which implies  $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}_{skew}^{N \times N}$  whenever condition b) holds true.

Eventually if condition c) holds true, by (3.33) we get

$$(3.40) \quad \int_{\Omega} \left| \nabla \mathbf{w}_j - \frac{\mathbf{R}_j - \mathbf{I}}{h_j t_j} \right|^2 d\mathbf{x} \leq \frac{A}{t_j^2} + \frac{C'}{t_j},$$

and by taking into account that  $h_j t_j \rightarrow +\infty$  and  $|\mathbf{R}_j - \mathbf{I}|$  is bounded, by (3.33) we get  $\nabla \mathbf{w}_{h_j} \rightarrow \mathbf{0}$  strongly in  $L^2(\Omega; M^{N \times N})$ , hence  $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}_{skew}^{N \times N}$  still by Lemma 3.2.

By summarizing, in all three cases if  $t_j := \|\mathbb{E}(\mathbf{v}_j)\|_{L^2} \rightarrow +\infty$  and  $\mathcal{F}_{h_j}(t_j \mathbf{w}_j) \leq C$  then  $\nabla(t_j^{-1} \mathbf{v}_j) = \nabla \mathbf{w}_j \rightarrow \nabla \mathbf{w}$  strongly in  $L^2(\Omega; \mathcal{M}^{N \times N})$  and  $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}_{skew}^{N \times N}$ .

Therefore  $\mathbb{E}(\mathbf{w}_j) \rightarrow \mathbb{E}(\mathbf{w})$  strongly in  $L^2(\Omega; \mathcal{M}^{N \times N})$ .

Since  $\tilde{\mathbf{w}}_j := \mathbf{w}_j - \mathbb{P} \mathbf{w}_j$  are equibounded in  $H^1(\Omega; \mathbb{R}^N)$ , every subsequence of  $\tilde{\mathbf{w}}_j$  has a weakly convergent subsequence and if  $\tilde{\mathbf{w}}$  is one of the limits we get  $\mathbb{E}(\tilde{\mathbf{w}}) = \mathbb{E}(\mathbf{w})$  hence by (2.24)  $\mathcal{L}(\tilde{\mathbf{w}}) = \mathcal{L}(\mathbf{w})$ . Therefore every subsequence of  $\mathcal{L}(\tilde{\mathbf{w}}_j)$  has a subsequence which converges to  $\mathcal{L}(\mathbf{w})$  that is the whole sequence  $\mathcal{L}(\tilde{\mathbf{w}}_j)$  converges to  $\mathcal{L}(\mathbf{w})$ , hence

$$(3.41) \quad -\mathcal{L}(\mathbf{w}) = -\limsup_{j \rightarrow +\infty} \mathcal{L}(\tilde{\mathbf{w}}_j) = -\limsup_{j \rightarrow +\infty} \mathcal{L}(\mathbf{w}_j) \leq \liminf_{j \rightarrow +\infty} t_j^{-1} \mathcal{F}_{h_j}(\mathbf{v}_j).$$

Since (3.32) entails  $\limsup_{j \rightarrow +\infty} t_j^{-1} \mathcal{F}_{h_j}(\mathbf{v}_j) \leq 0$ , by (3.41) we get  $\mathcal{L}(\mathbf{w}) \geq 0$ .

By taking into account that  $\nabla \mathbf{w} \in \mathbb{K} + \mathcal{M}_{skew}^{N \times N}$  then, either  $\nabla \mathbf{w} \in \mathcal{M}_{skew}^{N \times N}$  or

$$\mathbf{w}(\mathbf{x}) = \tau(\mathbf{R} - \mathbf{I})\mathbf{x} + \mathbf{A}\mathbf{x} + \mathbf{c}, \text{ for some } \tau > 0, \mathbf{R} \in SO(N), \mathbf{R} \neq \mathbf{I}, \mathbf{A} \in \mathcal{M}_{skew}^{N \times N}, \mathbf{c} \in \mathbb{R}^N.$$

The second case cannot occur since in such case by (2.2) there would exist  $\vartheta \in \mathbb{R}$  with  $\cos \vartheta < 1$  and  $\mathbf{W} \in \mathcal{M}_{sym}^{N \times N}$ ,  $\mathbf{W} \neq \mathbf{0}$  such that  $\mathbf{R} = \mathbf{I} + (1 - \cos \vartheta) \mathbf{W}^2 + (\sin \vartheta) \mathbf{W} \in SO(N)$  hence (2.24), (2.25) would entail the contradiction below

$$(3.42) \quad \begin{aligned} \mathcal{L}(\mathbf{w}) &= \tau \int_{\partial \Omega} \mathbf{f} \cdot (\mathbf{R} - \mathbf{I})\mathbf{x} d\mathcal{H}^{N-1} + \tau \int_{\Omega} \mathbf{g} \cdot (\mathbf{R} - \mathbf{I})\mathbf{x} = \\ &= \tau(1 - \cos \vartheta) \int_{\partial \Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} + \tau(1 - \cos \vartheta) \int_{\Omega} \mathbf{g} \cdot \mathbf{W}^2 \mathbf{x} < 0. \end{aligned}$$

Hence  $\nabla \mathbf{w} \in \mathcal{M}_{skew}^{N \times N}$  that is  $\mathbb{E}(\mathbf{w}) = \mathbf{0}$  which is again a contradiction since  $\|\mathbb{E}(\mathbf{w}_j)\|_{L^2} = 1$  and  $\mathbb{E}(\mathbf{w}_j) \rightarrow \mathbb{E}(\mathbf{w})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$ .  $\blacksquare$

*Proof of Theorem 2.2* - First we notice that minimizing sequences for  $\mathcal{F}_{h_j}$  do exist for every sequence of positive real numbers  $h_j$  converging to 0, thanks to Lemma 3.1.

Fix a sequence of real numbers  $h_j > 0$  converging to 0 and a minimizing sequence  $\mathbf{v}_j$  for  $\mathcal{F}_{h_j}$ .

Up to a preliminary extraction of a subsequence we can assume that  $h_j$  is decreasing.

Since  $-\infty < \inf \mathcal{F}_{h_j} \leq 0$  there exists  $C > 0$  such that  $\mathcal{F}_{h_j}(\mathbf{v}_j) \leq C$  hence by Lemma 3.6

$$\|\mathbb{E}(\mathbf{v}_j)\|_{L^2} \leq C$$

and by Lemma 3.2 there exists  $\mathbf{v}_0 \in H^1(\Omega; \mathbb{R}^N)$  such that, up to subsequences,  $\mathbb{E}(\mathbf{v}_j) \rightharpoonup \mathbb{E}(\mathbf{v}_0)$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  thus proving (2.26). By Lemma 3.4 we get

$$\liminf_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \mathcal{F}(\mathbf{v}_0) .$$

Still by Lemma 3.4, for every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  there exists  $\tilde{\mathbf{v}}_j \in H^1(\Omega; \mathbb{R}^N)$  such that  $\mathbb{E}(\tilde{\mathbf{v}}_j) \rightharpoonup \mathbb{E}(\mathbf{v})$  in  $L^2(\Omega; \mathcal{M}^{N \times N})$  and

$$\limsup_{j \rightarrow +\infty} \mathcal{F}_j(\tilde{\mathbf{v}}_{h_j}) \leq \mathcal{F}(\mathbf{v}) ,$$

hence

$$(3.43) \quad \mathcal{F}(\mathbf{v}_0) \leq \liminf_{n \rightarrow +\infty} \mathcal{F}_j(\mathbf{v}_{h_j}) \leq \liminf_{n \rightarrow +\infty} (\inf \mathcal{F}_{h_j} + o(1)) \leq \limsup_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\tilde{\mathbf{v}}_j) \leq \mathcal{F}(\mathbf{v})$$

and (2.28) is proven. Therefore (2.27) follows by (3.18).

Eventually we notice that (2.27) follows by labeling with  $\mathbf{W}_0$  the skew symmetric matrix  $\mathbf{W}$  in the convergence relationship (3.18) obtained in the proof of claim *i*) in Lemma 3.4.

So we have only to prove (2.29): to this aim, by arguing as in the proof of claim *i*) in Lemma 3.4, we have (3.24), hence we get

$$\mathcal{F}(\mathbf{v}_0) = \liminf_{j \rightarrow +\infty} \mathcal{F}_{h_j}(\mathbf{v}_j) \geq \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \frac{1}{2} \mathbf{W}_0^2) d\mathbf{x} \geq \mathcal{F}(\mathbf{v}_0)$$

thus proving (2.29). ■

#### 4. LIMIT PROBLEM AND LINEAR ELASTICITY

We denote by  $\mathcal{E} : H^1(\Omega; \mathbb{R}^N) \rightarrow \mathbb{R}$  the energy functional of classical linear elasticity

$$(4.1) \quad \mathcal{E}(\mathbf{v}) := \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v})) d\mathbf{x} - \mathcal{L}(\mathbf{v}) .$$

Notice that (1.6) is just a particular model case of (4.1) corresponding to (1.4).

As it was already emphasized, the inequality  $\mathcal{F} \leq \mathcal{E}$  always holds true. Moreover the two functionals cannot coincide: indeed  $\mathcal{F}(\mathbf{v}) < \mathcal{E}(\mathbf{v})$  whenever  $\mathbf{v}(\mathbf{x}) = \frac{1}{2} \mathbf{W}^2 \mathbf{x}$  with  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ . However we can show that the two functionals  $\mathcal{F}$  and  $\mathcal{E}$ , notwithstanding their differences, have the same minimum and same set of minimizers when the loads are equilibrated and compatible, that is they fulfil both (2.24) and (2.25).

Next results clarify the relationship between the minimizers of classical linear elasticity functional  $\mathcal{E}$  and the minimizers of functional  $\mathcal{F}$  defined in (1.8), and coincident with the Gamma limit of nonlinear energies  $\mathcal{F}_h$ , in the sense stated in Theorem 2.2.

**Theorem 4.1.** *Assume that (2.24) and (2.25) hold true. Then*

$$(4.2) \quad \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = \min_{\mathbf{w} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(\mathbf{w}) .$$

and

$$(4.3) \quad \operatorname{argmin}_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F} = \operatorname{argmin}_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}.$$

*Proof.* Both functionals  $\mathcal{F}$ ,  $\mathcal{E}$  do have minimizers under conditions (2.24), (2.25), thanks to Theorem 2.2. Taking into account that  $\mathcal{F}(\mathbf{v}) \leq \mathcal{E}(\mathbf{v})$  for every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$ , and setting  $\mathbf{z}_{\mathbf{W}}(\mathbf{x}) := \frac{1}{2} \mathbf{W}^2 \mathbf{x}$  for every  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ , we get  $\mathbb{E}(\mathbf{z}_{\mathbf{W}}) = \frac{1}{2} \mathbf{W}^2$  and

$$(4.4) \quad \begin{aligned} \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(\mathbf{v}) &\geq \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = \\ &\min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \left\{ \min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \left\{ \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) \right\} \right\} = \\ &\min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \left\{ \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) \right\} \right\} = \\ &\min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \left\{ \min_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \left\{ \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v} - \mathbf{z}_{\mathbf{W}})) d\mathbf{x} - \mathcal{L}(\mathbf{v} - \mathbf{z}_{\mathbf{W}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}}) \right\} \right\} = \\ &\min_{\mathbf{z} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}(\mathbf{z}) - \max_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \mathcal{L}(\mathbf{z}_{\mathbf{W}}) = \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E}. \end{aligned}$$

where last inequality follows by  $\mathcal{L}(\mathbf{z}_{\mathbf{W}}) \leq 0$ , due to (2.25) and  $\mathcal{L}(\mathbf{0})$ . Therefore (4.2) is proved and we are left to show (4.3).

First assume  $\mathbf{v} \in \operatorname{argmin}_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}$  and let

$$(4.5) \quad \mathbf{W}_{\mathbf{v}} \in \operatorname{argmin} \left\{ \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}) - \tfrac{1}{2} \mathbf{W}^2) d\mathbf{x} : \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \right\}.$$

If  $\mathbf{W}_{\mathbf{v}} \neq \mathbf{0}$  then, by setting  $\mathbf{z}_{\mathbf{W}_{\mathbf{v}}} = \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2 \mathbf{x}$  we get  $\mathbb{E}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) = \nabla \mathbf{z}_{\mathbf{W}_{\mathbf{v}}} = \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2$  and, by compatibility (2.25) we obtain

$$(4.6) \quad \begin{aligned} \min \mathcal{F} = \mathcal{F}(\mathbf{v}) &= \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}})) d\mathbf{x} - \mathcal{L}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) = \\ \mathcal{E}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) &\geq \min \mathcal{E} - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) > \min \mathcal{E}, \end{aligned}$$

say a contradiction. Therefore  $\mathbf{W}_{\mathbf{v}} = \mathbf{0}$ ,  $\mathbf{z}_{\mathbf{W}_{\mathbf{v}}} = \mathbf{0}$ , and all the inequalities in (4.6) turn out to be equalities, hence we get  $\mathcal{F}(\mathbf{v}) = \mathcal{E}(\mathbf{v}) = \min \mathcal{E} = \min \mathcal{F}$ , say  $\mathbf{v} \in \operatorname{argmin}_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$  and  $\operatorname{argmin}_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F} \subset \operatorname{argmin}_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$ .

In order to show the opposite inclusion, we assume  $\mathbf{v} \in \operatorname{argmin}_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{E}$  and still referring to the choice (4.5) we set  $\mathbf{z}_{\mathbf{W}_{\mathbf{v}}} = \frac{1}{2} \mathbf{W}_{\mathbf{v}}^2 \mathbf{x}$ . Then

$$(4.7) \quad \begin{aligned} \mathcal{F}(\mathbf{v}) &= \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}})) d\mathbf{x} - \mathcal{L}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) = \\ &= \mathcal{E}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) \geq \mathcal{F}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}}}). \end{aligned}$$

This leads to the contradiction  $\mathcal{F}(\mathbf{v}) > \mathcal{F}(\mathbf{v} - \mathbf{z}_{\mathbf{W}_{\mathbf{v}}})$  if  $\mathbf{z}_{\mathbf{W}_{\mathbf{v}}} \neq \mathbf{0}$ , due to (2.25); therefore  $\mathbf{z}_{\mathbf{W}_{\mathbf{v}}} = \mathbf{0}$  and we have equalities in place of inequalities in (4.7): therefore  $\mathcal{E}(\mathbf{v}) = \mathcal{F}(\mathbf{v})$  and  $\mathbf{v} \in \operatorname{argmin}_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F}$ .  $\blacksquare$

**Corollary 4.2.** *Assume that the standard structural assumptions hold true together with (2.24), (2.25), and  $\mathbf{W}_0 \in \mathcal{M}_{skew}^{N \times N}$  is the matrix whose existence is warranted by in Theorem 2.2.*

*Then  $\mathbf{W}_0 = \mathbf{0}$ .*

*Proof.* Let  $\mathbf{v}_0$  be in  $\text{argmin } \mathcal{F}$ ,  $\mathbf{W}_0$  be the skew symmetric matrix in the claim of Theorem 2.2 and assume by contradiction that  $\mathbf{W}_0 \neq \mathbf{0}$ . By (4.2) and (4.3) we get

$$\int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0) - \frac{1}{2} \mathbf{W}_0^2) d\mathbf{x} = \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0)) d\mathbf{x}$$

and by taking into account that  $\mathcal{V}_0$  is a quadratic form

$$\int_{\Omega} \mathcal{V}_0(\mathbf{x}, \frac{1}{2} \mathbf{W}_0^2) d\mathbf{x} - \frac{1}{2} \int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0)) \cdot \mathbf{W}_0^2 d\mathbf{x} = 0.$$

Since by (4.3)  $\mathbf{v}_0 \in \text{argmin } \mathcal{E}$ , the Euler-Lagrange equation yields

$$\int_{\Omega} D\mathcal{V}_0(\mathbf{x}, \mathbb{E}(\mathbf{v}_0)) \cdot \mathbf{W}_0^2 d\mathbf{x} = \mathcal{L}(\mathbf{W}_0^2 \mathbf{x}),$$

hence, by (2.25),

$$0 \leq \int_{\Omega} \mathcal{V}_0(\mathbf{x}, \frac{1}{2} \mathbf{W}_0^2) d\mathbf{x} = \frac{1}{2} \mathcal{L}(\mathbf{W}_0^2 \mathbf{x}) < 0$$

which is a contradiction.

Hence  $\mathbf{W}_0 = \mathbf{0}$ . ■

If strong inequality in (2.25) is replaced by a weak inequality, then Theorem 4.1 cannot hold true, as it is shown by the following general result.

**Proposition 4.3.** *If the structural assumptions together with (2.24) are fulfilled, but (2.25) is replaced by*

$$(4.8) \quad \mathcal{L}(\mathbf{W}^2 \mathbf{x}) \leq 0 \quad \forall \mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$$

*then  $\text{argmin } \mathcal{F}$  is still nonempty and*

$$(4.9) \quad \min \mathcal{F} = \min \mathcal{E},$$

*but the coincidence of minimizers sets is replaced by the inclusion*

$$(4.10) \quad \text{argmin } \mathcal{E} \subset \text{argmin } \mathcal{F}.$$

*If (4.8) holds true and there exists  $\mathbf{U} \in \mathcal{M}_{skew}^{N \times N}$ ,  $\mathbf{U} \neq \mathbf{0}$  such that  $\mathcal{L}(\mathbf{U}^2 \mathbf{x}) = 0$ , then  $\mathcal{F}$  admits infinitely many minimizers which are not minimizers of  $\mathcal{E}$ , precisely*

$$(4.11) \quad \text{argmin } \mathcal{E} \subsetneq \text{argmin } \mathcal{E} + \left\{ \mathbf{U}^2 \mathbf{x} : \mathbf{U} \in \mathcal{M}_{skew}^{N \times N}, \mathcal{L}(\mathbf{U}^2 \mathbf{x}) = 0 \right\} \subset \text{argmin } \mathcal{F},$$

*where the last inclusion is an equality in 2D:*

$$(4.12) \quad \text{argmin } \mathcal{E} \subsetneq \text{argmin } \mathcal{E} + \{ -t \mathbf{x} : t \geq 0 \} = \text{argmin } \mathcal{F}, \quad \text{if } N = 2.$$

*Proof.* The set  $\operatorname{argmin} \mathcal{E}$  is nonempty by classical arguments. Fix  $\mathbf{v}_* \in \operatorname{argmin} \mathcal{E}$ . Then for every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$  and for every  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$ , by setting  $\mathbf{z}_\mathbf{W} = \frac{1}{2} \mathbf{W}^2 \mathbf{x}$ , we get

$$\begin{aligned} \mathcal{F}(\mathbf{v}_*) &\leq \mathcal{E}(\mathbf{v}_*) \leq \mathcal{E}(\mathbf{v} - \mathbf{z}_\mathbf{W}) = \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v} - \mathbf{z}_\mathbf{W}) \leq \\ (4.13) \quad &\leq \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) \end{aligned}$$

hence for every  $\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$

$$(4.14) \quad \mathcal{F}(\mathbf{v}_*) \leq \min_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}) = \mathcal{F}(\mathbf{v})$$

thus proving that  $\operatorname{argmin} \mathcal{F}$  is nonempty and (4.10).

Moreover by setting

$$(4.15) \quad \mathbf{W}_\mathbf{v} \in \operatorname{argmin} \left\{ \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}) - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} : \mathbf{W} \in \mathcal{M}_{skew}^{N \times N} \right\}, \quad \forall \mathbf{v} \in H^1(\Omega; \mathbb{R}^N)$$

condition (4.8) entails for every  $\mathbf{v}_* \in \operatorname{argmin} \mathcal{E}$

$$(4.16) \quad \mathcal{F}(\mathbf{v}_*) = \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}_* - \mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}})) d\mathbf{x} - \mathcal{L}(\mathbf{v}_*) = \mathcal{E}(\mathbf{v}_* - \mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}}) - \mathcal{L}(\mathbf{z}_{\mathbf{W}_{\mathbf{v}_*}}) \geq \mathcal{E}(\mathbf{v}_*)$$

hence (4.9) follows by  $\mathcal{F} \leq \mathcal{E}$ .

If (4.8) holds true,  $\mathcal{L}(\mathbf{z}_\mathbf{U}) = 0$  for some  $\mathbf{0} \neq \mathbf{U} \in \mathcal{M}_{skew}^{N \times N}$  and  $\mathbf{v}^* \in \operatorname{argmin} \mathcal{E}$  then, by comparing the finite dimensional minimization over  $\mathbf{W}$  with evaluation at  $\mathbf{W} = \mathbf{U}$  and exploiting (4.9), we get

$$\begin{aligned} \mathcal{F}(\mathbf{v}_* + \mathbf{z}_\mathbf{U}) &= \min_{\mathbf{W}} \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}_*) + \frac{1}{2} \mathbf{U}^2 - \frac{1}{2} \mathbf{W}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}_* + \mathbf{z}_\mathbf{U}) \leq \\ (4.17) \quad &\leq \int_{\Omega} \mathcal{V}_0(x, \mathbb{E}(\mathbf{v}_*)) d\mathbf{x} - \mathcal{L}(\mathbf{v}_*) = \mathcal{E}(\mathbf{v}_*) = \min \mathcal{E} = \min \mathcal{F}, \end{aligned}$$

that is  $\mathbf{v}_* + \mathbf{z}_\mathbf{U} \in \operatorname{argmin} \mathcal{F}$ . Since  $\mathcal{V}_0$  is strictly convex we get  $\operatorname{argmin} \mathcal{E} = \{\mathbf{v}_* + \mathbf{z} : \mathbf{z} \in \mathcal{R}\}$  hence  $\mathcal{E}(\mathbf{v}_* + \mathbf{z}_\mathbf{U}) > \mathcal{E}(\mathbf{v}_*)$  thus proving the strict inclusion in (4.11).

Concerning last claim, if (4.8) holds true,  $\mathcal{L}(\mathbf{z}_\mathbf{U}) = 0$  for some  $\mathbf{0} \neq \mathbf{U} \in \mathcal{M}_{skew}^{N \times N}$ ,  $\mathbf{v}^* \in \operatorname{argmin} \mathcal{F}$  and  $N = 2$ , then  $\mathcal{M}_{skew}^{2 \times 2}$  is a 1D space, therefore we can assume  $\mathbf{U} = (\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1)$ ,  $\mathbf{U}^2 = -\mathbf{I}$ ,  $\mathcal{M}_{skew}^{2 \times 2} = \operatorname{span} \mathbf{U}$  and  $\mathbf{W}_{\mathbf{v}_*} = \lambda \mathbf{U}$  for some  $\lambda \in \mathbb{R}$ , and by (4.9)

$$\begin{aligned} \min \mathcal{E} &= \min \mathcal{F} = \mathcal{F}(\mathbf{v}_*) = \int_{\Omega} \mathcal{V}_0(\mathbb{E}(\mathbf{v}_*) - \frac{1}{2} \mathbf{W}_{\mathbf{v}_*}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}_*) = \\ &= \int_{\Omega} \mathcal{V}_0(\mathbb{E}(\mathbf{v}_*) - \frac{\lambda^2}{2} \mathbf{U}^2) d\mathbf{x} - \mathcal{L}(\mathbf{v}_* - \mathbf{z}_{\lambda \mathbf{U}}) = \mathcal{E}(\mathbf{v}_* - \mathbf{z}_{\lambda \mathbf{U}}), \end{aligned}$$

that is  $(\mathbf{v}_* - \mathbf{z}_{\lambda \mathbf{U}}) \in \operatorname{argmin} \mathcal{E}$  for every  $\mathbf{v}_* \in \operatorname{argmin} \mathcal{F}$ , therefore we get

$$\operatorname{argmin} \mathcal{F} - \{\mathbf{z}_{\lambda \mathbf{U}} : \lambda \in \mathbb{R}\} \subset \operatorname{argmin} \mathcal{E}, \quad \operatorname{argmin} \mathcal{F} \subset \operatorname{argmin} \mathcal{E} + \{\mathbf{z}_{\lambda \mathbf{U}} : \lambda \in \mathbb{R}\},$$

hence by  $\mathbf{z}_{\lambda \mathbf{U}} = \frac{\lambda^2}{2} \mathbf{U}^2 \mathbf{x} = -\frac{\lambda^2}{2} \mathbf{x}$  we obtain the equality in place of the last inclusion in (4.11), hence (4.12).  $\blacksquare$

Next example depicts the above Proposition in a simple explicit case.

**Example 4.4.** Let  $\Omega = (-1/2, 1/2)^2$ ,  $\mathbf{g} \equiv \mathbf{0}$ ,  $\mathbf{f} = (\mathbf{1}_{S_+} - \mathbf{1}_{S_-})\mathbf{e}_2 + (\mathbf{1}_{T_+} - \mathbf{1}_{T_-})\mathbf{e}_1$  where  $S_{\pm}$  denote respectively the right and the left side, and  $T_{\pm}$  the upper and the lower side of the square. A straightforward computation gives, for suitable  $\lambda \in \mathbb{R}$ ,

$$(4.18) \quad \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} = -\lambda^2 \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{x} d\mathcal{H}^{N-1} = 0 \quad \forall \mathbf{W} \in \mathcal{M}_{skew}^{2 \times 2},$$

Then, since (2.24) and (4.8) are fulfilled, by (4.12) in Proposition 4.3 we have the whole description of the set  $\operatorname{argmin} \mathcal{F}$ : for every choice of  $\mathcal{V}_0$  satisfying the standard structural hypotheses,  $\mathcal{F}$  has infinitely many minimizers  $\mathbf{v}$  which are not minimizers of  $\mathcal{E}$ , explicitly they are of the kind

$$\mathbf{v} = (\mathbf{v}^* - t\mathbf{x}) \in \operatorname{argmin} \mathcal{F} \setminus \operatorname{argmin} \mathcal{E} \quad \text{if } \mathbf{v}^* \in \operatorname{argmin} \mathcal{E}, \quad t > 0.$$

It is quite natural to ask whether condition (2.25), which is essential in the proof of Theorem 2.2, may be dropped in order to obtain at least existence of  $\min \mathcal{F}$ : the answer is negative. Actually next remark shows that, when inequality in compatibility condition (2.25) is reversed for at least one choice of the skew-symmetric matrix  $\mathbf{W}$ , then  $\mathcal{F}$  is unbounded from below.

**Remark 4.5.** *If*

$$(4.19) \quad \exists \mathbf{W}_* \in \mathcal{M}_{skew}^{N \times N} : \quad \mathcal{L}(\mathbf{z}\mathbf{W}_*) > 0, \quad \text{where } \mathbf{z}\mathbf{W}_* = \frac{1}{2}\mathbf{W}_*^2 \mathbf{x},$$

then

$$(4.20) \quad \inf_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = -\infty.$$

Indeed, by arguing as in (4.4) and replacing  $\min_{H^1}$  with  $\inf_{H^1}$ , we get

$$(4.21) \quad \inf_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F} = \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E} - \sup_{\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}} \mathcal{L}(\mathbf{z}\mathbf{W}) \quad \text{where } \mathbf{z}\mathbf{W} = \frac{1}{2}\mathbf{W}^2 \mathbf{x}.$$

Hence

$$\inf_{H^1(\Omega; \mathbb{R}^N)} \mathcal{F} \leq \min_{H^1(\Omega; \mathbb{R}^N)} \mathcal{E} - \tau \mathcal{L}(\mathbf{z}\mathbf{W}_*) \quad \forall \tau > 0,$$

which entails (4.20).

Next example shows that in case of uniform compression along the whole boundary functional  $\mathcal{F}$  is unbounded from below.

**Example 4.6.** Assume  $\Omega \subset \mathbb{R}^N$  is a Lipschitz, connected open set,  $N = 2, 3$ ,  $\mathbf{g} \equiv \mathbf{0}$ ,  $\mathbf{f} = -\mathbf{n}$ , where  $\mathbf{n}$  denotes the outer unit normal vector to  $\partial\Omega$ .

Then (4.19) holds true hence, by Remark 4.5,  $\inf_{\mathbf{v} \in H^1(\Omega; \mathbb{R}^N)} \mathcal{F}(\mathbf{v}) = -\infty$ .

Indeed, for every  $\mathbf{W} \in \mathcal{M}_{skew}^{N \times N}$  such that  $|\mathbf{W}|^2 = 2$  we obtain

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} = - \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{W}^2 \mathbf{x} d\mathcal{H}^{N-1} = - \int_{\Omega} \operatorname{div}(\mathbf{W}^2 \mathbf{x}) d\mathbf{x} = -|\Omega| \operatorname{Tr} \mathbf{W}^2 = 2|\Omega| > 0.$$

This means that any Lipschitz open set turns out to be always unstable when uniformly compressed in the direction of the inward normal vector along its boundary: therefore the linearized model proves inadequate for such case even for small load.



## REFERENCES

- [1] V. Agostiniani, T. Blass, K. Koumatos, *From nonlinear to linearized elasticity via Gamma-convergence: the case of multiwell energies satisfying weak coercivity conditions*, Math. Models Methods Appl. Sci., **25** 1 (2015), 1–38.
- [2] V. Agostiniani, G. Dal Maso, A. DeSimone, *Linear elasticity obtained from finite elasticity by Gamma-convergence under weak coerciveness conditions*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **29** 5 (2012), 715–735.
- [3] R. Alicandro, G. Dal Maso, G. Lazzaroni, M. Palombaro, *Derivation of a linearised elasticity model from singularly perturbed multiwell energy functionals*, Arch. Ration. Mech. Anal. , (2018) DOI: 10.1007/s00205-018-1240-6
- [4] G. Anzellotti, S. Baldo, D. Percivale, *Dimension reduction in variational problems, asymptotic development in  $\Gamma$ -convergence and thin structures in elasticity*. Asympt. Analysis, vol. 9, (1994) p.61-100, ISSN: 0921-7134.
- [5] B. Audoly, Y. Pomeau, *Elasticity and Geometry*, Oxford University Press, (2010).
- [6] C. Baiocchi, G. Buttazzo, F. Gastaldi, F. Tomarelli, *General existence results for unilateral problems in continuum mechanics*, Arch. Rational Mech. Anal. **100** (1988), 149–189.
- [7] G. Buttazzo, G. Dal Maso, *Singular perturbation problems in the calculus of variations*, Ann.Scuola Normale Sup., Cl.Sci., 4 ser., **11** 3 (1984) 395-430.
- [8] G. Buttazzo, F. Tomarelli, *Compatibility conditions for nonlinear Neumann problems*, Advances in Math. **89** (1991), 127-143.
- [9] M. Carriero, A. Leaci, F. Tomarelli, *Strong solution for an elastic-plastic plate*, Calc.Var.Partial Diff.Eq., **2**, 2, (1994) 219-240.
- [10] P. G. Ciarlet, *Mathematical Elasticity, Volume I: Three-Dimensional Elasticity*, Elsevier, (1988).
- [11] P. G. Ciarlet, P. Ciarlet Jr., *Another approach to linearized elasticity and Korn’s inequality*, C.R. Acad. Sci. Paris, Ser.I 339, (2004) 307-312.
- [12] G. Dal Maso, *An Introduction to Gamma Convergence*, Birkhäuser, PNLDE **8** (1993).
- [13] G. Dal Maso, M. Negri, D. Percivale, *Linearized elasticity as  $\Gamma$ -limit of finite elasticity*, Set-Valued Anal. **10** (2002), no. 2-3, 165-183.
- [14] E. De Giorgi, T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 58 (1975), no.6, 842–850.
- [15] D. De Tommasi, S. Marzano, *Small Strain and Moderate Rotation*, J. Elasticity 32, (1993) 37-50.
- [16] D. De Tommasi *On the Kinematics of Deformations with Small Strain and Moderate Rotation* , Math. And Mechanics of Solids **9**: (2004) 355-368.
- [17] G. Frieseke, R.D. James, S. Müller, *A theorem on geometric rigidity and the derivation of non linear plate theory from three dimensional elasticity*, Comm.Pure Appl. Math. 55 (2002), 1461–1506.
- [18] G. Frieseke, R.D. James, S. Müller, *A Hierarchy of Plate Models from Nonlinear Elasticity by Gamma-Convergence*, Arch. Rational Mech. Anal. 180 (2006), 183-236.
- [19] M.E.Gurtin, *The linear theory of elasticity*, in *Handbuch der Physik*, Vla/2, Springer, Berlin (1972).
- [20] B. Hall, *Lie Groups, Lie Algebras and Representations: An elementary introduction*, Springer Graduate Text in Math. 222 (2015).
- [21] M. Lecumberry, S. Müller, *Stability of slender bodies under compression and validity of von Kármán theory*, Arch. Rational Mech. Anal. 193 (2009), 255-310.
- [22] A.E. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, (1944).
- [23] F. Maddalena, D. Percivale, F. Tomarelli, *Adhesive flexible material structures*, Discr. Continuous Dynamic. Systems B, Vol. 17, Num. 2, March (2012), 553-574.
- [24] F. Maddalena, D. Percivale, F. Tomarelli, *Local and nonlocal energies in adhesive interaction*, IMA Journal of Applied Mathematics, **81**, 6, (2016) 1051–1075.
- [25] F. Maddalena, D. Percivale, F. Tomarelli, *Variational Problems for Föppl-von Kármán plates*, SIAM Journal on Mathematical Analysis, **50**, 1, (2018) 251–282, DOI 10.1137/17M1115502.
- [26] F. Maddalena, D. Percivale, F. Tomarelli, *A new variational approach to linearization of traction problems in elasticity*, *J.Optim.Theory Appl.*, (2019), DOI 10.1007/s10957-019-01533-8
- [27] D. Percivale, F. Tomarelli, *Scaled Korn-Poincaré inequality in BD and a model of elastic plastic cantilever*, Asymptot. Anal., **23**, 3-4, (2000). 291–311.

- [28] D. Percivale, F. Tomarelli, *From SBD to SBH: the elastic-plastic plate*, Interfaces Free Boundaries, **4**, 2, (2002) 137-165.
- [29] D. Percivale, F. Tomarelli, *A variational principle for plastic hinges in a beam*, Math. Models Methods Appl. Sci., 19, n.12, (2009), 2263-2297.
- [30] D. Percivale, F. Tomarelli, *Smooth and broken minimizers of some free discontinuity problems*, in: P. Colli et al. (eds.), *Solvability, Regularity, and Optimal Control of Boundary Value Problems for PDEs*, Springer INdAM Series, **22**, (2017), 431-468, DOI 10.1007/978-3-319-64489-9\_17.
- [31] P. Podio-Guidugli, *On the validation of theories of thin elastic structures*, Meccanica, **49** (6), (2014), 1343-1352.
- [32] C. Truesdell, W. Noll, *The non-linear field theories of mechanics*, In *Handbuch der Physik* 11113, Springer, Berlin (1965).

POLITECNICO DI BARI, DIPARTIMENTO DI MECCANICA, MATEMATICA, MANAGEMENT, VIA RE DAVID 200, 70125 BARI, ITALY

*E-mail address:* `francesco.maddalena@poliba.it`

UNIVERSITÀ DI GENOVA, DIPARTIMENTO DI INGEGNERIA MECCANICA, PIAZZALE KENNEDY, FIERA DEL MARE, PADIGLIONE D, 16129 GENOVA, ITALY

*E-mail address:* `percivale@diptem.unige.it`

POLITECNICO DI MILANO, DIPARTIMENTO DI MATEMATICA, PIAZZA LEONARDO DA VINCI 32, 20133 MILANO, ITALY

*E-mail address:* `franco.tomarelli@polimi.it`