

Normalized solutions for Nonlinear Schrödinger systems on bounded domains

Benedetta Noris, Hugo Tavares, Gianmaria Verzini

Abstract

We analyze L^2 -normalized solutions of nonlinear Schrödinger systems of Gross-Pitaevskii type, on bounded domains, with homogeneous Dirichlet boundary conditions. We provide sufficient conditions for the existence of orbitally stable standing waves. Such waves correspond to global minimizers of the associated energy in the L^2 -subcritical and critical cases, and to local ones in the L^2 -supercritical case. Notably, our study includes also the Sobolev-critical case.

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1 Introduction

In this paper, we carry on the study of normalized solutions for Nonlinear Schrödinger (NLS) equations and systems, started in [32, 33].

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded smooth domain, $\mu_1, \mu_2 > 0$ and $\beta \in \mathbb{R}$. We consider the following system of coupled Gross-Pitaevskii equations

$$\begin{cases} i\partial_t \Psi_1 + \Delta \Psi_1 + \Psi_1(\mu_1 |\Psi_1|^{p-1} + \beta |\Psi_1|^{(p-3)/2} |\Psi_2|^{(p+1)/2}) = 0 \\ i\partial_t \Psi_2 + \Delta \Psi_2 + \Psi_2(\mu_2 |\Psi_2|^{p-1} + \beta |\Psi_2|^{(p-3)/2} |\Psi_1|^{(p+1)/2}) = 0 \end{cases} \quad (1.1)$$

with $\Psi_i : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{C}$ and, for every $t > 0$, $\Psi_i(t, \cdot) \in H_0^1(\Omega; \mathbb{C})$ ($i = 1, 2$). Throughout the paper we will distinguish several cases in the range

$$\begin{cases} p > 1 & N = 1, 2, \\ 1 < p \leq 2^* - 1 & N \geq 3, \end{cases}$$

where $2^* = 2N/(N-2)$ denotes the Sobolev critical exponent.

NLS systems with power-type nonlinearities appear in several different physical models from quantum mechanics, in particular when $p = 3$ or $p = 5$. Such models include Bose–Einstein condensation in multiple hyperfine spin states [40, 14] and the propagation of mutually incoherent waves packets in nonlinear optics [1]. Moreover, both the cases $\Omega = \mathbb{R}^N$ and Ω bounded are of interest [19, 20], the latter one appearing also as a limiting case of the system on \mathbb{R}^N with (confining) trapping potential.

System (1.1) preserves, at least formally, both the masses

$$Q(\Psi_i) = \int_{\Omega} |\Psi_i|^2 \quad i = 1, 2,$$

and the energy

$$\mathcal{E}(\Psi_1, \Psi_2) := \frac{1}{2} \int_{\Omega} |\nabla \Psi_1|^2 + |\nabla \Psi_2|^2 - \frac{1}{p+1} \int_{\Omega} \mu_1 |\Psi_1|^{p+1} + 2\beta |\Psi_1|^{(p+1)/2} |\Psi_2|^{(p+1)/2} + \mu_2 |\Psi_2|^{p+1}.$$

We look for standing wave solutions $(\Psi_1(t, x), \Psi_2(t, x)) = (e^{i\omega_1 t} u_1(x), e^{i\omega_2 t} u_2(x))$ of (1.1) such that $(u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2)$ and

$$Q(u_i) = \rho_i, \quad i = 1, 2,$$

for some $\rho_1, \rho_2 \geq 0$ prescribed a priori. Then, (u_1, u_2) is a normalized solution of an elliptic system; namely, there exist $(\omega_1, \omega_2) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \mu_1 u_1 |u_1|^{p-1} + \beta u_1 |u_1|^{(p-3)/2} |u_2|^{(p+1)/2} \\ -\Delta u_2 + \omega_2 u_2 = \mu_2 u_2 |u_2|^{p-1} + \beta u_2 |u_2|^{(p-3)/2} |u_1|^{(p+1)/2} \\ \int_{\Omega} u_i^2 = \rho_i, \quad i = 1, 2, \\ (u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2). \end{cases} \quad (1.2)$$

Solutions of (1.2) can be seen as critical points of \mathcal{E} , constrained to the Hilbert manifold

$$\mathcal{M} = \mathcal{M}_{\rho_1, \rho_2} := \left\{ (u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2) : \int_{\Omega} u_i^2 = \rho_i, \quad i = 1, 2 \right\}, \quad (1.3)$$

in which case the unknowns ω_i play the role of Lagrange multipliers. Our main aim is to provide conditions on p and (ρ_1, ρ_2) (and also on μ_1, μ_2, β) so that $\mathcal{E}|_{\mathcal{M}}$ admits minima, either global or local. We call such solutions *least energy solutions*, or *ground states*. Secondly, we consider the stability properties of such ground states, with respect to the evolution system (1.1).

An alternative, non equivalent point of view —which we do not treat here— is that of considering the parameters ω_i in (1.2) as fixed, without any normalizing condition on the functions u_i . This leads to an alternative definition of ground states, that of *least action solutions*: for a detailed discussion of this topic we refer the interested reader to the introduction of [32]. Starting from [27, 28, 29, 8, 2, 36, 16, 18], the literature dealing with this approach is vast and we do not even make an attempt to summarize it here. As a matter of fact, the results for non-normalized solutions cannot be directly extended to the normalized ones: among the other reasons, because in the latter case the ambient space is the Hilbert manifold \mathcal{M} (rather than a vector space).

Going back to normalized solutions, the simplest case one can face is that of a single NLS equation on \mathbb{R}^N , with a pure power nonlinearity. In such case, the problem can be completely solved by simple scaling arguments. This structure breaks down whenever one considers a system, as well as non-homogeneous nonlinearities, bounded domains or confining potentials. Apart when global minimization can be applied, see [35], as far as we know the first result in the literature is due to Jeanjean [24], for the superlinear, Sobolev-subcritical NLS single equation on \mathbb{R}^N with a non-homogeneous nonlinearity. In recent years, other papers appeared, dealing with the NLS equation or system, always in the Sobolev subcritical regime, either on \mathbb{R}^N [5, 21, 7, 9, 4, 22, 6] or on a bounded domain [31, 32, 33, 15, 34, 10]. These two settings are rather different in nature: each one requires a specific approach, and the results are in general not comparable. A key difference is that \mathbb{R}^N is invariant under translations and dilations, which has pros and cons: on the one hand, translations are responsible for a loss of compactness; on the other hand, in the Sobolev subcritical case, dilations can be used to produce variations and eventually construct natural constraints such as the so-called Pohozaev manifold. This tool is not available when working in bounded domains, and also the gain of compactness is lost when we face the Sobolev critical case.

However, a common key tool in the study of normalized solution is the Gagliardo–Nirenberg inequality (see (1.15) below), which can be used to estimate the non-quadratic part in \mathcal{E} in terms of the quadratic one. As a consequence, the exponent p in (1.2) can be classified according to the following four cases:

(H1) superlinear, L^2 -subcritical: $1 < p < 1 + 4/N$;

(H2) L^2 -critical: $p = 1 + 4/N$;

(H3) L^2 -supercritical, Sobolev-subcritical: $1 + 4/N < p < 1 + 4/(N - 2)^+ = 2^* - 1$;

(H4) Sobolev-critical: $p = 2^* - 1$, for $N \geq 3$.

In the first three cases, the study of the single equation

$$\begin{cases} -\Delta u_1 + \omega_1 u = \mu_1 u_1 |u_1|^{p-1} \\ \int_{\Omega} u_1^2 = \rho_1, \quad u_1 \in H_0^1(\Omega), \end{cases} \quad (1.4)$$

has been carried on in [32, 34]. Notice that (1.4) is a particular case of (1.2), when $\rho_2 = 0$, with associated energy $u_1 \mapsto \mathcal{E}(u_1, 0)$. Summarizing, it is known that

- (H1) implies that (1.4) has a solution, which is a global minimizer, for $\rho_1 \geq 0$;
- (H2) implies that (1.4) has a solution, which is a global minimizer, for $0 \leq \rho_1 < \rho_*(\Omega, N, p, \mu_1) < +\infty$;
- (H3) implies that (1.4) has a solution, which is a local minimizer, for $0 \leq \rho_1 < \rho_*(\Omega, N, p, \mu_1) < +\infty$, and a second one of mountain pass type.

Moreover, all the minimizers above are associated to orbitally stable solitary waves of the corresponding evolutive equation.

Up to our knowledge, the only paper dealing with the NLS system (1.2) (with both $\rho_i > 0$) is [33]. Among other things, in that paper we deal with the L^2 -supercritical, Sobolev-subcritical case (H3), obtaining the existence of orbitally stable solitary waves, in case both ρ_1, ρ_2 are sufficiently small and ρ_1/ρ_2 is uniformly bounded away from 0 and $+\infty$. This result is perturbative in nature, the existence following by a multi-parametric extension of a Ambrosetti-Prodi-type reduction [3] and the stability by the Grillakis-Shatah-Strauss stability theory [23]. The corresponding solutions are close to suitably normalized first eigenfunctions of the Dirichlet laplacian.

The aim of the present paper is twofold: on the one hand, in the cases (H1)-(H2)-(H3), we extend to systems the above described results obtained in [32, 34] for the single equation; on the other hand, we treat for the first time the Sobolev critical case (H4), obtaining results which are new also in the case of a single equation. Now we describe in details our results.

In what follows, we take

$$\Omega \subset \mathbb{R}^N \text{ a Lipschitz bounded domain } (N \geq 1), \quad \mu_1, \mu_2 > 0, \quad \beta \in \mathbb{R},$$

and denote by C_N the best constant appearing in the Gagliardo-Nirenberg inequality in the L^2 -critical case (see (1.17) ahead) while S_N is the best constant appearing in the Sobolev inequality (see (1.18)).

To start with, as we already mentioned, both (H1) and (H2) can be treated in a quite standard way by using the Gagliardo-Nirenberg inequality. Even though this result is somewhat expected, we provide it here since we could not find a precise reference to cite.

Theorem 1.1 (L^2 -subcritical and L^2 -critical cases: existence and stability). *Suppose that one of the following cases occurs*

- (i) $1 < p < 1 + 4/N$ and $\rho_1, \rho_2 \geq 0$;

(ii) $p = 1 + 4/N$ and $\rho_1, \rho_2 \geq 0$ are such that (see Fig. 1)

$$\max \left\{ \mu_1 \rho_1^{2/N}, \mu_2 \rho_2^{2/N}, \mu_1 \rho_1^{2/N} + \mu_2 \rho_2^{2/N} + \frac{NC_N}{N+2} \left((\beta^+)^2 - \mu_1 \mu_2 \right) (\rho_1 \rho_2)^{2/N} \right\} < \frac{N+2}{NC_N} \quad (1.5)$$

Then:

- a) the level $\inf_{\mathcal{M}} \mathcal{E}$ is achieved at some $(u_1, u_2) \in \mathcal{M}$, which is a non-negative solution of (1.2) for some $(\omega_1, \omega_2) \in \mathbb{R}^2$ ($u_i > 0$ when $\rho_i > 0$);
- b) the set of ground states

$$G = \left\{ (u_1, u_2) \in H_0^1(\Omega; \mathbb{C}^2) : (|u_1|, |u_2|) \in \mathcal{M}, \mathcal{E}(u_1, u_2) = \inf_{\mathcal{M}} \mathcal{E} \right\}$$

is conditionally orbitally stable.

Remark 1.2. We recall the definition of orbital stability in Section 4. Actually, notice that we prove *conditional* orbital stability, where the condition is that the solution of system (1.1), with Cauchy datum $(\psi_1, \psi_2) \in H_0^1(\Omega; \mathbb{C}^2)$, exists locally in time for a time interval which is uniform in $\|(\psi_1, \psi_2)\|_{H_0^1}$, and that Q and \mathcal{E} are preserved along the solutions. This holds true under further assumptions on μ_i, β and Ω , see also [33] and references therein; however, being the field so vast, even a rough summary of well-posedness for Schrödinger systems on bounded domains is far beyond the scopes of this paper.

Remark 1.3. Observe that, for $\beta \leq 0$, (1.5) reduces to

$$\max \left\{ \mu_1 \rho_1^{2/N}, \mu_2 \rho_2^{2/N} \right\} < \frac{N+2}{NC_N}.$$

which is independent from β .

Now we turn to the case in which either (H3) or (H4) hold true, i.e. when $1 + 4/N < p \leq 2^* - 1$ (no upper bound if $N = 1, 2$). Contrarily to the previous cases, in this one it is known that $\mathcal{E}|_{\mathcal{M}}$ is not bounded below, see Lemma 3.1 ahead. Nonetheless we will show that, even though no global minima can exist, local ones do, in case (ρ_1, ρ_2) belongs to some explicit set. To detect the existence of such minima we need to introduce some auxiliary problem and further notations. Let, for $\alpha \geq \lambda_1(\Omega)$ (the first Dirichlet eigenvalue of $-\Delta$ in Ω),

$$\begin{aligned} \mathcal{B}_\alpha &:= \left\{ (u_1, u_2) \in \mathcal{M} : \int_{\Omega} |\nabla u_1|^2 + |\nabla u_2|^2 \leq (\rho_1 + \rho_2)\alpha \right\}, \\ \mathcal{U}_\alpha &:= \left\{ (u_1, u_2) \in \mathcal{M} : \int_{\Omega} |\nabla u_1|^2 + |\nabla u_2|^2 = (\rho_1 + \rho_2)\alpha \right\} \end{aligned} \quad (1.6)$$

Notice that \mathcal{B}_α is not empty, since it contains at least a pair of suitably normalized first eigenfunctions, and that \mathcal{U}_α is the topological boundary of \mathcal{B}_α in \mathcal{M} . Moreover, let us define

$$c_\alpha := \inf_{\mathcal{B}_\alpha} \mathcal{E}, \quad \hat{c}_\alpha := \inf_{\mathcal{U}_\alpha} \mathcal{E}. \quad (1.7)$$

Being \mathcal{B}_α weakly closed in \mathcal{M} , in the Sobolev subcritical case the level c_α is achieved for any $\alpha \geq \lambda_1(\Omega)$, possibly on $\mathcal{U}_\alpha \subset \mathcal{B}_\alpha$. Therefore, in order to find a solution of (1.2), it is sufficient to find α such that $c_\alpha < \hat{c}_\alpha$ (and this will be our strategy).

On the contrary, in the Sobolev critical case, it is also an issue to prove that c_α is achieved: indeed, since $H_0^1(\Omega)$ is not compactly embedded in $L^{2^*}(\Omega)$, $\mathcal{E}|_{\mathcal{M}}$ is no longer weakly lower semicontinuous. To overcome this difficulty, in the spirit of the celebrated paper by Brezis and Nirenberg [12], we are able to recover the compactness of the minimizing sequences associated to c_α by imposing a bound on the masses ρ_1, ρ_2 and on α . More precisely, we have the following key result.

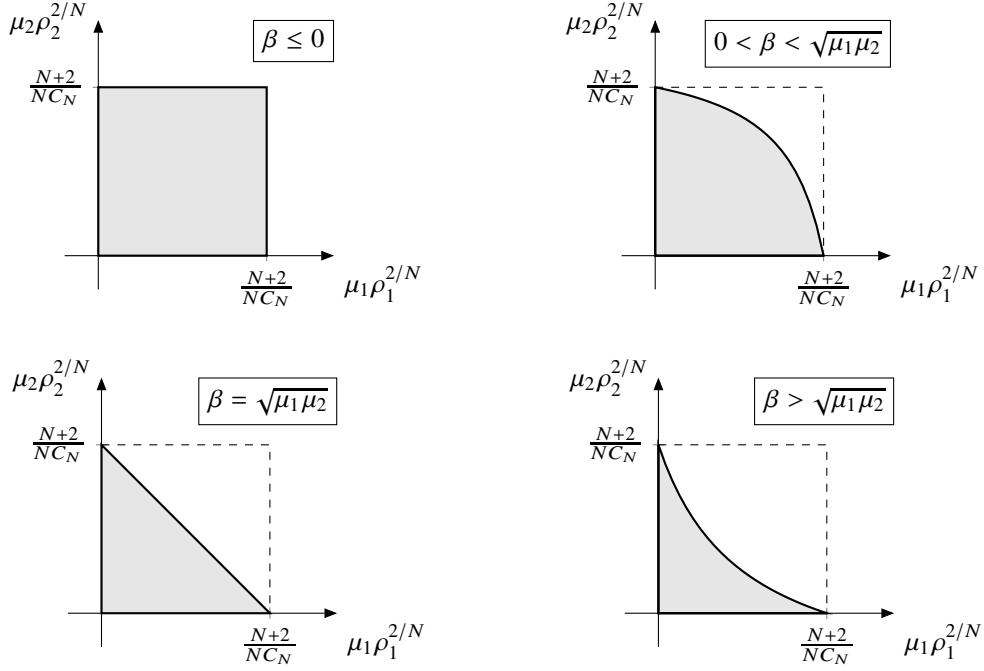


Figure 1: condition (1.5) read in terms of $(\mu_1 \rho_1^{2/N}, \mu_2 \rho_2^{2/N})$ (see also Remark 2.1 ahead).

Proposition 1.4. Consider $N \geq 3$ and $p = 2^* - 1$. Suppose that $\rho_1, \rho_2 \geq 0$ and $\alpha \geq \lambda_1(\Omega)$ are such that

$$(\rho_1 + \rho_2)(\alpha - \lambda_1(\Omega)) \leq \frac{1}{\Lambda^{(N-2)/2}}, \quad (1.8)$$

where

$$\Lambda := \frac{2S_N}{2^*} \max_{\{x^2+y^2=1\}} \left(\mu_1 |x|^{2^*} + \mu_2 |y|^{2^*} + 2\beta^+ |xy|^{2^*/2} \right). \quad (1.9)$$

Then any minimizing sequence associated to c_α is relatively compact in \mathcal{B}_α . In particular, c_α is achieved.

Based on the previous proposition, we introduce the following set of admissible masses

$$A := \left\{ (\rho_1, \rho_2) \in [0, \infty)^2 : \begin{array}{l} c_\alpha < \hat{c}_\alpha \text{ for some } \alpha \geq \lambda_1(\Omega), \\ \text{with } \alpha \text{ satisfying (1.8) if } p = 2^* - 1 \end{array} \right\} \cup \{(0, 0)\}. \quad (1.10)$$

Notice that, as a matter of fact, A depends on $\Omega, N, p, \mu_1, \mu_2$ and β . Moreover, if $(\rho_1, \rho_2) \in A$, then we can choose the local minimizer $(u_1, u_2) \in \mathcal{M}$ to be a non-negative solution of (1.2) for some $(\omega_1, \omega_2) \in \mathbb{R}^2$. Finally, we introduce the exponents a and r as

$$a = a(N, p) := \frac{N(p-1)}{4}, \quad r = r(N, p) := \frac{p+1}{4} - \frac{N(p-1)}{8}. \quad (1.11)$$

Notice that these two constants appear naturally in this context because (up to a suitable multiple) they enter in the Gagliardo-Nirenberg inequality. Observe also that

$$0 < a < 1 \text{ if } 1 < p < 1 + 4/N; \quad a = 1 \text{ if } p = 1 + 4/N; \quad a > 1 \text{ if } p = 1 + 4/N. \quad (1.12)$$

Theorem 1.5 (L^2 -supercritical cases: existence). *Let $1 + 4/N < p \leq 2^* - 1$. If A is defined as in (1.10), then*

A is star-shaped with respect to $(0, 0)$.

Moreover, there exists a positive constant $R = R(\Omega, N, p)$ such that if $\rho_1, \rho_2 \geq 0$ are such that

$$\left[\max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} + \beta^+ \rho_1^r \rho_2^r \right] \cdot (\rho_1 + \rho_2)^{a-1} \leq R(\Omega, N, p), \quad (1.13)$$

then $(\rho_1, \rho_2) \in A$. Here a and r are defined as in (1.11) and R is explicit (see (3.22) ahead).

Remark 1.6. When $N \geq 3$ and $p = 2^* - 1$, explicit calculations show that $a = N/(N-2)$, $r = 0$ and (1.13) rewrites as

$$\rho_1 + \rho_2 \leq \left[\frac{R(\Omega, N, 2^* - 1)}{\max\{\mu_1, \mu_2\} + \beta^+} \right]^{(N-2)/2}.$$

Remark 1.7. When $\beta \leq 0$, condition (1.13) is independent from β and reduces to:

$$\left[\max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} \right] \cdot (\rho_1 + \rho_2)^{a-1} \leq R(\Omega, N, p),$$

Remark 1.8. As we mentioned before, when $p > 1 + 4/N$ the functional \mathcal{E} is unbounded from below on \mathcal{M} . We deduce that, under the same assumptions of Theorem 1.5, \mathcal{E} has a mountain pass geometry on \mathcal{M} (see for instance [39, Thm. 4.2, Ch. II]). In a standard way, this implies that

if $1 + \frac{4}{N} < p < 2^* - 1$ and (ρ_1, ρ_2) satisfies (1.13), then $\mathcal{E}|_{\mathcal{M}}$ has two critical points: one local minimum and one mountain pass.

We cannot obtain the same result for $p = 2^* - 1$, since our compactness result in Proposition 1.4 holds only for minimizing sequences.

Remark 1.9. It is natural to expect that the set A is bounded in \mathbb{R}_+^2 . Actually, we know from [32] that this is the case, for the single equation, at least in the Sobolev subcritical case. The proof of this fact should follow by a careful blow-up analysis based on suitable pointwise a priori controls, along the lines of [32, Section 4], and will be the object of further investigation.

Since the solutions we found in the L^2 -supercritical cases are local minima of the energy, it is natural to expect that they correspond to orbitally stable solitary waves. The proof of this fact requires some modification of the standard arguments used for global minimizers. Notably, the lack of compactness of the embedding $H_0^1 \hookrightarrow L^2$ is an issue here, too. We have the following result.

Theorem 1.10 (L^2 -supercritical cases: stability). *Let $1 + 4/N < p \leq 2^* - 1$ and $(\rho_1, \rho_2) \in A$. Let $\bar{\alpha} \geq \lambda_1(\Omega)$ be such that*

$$c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}, \quad \text{and } \bar{\alpha} \text{ satisfies (1.8) if } p = 2^* - 1.$$

Then the set of local ground states

$$G_{\bar{\alpha}} := \{(u_1, u_2) \in H_0^1(\Omega; \mathbb{C}^2) : (|u_1|, |u_2|) \in \mathcal{B}_{\bar{\alpha}}, \mathcal{E}(u_1, u_2) = c_{\bar{\alpha}}\},$$

is (conditionally) orbitally stable.

As we noticed, in the Sobolev critical case our results are new also for the single equation. In particular, choosing $\rho_2 = 0$, $\rho_1 = \rho$, $\mu_1 = \mu$, we have the following direct consequence.

Theorem 1.11. *Let $\mu > 0$. If*

$$0 < \rho \leq \left[\frac{R(\Omega, N, 2^* - 1)}{\mu} \right]^{(N-2)/2}$$

then the problem

$$\begin{cases} -\Delta u + \omega u = \mu u |u|^{2^*-2} \\ \int_{\Omega} u^2 = \rho, \quad u \in H_0^1(\Omega) \end{cases}$$

admits a positive solution u , for a suitable $\omega \in (-\lambda_1(\Omega), 0)$, which is a local minimizer of the associated energy. Moreover, the corresponding set of local ground states is (conditionally) orbitally stable.

To conclude, we remark that all the assumptions in our results involve β^+ , the positive part of β . As a consequence, all our estimates are uniform in $\beta < 0$. To summarize, recalling Theorems 1.1 and 1.5 (see also and Remarks 1.3 and 1.7), we prove existence of solutions whenever $\beta < 0$ and (ρ_1, ρ_2) satisfy

$$\begin{cases} \rho_1, \rho_2 > 0 & \text{if } 1 < p < 1 + 4/N, \\ 0 < \mu_1 \rho_1^{\frac{2}{N}}, \mu_2 \rho_2^{\frac{2}{N}} < \frac{N+2}{NC_N} & \text{if } p = 1 + \frac{4}{N} \\ \max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} \cdot (\rho_1 + \rho_2)^{\alpha-1} \leq R(\Omega, N, p) & \text{if } 1 + \frac{4}{N} < p \leq 2^* - 1. \end{cases} \quad (1.14)$$

This allows to exploit results in [30, 38, 37] in order to perform a segregation analysis as $\beta \rightarrow -\infty$.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $\mu_1, \mu_2 > 0$, $\beta < 0$ and (ρ_1, ρ_2) be such that (1.14) holds. Let also $(u_{1,\beta}, u_{2,\beta})$ be a corresponding ground state of (1.2), with multipliers $(\omega_{1,\beta}, \omega_{2,\beta})$ and such that $u_{1,\beta}, u_{2,\beta} > 0$ in Ω . Then $\{(u_{1,\beta}, u_{2,\beta})\}_{\beta < 0}$ is uniformly bounded in $C^{0,\alpha}(\overline{\Omega})$ and, up to subsequences, $(u_{1,\beta}, u_{2,\beta}) \rightarrow (w^+, w^-)$ as $\beta \rightarrow -\infty$, in $C^{0,\alpha}(\overline{\Omega}) \cap H_0^1(\Omega)$, where $w \in C^{0,1}(\overline{\Omega})$ solves*

$$\begin{cases} -\Delta w + \omega_1 w^+ - \omega_2 w^- = \mu_1 (w^+)^p - \mu_2 (w^-)^p & \text{in } \Omega, \\ \int_{\Omega} (w^+)^2 = \rho_1, \int_{\Omega} (w^-)^2 = \rho_2, \quad w \in H_0^1(\Omega), \end{cases}$$

for $\omega_i := \lim_{\beta \rightarrow -\infty} \omega_{i,\beta}$.

The paper is structured as follows. In the next subsection we make some preliminary remarks and definitions which will be used in the text; in particular, we recall some facts about the Gagliardo-Nirenberg inequality and deduce some direct consequences.

Section 2 is devoted to the existence results under (H1)-(H2), i.e., to the proof of Theorem 1.1-a) as well as to a detailed explanation of condition (1.5) (which leads to Fig. 1).

The existence results under (H3)-(H4) (Theorem 1.5) are proved in Section 3. Therein, we provide lower estimates for \hat{c}_α (see Subsection 3.2), we prove a slightly more general version of Proposition 1.4 (Subsection 3.3), while in Subsection 3.4 we introduce an abstract criterium that guarantees that $(\rho_1, \rho_2) \in A$. Finally, Subsections 3.5 and 3.6 contain respectively the proofs of the qualitative properties of A and the deduction of condition (1.13).

Section 4 is concerned with the proof of the stability results, namely the proof of Theorems 1.1-b) and 1.10. Finally, Theorem 1.12 is proved in Section 5.

1.1 Notations and Preliminaries

Throughout the paper we denote by $\lambda_1(\Omega)$ the first eigenvalue of the Dirichlet Laplacian in Ω , and by φ_1 the corresponding first eigenfunction, which we assume normalized in $L^2(\Omega)$ and positive in Ω .

We use the following $L^q(\Omega)$ ($1 \leq q < \infty$) and $H_0^1(\Omega)$ -norms:

$$\|u\|_{L^q(\Omega)}^q := \int_{\Omega} |u|^q, \quad \|u\|_{H_0^1(\Omega)}^2 := \int_{\Omega} |\nabla u|^2.$$

Where there is no risk of confusion, we will denote $\|\cdot\|_{L^q(\Omega)}$ simply by $\|\cdot\|_q$.

The Gagliardo-Nirenberg inequality asserts that there exists a constant $C_{N,p}$ such that

$$\begin{aligned} \|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} &\leq C_{N,p} \|\nabla v\|_{L^2(\mathbb{R}^N)}^{N(p-1)/2} \|v\|_{L^2(\mathbb{R}^N)}^{p+1-N(p-1)/2} \\ &= C_{N,p} \|\nabla v\|_{L^2(\mathbb{R}^N)}^{2a} \|v\|_{L^2(\mathbb{R}^N)}^{4r}, \quad \forall v \in H^1(\mathbb{R}^N), \end{aligned} \quad (1.15)$$

where the exponents a and r were defined in (1.11). We remark that this inequality holds also in $H_0^1(\Omega)$, for any bounded domain Ω , with the same constant $C_{N,p}$. It is proved in [41] that

$$C_{N,p} := \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\|v\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}}{\|\nabla v\|_{L^2(\mathbb{R}^N)}^{2a} \|v\|_{L^2(\mathbb{R}^N)}^{4r}} = \frac{\|Z\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}}{\|\nabla Z\|_{L^2(\mathbb{R}^N)}^{2a} \|Z\|_{L^2(\mathbb{R}^N)}^{4r}},$$

where Z is, up to translations, the unique (see [25]) positive solution of

$$-\Delta Z + Z = Z^p, \quad Z \in H^1(\mathbb{R}^N). \quad (1.16)$$

In particular, the inequality on $H_0^1(\Omega)$ is strict unless v is trivial. In the special case $p = 1 + 4/N$ we denote

$$C_N := C_{N,1+4/N}, \quad (1.17)$$

while for $p = (N+2)/(N-2)$ and $N \geq 3$,

$$S_N := C_{N,(N+2)/(N-2)}. \quad (1.18)$$

Observe that S_N is just the best Sobolev constant of the embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2N/(N-2)}(\mathbb{R}^N)$:

$$\|v\|_{L^{2N/(N-2)}(\mathbb{R}^N)}^{2N/(N-2)} \leq S_N \|\nabla v\|_{L^2(\mathbb{R}^N)}^{2N/(N-2)}, \quad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

For $(u_1, u_2) \in \mathcal{M}$, defined as in (1.3), using the Hölder and Gagliardo-Nirenberg inequalities (on bounded domains) we have

$$\begin{aligned} &\int_{\Omega} \mu_1 |u_1|^{p+1} + 2\beta |u_1|^{(p+1)/2} |u_2|^{(p+1)/2} + \mu_2 |u_2|^{p+1} \\ &\leq \mu_1 \|u_1\|_{p+1}^{p+1} + \mu_2 \|u_2\|_{p+1}^{p+1} + 2\beta^+ \|u_1\|_{p+1}^{(p+1)/2} \|u_2\|_{p+1}^{(p+1)/2} \\ &< C_{N,p} \left(\mu_1 \rho_1^{2r} \|\nabla u_1\|_2^{2a} + \mu_2 \rho_2^{2r} \|\nabla u_2\|_2^{2a} + 2\beta^+ \rho_1^r \rho_2^r \|\nabla u_1\|_2^a \|\nabla u_2\|_2^a \right). \end{aligned} \quad (1.19)$$

where the exponents a and r are defined in (1.11). As a consequence we have, for $(u_1, u_2) \in \mathcal{M}$,

$$\begin{aligned} \mathcal{E}(u_1, u_2) &> \frac{1}{2} (\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2) \\ &\quad - \frac{C_{N,p}}{p+1} \left(\mu_1 \rho_1^{2r} \|\nabla u_1\|_2^{2a} + \mu_2 \rho_2^{2r} \|\nabla u_2\|_2^{2a} + 2\beta^+ \rho_1^r \rho_2^r \|\nabla u_1\|_2^a \|\nabla u_2\|_2^a \right). \end{aligned} \quad (1.20)$$

According to (1.12) and to the previous inequality, the L^2 -critical value $p = 1 + 4/N$ is the threshold for the coercivity of \mathcal{E} over \mathcal{M} , as we shall see more in detail in the following.

2 The L^2 -subcritical and L^2 -critical cases

In this section we deal with conditions (H1) and (H2), meaning that

$$1 < p \leq 1 + \frac{4}{N}.$$

Recall the definition of C_N in (1.17).

Proof of Theorem 1.1-a. Let us show in the two cases that \mathcal{E} restricted to \mathcal{M} is coercive. Then, by the direct method of the calculus of variations, $\inf_{\mathcal{M}} \mathcal{E}$ is achieved by a couple (u_1, u_2) (which belongs to \mathcal{M} because of the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$). By the Lagrange multipliers rule, (u_1, u_2) solves (1.2) for some $(\omega_1, \omega_2) \in \mathbb{R}^2$. By possibly taking $|u_i|$, we can suppose $u_i \geq 0$ and, if $\rho_1, \rho_2 > 0$, the maximum principle provides $u_i > 0$ (indeed, since Ω is Lipschitz, each u_i is continuous up to the boundary).

If (H1) holds, then $0 < N(p-1)/2 < 2$, so that, in (1.20), $a < 1$; we immediately deduce that \mathcal{E} restricted to \mathcal{M} is coercive for every $\rho_1, \rho_2 > 0$.

In case we have (H2), continuing from (1.20) and since in this case $a = 1, r = 1/N$, we have

$$\begin{aligned} \mathcal{E}(u_1, u_2) &> \frac{1}{2} (\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2) \\ &\quad - \frac{NC_N}{2(N+2)} \left(\mu_1 \rho_1^{2/N} \|\nabla u_1\|_2^2 + \mu_2 \rho_2^{2/N} \|\nabla u_2\|_2^2 + 2\beta^+ (\rho_1 \rho_2)^{1/N} \|\nabla u_1\|_2 \|\nabla u_2\|_2 \right) \\ &= \frac{1}{2} \|\nabla u_1\|_2^2 \left(1 - \frac{NC_N \mu_1 \rho_1^{2/N}}{N+2} \right) + \frac{1}{2} \|\nabla u_2\|_2^2 \left(1 - \frac{NC_N \mu_2 \rho_2^{2/N}}{N+2} \right) \\ &\quad - \frac{\beta^+ NC_N (\rho_1 \rho_2)^{1/N}}{N+2} \|\nabla u_1\|_2 \|\nabla u_2\|_2 \\ &= \frac{1}{2} \begin{bmatrix} \|\nabla u_1\|_2 & \|\nabla u_2\|_2 \end{bmatrix} \cdot A \cdot \begin{bmatrix} \|\nabla u_1\|_2 \\ \|\nabla u_2\|_2 \end{bmatrix}^T, \end{aligned} \quad (2.1)$$

where

$$A = \begin{bmatrix} 1 - \frac{NC_N \mu_1 \rho_1^{2/N}}{N+2} & -\frac{\beta^+ NC_N (\rho_1 \rho_2)^{1/N}}{N+2} \\ -\frac{\beta^+ NC_N (\rho_1 \rho_2)^{1/N}}{N+2} & 1 - \frac{NC_N \mu_2 \rho_2^{2/N}}{N+2} \end{bmatrix}$$

If A is positive definite then our result follows. Now A is positive definite if and only if the following inequalities are simultaneously satisfied

$$\begin{aligned} 1 - \frac{NC_N \mu_1 \rho_1^{2/N}}{N+2} &> 0, \quad 1 - \frac{NC_N \mu_2 \rho_2^{2/N}}{N+2} > 0, \\ \left(1 - \frac{NC_N \mu_1 \rho_1^{2/N}}{N+2} \right) \left(1 - \frac{NC_N \mu_2 \rho_2^{2/N}}{N+2} \right) - \left(\frac{NC_N}{N+2} \right)^2 (\beta^+)^2 (\rho_1 \rho_2)^{2/N} &> 0, \end{aligned}$$

that is to say (1.5) holds. \square

Remark 2.1. In this remark we interpret in the $(\mu_1 \rho_1^{2/N}, \mu_2 \rho_2^{2/N})$ -plane the condition (1.5) (see Fig. 1 for a visualization of this remark). Let $\bar{x} = \mu_1 \rho_1^{2/N}, \bar{y} = \mu_2 \rho_2^{2/N}$ so that (1.5) corresponds to $(\bar{x}, \bar{y}) \in C$, where

$$C := \left\{ (x, y) \in \mathbb{R}^2 : 0 < x, y < \frac{N+2}{NC_N} \text{ and } x + y + \frac{NC_N}{N+2} \frac{(\beta^+)^2 - \mu_1 \mu_2}{\mu_1 \mu_2} xy < \frac{N+2}{NC_N} \right\}$$

For $\beta \leq 0$, the condition reduces to (\bar{x}, \bar{y}) lying in the square

$$Q := \left\{ (x, y) \in \mathbb{R}^2 : 0 < x, y < \frac{N+2}{NC_N} \right\}$$

For $\beta = \sqrt{\mu_1 \mu_2}$, we have a half-square:

$$Q_1 := Q \cap \left\{ (x, y) \in \mathbb{R}^2 : x + y < \frac{N+2}{NC_N} \right\}.$$

For $\beta > 0$, $\beta \neq \sqrt{\mu_1 \mu_2}$, the curve

$$\left\{ (x, y) \in \mathbb{R}^2 : x + y + \frac{NC_N (\beta^+)^2 - \mu_1 \mu_2}{N+2} xy = \frac{N+2}{NC_N} \right\}$$

is an hyperbola which contains the points $(0, \frac{N+2}{NC_N})$, $(\frac{N+2}{NC_N}, 0)$. This hyperbola is the graph of

$$y = \left(\frac{N+2}{NC_N} - x \right) \left(1 + \frac{NC_N (\beta^+)^2 - \mu_1 \mu_2}{N+2} x \right)^{-1}$$

or, equivalently,

$$y = \frac{N+2}{NC_N} \left(-\frac{\mu_1 \mu_2}{(\beta^+)^2 - \mu_1 \mu_2} + \frac{N+2}{NC_N} \frac{(\beta^+)^2}{(\beta^+)^2 - \mu_1 \mu_2} \left(\frac{N+2}{NC_N} + \frac{(\beta^+)^2 - \mu_1 \mu_2}{\mu_1 \mu_2} x \right)^{-1} \right)$$

which has a vertical asymptote at $x = \frac{N+2}{NC_N} \frac{\mu_1 \mu_2}{\mu_1 \mu_2 - (\beta^+)^2}$. Thus, the set C always contains the sides of the square $Ox^+ \cap \bar{Q}$ and $Oy^+ \cap \bar{Q}$. When $0 < \beta < \sqrt{\mu_1 \mu_2}$ it contains Q_1 , and when $\beta > \sqrt{\mu_1 \mu_2}$ it is contained in Q_1 .

Remark 2.2. When $\beta \leq 0$, the condition reads as

$$\mu_1 \rho_1^{2/N}, \mu_2 \rho_2^{2/N} < \frac{N+2}{NC_N}.$$

Going to [32, p. 1833] we see that, as a consequence of Pohozaev's identity:

$$\frac{N+2}{NC_N} = \|Z\|_2^{2/N},$$

with Z defined in (1.16). Therefore the condition is equivalent to

$$\rho_1 < \|Z\|_2 \mu_1^{-N/2}, \quad \rho_2 < \|Z\|_2 \mu_2^{-N/2}.$$

This is consistent with the results in [19, 32], which correspond to the case $\beta = 0$ in (1.2).

3 The L^2 -supercritical and Sobolev-subcritical case. The Sobolev-critical case

3.1 Preliminaries

Assume from now on that p satisfies either (H3) or (H4), that is $p > 1 + 4/N$, with $p \leq (N+2)/(N-2)$ if $N \geq 3$. Along this section we do not make any distinction between the Sobolev-critical and the Sobolev-subcritical cases, unless otherwise specified.

In Proposition 1.1 we proved that \mathcal{E} restricted to \mathcal{M} is coercive for any $\rho_1, \rho_2 > 0$ under (H1) or for (ρ_1, ρ_2) satisfying (1.5) under (H2). Thus, solutions were found as global minimizers of $\mathcal{E}|_{\mathcal{M}}$. In the L^2 -supercritical case $p > 1 + 4/N$ the previous approach cannot work, since \mathcal{E} restricted to \mathcal{M} is not coercive for every value of (ρ_1, ρ_2) , as we show in the following lemma. Notice that this was already suggested by equation (1.20), since now $a = N(p-1)/4 > 1$.

Lemma 3.1. *Let $p > 1 + 4/N$. Then there exists $(U_{1,k}, U_{2,k}) \in \mathcal{M}$, with nonnegative components, such that, as $k \rightarrow \infty$,*

$$\|(U_{1,k}, U_{2,k})\|_{H_0^1(\Omega)} \rightarrow +\infty \quad \text{and} \quad \mathcal{E}(U_{1,k}, U_{2,k}) \rightarrow -\infty.$$

Proof. Let $\phi \in C_c^\infty(B_1)$ with $\phi > 0$ in B_1 and $\int_{B_1} \phi^2 = 1$, and $x_1, x_2 \in \Omega$ such that $x_1 \neq x_2$. For $k \in \mathbb{N}$ and $i = 1, 2$, we define

$$U_{i,k}(x) = \rho_i^{1/2} k^{N/2} \phi(k(x - x_i)), \quad x \in \Omega.$$

For k sufficiently large we have $\text{supp}(U_{i,k}) \subset B_{1/k}(x_i) \subset \Omega$, $i = 1, 2$, and $\text{supp}(U_{1,k}) \cap \text{supp}(U_{2,k}) = \emptyset$. Furthermore,

$$\int_{\Omega} U_{i,k}^2 = \rho_i \int_{B_1} \phi^2 = \rho_i,$$

so that $(U_{1,k}, U_{2,k}) \in \mathcal{M}$ for k sufficiently large. We compute

$$\|(U_{1,k}, U_{2,k})\|_{H_0^1(\Omega)} = k\sqrt{\rho_1 + \rho_2} \|\nabla \phi\|_{L^2(B_1)} \rightarrow +\infty$$

as $k \rightarrow +\infty$ and

$$\mathcal{E}(U_{1,k}, U_{2,k}) = k^2 \frac{\rho_1 + \rho_2}{2} \|\nabla \phi\|_{L^2(B_1)}^2 - k^{2a} \frac{\mu_1 \rho_1^{(p+1)/2} + \mu_2 \rho_2^{(p+1)/2}}{p+1} \|\phi\|_{L^{p+1}(B_1)}^{p+1} \rightarrow -\infty$$

as $k \rightarrow +\infty$, since $a > 1$. □

3.2 A basic estimate on \hat{c}_α

In order to prove the existence of a solution of (1.2) under (H3) or (H4) for certain values of ρ_1, ρ_2 , we use a different approach than the one used in Section 2. Recall that, for $\alpha \geq \lambda_1(\Omega)$, \mathcal{B}_α and \mathcal{U}_α are defined in (1.6), while c_α and \hat{c}_α are as in (1.7). Observe that $\mathcal{B}_\alpha \neq \emptyset$, since it contains at least $(\sqrt{\rho_1}\varphi_1, \sqrt{\rho_2}\varphi_1)$. Moreover

$$c_{\lambda_1(\Omega)} = \hat{c}_{\lambda_1(\Omega)} = \mathcal{E}(\sqrt{\rho_1}\varphi_1, \sqrt{\rho_2}\varphi_1).$$

Recalling (1.20) and using the identification $x = \|\nabla u_1\|_2$, $y = \|\nabla u_2\|_2$, we end up studying the function $\varphi : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ defined by

$$\Phi(x, y) = \frac{1}{2}(x^2 + y^2) - \frac{C_{N,p}}{p+1} \left(\mu_1 \rho_1^{2r} x^{2a} + \mu_2 \rho_2^{2r} y^{2a} + 2\beta^+ \rho_1^r \rho_2^r x^a y^a \right)$$

where now $a > 1$. Indeed, by (1.20) we obtain that

$$\mathcal{E}(u_1, u_2) \geq \Phi(\|\nabla u_1\|_2, \|\nabla u_2\|_2) \quad \text{for every } (u_1, u_2) \in \mathcal{M}. \quad (3.1)$$

In particular, this allows to estimate \hat{c}_α from below. To do that, let us define the following subsets of \mathbb{R}^2 :

$$U_\alpha = \{(x, y) \in \mathbb{R}_+^2 : x^2 + y^2 = (\rho_1 + \rho_2)\alpha\}, \quad V_\alpha = U_\alpha \cap \left\{ x \geq \sqrt{\rho_1 \lambda_1(\Omega)}, y \geq \sqrt{\rho_2 \lambda_1(\Omega)} \right\}.$$

The set U_α is obtained from \mathcal{U}_α through the identification $x = \|\nabla u_1\|_2$, $y = \|\nabla u_2\|_2$. The set V_α is motivated by the fact that, for $(u_1, u_2) \in \mathcal{M}$, $\|\nabla u_1\|^2 \geq \rho_1 \lambda_1(\Omega)$ and $\|\nabla u_2\|^2 \geq \rho_2 \lambda_1(\Omega)$. With this notation, using (3.1), we obtain

$$\hat{c}_\alpha \geq \min_{(x,y) \in V_\alpha} \Phi(x,y) = \frac{1}{2} \alpha - \frac{C_{N,p}}{p+1} \max_{(x,y) \in V_\alpha} \left(\mu_1 \rho_1^{2r} x^{2a} + \mu_2 \rho_2^{2r} y^{2a} + 2\beta^+ \rho_1^r \rho_2^r x^a y^a \right). \quad (3.2)$$

Now, due to the limitations in the definition of V_α , the last maximum can not be written explicitly in terms of α (except for a few particular cases). For this reason, we prefer the more rough estimate in which V_α is replaced with U_α . This allows more readable results, without modifying the qualitative structure of the estimates.

Lemma 3.2. *Let*

$$\Lambda = \Lambda(\rho_1, \rho_2) := \frac{2C_{N,p}}{p+1} \max_{t \in [0, \pi/2]} \left(\mu_1 \rho_1^{2r} \cos^{2a} t + \mu_2 \rho_2^{2r} \sin^{2a} t + 2\beta^+ \rho_1^r \rho_2^r \cos^a t \sin^a t \right). \quad (3.3)$$

Then, for every $\alpha > \lambda_1(\Omega)$,

$$\hat{c}_\alpha > \frac{1}{2} \left((\rho_1 + \rho_2) \alpha - \Lambda(\rho_1, \rho_2) (\rho_1 + \rho_2)^a \alpha^a \right).$$

Proof. Since

$$V_\alpha \subset U_\alpha = \left\{ (\cos t, \sin t) \sqrt{(\rho_1 + \rho_2) \alpha} : t \in [0, \pi/2] \right\},$$

the lemma follows by continuing the estimate in (3.2).

Remark 3.3. Notice that Λ depends on μ_1, μ_2, β , and also on p and N (via a and r). On the other hand, in case $N \geq 3$ and $p = 2^* - 1$, we have that $a = 2^*/2$, $r = 0$ and Λ does not depend on ρ_1, ρ_2 , and actually its definition coincides with that given in (1.9). Then we have, for any $(v_1, v_2) \in H_0^1(\Omega; \mathbb{R}^2)$,

$$\frac{2S_N}{2^*} \left(\mu_1 \|\nabla v_1\|_2^{2^*} + 2\beta^+ \|\nabla v_1\|_2^{2^*/2} \|\nabla v_2\|_2^{2^*/2} + \mu_2 \|\nabla v_2\|_2^{2^*} \right) \leq \Lambda \left(\|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right)^{2^*/2} \quad (3.4)$$

(recall the definition of $S_N = C_{N, 2^*-1}$ given in (1.18)). To see this, we notice that for any (v_1, v_2) one can find $t \in [0, \pi/2]$ such that

$$\|\nabla v_1\|_2 = \left(\|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right)^{1/2} \cos t, \quad \|\nabla v_2\|_2 = \left(\|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right)^{1/2} \sin t,$$

and we substitute in (3.3).

3.3 The level c_α is achieved

As we mentioned, we will look for local minimizers of \mathcal{E} on \mathcal{B}_α , hence at level c_α , for suitable values of α . A first necessary step is to prove that c_α is achieved (possibly on \mathcal{U}_α , the topological boundary of \mathcal{B}_α). This is easily obtained, for every $\alpha \geq \lambda_1(\Omega)$, in the Sobolev subcritical case: indeed, in such situation, \mathcal{B}_α is weakly compact and \mathcal{E} weakly lower semicontinuous. On the other hand, if $N \geq 3$ and $p = 2^* - 1$, \mathcal{E} is no longer weakly lower semicontinuous. In this situation, inspired by the celebrated paper by Brezis and Nirenberg [12], we can recover compactness of the minimizing sequences by imposing a smallness condition on the masses ρ_1, ρ_2 and on α , as stated in Proposition 1.4. Actually, here we will prove a slightly more general result, considering sequences in which also the masses are not fixed; this will be useful when dealing with stability issues.

In the following, recall that Λ has been introduced in (3.3) (or, equivalently, in (1.9)) and that, in the Sobolev critical case, it does not depend on ρ_1, ρ_2 .

Proposition 3.4. Let $\alpha > \lambda_1(\Omega)$, $\rho_1, \rho_2 > 0$ satisfy

$$(\rho_1 + \rho_2)(\alpha - \lambda_1(\Omega)) < \frac{1}{\Lambda^{(N-2)/2}},$$

and let $(u_{1,n}, u_{2,n})_n$ be such that

$$\begin{cases} \|u_{i,n}\|_2^2 = \rho_i + o(1) & \text{for } i = 1, 2, \\ \|\nabla u_{1,n}\|_2^2 + \|\nabla u_{2,n}\|_2^2 \leq \alpha(\rho_1 + \rho_2) + o(1) \\ c_\alpha \leq \mathcal{E}(u_{1,n}, u_{2,n}) \leq c_\alpha + o(1) \end{cases} \quad (3.5)$$

as $n \rightarrow \infty$. Then, up to subsequences,

$$(u_{1,n}, u_{2,n}) \rightarrow (\bar{u}_1, \bar{u}_2), \quad \text{strongly in } H_0^1(\Omega).$$

In particular, c_α is achieved.

Proof of Proposition 1.4. The proposition follows as a particular case of Proposition 3.4, when in (3.5) both $\|u_{i,n}\|_2^2 = \rho_i$, $i = 1, 2$, and $\|\nabla u_{1,n}\|_2^2 + \|\nabla u_{2,n}\|_2^2 \leq \alpha(\rho_1 + \rho_2)$. \square

Proof of Proposition 3.4. By assumption there exists $(\bar{u}_1, \bar{u}_2) \in H_0^1(\Omega; \mathbb{R}^2)$ such that, up to subsequences,

$$\begin{cases} \|\bar{u}_i\|_2^2 = \rho_i & \text{for } i = 1, 2 \\ u_{i,n} \rightharpoonup \bar{u}_i & H_0^1(\Omega)\text{-weak for } i = 1, 2 \\ \|\nabla \bar{u}_i\|_2^2 \leq \liminf_{n \rightarrow \infty} \|\nabla u_{i,n}\|_2^2 & \text{for } i = 1, 2. \end{cases}$$

Notice that (\bar{u}_1, \bar{u}_2) is admissible for the minimization problem c_α , whence

$$\mathcal{E}(\bar{u}_1, \bar{u}_2) \geq c_\alpha. \quad (3.6)$$

Let $v_{i,n} = u_{i,n} - \bar{u}_i$ and notice that, for $i = 1, 2$,

$$v_{i,n} \rightarrow 0 \quad \text{weakly both } H_0^1(\Omega) \text{ and } L^{2^*}(\Omega), \quad v_{i,n} \rightarrow 0 \quad \text{strongly } L^2(\Omega),$$

where $2^* = 2N/(N-2)$.

Notice that the strong convergence of a subsequence of $(u_{1,n}, u_{2,n})$ is equivalent to the statement:

$$\text{there exists a subsequence } (v_{1,n_k}, v_{2,n_k}) \text{ such that } \|\nabla v_{1,n_k}\|_2^2 + \|\nabla v_{2,n_k}\|_2^2 \rightarrow 0. \quad (3.7)$$

In such a case, by continuity of the Sobolev embeddings, we have that $c_\alpha = \mathcal{E}(\bar{u}_1, \bar{u}_2)$. Since a minimizing sequence for c_α exists, and it satisfies (3.5), we deduce that c_α is achieved.

To conclude the proof, suppose by contradiction that (3.7) does not hold, so that

$$\|\nabla v_{1,n}\|_2^2 + \|\nabla v_{2,n}\|_2^2 \geq K > 0 \quad \text{eventually.} \quad (3.8)$$

We can write

$$\begin{aligned} \mathcal{E}(u_{1,n}, u_{2,n}) &= \frac{1}{2} \left(\|\nabla(\bar{u}_1 + v_{1,n})\|_2^2 + \|\nabla(\bar{u}_2 + v_{2,n})\|_2^2 \right) \\ &\quad - \frac{1}{2^*} \left(\mu_1 \|\bar{u}_1 + v_{1,n}\|_{2^*}^{2^*} + 2\beta \|(\bar{u}_1 + v_{1,n})(\bar{u}_2 + v_{2,n})\|_{2^*/2}^{2^*/2} + \mu_2 \|\bar{u}_2 + v_{2,n}\|_{2^*}^{2^*} \right). \end{aligned} \quad (3.9)$$

Notice that, by weak convergence, we have, for $i = 1, 2$,

$$\|\nabla(\bar{u}_i + v_{i,n})\|_2^2 = \|\nabla\bar{u}_i\|_2^2 + \|\nabla v_{i,n}\|_2^2 + o(1) \quad \text{as } n \rightarrow \infty. \quad (3.10)$$

In order to estimate the remaining terms of (3.9), we recall the following Lemma by Brezis and Lieb [11]: given $1 \leq q < \infty$, if $\{f_n\}_n \subset L^q(\Omega)$ is a sequence bounded in $L^q(\Omega)$, such that $f_n \rightarrow f$ almost everywhere, then

$$\|f_n\|_q^q = \|f\|_q^q + \|f_n - f\|_q^q + o(1) \quad \text{as } n \rightarrow \infty. \quad (3.11)$$

We apply (3.11) first with $f_n = u_{i,n} = \bar{u}_i + v_{i,n}$ and $q = 2^*$ to get

$$\|\bar{u}_i + v_{i,n}\|_{2^*}^{2^*} = \|\bar{u}_i\|_{2^*}^{2^*} + \|v_{i,n}\|_{2^*}^{2^*} + o(1) \quad \text{as } n \rightarrow \infty, \quad (3.12)$$

then we apply it with $f_n = (\bar{u}_1 + v_{1,n})(\bar{u}_2 + v_{2,n})$ and $q = 2^*/2$ to obtain

$$\|(\bar{u}_1 + v_{1,n})(\bar{u}_2 + v_{2,n})\|_{2^*/2}^{2^*/2} = \|\bar{u}_1\bar{u}_2\|_{2^*/2}^{2^*/2} + \|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n} + v_{1,n}v_{2,n}\|_{2^*/2}^{2^*/2} + o(1) \quad (3.13)$$

as $n \rightarrow \infty$. In order to estimate the second term in the right hand side of (3.13), we shall need two inequalities. For every $q > 1$ and for every $a, b \in \mathbb{R}$ it holds

$$|a + b|^q \leq 2^{q-1}(|a|^q + |b|^q); \quad (3.14)$$

$$|a + b|^q - |a|^q \leq C(|a|^{q-1}|b| + |b|^q), \quad (3.15)$$

for a constant C not depending on a and b . By (3.14), we have

$$\|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}\|_{2^*/2}^{2^*/2} \leq 2^{(2^*-2)/2} \left(\|\bar{u}_1 v_{2,n}\|_{2^*/2}^{2^*/2} + \|\bar{u}_2 v_{1,n}\|_{2^*/2}^{2^*/2} \right) = o(1) \quad \text{as } n \rightarrow \infty, \quad (3.16)$$

because $|v_{i,n}|^{2^*/2} \rightharpoonup 0$ in $L^2(\Omega)$ -weak as $n \rightarrow +\infty$, for $i = 1, 2$. Then using (3.15), the Hölder inequality and (3.16), we compute

$$\begin{aligned} & \left| \|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n} + v_{1,n}v_{2,n}\|_{2^*/2}^{2^*/2} - \|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}\|_{2^*/2}^{2^*/2} \right| \\ & \leq C \int_{\Omega} \left(|v_{1,n}v_{2,n}|^{(2^*-2)/2} |\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}| + |\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}|^{2^*/2} \right) dx \\ & \leq C \|v_{1,n}\|_{2^*}^{(2^*-2)/2} \|v_{2,n}\|_{2^*}^{(2^*-2)/2} \|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}\|_{2^*/2} + \|\bar{u}_1 v_{2,n} + \bar{u}_2 v_{1,n}\|_{2^*/2}^{2^*/2} = o(1) \end{aligned}$$

as $n \rightarrow +\infty$. This last estimate, replaced into (3.13), provides

$$\|(\bar{u}_1 + v_{1,n})(\bar{u}_2 + v_{2,n})\|_{2^*/2}^{2^*/2} = \|\bar{u}_1\bar{u}_2\|_{2^*/2}^{2^*/2} + \|v_{1,n}v_{2,n}\|_{2^*/2}^{2^*/2} + o(1) \quad (3.17)$$

as $n \rightarrow +\infty$.

By replacing (3.10), (3.12) and (3.17) into (3.9), we see that

$$\mathcal{E}(u_{1,n}, u_{2,n}) = \mathcal{E}(\bar{u}_1, \bar{u}_2) + \mathcal{E}(v_{1,n}, v_{2,n}) + o(1) \quad \text{as } n \rightarrow \infty.$$

The last expression, together with (3.5) and (3.6), implies

$$\mathcal{E}(v_{1,n}, v_{2,n}) \leq o(1) \quad \text{as } n \rightarrow \infty,$$

whence, using (1.19) (with $r = 0$, $a = 2^*/2$ and $C_{N,2^*-1} = S_N$) and (3.4),

$$\begin{aligned} \|\nabla v_{1,n}\|_2^2 + \|\nabla v_{2,n}\|_2^2 & \leq \frac{2S_N}{2^*} \left(\mu_1 \|\nabla v_{1,n}\|_2^{2^*} + 2\beta^+ \|\nabla v_{1,n}\|_2^{2^*/2} \|\nabla v_{2,n}\|_2^{2^*/2} + \mu_2 \|\nabla v_{2,n}\|_2^{2^*} \right) + o(1) \\ & \leq \Lambda \left(\|\nabla v_1\|_2^2 + \|\nabla v_2\|_2^2 \right)^{2^*/2} + o(1). \end{aligned}$$

Now, we can use (3.8) to rewrite the last inequality as

$$\left(\|\nabla v_{1,n}\|_2^2 + \|\nabla v_{2,n}\|_2^2 \right)^{(2^*-2)/2} \geq \frac{1}{\Lambda} + o(1).$$

We combine the previous inequality with (3.10) to obtain

$$\begin{aligned} \left(\frac{1}{\Lambda} + o(1) \right)^{(N-2)/2} &\leq \|\nabla v_{1,n}\|_2^2 + \|\nabla v_{2,n}\|_2^2 = \|\nabla u_{1,n}\|_2^2 + \|\nabla u_{2,n}\|_2^2 - (\|\nabla \bar{u}_1\|_2^2 + \|\nabla \bar{u}_2\|_2^2) + o(1) \\ &\leq (\rho_1 + \rho_2)\alpha - \lambda_1(\Omega)(\rho_1 + \rho_2) + o(1), \end{aligned}$$

as $n \rightarrow +\infty$, which contradicts the assumption.

3.4 Existence of ground states

This section is devoted to prove the following result.

Theorem 3.5. *Let $\rho_1, \rho_2 \geq 0$ be such that*

$$\Lambda(\rho_1, \rho_2) \cdot (\rho_1 + \rho_2)^{a-1} \leq \frac{(a-1)^{a-1}}{a^a} \lambda_j(\Omega)^{-(a-1)}, \quad (3.18)$$

where $j = 1$ if $\beta \geq -\sqrt{\mu_1\mu_2}$, $j = 2$ otherwise. Let $\bar{\alpha} = \frac{a}{a-1} \lambda_i(\Omega)$.

Then $c_{\bar{\alpha}}$ is achieved by $(\bar{u}_1, \bar{u}_2) \in \mathcal{B}_{\bar{\alpha}} \setminus \mathcal{U}_{\bar{\alpha}}$ such that $\mathcal{E}(\bar{u}_1, \bar{u}_2) = c_{\bar{\alpha}}$, which implies that (\bar{u}_1, \bar{u}_2) is a local minimum of $\mathcal{E}|_{\mathcal{M}}$, corresponding to a positive solution of (1.2) for some $(\omega_1, \omega_2) \in \mathbb{R}^2$. Equivalently, $(\rho_1, \rho_2) \in A$ as defined in (1.10).

First of all, we state a sufficient condition for the above theorem to hold, in terms of \hat{c}_α .

Lemma 3.6. *Let us assume that $\rho_1, \rho_2 > 0$ are such that, for some α_1, α_2 ,*

$$\lambda_1(\Omega) \leq \alpha_1 < \alpha_2 \quad \text{and} \quad \hat{c}_{\alpha_1} < \hat{c}_{\alpha_2};$$

furthermore, in the Sobolev critical case $N \geq 3$, $p = 2^* - 1$, let us also assume that

$$\alpha_2 < \lambda_1(\Omega) + \frac{\Lambda^{-(N-2)/2}}{\rho_1 + \rho_2}.$$

Then $c_{\alpha_2} < \hat{c}_{\alpha_2}$, and c_{α_2} is achieved by a positive solution of (1.2).

Proof. Firstly, c_{α_2} is achieved by some $(\bar{u}_1, \bar{u}_2) \in \mathcal{B}_{\alpha_2}$: as we already observed, this is trivial in the Sobolev subcritical case, while in the critical one it follows by Proposition 3.4. Next we observe that

$$c_{\alpha_2} = \min \{ \hat{c}_\alpha : \lambda_1(\Omega) \leq \alpha \leq \alpha_2 \} \leq \hat{c}_{\alpha_1} < \hat{c}_{\alpha_2}.$$

We deduce that $(\bar{u}_1, \bar{u}_2) \in \mathcal{B}_{\alpha_2} \setminus \mathcal{U}_{\alpha_2}$, and the lemma follows. \square

Let us denote by $\lambda_2(\Omega)$ the second eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and by φ_2 a corresponding eigenfunction.

Lemma 3.7. *We have*

$$\left(\sqrt{\rho_1} \varphi_1, \sqrt{\rho_2} \varphi_1 \right) \in \mathcal{U}_{\lambda_1(\Omega)}, \quad \left(\sqrt{\rho_1} \frac{\varphi_2^+}{\|\varphi_2^+\|_2}, \sqrt{\rho_2} \frac{\varphi_2^-}{\|\varphi_2^-\|_2} \right) \in \mathcal{U}_{\lambda_2(\Omega)}.$$

In particular

$$\hat{c}_{\lambda_j(\Omega)} \leq \frac{\rho_1 + \rho_2}{2} \lambda_j(\Omega), \quad \text{for } j = \begin{cases} 1 & \text{if } \beta \geq -\sqrt{\mu_1\mu_2}, \\ 2 & \text{if } \beta < -\sqrt{\mu_1\mu_2}. \end{cases}$$

Proof. The first assertion is direct. Then

$$\begin{aligned}\hat{c}_{\lambda_1(\Omega)} &\leq \mathcal{E}(\sqrt{\rho_1}\varphi_1, \sqrt{\rho_2}\varphi_1) \\ &= \frac{\rho_1 + \rho_2}{2}\lambda_1(\Omega) - \frac{\mu_1\rho_1^{p+1} + 2\beta(\rho_1\rho_2)^{\frac{p+1}{2}} + \mu_2\rho_2^{p+1}}{p+1} \int_{\Omega} \varphi_1^4 \leq \frac{\rho_1 + \rho_2}{2}\lambda_1(\Omega),\end{aligned}$$

since $\beta \geq -\sqrt{\mu_1\mu_2}$ implies that $\mu_1\rho_1^{p+1} + 2\beta(\rho_1\rho_2)^{\frac{p+1}{2}} + \mu_2\rho_2^{p+1} \geq 0$ for every $\rho_1, \rho_2 > 0$.

On the other hand,

$$\begin{aligned}\hat{c}_{\lambda_2(\Omega)} &\leq \mathcal{E}\left(\sqrt{\rho_1}\frac{\varphi_2^+}{\|\varphi_2^+\|_{L^2(\Omega)}}, \sqrt{\rho_2}\frac{\varphi_2^-}{\|\varphi_2^-\|_{L^2(\Omega)}}\right) \\ &\leq \frac{\rho_1 + \rho_2}{2}\lambda_2(\Omega) - \frac{\mu_1\rho_1^{p+1}}{(p+1)\|\varphi_2^+\|_{L^2(\Omega)}^{p+1}} \int_{\Omega} (\varphi_2^+)^{p+1} dx - \frac{\mu_2\rho_2^{p+1}}{(p+1)\|\varphi_2^-\|_{L^2(\Omega)}^{p+1}} \int_{\Omega} (\varphi_2^-)^{p+1} dx \\ &\leq \frac{\rho_1 + \rho_2}{2}\lambda_2(\Omega).\end{aligned}$$

Proof of Theorem 3.5. In the following let $j = 1$ if $\beta \geq -\sqrt{\mu_1\mu_2}$ and $j = 2$ otherwise. In view of the application of Lemma 3.6, our aim is to find $\bar{\alpha} > \lambda_j(\Omega)$ such that

$$\hat{c}_{\lambda_j(\Omega)} < \hat{c}_{\bar{\alpha}}. \quad (3.19)$$

To start with, we look for a sufficient condition implying (3.19). Using Lemmas 3.2 and 3.7, it is sufficient to find $\bar{\alpha} > \lambda_j(\Omega)$ such that

$$\frac{\rho_1 + \rho_2}{2}\lambda_j(\Omega) \leq \frac{1}{2}(\bar{\alpha}(\rho_1 + \rho_2) - \Lambda(\rho_1, \rho_2)(\rho_1 + \rho_2)^a \bar{\alpha}^a)$$

(by Lemma 3.2, the right hand side is strictly less than $\hat{c}_{\bar{\alpha}}$). Equivalently,

$$\Lambda(\rho_1, \rho_2)(\rho_1 + \rho_2)^{a-1} \leq \frac{\bar{\alpha} - \lambda_j(\Omega)}{\bar{\alpha}^a}. \quad (3.20)$$

By a direct computation, recalling that $a > 1$, the best possible choice for the right hand side is

$$\max_{\alpha \geq \lambda_j(\Omega)} \frac{\alpha - \lambda_j(\Omega)}{\alpha^a} = \frac{(a-1)^{a-1}}{a^a} \lambda_j(\Omega)^{-(a-1)}, \quad \text{achieved by } \bar{\alpha} = \frac{a}{a-1} \lambda_j(\Omega).$$

This choice of $\bar{\alpha}$ is possible, since it makes (3.20) equivalent to (3.18), the assumption of the theorem. Furthermore, it is clear that $\bar{\alpha} > \lambda_j(\Omega)$. Then, in order to apply Lemma 3.6 and conclude the proof, we only need to check that, in case $N \geq 3$ and $p = 2^* - 1$, the additional assumption

$$\bar{\alpha} < \lambda_1(\Omega) + \frac{1}{\Lambda^{(N-2)/2}(\rho_1 + \rho_2)} \quad (3.21)$$

holds true. This is straightforward since, being $a = N/(N-2)$, relation (3.20) provides

$$\Lambda(\rho_1 + \rho_2)^{2/(N-2)} \leq \frac{\bar{\alpha} - \lambda_j(\Omega)}{\bar{\alpha}^{N/(N-2)}} < \frac{\bar{\alpha} - \lambda_1(\Omega)}{(\bar{\alpha} - \lambda_1(\Omega))^{N/(N-2)}} = \frac{1}{(\bar{\alpha} - \lambda_1(\Omega))^{2/(N-2)}},$$

which is equivalent to (3.21).

Remark 3.8. The solution (\bar{u}_1, \bar{u}_2) does not coincide with $(\sqrt{\rho_1}\varphi_1, \sqrt{\rho_2}\varphi_1)$, unless $\beta = -\mu_1 = -\mu_2$. Indeed, this last pair solves (1.2) if and only if, for every $i = 1, 2$,

$$\begin{aligned}(\lambda_1(\Omega) + \omega_i)\sqrt{\rho_i}\varphi_1 &= (\mu_i + \beta)\rho_i^{\frac{p}{2}}\varphi_1^p \iff \lambda_1(\Omega) + \omega_i = (\mu_i + \beta)\rho_i^{\frac{p-1}{2}}\varphi_1^{p-1} \\ &\iff \lambda_1(\Omega) = -\omega_i, \beta = -\mu_1 = -\mu_2.\end{aligned}$$

3.5 The set A is star-shaped

This section is devoted to the proof of the following result.

Proposition 3.9. *Let A be defined as in (1.10). Then A is star-shaped with respect to $(0, 0)$.*

We follow a strategy inspired by [33] but, since such paper does not extend directly to the Sobolev critical case, we provide here a self-contained argument. In this section it is convenient to make explicit the dependence of some quantities with respect to ρ_1, ρ_2 : in view of this, we write $c_\alpha(\rho_1, \rho_2)$, $\hat{c}_\alpha(\rho_1, \rho_2)$, $\mathcal{B}_\alpha(\rho_1, \rho_2)$, $\mathcal{U}_\alpha(\rho_1, \rho_2)$. For shorter notation, we define

$$F(u_1, u_2) := \int_{\Omega} \mu_1 |u_1|^{p+1} + 2\beta |u_1|^{(p+1)/2} |u_2|^{(p+1)/2} + \mu_2 |u_2|^{p+1}$$

and we introduce the optimization problem

$$M_\alpha(\rho_1, \rho_2) := \sup_{\mathcal{U}_\alpha(\rho_1, \rho_2)} F$$

(a quantity thoroughly investigated in [33]). Notice that

$$\hat{c}_\alpha(\rho_1, \rho_2) = \frac{1}{2} \alpha (\rho_1 + \rho_2) - \frac{1}{p+1} M_\alpha(\rho_1, \rho_2),$$

and that $\hat{c}_\alpha(\rho_1, \rho_2)$ is achieved at $(u_1, u_2) \in \mathcal{U}_\alpha(\rho_1, \rho_2)$ if, and only if, $M_\alpha(\rho_1, \rho_2)$ is achieved at the same pair.

Fix, if any, $(\rho_1, \rho_2) \in A \setminus \{(0, 0)\}$. By definition of A , there exist $\alpha > \lambda_1(\Omega)$ and $(\bar{u}_1, \bar{u}_2) \in \mathcal{B}_\alpha$, a solution of (1.2), such that $\mathcal{E}(\bar{u}_1, \bar{u}_2) = c_\alpha < \hat{c}_\alpha$, and α satisfies (1.8) in case $N \geq 3$, $p = 2^* - 1$. Notice that the assumption $c_\alpha < \hat{c}_\alpha$ implies $\int_{\Omega} |\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 < (\rho_1 + \rho_2) \alpha$, so that

$$\bar{\alpha} := \frac{1}{\rho_1 + \rho_2} \int_{\Omega} |\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 < \alpha.$$

As a consequence, $(\bar{u}_1, \bar{u}_2) \in \mathcal{U}_{\bar{\alpha}}(\rho_1, \rho_2)$ achieves $\hat{c}_{\bar{\alpha}} = c_\alpha$.

Lemma 3.10. *If $s > 0$ then $(s\bar{u}_1, s\bar{u}_2) \in \mathcal{U}_{\bar{\alpha}}(s^2\rho_1, s^2\rho_2)$ achieves*

$$\hat{c}_{\bar{\alpha}}(s^2\rho_1, s^2\rho_2) = \frac{s^2}{2} \bar{\alpha} (\rho_1 + \rho_2) - \frac{s^{p+1}}{p+1} F(\bar{u}_1, \bar{u}_2).$$

Proof. This follows by noticing that

$$(u_1, u_2) \in \mathcal{U}_{\bar{\alpha}}(\rho_1, \rho_2) \iff (su_1, su_2) \in \mathcal{U}_{\bar{\alpha}}(s^2\rho_1, s^2\rho_2),$$

with

$$F(su_1, su_2) = s^{p+1} F(u_1, u_2).$$

Then $M_\alpha(s^2\rho_1, s^2\rho_2) = s^{p+1} M_\alpha(\rho_1, \rho_2)$ and the lemma follows.

Lemma 3.11. *Let $s \in (0, 1)$ and $(v_1, v_2) \in H_0^1(\Omega, \mathbb{R}^2)$ be such that*

$$\int_{\Omega} \bar{u}_1 v_1 = \int_{\Omega} \bar{u}_2 v_2 = 0, \quad \int_{\Omega} \nabla \bar{u}_1 \cdot \nabla v_1 + \nabla \bar{u}_2 \cdot \nabla v_2 < 0.$$

Let, for $|t|$ small,

$$(U_1(t), U_2(t)) := \left(s\sqrt{\rho_1} \frac{\bar{u}_1 + tv_1}{\|\bar{u}_1 + tv_1\|_2}, s\sqrt{\rho_2} \frac{\bar{u}_2 + tv_2}{\|\bar{u}_2 + tv_2\|_2} \right).$$

Then $(U_1(t), U_2(t)) \in \mathcal{M}_{s^2\rho_1, s^2\rho_2}$ for every t and

$$\left. \frac{d}{dt} \|(U_1(t), U_2(t))\|_{H_0^1(\Omega)}^2 \right|_{t=0} < 0, \quad \left. \frac{d}{dt} \mathcal{E}(U_1(t), U_2(t)) \right|_{t=0} < 0.$$

Proof. By direct inspection we have that $(U_1(t), U_2(t)) \in \mathcal{M}_{s^2\rho_1, s^2\rho_2}$ for every t , and that

$$\left. \frac{d}{dt}(U_1(t), U_2(t)) \right|_{t=0} = (sv_1, sv_2).$$

Then

$$\left. \frac{d}{dt} \|(U_1(t), U_2(t))\|_{H_0^1(\Omega)}^2 \right|_{t=0} = 2s^2 \int_{\Omega} \nabla \bar{u}_1 \cdot \nabla v_1 + \nabla \bar{u}_2 \cdot \nabla v_2 < 0$$

by assumption. On the other hand, recalling that (\bar{u}_1, \bar{u}_2) solves (1.2), we have that

$$\left. \frac{d}{dt} F(U_1(t), U_2(t)) \right|_{t=0} = s^{p+1} F'(\bar{u}_1, \bar{u}_2)[v_1, v_2] = s^{p+1} \int_{\Omega} \nabla \bar{u}_1 \cdot \nabla v_1 + \nabla \bar{u}_2 \cdot \nabla v_2$$

and

$$\left. \frac{d}{dt} \mathcal{E}(U_1(t), U_2(t)) \right|_{t=0} = (s^2 - s^{p+1}) \int_{\Omega} \nabla \bar{u}_1 \cdot \nabla v_1 + \nabla \bar{u}_2 \cdot \nabla v_2 < 0,$$

as $0 < s < 1$ and $p > 1$. \square

End of the proof of Proposition 3.9. With the notation of Lemma 3.11, being $(U_1(0), U_2(0)) = (s\bar{u}_1, s\bar{u}_2) \in \mathcal{U}_{\bar{\alpha}}(s^2\rho_1, s^2\rho_2)$, there exist positive and small constants ε, τ such that

$$(U_1(\tau), U_2(\tau)) \in \mathcal{U}_{\bar{\alpha}-\varepsilon}(s^2\rho_1, s^2\rho_2)$$

and

$$\hat{c}_{\bar{\alpha}-\varepsilon}(s^2\rho_1, s^2\rho_2) \leq \mathcal{E}(U_1(\tau), U_2(\tau)) < \mathcal{E}(U_1(0), U_2(0)) = \hat{c}_{\bar{\alpha}}(s^2\rho_1, s^2\rho_2)$$

(the last equality following by Lemma 3.10). Then we can apply Lemma 3.6, with $\alpha_1 = \bar{\alpha} - \varepsilon$ and $\alpha_2 = \bar{\alpha}$, obtaining that $(s^2\rho_1, s^2\rho_2) \in A$. Since this holds true for any $s \in (0, 1)$, the proposition follows. \square

Remark 3.12. In [33, Theorem 1.1], we show that, in the Sobolev subcritical case, $M_{\alpha}(\rho_1, \rho_2)$ is achieved and that an associated maximum point (u_1, u_2) satisfies

$$\begin{cases} -\Delta u_1 + \omega_1 u_1 = \gamma(\mu_1 u_1 |u_1|^{p-1} + \beta u_1 |u_1|^{(p-3)/2} |u_2|^{(p+1)/2}) \\ -\Delta u_2 + \omega_2 u_2 = \gamma(\mu_2 u_2 |u_2|^{p-1} + \beta u_2 |u_2|^{(p-3)/2} |u_1|^{(p+1)/2}) \\ \int_{\Omega} u_i^2 = \rho_i, \quad i = 1, 2, \quad (u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2). \end{cases}$$

for a suitable Lagrange multiplier $\gamma > 0$. Then, repeating the above arguments, we obtain that

$$\gamma > 1 \text{ for some } \alpha \implies \hat{c}_{\alpha-\varepsilon} < \hat{c}_{\alpha},$$

so that Theorem 3.5 applies.

3.6 Explicit estimates for Λ

At this point, the main assumption in Theorem 3.5 is written in terms of the function $\Lambda(\rho_1, \rho_2)$ defined in (3.3), which we recall here for the reader's convenience

$$\Lambda(\rho_1, \rho_2) = \frac{2C_{N,p}}{p+1} \max_{t \in [0, \pi/2]} \left(\mu_1 \rho_1^{2r} \cos^{2a} t + \mu_2 \rho_2^{2r} \sin^{2a} t + 2\beta^+ \rho_1^r \rho_2^r \cos^a t \sin^a t \right),$$

where

$$a = \frac{N(p-1)}{4} \in \left(1, \frac{N}{N-2} \right], \quad r = \frac{p+1}{4} - \frac{N(p-1)}{8} \in \left[0, \frac{1}{N} \right).$$

It is clear that Λ is a r -homogeneous polynomial of (ρ_1, ρ_2) , but its explicit expression can be derived only in few particular cases. The aim of this subsection is to prove Theorem 1.5 by showing that condition (1.13) in Theorem 1.5, with $R = R(\Omega, N, p)$ defined as

$$R(\Omega, N, p) = \frac{p+1}{2C_{N,p}} \frac{(a-1)^{a-1}}{a^a} \lambda_j(\Omega)^{-(a-1)}, \quad (3.22)$$

implies assumption (3.18) in Theorem 3.5. Here, as usual, $j = 1$ if $\beta \geq -\sqrt{\mu_1 \mu_2}$ and $j = 2$ otherwise (or simply $j = 2$ for any β , in case one wants to avoid this weak dependence of R on β). The advantage of (1.13) with respect to (3.18) is that of being more explicit; furthermore, the two conditions coincide in the case $\beta \leq 0$, as proved in Remark 3.13 below.

End of the proof of Theorem 1.5. We proved that A is star-shaped with respect to the origin in Proposition 3.9. We estimate Λ from above noticing that, as $a > 1$, we have

$$\begin{aligned} \Lambda(\rho_1, \rho_2) &\leq \Lambda'(\rho_1, \rho_2) := \frac{2C_{N,p}}{p+1} \max_{t \in [0, \pi/2]} \left(\mu_1 \rho_1^{2r} \cos^2 t + \mu_2 \rho_2^{2r} \sin^2 t + 2\beta^+ \rho_1^r \rho_2^r \cos t \sin t \right) \\ &= \frac{C_{N,p}}{p+1} \max_{t \in [0, \pi/2]} \left[\left(\mu_1 \rho_1^{2r} + \mu_2 \rho_2^{2r} \right) + \left(\mu_1 \rho_1^{2r} - \mu_2 \rho_2^{2r} \right) \cos 2t + 2\beta^+ \rho_1^r \rho_2^r \sin 2t \right] \\ &= \frac{C_{N,p}}{p+1} \max_{x^2+y^2=1} \left[\left(\mu_1 \rho_1^{2r} + \mu_2 \rho_2^{2r} \right) + \left(\mu_1 \rho_1^{2r} - \mu_2 \rho_2^{2r} \right) x + 2\beta^+ \rho_1^r \rho_2^r y \right]. \end{aligned}$$

Next, explicit computations show that

$$\begin{aligned} \Lambda'(\rho_1, \rho_2) &= \frac{C_{N,p}}{p+1} \left(\mu_1 \rho_1^{2r} + \mu_2 \rho_2^{2r} + \sqrt{(\mu_1 \rho_1^{2r} - \mu_2 \rho_2^{2r})^2 + 4(\beta^+ \rho_1^r \rho_2^r)^2} \right) \\ &\leq \frac{C_{N,p}}{p+1} \left(\mu_1 \rho_1^{2r} + \mu_2 \rho_2^{2r} + |\mu_1 \rho_1^{2r} - \mu_2 \rho_2^{2r}| + 2\beta^+ \rho_1^r \rho_2^r \right) \\ &= \frac{2C_{N,p}}{p+1} \left[\max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} + \beta^+ \rho_1^r \rho_2^r \right]. \end{aligned}$$

Therefore assumption (1.13), with R as in (3.22), implies (3.18), so that we can apply Theorem 3.5 to conclude. \square

Remark 3.13. Relations (1.13) and (3.18) coincide for $\beta \leq 0$. Indeed, in such case, the maximum in the definition of Λ' is achieved when either $t = 0$ or $t = \pi/2$, and for such values the estimate is an equality:

$$\Lambda(\rho_1, \rho_2) = \Lambda'(\rho_1, \rho_2) = \frac{2C_{N,p}}{p+1} \max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} \quad \text{for } \beta \leq 0.$$

4 Orbital stability of the set of ground states

This section is devoted to the proof of the stability statements, namely of Theorems 1.1-b) and 1.10. Our aim is to prove the stability of the sets G and $G_{\bar{\alpha}}$ defined in the statements. Actually, in the case of global minimizers (i.e. $1 < p \leq 1 + 4/N$), the stability follows from the conservation of the energy and masses, and from the compactness of any minimizing sequence, see e.g. [13, Remark 8.3.9]. The case of local minimizers (i.e. $1 + 4/N < p \leq 2^* - 1$), however, requires an adaptation of such arguments, in particular in the Sobolev critical case.

In order to provide a unified proof for all the cases, we first observe that global minimizers are also local ones. Recall the definitions of B_{α} , \mathcal{U}_{α} in (1.6), and those of c_{α} , \hat{c}_{α} in (1.7). For $p \leq 1 + 4/N$ and

(ρ_1, ρ_2) satisfying the assumptions of Theorem 1.1, by the inequalities (1.20) and (2.1), we readily infer the existence of $\bar{\alpha} > \lambda_1(\Omega)$ such that $\{(|u_1|, |u_2|) \in H_0^1(\Omega; \mathbb{R}^2) : (u_1, u_2) \in G\} \subseteq \mathcal{B}_{\bar{\alpha}}$ and $\inf_{\mathcal{M}} \mathcal{E} = c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$; in particular, $G = G_{\bar{\alpha}}$. For $p > 1 + 4/N$ and $(\rho_1, \rho_2) \in A$, take $\bar{\alpha} \geq \lambda_1(\Omega)$ such that $c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$, as in the statement of Theorem 1.10, and satisfying moreover (1.8) in the case $p = 2^* - 1$. Therefore, for the previous choice of $\bar{\alpha}$, we are reduced in all cases to prove the stability of the set $G_{\bar{\alpha}}$.

To this aim, we recall that a set $\mathcal{G} \subset H_0^1(\Omega; \mathbb{C}^2)$ is orbitally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $(\psi_1, \psi_2) \in H_0^1(\Omega; \mathbb{C}^2)$ satisfies $\text{dist}_{H_0^1}((\psi_1, \psi_2), \mathcal{G}) < \delta$, $\text{dist}_{H_0^1}$ denoting the H_0^1 -distance, then the solution (Ψ_1, Ψ_2) of

$$\begin{cases} i\partial_t \Psi_1 + \Delta \Psi_1 + \Psi_1(\mu_1 |\Psi_1|^{p-1} + \beta |\Psi_1|^{(p-3)/2} |\Psi_2|^{(p+1)/2}) = 0 \\ i\partial_t \Psi_2 + \Delta \Psi_2 + \Psi_2(\mu_2 |\Psi_2|^{p-1} + \beta |\Psi_2|^{(p-3)/2} |\Psi_1|^{(p+1)/2}) = 0 \\ \Psi_i(0, \cdot) = \psi_i(\cdot), \quad \Psi_i(t, \cdot) \in H_0^1(\Omega; \mathbb{C}^2). \end{cases}$$

is such that

$$(\Psi_1(t, \cdot), \Psi_2(t, \cdot)) \text{ can be continued to a solution in } 0 \leq t < +\infty \quad (4.1)$$

and

$$\sup_{t>0} \text{dist}_{H_0^1}((\Psi_1(t, \cdot), \Psi_2(t, \cdot)), \mathcal{G}) < \varepsilon, \quad (4.2)$$

As we mentioned, we prove the orbital stability of $G_{\bar{\alpha}}$ under the condition that for every $M > 0$ there exists $T_0 = T_0(M)$ such that if $\|(\psi_1, \psi_2)\|_{H_0^1(\Omega; \mathbb{C}^2)} \leq M$ then the above Cauchy problem admits a unique solution on $[0, T_0)$, and that both \mathcal{Q} and \mathcal{E} are preserved along the solutions. Notice that, under these conditions, the failure of (4.1) implies the failure of (4.2). Indeed, if (4.1) does not hold, then since T_0 depends on the norm of the initial data we necessarily have $\|(\Psi_1(t, \cdot), \Psi_2(t, \cdot))\|_{H_0^1(\Omega; \mathbb{C}^2)} \rightarrow +\infty$ as t approaches a finite endpoint of the maximal existence time-interval. Since $G_{\bar{\alpha}}$ is bounded in $H_0^1(\Omega; \mathbb{C}^2)$, (4.2) cannot hold.

We start with the following preliminary considerations.

Lemma 4.1. *Let $(u_1, u_2) \in G_{\bar{\alpha}}$. Then there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that $(u_1, u_2) = (e^{i\theta_1}|u_1|, e^{i\theta_2}|u_2|)$. In particular,*

$$\inf \{ \mathcal{E}(v_1, v_2) : (v_1, v_2) \in H_0^1(\Omega; \mathbb{C}^2), (|v_1|, |v_2|) \in \mathcal{B}_{\bar{\alpha}} \} = c_{\bar{\alpha}},$$

while

$$\inf \{ \mathcal{E}(v_1, v_2) : (v_1, v_2) \in H_0^1(\Omega; \mathbb{C}^2), (|v_1|, |v_2|) \in \mathcal{U}_{\bar{\alpha}} \} =: \tilde{c}_{\bar{\alpha}} \leq \hat{c}_{\bar{\alpha}}.$$

Proof. Given $(v_1, v_2) \in \mathcal{U}_{\bar{\alpha}}$, we have clearly that $(v_1, v_2) \in H_0^1(\Omega; \mathbb{C}^2)$ and that $(|v_1|, |v_2|) \in \mathcal{U}_{\bar{\alpha}}$, so that $\tilde{c}_{\bar{\alpha}} \leq \hat{c}_{\bar{\alpha}}$.

Now let $(u_1, u_2) \in G_{\bar{\alpha}}$. By the diamagnetic inequality [26, Theorem 7.21], we have $\int_{\Omega} |\nabla |u_i||^2 \leq \int_{\Omega} |\nabla u_i|^2$, for $i = 1, 2$, so that $c_{\bar{\alpha}} \leq \mathcal{E}(|u_1|, |u_2|) \leq \mathcal{E}(u_1, u_2) = c_{\bar{\alpha}}$. As a consequence, $\int_{\Omega} |\nabla |u_i||^2 = \int_{\Omega} |\nabla u_i|^2$, for $i = 1, 2$ so that equality holds in the diamagnetic inequality, whence u_i is a complex multiple of $|u_i|$, that is to say $u_i = e^{i\theta_i} |u_i|$ for some $\theta_i \in \mathbb{R}$, and the rest of the lemma follows.

Our general criterion for stability is the following.

Proposition 4.2. *Let $\bar{\alpha}$ be as above. If $c_{\bar{\alpha}} < \tilde{c}_{\bar{\alpha}}$, then $G_{\bar{\alpha}}$ is (conditionally) orbitally stable.*

The proof is presented after the following lemma.

Lemma 4.3. Let $\bar{\alpha}$ be as above. Let $\{(\psi_{1,n}, \psi_{2,n})\} \subset H_0^1(\Omega; \mathbb{C}^2)$ satisfy, as $n \rightarrow \infty$,

$$\int_{\Omega} |\psi_{i,n}|^2 \rightarrow \rho_i \quad \text{for } i = 1, 2, \quad \mathcal{E}(\psi_{1,n}, \psi_{2,n}) \rightarrow c_{\bar{\alpha}} \quad (4.3)$$

and, for every n sufficiently large,

$$\int_{\Omega} |\nabla \psi_{1,n}|^2 + |\nabla \psi_{2,n}|^2 \leq (\rho_1 + \rho_2)\bar{\alpha} + o(1). \quad (4.4)$$

Then there exists $(u_1, u_2) \in G_{\bar{\alpha}}$ such that, up to a subsequence, $(\psi_{1,n}, \psi_{2,n}) \rightarrow (u_1, u_2)$, strongly in $H_0^1(\Omega; \mathbb{C}^2)$.

Proof. By (4.4) there exists $(\bar{\psi}_1, \bar{\psi}_2) \in H_0^1(\Omega; \mathbb{C}^2)$ such that, up to a subsequence, $\psi_{i,n} \rightharpoonup u_i$ weakly in $H_0^1(\Omega; \mathbb{C})$ and $\psi_{i,n} \rightarrow u_i$ in $L^2(\Omega; \mathbb{C})$ for $i = 1, 2$, as $n \rightarrow +\infty$. Then (4.3), (4.4) and Lemma 4.1 provide, for $i = 1, 2$,

$$\int_{\Omega} |u_i|^2 = \rho_i, \quad \int_{\Omega} |\nabla u_1|^2 + |\nabla u_2|^2 \leq (\rho_1 + \rho_2)\bar{\alpha}, \quad \mathcal{E}(u_1, u_2) \geq c_{\bar{\alpha}}.$$

Now, in case $p < 2^* - 1$, we have that $(\psi_{1,n}, \psi_{2,n}) \rightarrow (u_1, u_2)$ also in $L^{p+1}(\Omega; \mathbb{C}^2)$. Then

$$c_{\bar{\alpha}} \leq \mathcal{E}(u_1, u_2) \leq \liminf_{n \rightarrow +\infty} \mathcal{E}(\psi_{1,n}, \psi_{2,n}) = c_{\bar{\alpha}},$$

and the strong H_0^1 convergence follows, together with the fact that $(u_1, u_2) \in G_{\bar{\alpha}}$.

On the other hand, in case $p = 2^* - 1$, the result follows by Proposition 3.4: actually, such proposition is stated for real valued functions, but after Lemma 4.1 it is straightforward to check that its proof holds also for complex valued ones. \square

Proof of Proposition 4.2. Suppose by contradiction that $\{(\psi_{1,n}, \psi_{2,n})\} \subset H_0^1(\Omega; \mathbb{C}^2)$, $(u_{1,n}, u_{2,n}) \in G_{\bar{\alpha}}$ and $\bar{\varepsilon} > 0$ are such that

$$\lim_{n \rightarrow \infty} \|(\psi_{1,n}, \psi_{2,n}) - (u_{1,n}, u_{2,n})\|_{H_0^1(\Omega; \mathbb{C}^2)} = 0 \quad (4.5)$$

and

$$\sup_{t > 0} \text{dist}_{H_0^1}((\Psi_{1,n}(t, \cdot), \Psi_{2,n}(t, \cdot)), G_{\bar{\alpha}}) \geq 2\bar{\varepsilon},$$

where $(\Psi_{1,n}, \Psi_{2,n})$ is the solution of (1.1) with initial condition $(\psi_{1,n}, \psi_{2,n})$. Then there exists $\{t_n\}$ such that, letting $\phi_{i,n}(x) := \Psi_{i,n}(t_n, x)$, $i = 1, 2$,

$$\text{dist}_{H_0^1}((\phi_{1,n}, \phi_{2,n}), G_{\bar{\alpha}}) \geq \bar{\varepsilon}. \quad (4.6)$$

Let us prove that $\{(\phi_{1,n}, \phi_{2,n})\}$ satisfies (4.3) and (4.4). Then Lemma 4.3 provides a contradiction to (4.6), thus concluding the proof.

By Lemma 4.3, $G_{\bar{\alpha}}$ is compact. Therefore, (4.5) implies the existence of $(u_1, u_2) \in G_{\bar{\alpha}}$ such that, up to a subsequence,

$$(\psi_{1,n}, \psi_{2,n}) \rightarrow (u_1, u_2) \quad \text{in } H_0^1(\Omega; \mathbb{C}^2).$$

This, combined with the continuity of Sobolev embeddings, implies that $(\psi_{1,n}, \psi_{2,n})$ satisfies (4.3). Then the conservation of the mass and of the energy imply that

$$\int_{\Omega} |\phi_{i,n}|^2 = \int_{\Omega} |\psi_{i,n}|^2 \rightarrow \rho_i \quad \text{for } i = 1, 2, \quad \text{and} \quad \mathcal{E}(\phi_{1,n}, \phi_{2,n}) = \mathcal{E}(\psi_{1,n}, \psi_{2,n}) \rightarrow c_{\bar{\alpha}},$$

as $n \rightarrow +\infty$, so that $(\phi_{1,n}, \phi_{2,n})$ also satisfies (4.3).

To conclude the proof, let us check that, at least for a subsequence, $(\phi_{1,n}, \phi_{2,n})$ satisfies (4.4), that is, we claim that, for n sufficiently large,

$$\int_{\Omega} |\nabla \phi_{1,n}|^2 + |\nabla \phi_{2,n}|^2 \leq (\rho_1 + \rho_2)\bar{\alpha} + o(1)$$

By contradiction, assume there exists $\bar{n} \in \mathbb{N}$ and $\bar{\varepsilon} > 0$ such that

$$\int_{\Omega} |\nabla \phi_{1,n}|^2 + |\nabla \phi_{2,n}|^2 \geq (\rho_1 + \rho_2)\bar{\alpha} + \bar{\varepsilon}.$$

Since

$$\begin{aligned} \int_{\Omega} |\nabla \Psi_{1,n}(0, \cdot)|^2 + |\nabla \Psi_{2,n}(0, \cdot)|^2 &= \int_{\Omega} |\nabla \psi_{1,n}|^2 + |\nabla \psi_{2,n}|^2 \\ &\leq \int_{\Omega} |\nabla u_{1,n}|^2 + |\nabla u_{2,n}|^2 + o(1) \leq (\rho_1 + \rho_2)\bar{\alpha} + o(1) \end{aligned}$$

for n large, then there exists $\bar{t}_n \in (0, t_n)$ such that $(\Psi_{1,n}(\bar{t}_n, \cdot), \Psi_{2,n}(\bar{t}_n, \cdot))$ satisfies (4.3) and

$$\int_{\Omega} |\nabla \Psi_{1,n}(\bar{t}_n, \cdot)|^2 + |\nabla \Psi_{2,n}(\bar{t}_n, \cdot)|^2 = (\rho_1 + \rho_2)\bar{\alpha} + o(1)$$

and in particular (4.4). By Lemma 4.3 there exists $(\bar{u}_1, \bar{u}_2) \in G_{\bar{\alpha}}$ such that

$$\int_{\Omega} |\nabla \bar{u}_1|^2 + |\nabla \bar{u}_2|^2 = (\rho_1 + \rho_2)\bar{\alpha},$$

which contradicts the assumption $c_{\bar{\alpha}} < \bar{c}_{\bar{\alpha}}$.

We proved Proposition 4.2 assuming that $c_{\bar{\alpha}} < \bar{c}_{\bar{\alpha}}$. We now check that, since $c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$, this assumption is satisfied.

Lemma 4.4. *Let $\bar{\alpha}$ be as above. Then $c_{\bar{\alpha}} < \bar{c}_{\bar{\alpha}}$.*

Proof. If by contradiction $\bar{c} = c$, then there exists $\varepsilon_n \rightarrow 0$ and $(v_{1,n}, v_{2,n}) \in H_0^1(\Omega; \mathbb{C}^2)$ such that

$$\|(v_{1,n}, v_{2,n})\|_{H_0^1(\Omega; \mathbb{C}^2)}^2 = \bar{\alpha}(\rho_1 + \rho_2), \quad \int_{\Omega} |v_{i,n}|^2 = \rho_i, \quad c_{\bar{\alpha}} \leq \mathcal{E}(v_{1,n}, v_{2,n}) \leq c_{\bar{\alpha}} + \varepsilon_n,$$

for every $n, i = 1, 2$. Letting $u_{i,n} := |v_{i,n}|$, $i = 1, 2$, the diamagnetic inequality implies

$$\|(u_{1,n}, u_{2,n})\|_{H_0^1(\Omega; \mathbb{R}^2)}^2 \leq \|(v_{1,n}, v_{2,n})\|_{H_0^1(\Omega; \mathbb{C}^2)}^2 = (\rho_1 + \rho_2)\bar{\alpha},$$

so that $(u_{1,n}, u_{2,n})$ is an admissible couple for the minimization problem c and then

$$c_{\bar{\alpha}} \leq \mathcal{E}(u_{1,n}, u_{2,n}) \leq \mathcal{E}(v_{1,n}, v_{2,n}) \leq c_{\bar{\alpha}} + \varepsilon_n. \quad (4.7)$$

In particular,

$$\frac{1}{2} \left(\|(v_{1,n}, v_{2,n})\|_{H_0^1(\Omega; \mathbb{C}^2)}^2 - \|(u_{1,n}, u_{2,n})\|_{H_0^1(\Omega; \mathbb{R}^2)}^2 \right) = \mathcal{E}(v_{1,n}, v_{2,n}) - \mathcal{E}(u_{1,n}, u_{2,n}) \leq \varepsilon_n. \quad (4.8)$$

Then Lemma 4.3 applies to both sequences, yielding both $(v_{1,n}, v_{2,n}) \rightarrow (v_{1,\infty}, v_{2,\infty})$ and $(u_{1,n}, u_{2,n}) \rightarrow (u_{1,\infty}, u_{2,\infty})$, strongly in H_0^1 . Passing to the limit in (4.7) and (4.8), we infer

$$\mathcal{E}(u_{1,\infty}, u_{2,\infty}) = c_{\bar{\alpha}} \quad \text{and} \quad \|(u_{1,\infty}, u_{2,\infty})\|_{H_0^1(\Omega; \mathbb{R}^2)}^2 = \|(v_{1,\infty}, v_{2,\infty})\|_{H_0^1(\Omega; \mathbb{C}^2)}^2 = \bar{\alpha}(\rho_1 + \rho_2).$$

Then $(u_{1,\infty}, u_{2,\infty}) \in \mathcal{U}_{\bar{\alpha}}$, contradicting the fact that $c_{\bar{\alpha}} < \hat{c}_{\bar{\alpha}}$.

End of the proof of Theorems 1.4-b) and 1.10. Recalling the first paragraph of this section, we have to prove that the set $G_{\bar{\alpha}}$ is (conditionally) orbitally stable. This is a direct consequence of Proposition 4.2 together with Lemma 4.4. \square

5 Asymptotic study as $\beta \rightarrow -\infty$

In this section we prove Theorem 1.12. Let $\mu_1, \mu_2 > 0$, and take $\rho_1, \rho_2 > 0$ satisfying

$$\begin{cases} \rho_1, \rho_2 > 0 & \text{if } 1 < p < 1 + 4/N, \\ 0 < \mu_1 \rho_1^{\frac{2}{N}}, \mu_2 \rho_2^{\frac{2}{N}} < \frac{N+2}{NC_N} & \text{if } p = 1 + \frac{4}{N} \\ \max\{\mu_1 \rho_1^{2r}, \mu_2 \rho_2^{2r}\} \cdot (\rho_1 + \rho_2)^{a-1} \leq \frac{(a-1)^{a-1}}{a^a} \lambda_2(\Omega)^{-(a-1)} & \text{if } 1 + \frac{4}{N} < p \leq 2^* - 1 \end{cases}$$

Observe that all these conditions are independent from β . Combining Theorems 1.1, 1.5 and 3.5 (see also Remarks 1.3 and 1.7) with the definition of R in (3.22), we deduce that, given $\beta < 0$, there exist positive functions $u_{1,\beta}, u_{2,\beta}$ and $\omega_{1,\beta}, \omega_{2,\beta} \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u_{1,\beta} + \omega_{1,\beta} u_{1,\beta} = \mu_1 u_{1,\beta}^p + \beta u_{1,\beta}^{(p-1)/2} u_{2,\beta}^{(p+1)/2} \\ -\Delta u_{2,\beta} + \omega_{2,\beta} u_{2,\beta} = \mu_2 u_{2,\beta}^p + \beta u_{2,\beta}^{(p-1)/2} u_{1,\beta}^{(p+1)/2} \\ \int_{\Omega} u_i^2 = \rho_i, \quad i = 1, 2, \\ (u_1, u_2) \in H_0^1(\Omega; \mathbb{R}^2) \end{cases} \quad (5.1)$$

while

$$\mathcal{E}(u_{1,\beta}, u_{2,\beta}) = \inf_{\mathcal{M}} \mathcal{E} \quad \text{if } 1 < p \leq 1 + \frac{4}{N} \quad (5.2)$$

$$\mathcal{E}(u_{1,\beta}, u_{2,\beta}) = \inf_{\mathcal{B}_{\bar{\alpha}}} \mathcal{E}, \quad (u_{1,\beta}, u_{2,\beta}) \in \mathcal{B}_{\bar{\alpha}} \setminus \mathcal{U}_{\bar{\alpha}} \quad \text{if } 1 + \frac{4}{N} < p \leq 2^* - 1 \quad (5.3)$$

where $\bar{\alpha} := \frac{a}{a-1} \lambda_2(\Omega)$ in (5.3).

Lemma 5.1. *Under the previous assumptions, there exists a constant $C > 0$, independent of β , such that*

$$\|u_{i,\beta}\|_{H_0^1(\Omega)} + \|u_{i,\beta}\|_{L^\infty(\Omega)} + |\omega_{i,\beta}| \leq C \quad \text{for every } \beta < 0, i = 1, 2.$$

Proof. Take $(\xi_1, \xi_2) \in \mathcal{M}$ (if $p \leq 1 + \frac{4}{N}$) or $(\xi_1, \xi_2) \in \mathcal{B}_{\bar{\alpha}}$ (if $p > 1 + \frac{4}{N}$), with $\xi_1 \cdot \xi_2 \equiv 0$ in either case. From (5.2)–(5.3) we have

$$\mathcal{E}(u_{1,\beta}, u_{2,\beta}) \leq \mathcal{E}(\xi_1, \xi_2) = \frac{1}{2} \int_{\Omega} (|\nabla \xi_1|^2 + |\nabla \xi_2|^2) - \frac{1}{p+1} \int_{\Omega} (\mu_1 |\xi_1|^{p+1} + \mu_2 |\xi_2|^{p+1}) =: C_1 \quad (5.4)$$

where C_1 is independent of $\beta < 0$.

From the first statement in (5.3) we deduce that $\{(u_{1,\beta}, u_{2,\beta})\}_{\beta < 0}$ is uniformly bounded in $H_0^1(\Omega)$ for $p > 1 + \frac{4}{N}$. In case $1 < p < 1 + \frac{4}{N}$, the H_0^1 -boundedness follows combining (5.4) with the estimate

$$\mathcal{E}(u_{1,\beta}, u_{2,\beta}) \geq \|\nabla u_{1,\beta}\|_2^{2a} \left(\frac{1}{2} \|\nabla u_{1,\beta}\|_2^{2-2a} - \frac{C_{N,p}}{p+1} \mu_1 \rho_1^{2r} \right) + \|\nabla u_{2,\beta}\|_2^{2a} \left(\frac{1}{2} \|\nabla u_{2,\beta}\|_2^{2-2a} - \frac{C_{N,p}}{p+1} \mu_2 \rho_2^{2r} \right)$$

($a := N(p-1)/4 < 1$), which corresponds to (1.20) for $\beta < 0$, while for $p = 1 + \frac{4}{N}$ it follows from (5.4) and

$$\mathcal{E}(u_{1,\beta}, u_{2,\beta}) \geq \frac{1}{2} \left(1 - \frac{NC_N \mu_1 \rho_1^{2/N}}{N+2} \right) \|\nabla u_{1,\beta}\|_2^2 + \frac{1}{2} \left(1 - \frac{NC_N \mu_2 \rho_2^{2/N}}{N+2} \right) \|\nabla u_{2,\beta}\|_2^2.$$

(see (2.1) with $\beta < 0$).

By the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$, we have that $\{(u_{1,\beta}, u_{2,\beta})\}_{\beta < 0}$ is uniformly bounded in the L^{p+1} -norm. In particular,

$$0 \leq \frac{2(-\beta)}{p+1} \int_{\Omega} (u_{1,\beta} u_{2,\beta})^{(p+1)/2} \leq \mathcal{E}(u_{1,\beta}, u_{2,\beta}) + \frac{1}{p+1} \int_{\Omega} \mu_1 u_{1,\beta}^{p+1} + \mu_2 u_{2,\beta}^{p+1} \leq C_2.$$

By testing the first equation in (5.1) by $u_{1,\beta}$ and the second one by $u_{2,\beta}$, and using the previous estimates, we have, for $i, j \in \{1, 2\}$, $i \neq j$,

$$\begin{aligned} \rho_i |\omega_{i,\beta}| &= \left| \int_{\Omega} \mu_i u_{i,\beta}^{p+1} + \beta (u_{1,\beta} u_{2,\beta})^{(p+1)/2} - |\nabla u_{i,\beta}|^2 \right| \\ &\leq \int_{\Omega} \mu_i u_{i,\beta}^{p+1} + |\beta| (u_{1,\beta} u_{2,\beta})^{(p+1)/2} + |\nabla u_{i,\beta}|^2 \leq C_3. \end{aligned}$$

Now we can use a Brezis-Kato-Moser type argument exactly as in [31, pp. 1264–1265], obtaining uniform L^∞ -bounds for $\{(u_{1,\beta}, u_{2,\beta})\}_{\beta < 0}$.

Proof of Theorem 1.12. By Lemma 5.1, $\{(u_{1,\beta}, v_{2,\beta})\}_{\beta < 0}$ satisfies the assumptions of [37, Theorems 1.3 and 1.5]. Therefore this sequence is uniformly bounded in $C^{0,\alpha}(\overline{\Omega})$ for every $0 < \alpha < 1$, and there exist $(u_1, u_2) \in C^{0,1}(\overline{\Omega})$ with $u_1, u_2 \geq 0$ in Ω , and $(\omega_1, \omega_2) \in \mathbb{R}^2$ such that, up to subsequences, as $\beta \rightarrow -\infty$ we have

$$u_{i,\beta} \rightarrow u_i \text{ in } C^{0,\alpha}(\overline{\Omega}) \cap H_0^1(\Omega), \quad \omega_i \rightarrow \omega_i.$$

(see also [30, 38]). By [17, Theorem 1.2] (which is stated for $p = 3$, but holds also for a general p without any extra effort), we have

$$-\Delta(u_1 - u_2) + \omega_1 u_1 - \omega_2 u_2 \geq \mu_1 u_1^p - \mu_2 u_2^p \quad \text{and} \quad -\Delta(u_2 - u_1) + \omega_2 u_2 - \omega_1 u_1 \geq \mu_2 u_2^p - \mu_1 u_1^p \quad \text{in } \Omega.$$

We can now conclude by taking $w := u_1 - u_2$.

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`benedetta.noris@u-picardie.fr`
Laboratoire Amiénois de Mathématique Fondamentale et Appliquée
Université de Picardie Jules Verne
33 Rue Saint-Leu, 80039 Amiens (France)

`hrtavares@fc.ul.pt`
CMAFCIO & Departamento de Matemática
Faculdade de Ciências da Universidade de Lisboa
Edifício C6, Piso 1, Campo Grande 1749–016 Lisboa (Portugal)

`gianmaria.verzini@polimi.it`
Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano (Italy)