$\mathcal{H}_2$ and $\mathcal{H}_\infty$ control of time-varying delay switched linear systems with application to sampled-data control[☆]

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This paper deals with switched linear systems subject to time-varying delay. The main goal is to design state and output feedback switching strategies preserving closed-loop stability and a guaranteed $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance. The switching strategies are based on a generalization of a recent extended version of the small gain theorem and do not require any assumption on the continuity of the delay and its time-variation rate. The key point to obtain the design conditions is the adoption of an equivalent switched linear system where the time-varying delay is modeled as a norm-bounded perturbation. Moreover, with this approach, it is possible to deal with sampled-data control systems. All conditions are formulated in terms of Lyapunov–Metzler inequalities, which allow the maximization of an upper bound on the time-delay preserving stability and guaranteed performance. Numerical examples are discussed in order to illustrate the effectiveness of the design approach.

Keywords:
Switched systems
Time-varying delay
Sampled-data control LMI

1. Introduction

The recent literature displays an increasing interest in the study of switched systems subject to time-delays, which can be used to represent many classes of real-world situations including measurement and actuators delays [1], information transmission delays [2,3], neural networks [4], among others. Relevant books in the area of time-delay systems and switched systems are [5,6], among others.

In the control community, consistent attention has been devoted to the special class of switched linear systems with time-delays, see for instance [7–9]. When the feedback stabilization is considered, most of the contributions proposed so far are related to the case where the delay is constant. Among the recent papers, [10] proposed a stabilizing switching rule based on a Riccati-type common Lyapunov functional approach and assuming a condition on the time-delay. In [11], both delay-independent and delay-dependent strategies for the state-feedback $\mathcal{H}_\infty$ control of switched linear systems were worked out, relying on suitable Lyapunov–Krasovskii functionals. Further advances were achieved in [12], where output-feedback delay-independent switching laws are proposed, and [13], which provides a new perspective based on an extended small-gain theorem. On the other side, new research efforts are being made to deal with time-varying delays. In [14], exponential

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stability and $\mathcal{L}_2$-gain were studied, considering switching signals with average dwell time. Ref. [15] addressed stability analysis of systems with uncertain time-varying delays. In [16], a time-varying delay acting on both the state and the control was considered for the nonlinear case by an approach based on Lyapunov–Krasovskii functionals.

Differently from [17,18], where the switching is considered as a given exogenous perturbation characterized by presenting some dwell time and/or average dwell time, switched linear systems subject to time-varying delays, for which the switching is a control variable, are the main focus of this paper. It is devoted to the use of switching for stabilization and $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed performance optimization by either state and output feedback, under the assumption that the time-varying delay evolves within specified bounds. Our results are based on the concept of Lyapunov–Metzler inequalities, originally proposed in [19,20], as a tool to derive stabilizing switching strategies. Beyond stabilization, controlled switching has been exploited to improve performance when compared to the non-switched situation, see [20–22], or to enhance robustness of the feedback control system [13]. In particular, the present contribution stands as a generalization of Ref. [13], which has provided delay-dependent stability conditions for the design of a stabilizing switching rule, but only when the delay is time-invariant and without considering any performance index. Furthermore, here, we extend the results of [23] by including the determination of switching strategies ensuring prescribed $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed performances which, to the best of the author's knowledge, is a problem not treated in the literature to date.

In order to formulate our results, we resort to the extended formulation of the small-gain theorem originally provided in [13]. In this way, the original switched linear system is reformulated as a feedback interconnection between a delay-free switched linear subsystem and a norm-bounded perturbation block. Then, the stability of the overall system is imposed by designing the switching law so that the $\mathcal{L}_2$-induced norm of the delay-free subsystem is less than a prescribed value. Finally, we specify $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance requirements accordingly.

An interesting feature about the problem considered in this paper is its possible application to control design of sampled-data systems. In fact, it is well known that sampling can be represented in terms of delays, see for instance [24,25] and the references therein. In particular, nonuniform sampling can be equivalently represented as a time-varying delay acting on the input variables of the system [26]. The recent results on state feedback sampled-data control design for linear time invariant systems reported in [25] are used to certificate the ones proposed in this paper.

The paper is organized as follows: in Section 2 we formulate the control problem and review relevant preliminary results, specially related to the use of the small gain theorem for the class of problems under consideration. Section 3 introduces a way to rewrite the system equations enabling its study from the mentioned perspective. Performance optimization is the subject of Section 4, where we introduce both state and output feedback control synthesis results. Section 5 discusses the application of our techniques to sampled-data control systems design with the support of a numerical example. Finally, there are some concluding remarks.

The notation is standard. The identity matrix of any dimension is denoted by I. For real matrices or vectors, the symbol $(')$ indicates transpose. For any square matrix $\mathrm{Tr}(\cdot)$ represents its trace. For a symmetric matrix, the symbol $(\bullet)$ denotes each of its symmetric blocks and $Q > 0$ ($Q < 0$) indicates that the symmetric, real matrix Q is positive definite (negative definite). The set of natural numbers is $\mathbb{N}$ and $\mathbb{K} = \{1, 2, \dots, N\}$. The squared norm of a signal $\xi(t)$ defined for all $t \geq 0$, denoted by $\|\xi\|_2^2$, is equal to $\int_0^\infty \xi(t)'\xi(t)dt$. The set of all signals such that $\|\xi\|_2^2 < \infty$ is denoted by $\mathcal{L}_2$. For a real matrix M, the Hermitian operator $\mathrm{H_e}\{\cdot\}$ is defined as $\mathrm{H_e}\{M\} = M + M'$. The set $\mathcal{M}$ is composed by all Metzler matrices $\Pi = \{\pi_{ji}\} \in \mathbb{R}^{N\times N}$, with non-negative off-diagonal elements satisfying the constraints $\sum_{j\in\mathbb{K}} \pi_{ji} = 0,\ \forall i \in \mathbb{K}$. Given a continuous (not necessarily differentiable) function $f(t)$, the Dini derivative is defined as $D^+f(t) = \limsup_{\Delta t\to 0^+} \left(f(t+\Delta t) - f(t)\right)/\Delta t$. Finally, the symbol "$\circ$" indicates the application of a linear input–output operator to a signal.

2. Problem formulation and preliminaries

Consider a switched linear system described by

$$\dot{x}(t) = A_\sigma x(t) + A_{d\sigma} x(t-h(t)) + H_\sigma w(t) \tag{1}$$

$$z(t) = E_\sigma x(t) + E_{d\sigma} x(t-h(t)) + G_\sigma w(t) \tag{2}$$

$$y(t) = C_\sigma x(t) + C_{d\sigma} x(t-h(t)) + D_\sigma w(t) \tag{3}$$

where $h(t)$ is the time-varying delay and the vectors $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, $y \in \mathbb{R}^{n_y}$ and $z \in \mathbb{R}^{n_z}$ are the state, the external input, the measured output and the controlled output, respectively. It is supposed that $h(t)$ satisfies the constraint $0 \leq h(t) \leq h_m$ for all $t \geq 0$ and the system evolves from zero initial condition, that is $x(t) = 0$, $-h_m \leq t \leq 0$. No assumption on continuity and on the time derivative $\dot{h}(t)$ is required. We only suppose that $h(t)$ is piecewise continuous. The switching function, denoted by $\sigma(\cdot)$, is the unique control variable to be determined. It selects at each instant of time $t \geq 0$ a subsystem $\mathcal{P}_i$ among the set $\{\mathcal{P}_1, \dots, \mathcal{P}_N\}$ of available ones defined by matrices

$$\mathcal{P}_i := \begin{bmatrix} A_i & A_{di} & H_i \\ E_i & E_{di} & G_i \\ C_i & C_{di} & D_i \end{bmatrix} \tag{4}$$

of compatible block dimensions.

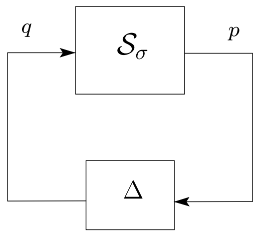

Fig. 1. Switched closed-loop system.

Remark 1. Eq. (1) describes a differential equation with discontinuous right hand side. If the switching signal $\sigma(\cdot)$ is piecewise continuous, then a solution obtained from concatenating the motion of linear subsystems exists, it is unique and absolutely continuous. Furthermore, for the case of switching with unbounded frequency along a commutation surface the usual notion of Filippov solution is adopted, see [27,28].

In a first moment, we consider that the state $x \in \mathbb{R}^{n_x}$ is available and the switching rule is of the form $\sigma(t) = g(x(t))$ where the mapping $g(x) : \mathbb{R}^{n_x} \to \mathbb{K}$ must be determined. Afterwards, the result will be generalized to treat the control design problem where only the measured output $y \in \mathbb{R}^{n_y}$ is available. In this case, a set of delay-free full order switched linear filters

$$\dot{\hat{x}}(t) = \hat{A}_\sigma \hat{x}(t) + \hat{B}_\sigma y(t) \tag{5}$$

with matrices $(\hat{A}_i, \hat{B}_i)$, $\forall i \in \mathbb{K}$ to be determined, provides information for the switching rule $\sigma(t) = g(\hat{x}(t)) : \mathbb{R}^{n_x} \to \mathbb{K}$, which depends only on the measured output $y(t)$ through the filter state variables $\hat{x}(t)$. Moreover, the control design must take into account $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performances. They were defined in [22] and are provided here for convenience:

- $\mathcal{H}_2$ **performance**: For strictly proper subsystems ($G_i = 0, \forall i \in \mathbb{K}$), the controlled output $z_l(t)$ associated with disturbances of the form $w(t) = e_l \delta(t)$, where $e_l \in \mathbb{R}^{n_w}$ is the lth column of the identity matrix and $\delta(t)$ is the impulsive function, provides the index

$$J_2(\sigma) = \sum_{l=1}^{n_w} \|z_l\|_2^2 \tag{6}$$

- $\mathcal{H}_\infty$ **performance**: The controlled output $z(t)$ associated with arbitrary square integrable disturbances $w \in \mathcal{L}_2$ provides the index

$$J_\infty(\sigma) = \sup_{0 \neq w \in \mathcal{L}_2} \frac{\|z\|_2^2}{\|w\|_2^2}. \tag{7}$$

The rationale behind these definitions is that whenever the switching rule is kept constant, that is, $\sigma(t) = i \in \mathbb{K}$ for all $t \geq 0$, the indexes equal the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ squared norms of the ith subsystem transfer function from the input w to the controlled output z. Notice that they depend on σ being, therefore, highly nonlinear and difficult to determine. Our aim is to design switching strategies in order to assure suitable $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed performance levels.

This paper generalizes the results of [13] in order to cope with time-varying delay and the previously defined $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes. Indeed, [13] has provided delay-dependent stability conditions for the design of a stabilizing switching rule, but only in the case where the delay is constant with respect to time and without considering any performance index. According to [13] the time-invariant delay switched linear system

$$\dot{x}(t) = A_\sigma x(t) + A_{d\sigma} x(t-h) \tag{8}$$

can be, alternatively, rewritten as the feedback interconnection, shown in Fig. 1, of the delay-free subsystem $\mathcal{S}_\sigma$

$$\dot{x}(t) = (A_\sigma + A_{d\sigma})x(t) + A_{d\sigma} M^{-1} q(t) \tag{9}$$

$$p(t) = M\left((A_\sigma + A_{d\sigma})x(t) + A_{d\sigma} M^{-1} q(t)\right) \tag{10}$$

where M is an arbitrary nonsingular square matrix, and the linear subsystem

$$q(t) = \Delta \circ p(t) \tag{11}$$

defined by an operator Δ, whose Laplace transform is $\hat{\Delta}(s) = ((e^{-hs} - 1)/s)I$ which yields $\|\Delta\|_\infty = \sup_{0 \neq p \in \mathcal{L}_2} \|q\|_2/\|p\|_2 = h \geq 0$. Based on an extended version of the small gain theorem, also proposed in [13], the stability of the overall system is assured if the switching rule is designed so that the $\mathcal{L}_2$-induced norm of $\mathcal{S}_\sigma$ is less than $1/h$. The main difficulty in generalizing this result to deal with time-varying delay $h(t)$ is to obtain an equivalent system with a structure that allows to use the extended small gain theorem approach. Notice that this Δ operator is not valid anymore whenever the delay becomes time-varying. However, it will be shown in the next section that this generalization can be made by adopting a more suitable equivalent system. It is worth mentioning that the conditions obtained in this paper only require boundedness and piecewise continuity of the time-delay, but no assumption on its time variation rate. Moreover, as it will be clear in Section 5, for a special subclass of time-varying delay, it is possible to design suitable switching strategies to sampled-data control systems.

3. Equivalent system

As already mentioned, the key point to obtain the results of this paper is to write the time-varying delay system (1)–(2) with the structure of Fig. 1, so that the small gain theorem of [13] can be applied. In order to ease the notation, let us define the matrices $A_{0i} = A_i + A_{di}$ and $E_{0i} = E_i + E_{di}$ for all $i \in \mathbb{K}$.

Lemma 1. *The switched linear system* (1)–(2) *can be, alternatively, rewritten as*

$$\dot{x}(t) = A_{0\sigma}x(t) + A_{d\sigma}M^{-1}q(t) + H_\sigma w(t) \tag{12}$$

$$p(t) = M\left(A_{0\sigma}x(t) + A_{d\sigma}M^{-1}q(t) + H_\sigma w(t)\right) \tag{13}$$

$$q(t) = \Delta \circ p(t) = -\int_{t-h(t)}^{t} p(\tau)d\tau \tag{14}$$

$$z(t) = E_{0\sigma}x(t) + E_{d\sigma}M^{-1}q(t) + G_\sigma w(t) \tag{15}$$

where M is an arbitrary nonsingular square matrix and $\|\Delta\|_\infty \le h_m$.

Proof. The proof comes from the observation that $p(t) = M\dot{x}(t)$ and, therefore

$$q(t) = -\int_{t-h(t)}^{t} p(\tau)d\tau = -M(x(t) - x(t-h(t))) \tag{16}$$

which when plugged into (12)–(13) provides (1)–(2). Now, our goal is to evaluate the $\mathcal{L}_2$ gain of the linear operator Δ. In this respect, using [29], we conclude that $\|\Delta\|_\infty \le h_m$ which completes the proof.

The time varying delay $h(t)$ is considered as a perturbation which is embedded in the operator Δ. Only for stability analysis, making $w(t) = 0$, $\forall t \ge 0$, the state space realization (12)–(14) is exactly the same reported in [13] and provided in (9)–(11). It is important to notice that the same conditions provided in Ref. [13] for the time-invariant delay case can be used to treat a much more general problem characterized by the time-varying delay. Moreover, taking into account that the upper bound $\|\Delta\|_\infty \le h_m$ does not depend on a possible lower bound $h(t) \ge h_{\min} \ge 0$, then the maximum delay interval is obtained whenever we set $h_{\min} = 0$ which improves the result of [23]. Hence, the result of [13] is actually an upper bound valid for a larger class of time-varying delay, not necessarily continuous and without any limitation on its time variation rate. Indeed, in order to apply small gain arguments, the operator Δ is embedded in a larger set of perturbations with bounded norm where only the $\mathcal{L}_2$-induced norm of Δ is relevant. In addition, as discussed in [13], the approach of the extended small gain theorem to treat stabilization of time-delay switched systems has provided better results than the ones available in the literature, as for instance, by the technique based on the Lyapunov–Krasovskii functionals, see [12]. This indicates that the extended small gain theorem is a less conservative and useful tool to treat the problem of interest.

Remark 2. It is clear that for a specific time-varying delay, the upper bound $\|\Delta\|_\infty \le h_m$ can be significantly improved. For instance, consider the sawtooth time-varying delay involved in sampled-data control systems, see [26,30], that is $h(t) = t - t_k$ for all $t \in [t_k, t_{k+1})$, where $\{t_k\}_{k=0}^{\infty}$ is a sequence of time instants such that $t_0 = 0$, $h_m \ge t_{k+1} - t_k \ge 0$, $\forall k \in \mathbb{N}$, and $t_\infty = +\infty$. Compared to [31], we propose an alternative way to calculate the exact value of $\|\Delta\|_\infty$. To this end, we observe that the smallest $\gamma > 0$, valid for all $k \in \mathbb{N}$, such that

$$\sup_p \left\{ \int_{t_k}^{t_{k+1}} (q'q - \gamma^2 p'p)dt \ : \dot{q} = p,\ q(t_k) = 0 \right\} = 0 \tag{17}$$

provides $\|\Delta\|_\infty = \gamma$ and can be calculated from the existence of a solution to the differential Riccati equation $-\dot{R} = (R/\gamma)^2 + I$ for all $t \in [t_k, t_{k+1})$, subject to the final time condition $R(t_{k+1}) = 0$ which is uniquely given by

$$R(t) = \gamma \ \mathrm{tg}\left(\gamma^{-1}(t_{k+1} - t)\right) I \tag{18}$$

for all $t \in [t_k, t_{k+1})$ provided that $\gamma = (2/\pi)h_m \ge (2/\pi)(t_{k+1} - t_k)$ for all $k \in \mathbb{N}$. This simpler calculation reproduces exactly the norm of the operator corresponding to the sawtooth reported in [31] as being $\|\Delta\|_\infty = (2/\pi)h_m$.

4. Performance optimization

In this section, our main goal is to include the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs in the previous calculations. Notice that, in both cases, the results follow from the adoption of a min-type Lyapunov function candidate which also provides the associated switching strategy.

4.1. State feedback control design

Let us first consider the $\mathcal{H}_2$ guaranteed performance. To this end, as usual, we have to impose $G_i = 0$ for all $i \in \mathbb{K}$ in (12)–(15), making all subsystems strictly proper. Moreover, considering an external perturbation of impulsive type $w(t) = e_l\delta(t)$ the system (12)–(15) can be alternatively rewritten as

$$\dot{x}(t) = A_{0\sigma}x(t) + A_{d\sigma}M^{-1}q(t), \qquad x(0) = H_{\sigma(0)}e_l \tag{19}$$

$$p(t) = M\left(A_{0\sigma}x(t) + A_{d\sigma}M^{-1}q(t)\right) \tag{20}$$

$$q(t) = \Delta \circ p(t) \tag{21}$$

$$z(t) = E_{0\sigma}x(t) + E_{d\sigma}M^{-1}q(t) \tag{22}$$

with $\|\Delta\|_\infty \leq h_m$. We adopt the min-type Lyapunov function $v(x) = \min_{i\in\mathbb{K}} x'P_ix$, with $P_i > 0$ for all $i \in \mathbb{K}$, in order to assure that $v(x) > 0$ for all $0 \neq x \in \mathbb{R}^{n_x}$, and the associated min-type switching strategy $\sigma(t) = g(x(t))$, defined as

$$g(x) = \arg\min_{i\in\mathbb{K}} x'P_ix. \tag{23}$$

Whenever the minimum is not unique, any minimizer can be adopted as, for instance, the one corresponding to the smallest index value. The next theorem presents conditions to ensure an $\mathcal{H}_2$ guaranteed level of performance. Although omitted for convenience, it is assumed that all matrices are of compatible dimensions.

Theorem 1. *Consider system (1)–(2) with $G_i = 0$ for all $i \in \mathbb{K}$, set $\mu = 1/h_m^2$ and assume that there exist symmetric matrices $Q > 0$, $P_i > 0$ and a Metzler matrix $\Pi \in \mathcal{M}$ satisfying the Riccati–Metzler inequalities*

$$\begin{bmatrix} \mathrm{He}\{A_{0i}'P_i\} + \sum_{j\in\mathbb{K}} \pi_{ji}P_j & \bullet & \bullet & \bullet \\ A_{di}'P_i & -\mu Q & \bullet & \bullet \\ QA_{0i} & QA_{di} & -Q & \bullet \\ E_{0i} & E_{di} & 0 & -I \end{bmatrix} < 0, \quad i \in \mathbb{K}. \tag{24}$$

The switching strategy $\sigma(t) = g(x(t))$ with $g(x)$ given by (23) is globally asymptotically stabilizing for any time-varying delay such that $0 \leq h(t) \leq h_m$ and satisfies the guaranteed performance $J_2(\sigma) < \min_{i\in\mathbb{K}} \mathrm{Tr}(H_{\sigma(0)}'P_iH_{\sigma(0)})$.

Proof. Suppose inequalities (24) are satisfied. Adopting the Lyapunov function $v(x) = \min_{i\in\mathbb{K}} x'P_ix$ and writing the system as in (19)–(22), we can follow the same steps of Theorem 2 from [13] to conclude that the feasibility of inequalities

$$\begin{bmatrix} \mathrm{He}\{A_{0i}'P_i\} + \sum_{j\in\mathbb{K}} \pi_{ji}P_j & \bullet & \bullet \\ A_{di}'P_i & -\mu Q & \bullet \\ QA_{0i} & QA_{di} & -Q \end{bmatrix} < 0, \quad i \in \mathbb{K} \tag{25}$$

assures that the Dini derivative of $v(x(t))$ satisfies the condition $D^+v(x(t)) < -p(t)'p(t) + \mu q(t)'q(t)$. Details on Dini derivative calculation can be found in [12]. From this point, performing the Schur complement with respect to the bottom right element of (24), it follows that

$$D^+v(x(t)) < -p(t)'p(t) + \mu q(t)'q(t) - z(t)'z(t) \tag{26}$$

which integrating both sides from zero to infinity provides

$$v(x(\infty)) - v(x(0)) < -\int_0^\infty \left(p(\tau)'p(\tau) - \mu q(\tau)'q(\tau)\right)d\tau - \int_0^\infty z(\tau)'z(\tau)d\tau. \tag{27}$$

Hence, observing that $\|\Delta\|_\infty^2 \leq h_m^2 = \mu^{-1}$ and $q(t) = \Delta \circ p(t)$ yield

$$\|\Delta\|_\infty^2 = \sup_{0\neq p\in\mathcal{L}_2} \frac{\int_0^\infty q(\tau)'q(\tau)d\tau}{\int_0^\infty p(\tau)'p(\tau)d\tau} \leq \mu^{-1} \tag{28}$$

then, taking into account that the first diagonal block of (24) assures the system is asymptotically stable, we conclude that $v(x(\infty)) = \lim_{t\to\infty} v(x(t)) = 0$ and, consequently

$$\int_0^\infty z(\tau)'z(\tau)d\tau < v(x(0)) = \min_{i\in\mathbb{K}} x(0)'P_ix(0) \tag{29}$$

is valid for all $x(0) \neq 0$. Using this inequality successively for each initial condition of the form $x(0) = H_{\sigma(0)}e_l$ and summing up both sides, we obtain

$$\begin{aligned}\sum_{l=1}^{n_w} \|z_l\|_2^2 &< \sum_{l=1}^{n_w} \min_{i\in\mathbb{K}} e_l'H_{\sigma(0)}'P_iH_{\sigma(0)}e_l \\ &< \min_{i\in\mathbb{K}} \sum_{l=1}^{n_w} e_l'H_{\sigma(0)}'P_iH_{\sigma(0)}e_l \\ &= \min_{i\in\mathbb{K}} \mathrm{Tr}\left(H_{\sigma(0)}'P_iH_{\sigma(0)}\right)\end{aligned} \tag{30}$$

which concludes the proof.

Remark 3. The proof of Theorem 1 strongly depends on the calculation of the Dini derivative of the continuous (but not differentiable everywhere) function $v(x)$. From [12], the Dini derivative of $v(x)$ along a generic trajectory of $\dot{x} = A_\sigma x$ is $D^+v(x) = \min_{j\in\Omega(x)} x'(A_\sigma P_j + P_jA_\sigma)x$ where $\Omega(x) = \{i \in \mathbb{K} : x'P_ix = v(x)\}$. Then, for any $\sigma = i \in \Omega(x)$ the Dini derivative is negative because $D^+v(x) = \min_{j\in\Omega(x)} x'(A_iP_j + P_jA_i)x \le x'(A_iP_i + P_iA_i)x < 0$ and by consequence $v(x)$ is strictly decreasing for any switching strategy provided by $\sigma(x) = \arg\min_{i\in\mathbb{K}} x'P_ix$. Hence, chattering (high frequency switching) whenever occurs, it does not cause instability. Notice however that, in this situation, the differential equation must be analyzed through a different notion of solution, namely the Filippov solution, see [28,32] for time-delay systems. Roughly speaking, a Filippov solution provides a description of the sliding motion of $x(t)$ along a commutation surface in terms of a smooth linear combination of the corresponding subsystems equations. It is remarkable that the min-type switching strategy ensures global asymptotic stability of any possible sliding mode, so generalizing the validity of the results. This point has been fully addressed in Remark 1 of [12] and is a well-established property of the min-type switching law adopted in this paper, see e.g. Section 1.2.3 and Section 3.4.2 of [6] and Remark 2 of [19].

The guaranteed cost provided by Theorem 1 depends on the switching function $\sigma(0)$ evaluated at time $t = 0$ that may be defined by the designer. A good choice is $\sigma(0) = i^*$ that follows from the optimal solution of the optimization problem

$$J_2(\sigma) < \min_{i\in\mathbb{K}} \inf_{\{P_i>0, Q>0, \Pi\in\mathcal{M}\}\in\Phi} \mathrm{Tr}(H_i'P_iH_i) \tag{31}$$

where Φ is the set of all feasible solutions of (24). It provides matrices P_i, $\forall i \in \mathbb{K}$, and the minimum guaranteed $\mathcal{H}_2$ cost that can be achieved by the min-type switching strategy (23). Note that since (31) is nonconvex a possible way to solve it is to search $\Pi \in \mathcal{M}$ iteratively in order to take advantage to the fact that whenever $\Pi \in \mathcal{M}$ is fixed the remainder problem becomes convex, see [13].

We now move our attention to the $\mathcal{H}_\infty$ guaranteed cost determination by considering the system (12)–(14) with external perturbations $w(t) \in \mathcal{L}_2$. The next theorem presents a switching strategy of the form (23) that imposes a certain pre-specified $\mathcal{H}_\infty$ performance level to the switching system under consideration.

Theorem 2. *Consider system (1)–(2), set $\mu = 1/h_m^2$ and assume that there exist symmetric matrices $Q > 0$, $P_i > 0$, a Metzler matrix $\Pi \in \mathcal{M}$ and a scalar $\rho > 0$ satisfying the Riccati–Metzler inequalities*

$$\begin{bmatrix} \mathrm{He}\{A_{0i}'P_i\} + \sum_{j\in\mathbb{K}} \pi_{ji}P_j & \bullet & \bullet & \bullet & \bullet \\ A_{di}'P_i & -\mu Q & \bullet & \bullet & \bullet \\ H_i'P_i & 0 & -\rho I & \bullet & \bullet \\ QA_{0i} & QA_{di} & QH_i & -Q & \bullet \\ E_{0i} & E_{di} & G_i & 0 & -I \end{bmatrix} < 0, \quad i \in \mathbb{K}. \tag{32}$$

The switching strategy $\sigma(t) = g(x(t))$ with $g(x)$ given by (23) is globally asymptotically stabilizing for any time-varying delay such that $0 \le h(t) \le h_m$ and satisfies the guaranteed performance $J_\infty(\sigma) < \rho$.

Proof. Supposing that inequalities (32) are satisfied, adopting the same procedure as in [13,21], it follows that when the switching strategy $\sigma(t) = g(x(t))$ is applied, it results that

$$D^+v(x(t)) < -p(t)'p(t) + \mu q(t)'q(t) - z(t)'z(t) + \rho w(t)'w(t). \tag{33}$$

Integrating both sides from zero to infinity, remembering that $v(x(0)) = v(x(\infty)) = 0$, since the feasibility of the first diagonal block implies the system is asymptotically stable, we have

$$\int_0^\infty \left(z(\tau)'z(\tau) - \rho w(\tau)'w(\tau)\right) d\tau < \int_0^\infty \left(-p(\tau)'p(\tau) + \mu q(\tau)'q(\tau)\right) d\tau$$
$$< 0 \tag{34}$$

due to the fact that inequality (28) holds. Hence, $J_\infty(\sigma) < \rho$ is obtained.

The results we have just presented generalize the ones available in the literature, since even including $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance indexes, we are still able to cope with time-varying delay. To the best of the author's knowledge, in the context of switched linear systems, this is a problem that was not treated up to date. It is interesting to observe that, in both cases, there is a tradeoff between performance and time-varying delay range, because the conditions of Theorems 1 and 2 become more restrictive whenever $\mu = 1/h_m^2$ decreases. This aspect will be illustrated afterwards by means of a numerical example.

4.2. Output feedback control design

The main purpose of this section is to generalize the conditions of Theorems 1 and 2 to cope with dynamic output feedback switching control design. Connecting the delay-free full order switched linear filter (5) to the time-varying delay switched linear system (1)–(3), we obtain

$$\dot{\tilde{x}}(t) = \tilde{A}_\sigma \tilde{x}(t) + \tilde{A}_{d\sigma}\tilde{x}(t-h(t)) + \tilde{H}_\sigma w(t) \tag{35}$$

$$z(t) = \tilde{E}_\sigma \tilde{x}(t) + \tilde{E}_{d\sigma}\tilde{x}(t-h(t)) + \tilde{G}_\sigma w(t) \tag{36}$$

where $\tilde{x} = [x' \ \hat{x}']' \in \mathbb{R}^{2n_x}$ and

$$\tilde{A}_\sigma = \begin{bmatrix} A_\sigma & 0 \\ \hat{B}_\sigma C_\sigma & \hat{A}_\sigma \end{bmatrix}, \qquad \tilde{A}_{d\sigma} = \begin{bmatrix} A_{d\sigma} & 0 \\ \hat{B}_\sigma C_{d\sigma} & 0 \end{bmatrix}, \qquad \tilde{H}_\sigma = \begin{bmatrix} H_\sigma \\ \hat{B}_\sigma D_\sigma \end{bmatrix} \tag{37}$$

with output matrices

$$\tilde{E}_\sigma = \begin{bmatrix} E_\sigma & 0 \end{bmatrix}, \qquad \tilde{E}_{d\sigma} = \begin{bmatrix} E_{d\sigma} & 0 \end{bmatrix}, \qquad \tilde{G}_\sigma = G_\sigma. \tag{38}$$

Our goal is to determine $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs for the augmented system (35)–(36) by applying the conditions of Theorems 1 and 2 with a matrix $\tilde{P}_i > 0$ exhibiting the special structure

$$\tilde{P}_i = \begin{bmatrix} Y & V \\ V' & \hat{Y}_i \end{bmatrix} \tag{39}$$

where the four block matrices are square and have the same dimensions. This structure is crucial to make the switched system dependent only on the measured output through the state variable of the filter. This is verified by letting $\sigma(t) = g(\hat{x}(t))$ with

$$g(\hat{x}) = \arg\min_{i\in\mathbb{K}} \hat{x}'\hat{Y}_i\hat{x} = \arg\min_{i\in\mathbb{K}} \tilde{x}'\tilde{P}_i\tilde{x}. \tag{40}$$

The next theorem presents the conditions for the $\mathcal{H}_2$ output feedback control design of the system (35)–(36). In order to ease the notation, let us define matrices $C_{0i} = C_i + C_{di}$ for all $i \in \mathbb{K}$.

Theorem 3. *Consider system (1)–(3) with $G_i = 0$ for all $i \in \mathbb{K}$, set $\mu = 1/h_m^2$ and assume that there exist symmetric matrices $Y > 0$, $Q > 0$, $Z_i > 0$, $W_i > 0$ and R_{ij}, matrices N_i, L_i, and a Metzler matrix $\Pi \in \mathcal{M}$ satisfying the inequalities*

$$\begin{bmatrix} \mathrm{He}\{A_{0i}'Y + C_{0i}'L_i'\} & \bullet & \bullet & \bullet & \bullet \\ Z_iA_{0i} + N_i' & \mathrm{He}\{A_{0i}'Z_i\} + P_{Ri} & \bullet & \bullet & \bullet \\ A_{di}'Y + C_{di}'L_i' & A_{di}'Z_i & -\mu Q & \bullet & \bullet \\ QA_{0i} & QA_{0i} & QA_{di} & -Q & \bullet \\ E_{0i} & E_{0i} & E_{di} & 0 & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \tag{41}$$

where $P_{Ri} = \sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij}$ and

$$\begin{bmatrix} R_{ij} + Z_i & \bullet & \bullet \\ Z_j & Z_j & \bullet \\ Z_i & Z_j & Y \end{bmatrix} > 0, \quad j \neq i \in \mathbb{K} \tag{42}$$

$$\begin{bmatrix} W_i & \bullet & \bullet \\ YH_i + L_iD_i & Y & \bullet \\ Z_iH_i & Z_i & Z_i \end{bmatrix} > 0, \quad i \in \mathbb{K}. \tag{43}$$

Taking an arbitrary nonsingular matrix V and defining the filter (5) *with matrices*

$$\hat{A}_i = V^{-1}(N_i - YA_{0i} - L_iC_{0i})(Z_i - Y)^{-1}V \tag{44}$$

$$\hat{B}_i = V^{-1}L_i \tag{45}$$

the switching law $\sigma(t) = g(\hat{x}(t))$ with $g(\hat{x}) = \arg\min_{i\in\mathbb{K}} \hat{x}'V'(Y - Z_i)^{-1}V\hat{x}$ is globally asymptotically stabilizing for any time-varying delay such that $0 \le h(t) \le h_m$ and satisfies the guaranteed performance $J_2(\sigma) < \min_{i\in\mathbb{K}} \mathrm{Tr}(W_i)$.

Proof. In view of Theorem 1, for the augmented system (35)–(36), we have to determine a feasible solution $\tilde{P}_i$ to the matrix inequalities (24) where $\tilde{P}_i = \tilde{S}_i^{-1}$ has the structure (39) and

$$\tilde{S}_i = \begin{bmatrix} X_i & U_i \\ U_i' & \hat{X}_i \end{bmatrix}, \qquad \tilde{\Gamma} = \begin{bmatrix} Y & I \\ V' & 0 \end{bmatrix}, \qquad \tilde{Q} = \begin{bmatrix} Q & 0 \\ 0 & \hat{Q} \end{bmatrix}. \tag{46}$$

To this end, multiply inequality (24) to the left by $\mathrm{diag}\{\tilde{\Gamma}'\tilde{S}_i, I, \tilde{\Gamma}'\tilde{Q}^{-1}, I\}$ and to the right by its transpose and take into account the identities

$$\tilde{\Gamma}'\tilde{S}_i\tilde{A}_{0i}'\tilde{\Gamma} = \begin{bmatrix} A_{0i}'Y + C_{0i}'L_i' & A_{0i}' \\ M_i' & X_iA_{0i}' \end{bmatrix} \tag{47}$$

$$\tilde{A}_{di}'\tilde{\Gamma} = \begin{bmatrix} A_{di}'Y + C_{di}'L_i' & A_{di}' \\ 0 & 0 \end{bmatrix} \tag{48}$$

$$\tilde{\Gamma}'\tilde{Q}^{-1}\tilde{\Gamma} = \begin{bmatrix} YQ^{-1}Y + V\hat{Q}^{-1}V' & YQ^{-1} \\ Q^{-1}Y & Q^{-1} \end{bmatrix} \tag{49}$$

$$\tilde{E}_{0i}\tilde{S}_i\tilde{\Gamma} = \begin{bmatrix} E_{0i} & E_{0i}X_i \end{bmatrix} \tag{50}$$

$$\tilde{\Gamma}'\tilde{H}_i = \begin{bmatrix} YH_i + L_iD_i \\ H_i \end{bmatrix} \tag{51}$$

where we have used the change of variables $M_i = (YA_{0i} + L_iC_{0i})X_i + V\hat{A}_iU_i'$, $L_i = V\hat{B}_i$ from (44)–(45) with $N_i = M_iX_i^{-1}$, $Z_i = X_i^{-1}$ and the equality $U_i = (I - X_iY)V'^{-1}$ obtained from the fact that $\tilde{S}_i^{-1} = \tilde{P}_i$. Then, making $\hat{Q} \to 0$, it is possible to eliminate two rows and columns. Multiplying the result both sides by $\mathrm{diag}\{I, X_i^{-1}, I, Q, I\}$, setting $Z_i = X_i^{-1}$, $N_i = M_iX_i^{-1}$ and recalling that $P_{Ri} = \sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij}$ satisfies (42), we get inequalities (41). Moreover, applying the Schur Complement to $W_i > \tilde{H}_i'\tilde{P}_i\tilde{H}_i$ it is immediate to see that

$$\begin{bmatrix} W_i & \bullet \\ \tilde{P}_i\tilde{H}_i & \tilde{P}_i \end{bmatrix} > 0, \quad i \in \mathbb{K}. \tag{52}$$

Multiplying (52) to the left by $\mathrm{diag}\{I, \tilde{\Gamma}'\tilde{S}_i\}$, to the right by its transpose, and the result both sides by $\mathrm{diag}\{I, I, X_i^{-1}\}$, we obtain (43). Finally, observe that matrices $\hat{A}_i$ and $\hat{B}_i$ can be uniquely determined from the identities (44)–(45). Moreover, as presented in [21], the equality $\tilde{S}_i^{-1} = \tilde{P}_i$ implies $\hat{Y}_i = V'(Y - Z_i)^{-1}V$ and, therefore, the switching function follows from (40). The proof is concluded.

We have considered $\sigma(0) = i^*$ that has been obtained from the optimal solution of the problem that provides the minimum $\mathcal{H}_2$ guaranteed cost. Notice that the previous conditions allow the design of an output feedback switching strategy without any requirement on the time variation rate of the time-delay which is an important point to be explored in the context of sampled-data control system. Notice that conditions (41) are not simple to solve due to the nonconvexity inherited by the product of variables $\pi_{ji}R_{ij}$. However, as already mentioned, considering matrix $\Pi \in \mathcal{M}$ fixed, the conditions of Theorem 3 become LMIs which open the possibility to determine the minimum $\mathcal{H}_2$ guaranteed cost iteratively by an appropriate nonlinear programming method. The next theorem presents a similar result for $\mathcal{H}_\infty$ performance.

Theorem 4. *Consider system* (1)–(3), *set $\mu = 1/h_m^2$ and assume that there exist symmetric matrices $Y > 0$, $Q > 0$, $Z_i > 0$ and R_{ij}, matrices N_i, L_i, a Metzler matrix $\Pi \in \mathcal{M}$ and a scalar $\rho > 0$ satisfying the inequalities* (42) *and*

$$\begin{bmatrix} \mathrm{He}\{A_{0i}'Y + C_{0i}'L_i'\} & \bullet & \bullet & \bullet & \bullet & \bullet \\ Z_iA_{0i} + N_i' & \mathrm{He}\{A_{0i}'Z_i\} + P_{Ri} & \bullet & \bullet & \bullet & \bullet \\ A_{di}'Y + C_{di}'L_i' & A_{di}'Z_i & -\mu Q & \bullet & \bullet & \bullet \\ H_i'Y + D_i'L_i' & H_i'Z_i & 0 & -\rho I & \bullet & \bullet \\ QA_{0i} & QA_{0i} & QA_{di} & QH_i & -Q & \bullet \\ E_{0i} & E_{0i} & E_{di} & G_i & 0 & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \tag{53}$$

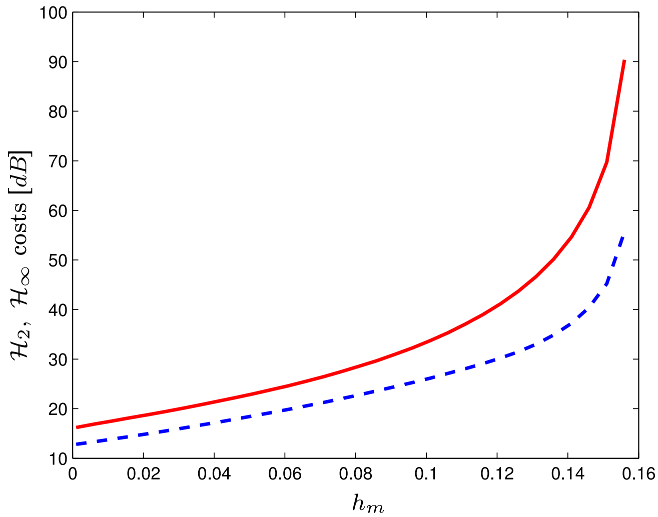

Fig. 2. $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed costs versus h_m.

with $P_{Ri} = \sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij}$. *Taking an arbitrary nonsingular matrix V and defining the filter* (5) *with matrices* (44)–(45)*, the switching law* $\sigma(t) = g(\hat{x}(t))$ *with* $g(\hat{x}) = \arg\min_{i\in\mathbb{K}} \hat{x}'V'(Y - Z_i)^{-1}V\hat{x}$ *is globally asymptotically stabilizing for any time-varying delay such that* $0 \le h(t) \le h_m$ *and satisfies the guaranteed performance* $J_\infty(\sigma) < \rho$.

Proof. The proof is similar to that of Theorem 3 and, for this reason, it is omitted.

About solvability, the same comments made after Theorem 3 apply as well. Notice that in both Theorems 3 and 4, it is not required that matrices A_{0i} for all $i \in \mathbb{K}$ be Hurwitz as a necessary condition for feasibility. Indeed, this fact can be easily checked in the second diagonal block of inequalities (41) and (53), by taking into account that matrices R_{ij} and, consequently, P_{Ri} are sign indefinite for all $i \neq j \in \mathbb{K} \times \mathbb{K}$. The only necessary condition for solvability (see the first diagonal block of the same inequalities) is the existence of gain matrices K_i rendering $A_{0i} + K_iC_{0i}$ quadratically stable for all $i \in \mathbb{K}$, meaning that they have to share the same Lyapunov matrix $Y > 0$. However, the fact that the gain matrices K_i, $\forall i \in \mathbb{K}$, are index dependent reduces the conservatism of this requirement present in both theorems.

4.3. Numerical example

Consider the time-varying delay switched linear system, inspired in [13], with state space realization (1)–(3) defined by matrices

$$A_1 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}, \qquad H_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \qquad H_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and outputs given by

$$C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad C_{d1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad C_{d2} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad E_{d1} = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \qquad E_{d2} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

with $G_1 = G_2 = 0$ and $D_1 = D_2 = 0$. Notice that neither subsystem is stable with zero delay, since A_{01} and A_{02} are not Hurwitz stable. We have applied the output feedback stability conditions in order to calculate, by gridding inside the box $15 \le \pi_{12} \le 45$ and $15 \le \pi_{21} \le 45$, the $\mathcal{H}_2$ (Theorem 3) and $\mathcal{H}_\infty$ (Theorem 4) guaranteed costs as a function of the delay upper bound h_m. As determined in [13] the maximum admissible bound for the delay is $h_m = 0.1560$. The plot of Fig. 2 presents the $\mathcal{H}_2$ guaranteed cost (in dashed line) and the $\mathcal{H}_\infty$ guaranteed cost (in continuous line) as functions of h_m, both expressed in decibels. The plot shows that there is a tradeoff between the amplitude of the delay bound and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance enhancement. It puts in evidence that a small increase in the time-delay may impose a severe loss of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performances.

5. Sampled-data control systems design

It is well known that a time-varying delay approach can be used in the analysis of sampled-data control systems under nonuniform sampling, see [26,33,34] and the references therein. Consider a switched delay-free linear system described by

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + H_{\sigma(t)}w(t) \tag{54}$$

$$z(t) = E_{\sigma(t)}x(t) + F_{\sigma(t)}u(t) + G_{\sigma(t)}w(t) \tag{55}$$

and the discrete-time instants $\{t_k\}_{k=0}^{\infty}$, such that $t_0 = 0$, $t_\infty = +\infty$ and $h_m \geq t_{k+1} - t_k \geq 0$, $\forall k \in \mathbb{N}$. Hence, h_m represents the maximum distance between two successive sampling instants. Moreover, assume that system (54)–(55) is controlled by means of a state-feedback control law of the form

$$u(t) = K_{\sigma(t)}x(t_k), \quad \forall t \in [t_k, t_{k+1}) \tag{56}$$

where it is clear that a sample-and-hold device has been included in the measurement channel that provides the state variable at each sampling instant t_k for all $k \in \mathbb{N}$. On the contrary, at this stage, we assume that the switching can be activated continuously, see [18]. This control asymmetry might occur in a supervised sampled-data control scheme where the channel between the plant and the controller delivering $u(t)$ has limited bandwidth, see [35], while the communication between the plant and the supervisor which selects the value of $\sigma(t)$ has no limitation.

As explained in Remark 2, the closed-loop system can be modeled as in (1)–(3), by defining the sawtooth time-varying delay $h(t) = t - t_k$, $\forall t \in [t_k, t_{k+1})$, and letting $A_{d\sigma} = B_\sigma K_\sigma$ and $E_{d\sigma} = F_\sigma K_\sigma$. As a consequence, the results of Section 4 can be applied to derive suitable switching strategies with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ guaranteed performances under any nonuniform sampling pattern bounded by h_m. In so doing, one can exploit the exact value of the gain $\|\Delta\|_\infty = 2h_m/\pi$ as discussed in Remark 2. The next corollary to Theorem 2 provides conditions for the design of the state feedback gains K_i and the matrices $P_i > 0$, such that the sampled-data control law (56) imposes to the closed-loop switched linear system a pre-specified $\mathcal{H}_\infty$ performance level.

Corollary 1. *Consider system* (54)–(55)*, set* $\mu = (\pi/(2h_m))^2$ *and assume that there exist symmetric matrices* $Q > 0$, $Z_i > 0$ *and* R_{ij}*, matrices* L_i*, a Metzler matrix* $\Pi \in \mathcal{M}$ *and a scalar* $\rho > 0$ *satisfying the Riccati–Metzler inequalities*

$$\begin{bmatrix} \mathrm{He}\{A_iZ_i + B_iL_i\} + \sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij} & \bullet & \bullet & \bullet & \bullet \\ L_i'B_i' & -2\mu Z_i + \mu Q & \bullet & \bullet & \bullet \\ H_i' & 0 & -\rho I & \bullet & \bullet \\ A_iZ_i + B_iL_i & B_iL_i & H_i & -Q & \bullet \\ E_iZ_i + F_iL_i & F_iL_i & G_i & 0 & -I \end{bmatrix} < 0, \quad i \in \mathbb{K} \tag{57}$$

and

$$\begin{bmatrix} R_{ij} + Z_i & \bullet \\ Z_i & Z_j \end{bmatrix} > 0, \quad i \neq j \in \mathbb{K}. \tag{58}$$

The min-type switching strategy $\sigma(t) = g(x(t))$ *with* $g(x) = \arg\min_{i\in\mathbb{K}} x'Z_i^{-1}x$ *and the switched sampled-data control* (56) *with the gains* $K_i = L_iZ_i^{-1}$, $i \in \mathbb{K}$, *and* $0 \leq t_{k+1} - t_k \leq h_m$ *are globally asymptotically stabilizing and satisfy the guaranteed cost* $J_\infty(\sigma) < \rho$.

Proof. Using the fact that $\Pi \in \mathcal{M}$ and denoting $P_i = Z_i^{-1} > 0$ from (58), we obtain

$$\begin{aligned} Z_i^{-1}\left(\sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij}\right)Z_i^{-1} &> \sum_{j\neq i\in\mathbb{K}} \pi_{ji}\left(Z_j^{-1} - Z_i^{-1}\right) \\ &= \left(\sum_{j\in\mathbb{K}} \pi_{ji}P_j\right) \end{aligned} \tag{59}$$

and, in addition, the inequality $(Z_i - Q)Q^{-1}(Z_i - Q) \geq 0$ yields $Z_iQ^{-1}Z_i \geq 2Z_i - Q$ which implies that $Z_i^{-1}(-2\mu Z_i + \mu Q)Z_i^{-1} \geq -\mu Q^{-1}$ holds for all $i \in \mathbb{K}$. The corollary follows from the fact that multiplying both sides of (57) by $\mathrm{diag}\{Z_i^{-1}, Z_i^{-1}, I, Q^{-1}, I\}$, using inequality (59) and setting $P_i = Z_i^{-1}$, $K_i = L_iZ_i^{-1}$, $A_{di} = B_iK_i$ and $E_{di} = F_iK_i$, it is seen that the conditions of Theorem 2 are fulfilled with $Q > 0$ replaced by its inverse.

The same algebraic manipulations yield a similar corollary to Theorem 1 providing, thus, an $\mathcal{H}_2$ guaranteed performance. Moreover, a particular and important case follows by setting $N = 1$, which allows us to design a sampled-data state feedback control for an LTI system of the form $u(t) = Kx(t_k)$, $\forall t \in [t_k, t_{k+1})$ whenever $h_m \geq t_{k+1} - t_k \geq 0$, for all $k \in \mathbb{N}$. It is important to stress that in this particular framework, the inequality (58) as well as the nonlinear term $\sum_{j\neq i\in\mathbb{K}} \pi_{ji}R_{ij}$ in the first diagonal element of inequality (57) have to be eliminated. The consequence is that the conditions to be solved reduce to LMIs. The next examples illustrate the theoretical results obtained so far.

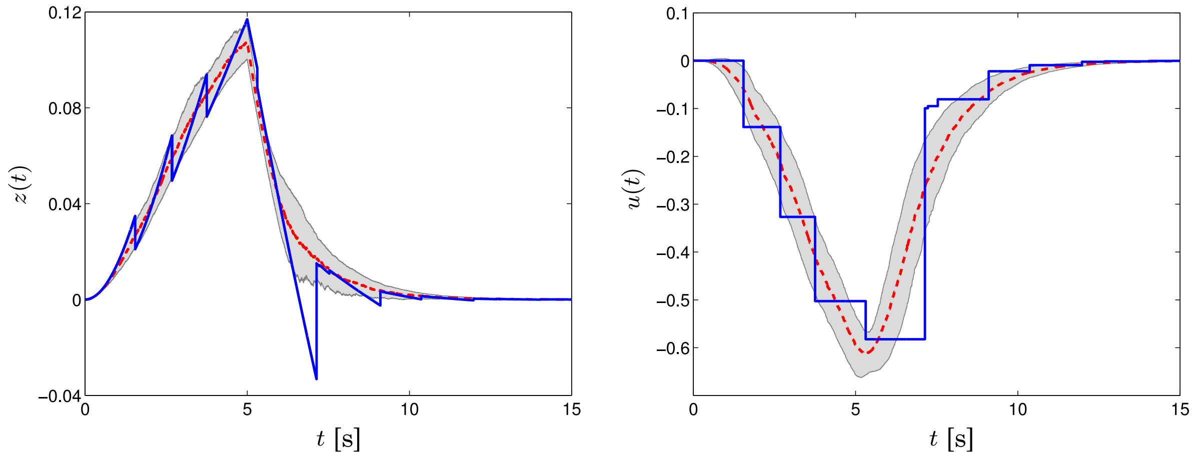

Fig. 3. Time simulation.

5.1. Numerical example—LTI system sampled-data control

Let us consider the Example 2 from [25]. The goal is to design a sampled-data state feedback control for a time invariant linear system with state space realization (54)–(55) and $N = 1$. To ease the notation, the matrix index $i \in \mathbb{K} = \{1\}$ has been dropped.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$
$$E = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0.1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \end{bmatrix}.$$

Since the design conditions provided in [25] are not LMIs with respect to the sampled-data state feedback gain, it is reported in the mentioned reference that with the fixed gain $K = -[3.75\ 11.50]$ the $\mathcal{H}_\infty$ guaranteed cost remains equal to $J_\infty = 0.0441$ for all $0 \le h_m \le 0.98$. From Corollary 1 we have determined the sampled-data state feedback gain $K = -[10^{-6}\ 3.9692]$ for which the $\mathcal{H}_\infty$ guaranteed cost remains equal to $J_\infty = 0.0441$ in the substantially larger interval $0 \le h_m \le 1.89$.

Fig. 3 presents the time simulation for this sampled-data state feedback control system evolving from zero initial condition and input $w(t) = \sin(\pi t/10)$ for all $0 \le t \le 5$ and zero elsewhere. At any t_k, the next sampling instant has been calculated from $t_{k+1} = t_k + h_k$ for all $k \in \mathbb{N}$ with h_k being a random variable uniformly distributed in the time interval [0, 1.89]. The controlled output $z(t)$ appears on the left hand side and the control signal $u(t)$ appears on the right hand side of the figure, where in both sides, dotted lines represent the time evolution of the mean inside a shadow area defined by a standard deviation calculated from 500 runs. Accordingly, in solid lines, the trajectories of a chosen run are shown. It can be verified that the designed sampled-data control is very effective to ensure stability and $\mathcal{H}_\infty$ guaranteed performance.

In this particular example, our results outperform the ones of [25] in two aspects. First, our conditions are LMIs that allow to calculate the gain K to obtain better performance. Second, with approximately the same guaranteed cost the maximum delay interval h_m can be enlarged. Indeed, from the same design conditions, we have obtained $K = -[10^{-6}\ 3.7482]$ which imposes $J_\infty = 0.0498$ for $h_m = 1.99$. In each case, the value of the first element of the gain matrix reached the numerical precision.

5.2. Numerical example—switched system sampled-data control

We have considered a switched linear system with $N = 2$ subsystems, both identical to the previous ones, but with different output matrices, given by $E_1 = [0\ 1]$ and $E_2 = [1\ 0]$, respectively. Using Corollary 1 with $\pi_{12} = 100$ and $\pi_{21} = 40$, we have determined for $h_m = 0.5$ the sampled-data state feedback gains

$$K_1 = -\begin{bmatrix} 1.8371 & 11.9205 \end{bmatrix}, \quad K_2 = -\begin{bmatrix} 9.4399 & 4.3374 \end{bmatrix}$$

and the switching function matrices

$$P_1 = \begin{bmatrix} 0.2248 & 0.1696 \\ 0.1696 & 0.3522 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.2239 & 0.1678 \\ 0.1678 & 0.3565 \end{bmatrix}$$

which impose to the closed-loop system, the $\mathcal{H}_\infty$ guaranteed cost $J_\infty = 0.0254$. Adopting the same reasoning as before, we have determined $J_\infty = 0.0441$ and $J_\infty = 0.0529$ for the LTI systems with output matrices E_1 and E_2, respectively. This is a consistent performance improvement achieved by the proposed switching strategy. Notice that, in this case, two different criteria are jointly considered in such a manner that the performance of the closed-loop system is strictly better than the ones associated with each isolated subsystem, see [22].

6. Conclusion

In this paper we have contributed to state and output feedback control design of time-delay switched linear systems. The delay class we dealt with is quite general being characterized as time-varying, piecewise continuous and bounded. Under this framework the switching function is designed in order to impose global asymptotic stability and a pre-specified level of $\mathcal{H}_2$ or $\mathcal{H}_\infty$ guaranteed performance. The design conditions have successfully been applied to the development of a design procedure for sampled-data state feedback control that outperforms the ones available in the literature to date. In particular, we have shown by means of a simple numerical example, how to determine a sampled-data state feedback control for a time invariant system by the adoption of several different and possibly conflicting criteria. It is important to notice that for this particular design problem, the conditions involving all variables are expressed in terms of LMIs so being simple to be solved. We believe that the theoretical results reported in this paper, mainly those based on an extended version of the Small Gain Theorem may be useful to face different control design problems as, for instance, those involving multi-objective optimization.

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