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Integer & Nonlinear Optimization

Linear inequalities for bounded products of variables

Pietro Belotti

Dept. of Mathematical Sciences, Clemson University (pbelott@clemson.edu)

Andrew J. Miller

Institut de Mathématiques de Bordeaux (IMB), Talence, France (Andrew.Miller@math.u-bordeaux1.fr)

Mahdi Namazifar

Dept. of Industrial and Systems Engineering, University of Wisconsin at Madison (namazifar@wisc.edu)

Mixed-integer nonlinear programming (MINLP) is a vast class of optimization problems with a broad range of applications. In its most general form, a MINLP problem can be formulated as

$$\textbf{MINLP}: \begin{array}{lll} \min & g_0(x) & \\ \text{s.t.} & g_i(x) \leq 0 & \forall i = 1, 2 \ldots, m \\ & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}, & \end{array}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ is, in general, a nonlinear function for all $i = 0, 1, \ldots, m$ and may be nonconvex. MINLP problems subsume two major difficulties of optimization problems, namely nonlinear g_i's and integrality of a set of variables. Some well-known subclasses of MINLP are NP-hard: relaxing integrality on x yields a nonconvex (in general) nonlinear optimization problem, while assuming that both the objective function $g_0(x)$ and all constraints $g_i(x) \leq 0$ are convex yields the subclass of *convex* MINLP.

Global optima of MINLP problems can be computed by implicit enumeration schemes such as branch-and-bound [9], which relies on lower bounds obtained from a relaxation of the problem. Because a large lower bound can reduce the solution time, it is crucial to find a tight relaxation. Several MINLP solvers use Linear Programming (LP) relaxations computed by *reformulating* a MINLP into an equivalent problem with constraints of the form $x_k = f_k(x_1, x_2 \ldots, x_{k-1})$, where f_k is a nonlinear function, and replacing each such constraint with a system of linear inequalities $A^k x \leq b^k$ [3, 18, 20].

Multilinear functions are an important class used in MINLP models. They are n-variate functions that are linear in each variable x_i, i.e., when the remaining $n-1$ variables are fixed. Among multilinear functions, the linear combination of products $\sum_{i=1}^k a_i \prod_{j \in S_i} x_j$, where $S_i \subseteq \{1, 2 \ldots, n\}$, is widely used in modeling practical MINLPs. Several practical applications arise in the *bilinear* case, where functions $\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$ are used: for instance, *pooling* and scheduling problems in Chemical Engineering [14, 17] and bidimensional bin packing [6].

This paper focuses on polyhedral relaxations of *monomials*, i.e., products of a set of variables: we aim to find valid linear inequalities for

$$M_n = \{x \in \mathbb{R}^{n+1} : x_{n+1} = \textstyle\prod_{i=1}^n x_i, x \in [\ell, u]\},$$

where $\ell, u \in \mathbb{R}^{n+1}$. We assume $0 \leq \ell_i < u_i < +\infty$ for $i = 1, 2 \ldots, n+1$. M_n is bounded and nonconvex as the function $\xi(x) = \prod_{i=1}^n x_i$ is nonconvex.

Define $N = \{1, 2 \ldots, n\}$. The assumption $\ell \geq 0$ implies that trivial bounds on x_{n+1} are $\bar{\ell}_{n+1} = \prod_{i \in N} \ell_i$ and $\bar{u}_{n+1} = \prod_{i \in N} u_i$. In general, $\bar{\ell}_{n+1} \leq \ell_{n+1} < u_{n+1} \leq \bar{u}_{n+1}$; in the remainder, we denote as $M_n^\star$ the special case of M_n where $\ell_{n+1} = \bar{\ell}_{n+1}$ and $u_{n+1} = \bar{u}_{n+1}$. We are interested in developing a convex set enclosing M_n, defined by a system of linear inequalities. This would also allow us to approximate rational terms:

$$Q_2 = \{x \in \mathbb{R}^3 : x_1 = \tfrac{x_3}{x_2}, x \in [\ell, u]\},$$

and, in general, quotients with products as denominator: $Q_n = \{x \in \mathbb{R}^{n+1} : x_1 = \frac{x_{n+1}}{\prod_{k=2}^n x_k}, x \in [\ell, u]\}$.

1. Linear Inequalities for M_2

The following linear relaxation of $M_2^\star$ was introduced by McCormick [12] and shown to be its tightest convex relaxation by Al-Khayyal and Falk [1]:

$$\begin{array}{rcccccc} x_3 & \geq & \ell_2 x_1 & + & \ell_1 x_2 & - & \ell_1 \ell_2 \\ x_3 & \geq & u_2 x_1 & + & u_1 x_2 & - & u_1 u_2 \\ x_3 & \leq & \ell_2 x_1 & + & u_1 x_2 & - & \ell_1 u_2 \\ x_3 & \leq & u_2 x_1 & + & \ell_1 x_2 & - & u_1 \ell_2. \end{array} \tag{1}$$

$M_2^\star$ and its convex hull are depicted in Figure 1.

As regards M_2, Tawarmalani et al. [19] describe the convex hull of $\{x \in \mathbb{R}^3 : x_1 x_2 + x_3 \geq c, \ell_i \leq x_i \leq u_i, i = 1, 2, 3\}$. This is a special case of M_2, as $x_1 x_2 + x_3 \geq c$ implies a lower bound ℓ_3 on $x_1 x_2$ that is larger than $\bar{\ell}_3 = \ell_1 \ell_2$ if $\ell_1 \ell_2 + u_3 < c$. Tawarmalani and Sahinidis [20] describe the convex hull of

$$\begin{array}{rcl} D_3 & = & \{x \in \mathbb{R}^3 : x_1 = \frac{x_3}{x_2}, \\ & & 0 < \ell_2 \leq x_2 \leq u_2,\ 0 \leq \ell_3 \leq x_3 \leq u_3\}, \end{array}$$

again a special case of M_2 where $\ell_1 = \frac{\ell_3}{u_2}$, $u_1 = \frac{u_3}{\ell_2}$. The set D_3 is also studied by Jach et al. [8], who generalize the approach of [20] to find the convex hull of $(n-1)$-*convex functions*, i.e., nonconvex functions that are convex when any of their variables is fixed.

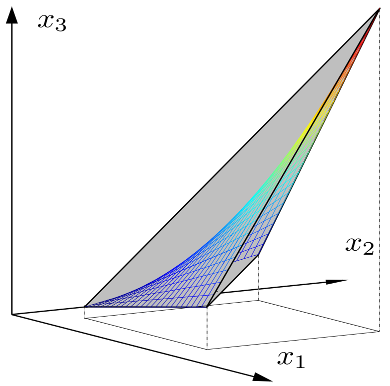


Figure 1: $M_2^\star$ and its convex hull (1).

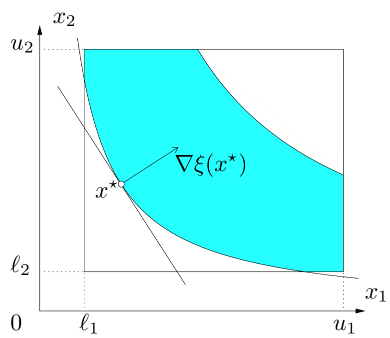


Figure 2: Projection of M_2 onto (x_1, x_2).

In order to find valid inequalities for the more general M_2, consider its projection onto (x_1, x_2): $P_2 = \{(x_1, x_2) \in \mathbb{R}^2 : \ell_i \leq x_i \leq u_i, i = 1, 2, \ell_3 \leq x_1 x_2 \leq u_3\}$ (see Figure 2). It is safe to assume here that $\ell_3 \leq \ell_1 u_2$ and $\ell_3 \leq u_1 \ell_2$, as otherwise a tighter valid lower bound for x_1 (resp. x_2) would be $\ell_3/u_2 > \ell_1$ (resp. $\ell_3/u_1 > \ell_2$), or equivalently, the upper left (resp. lower right) corner of the bounding box would be cut out by $x_1 x_2 \geq \ell_3$. Similarly, we assume that $u_3 \geq \ell_1 u_2$ and $u_3 \geq u_1 \ell_2$.

Before describing a valid linear inequality for M_2, it is worth to briefly mention the particular case where $\ell_1 = \ell_2 = 0$ and $u_1 = u_2 = +\infty$. In that case, the convex hull is easily proved to be the intersection of $\{x \in \mathbb{R}^3 : \ell_3 \leq x_3 \leq u_3\}$ with the second order cone $\{x \in \mathbb{R}^3 : (x_3 + \sqrt{\ell_3 u_3})^2 \leq (\sqrt{\ell_3} + \sqrt{u_3})^2 x_1 x_2\}$.

Lifted Tangent Inequalities. We provide a more detailed derivation of the results below in [4]. Consider a point $x^\star \in [\ell_1, u_1] \times [\ell_2, u_2]$ such that $x_1^\star x_2^\star = \ell_3$, therefore $\ell_1 \leq x_1^\star \leq \min\{u_1, \ell_3/\ell_2\}$ and $\ell_2 \leq x_2^\star = \ell_3/x_1^\star \leq \min\{u_2, \ell_3/\ell_1\}$. The tangent to the curve $x_1x_2 = \ell_3$ at $x^\star$ gives a linear inequality $a_1(x_1 - x_1^\star) + a_2(x_2 - x_2^\star) \geq 0$ that is valid for P_2 (see Figure 2). The coefficients a_1 and a_2 are given by the gradient of the function $\xi(x) = x_1x_2$ at $x^\star$, i.e., $a_1 = \frac{\partial \xi}{\partial x_1}(x^\star) = x_2^\star$ and $a_2 = \frac{\partial \xi}{\partial x_2}(x^\star) = x_1^\star$. Hence the inequality, which we call *tangent inequality*, is

$$x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) \geq 0. \qquad (2)$$

As this inequality is valid within P_2 and is independent from x_3, it is also valid for M_2.

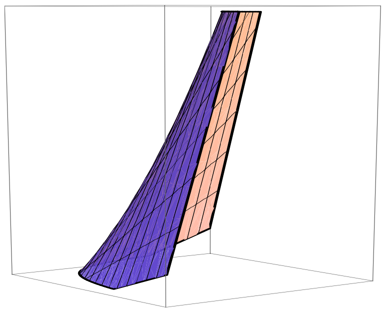


Figure 3: A representation of M_2.

Consider now M_2 (depicted in Figure 3) rather than its projection. To give a hint as to why McCormick inequalities (1) are not sufficient in this case, consider the set Y obtained by intersecting M_2 with the set $\{x \in \mathbb{R}^3 : x_1 = x_2\}$, and suppose $\ell_1 = \ell_2 = 0$ and $u_1 = u_2 = 10$. Then Y can be represented as $\{(\lambda, y) \in \mathbb{R}^2 : y = \lambda^2\}$. The McCormick inequalities imply $y \leq 100\lambda$, which yields the convex relaxation given by the shaded area (both light and dark) in Figure 4, clearly not the tightest relaxation given that (2) tightens it. Furthermore, lifting (2) would restrict the relaxation to the darker area in Figure 4. We lift (2) as follows: the inequality

$$x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + b(x_3 - \ell_3) \geq 0 \qquad (3)$$

is clearly valid for $x_3 = \ell_3$. Validity for M_2 requires

$$g(b) = \min\{x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + b(x_3 - \ell_3) : (x_1, x_2, x_3) \in M_2\} \geq 0.$$

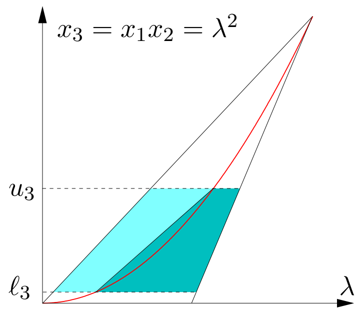


Figure 4: The relaxation of M_2 intersected with $\{x \in \mathbb{R}^3 : x_1 = x_2\}$ using McCormick inequalities only (the light shaded area) and with the lifted inequality (the dark shaded area).

Clearly, $g(b) = 0$ if $b \geq 0$ (a global optimum is given by $(x_1^\star, x_2^\star)$), hence we aim at finding the minimum $b < 0$ such that (3) is valid.

Observe that validity of (3) requires that it be satisfied by all points of $UC_2 = \{x \in [\ell, u] : x_3 = x_1x_2 = u_3\}$. If we relax the bounds u_1 and u_2 on x_1 and x_2, the line $T = \{x \in \mathbb{R}^3 : x_3 = u_3, x_2^\star(x_1 - x_1^\star) + x_1^\star(x_2 - x_2^\star) + b(u_3 - \ell_3) = 0\}$ intersects UC_2 in either (i) none, (ii) one, or (iii) two points. The first two cases imply validity of the inequality, unlike the third one.

Consider case (ii) and denote $\bar{x} = (\bar{x}_1, \bar{x}_2)$ the only intersection; T is then tangent to UC_2 at $\bar{x}$. Thus, the gradient of ξ at $\bar{x}$ must be parallel to $\nabla\xi(x^\star)$, i.e., $\nabla\xi(\bar{x}) = \alpha\nabla\xi(x^\star)$ for some $\alpha > 0$, hence $(\bar{x}_2, \bar{x}_1) = (\alpha x_2^\star, \alpha x_1^\star)$ and $\bar{x}_1\bar{x}_2 = u_3 = \alpha^2 x_1^\star x_2^\star = \alpha^2\ell_3$, therefore $\alpha = \sqrt{\frac{u_3}{\ell_3}}$. Since $(\bar{x}_1, \bar{x}_2)$ satisfies (3) at equality,

$$\begin{aligned} x_2^\star(\bar{x}_1 - x_1^\star) + x_1^\star(\bar{x}_2 - x_2^\star) + b(u_3 - \ell_3) &= \\ x_2^\star(\alpha x_1^\star - x_1^\star) + x_1^\star(\alpha x_2^\star - x_2^\star) + b(u_3 - \ell_3) &= 0, \end{aligned}$$

and as a result $b = \frac{2\left(1 - \sqrt{\frac{u_3}{\ell_3}}\right)\ell_3}{u_3 - \ell_3}$.

The procedure outlined above does not work in general as $\bar{x} = \alpha x^\star$ may exceed one of the upper bounds on x_1 or x_2. To this purpose, consider the parametric vector $\hat{x}(t)$ with $\hat{x}_i(t) = \min\{u_i, tx_i^\star\}$. The set $\Gamma(x^\star) = \{x \in \mathbb{R}^2 : x_i = \min\{u_i, tx_i^\star\}, i = 1, 2, t \geq 1\}$, depicted in Figure 5 for two distinct vectors $x^\star$, is a piecewise linear set. The function $\xi(t) = \hat{x}_1(t)\hat{x}_2(t)$ is monotonically non-decreasing and piecewise convex, and hence there exists a $\hat{t}$

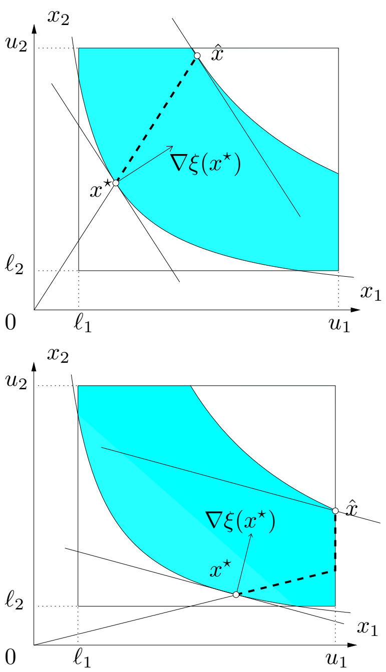


Figure 5: Construction of $\Gamma(x^\star)$, $\hat{t}$, and $\hat{x}$. The set $\Gamma(x^\star)$ is represented by the dashed line.

such that $\xi(\hat{t}) = u_3$. In order to compute $\hat{t}$, assume w.l.o.g. that $\frac{u_1}{x_1^\star} \leq \frac{u_2}{x_2^\star}$. Then

$$\xi(t) = \begin{cases} x_1^\star x_2^\star t^2 = \ell_3 t^2 & \text{if } 1 \leq t \leq \frac{u_1}{x_1^\star} \\ u_1 x_2^\star t & \text{if } \frac{u_1}{x_1^\star} \leq t \leq \frac{u_2}{x_2^\star} \\ u_1 u_2 & \text{if } t \geq \frac{u_2}{x_2^\star}, \end{cases}$$

and $\hat{t} = \xi^{-1}(u_3)$ is computed as follows: if $x_1^\star x_2^\star \left(\frac{u_1}{x_1^\star}\right)^2 = \frac{u_1^2 x_2^\star}{x_1^\star} \geq u_3$, then $\hat{t} = \sqrt{\frac{u_3}{\ell_3}}$; otherwise, if $u_1 x_2^\star \frac{u_2}{x_2^\star} = u_1 u_2 \geq u_3$, $\hat{t} = \frac{u_3}{u_1 x_1^\star}$. Note that these two cases exhaust all values of t as we assume $u_1 u_2 \geq u_3$.

Denote $\hat{x} = \hat{x}(\hat{t})$. Clearly $\hat{x} = (\hat{x}_1, \hat{x}_2, u_3) \in M_2$, and it satisfies (3) at equality if

$$b = \bar{b} := -\frac{x_2^\star(\hat{x}_1 - x_1^\star) + x_1^\star(\hat{x}_2 - x_2^\star)}{u_3 - \ell_3},$$

which yields a valid inequality (3) for M_2 that we call *lifted tangent inequality* (LTI) – note that it only depends on $x^\star$. The generalization to M_n is given in Section 3. LTIs are easily proven to be disjunctive cuts obtained from intersecting M_2 with the disjunction $x_3 = \ell_3 \vee x_3 = u_3$.

2. Linear Inequalities for $M_n^\star$

The convex hull of sets defined by products of more than two terms has attracted interest for some decades. Meyer and Floudas [13] provide a set of linear inequalities describing the convex hull of a more general case of $M_3^\star$, where lower and upper bounds can also be negative.

Ryoo and Sahinidis [16] construct polyhedral relaxations of $M_n^\star$ with $n > 2$ as follows: given an index set $I = \{i_1, i_2 \ldots \ldots i_K\}$ and the product of $K > 2$ variables $\prod_{i \in I} x_i$, add *auxiliary* variables $y_2, y_3 \ldots, y_K$ defined as

$$\begin{aligned} y_2 &= x_{i_1} x_{i_2} \\ y_3 &= y_2 x_{i_3} \\ y_4 &= y_3 x_{i_4} \\ &\vdots \\ y_K &= y_{K-1} x_{i_K}, \end{aligned}$$

where the bounds on y_k are determined by the bounds on the factors of the product. Then, add McCormick inequalities for $M_2^{(2)} = \{(x_{i_1}, x_{i_2}, y_2) \in [\ell_{i_1}, u_{i_1}] \times [\ell_{i_2}, u_{i_2}] \times [\ell(y_2), u(y_2)] : y_2 = x_{i_1} x_{i_2}\}$ and for each set $M_2^{(k)} = \{(y_{k-1}, x_{i_k}, y_k) \in [\ell(y_{k-1}), u(y_{k-1})] \times [\ell_{i_k}, u_{i_k}] \times [\ell(y_k), u(y_k)] : y_k = y_{k-1} x_{i_k}\}$, with $3 \leq k \leq K$. We define $\ell(y_k) := \ell(y_{k-1})\ell_{i_k}$, with $\ell(y_2) = \ell_{i_1}\ell_{i_2}$, and analogously define the upper bounds $u(y_k)$.

A convex estimator can thus be obtained with $4(n-1)$ linear inequalities. This procedure, called *Recursive Arithmetic Intervals* (rAI), is shown by [16] to yield the convex hull of $M_n^\star$ when $\ell = 0$. Luedtke et al. [11] prove that this result also holds in the case where $\ell = -u$, and compare the tightness of the convex hull of bilinear functions to that of the McCormick relaxations.

A central result has been proved by Rikun [15] on the more general multilinear functions defined on polyhedra. For such functions, the validity of an inequality only needs to be checked at the vertices of the polyhedron on which they are defined, hence the convex hull of $M_n^\star$ is polyhedral. However, said convex hull contains an exponential number of inequalities, which makes it impractical for use in global optimization solvers except for small n (see e.g. [2]). Inequalities for $M_4^\star$ have been proposed by Cafieri et al. [5] by "composing" the convex hulls of bilinear and trilinear terms.

3. Linear Inequalities for M_n

The derivation of valid inequalities for M_n is a straightforward generalization of the method described in Section 1. Similar to M_2, we assume $\ell_{n+1} \leq \min_{i\in N}\{\ell_i \prod_{j\in N\setminus\{i\}} u_j\}$ (resp. $u_{n+1} \geq \max_{i\in N}\{u_i \prod_{j\in N\setminus\{i\}} \ell_j\}$), as otherwise we can tighten one of the lower (resp. upper) bounds. Specifically, for i such that $\ell_{n+1} > \ell_i \prod_{j\in N\setminus\{i\}} u_j$, ℓ_i is increased to $\frac{\ell_{n+1}}{\prod_{j\in N\setminus\{i\}} u_j} > \ell_i$, and for all i such that $u_{n+1} < u_i \prod_{j\in N\setminus\{i\}} \ell_j$, u_i is reduced to $\frac{u_{n+1}}{\prod_{j\in N\setminus\{i\}} \ell_j} < u_i$.

Tangent Inequalities for M_n. Let us denote P_n the projection of M_n onto $\mathbb{R}^n$: $P_n = \{x \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i, i \in N, \ell_{n+1} \leq \prod_{i\in N} x_i \leq u_{n+1}\}$. Define

$$LC_n := \{x \in P_n : \textstyle\prod_{i\in N} x_i = \ell_{n+1}\};$$
$$UC_n := \{x \in P_n : \textstyle\prod_{i\in N} x_i = u_{n+1}\}.$$

The following simple result generalizes the validity of the tangent inequality (2).

Lemma 1 *For any $x^\star \in LC_n$, the inequality:*

$$\sum_{i\in N} a_i(x_i - x_i^\star) \geq 0, \tag{4}$$

where $a_i := \prod_{j\in N\setminus\{i\}} x_j^\star$, is valid for P_n.

For $n = 2$, (4) reduces to (2). We lift (4) to obtain an inequality satisfied by a point on UC_n. To this purpose, consider the parametric point $\hat{x}(t)$ with components $\hat{x}_i(t) = \min\{u_i, tx_i^\star\}$ and the set

$$\Gamma(x^\star) = \{x \in \mathbb{R}^n : x_i = \min\{u_i, tx_i^\star\}\ \forall i \in N, t \geq 1\},$$

where $x^\star$ corresponds to $t = 1$. Also, consider the function $\xi(t) = \prod_{i\in N} \hat{x}_i(t)$, defined for all $t \geq 1$. Define $\hat{t} = \min\{\tau \geq 1 : \xi(\tau) = u_{n+1}\}$, i.e. the minimum t attaining a point in UC_n, and denote $\hat{x} = x(\hat{t})$. It can be shown that such a $\hat{t}$, in the general case, can be computed in $O(n)$. Note that, for small values of t, the gradient of ξ at $\hat{x}(t)$ is proportional to $\nabla\xi(x^\star)$.

Lifted Tangent Inequalities. A lifting of (4) that satisfies $\hat{x}$ at equality yields a valid inequality for M_n. Then the inequality

$$\sum_{i\in N} a_i(x_i - x_i^\star) + b(x_{n+1} - \ell_{n+1}) \geq 0 \tag{5}$$

holds at equality at $x^\star$ for any b, while it does at $\hat{x}$ if $\sum_{i\in N} a_i(\hat{x}_i - x_i^\star) + b(u_{n+1} - \ell_{n+1}) = 0$, or

$$b = \bar{b} := -\frac{\sum_{i\in N} a_i(\hat{x}_i - x_i^\star)}{u_{n+1} - \ell_{n+1}}.$$

Note that, as for M_2, $\bar{b}$ is negative (a positive value yields a redundant inequality) and depends on $x^\star$.

Theorem 1 *Inequality (5) is valid for any $b \geq \bar{b}$.*

A similar result can be proved when starting from any point $x^\star$ of UC_n, though the analogous inequality (4) is *not* valid unless lifted. The derivation is similar to the one above and is thus omitted.

Note that LTIs have to be *amended* to the LP relaxation; they do not dominate McCormick inequalities, and are thus not sufficient to describe the convex hull of M_n. For instance, the convex hull of M_2 is obtained by considering both McCormick inequalities and LTIs [4].

4. Computational Results

In order to assess the utility of the lifted tangent inequalities introduced above in the context of MINLP solvers, we have developed a procedure for generating LTIs and tested it on a set of MINLP problems.

We have used Couenne [7], an open-source software package included in the Coin-OR infrastructure [10], for all experiments. Couenne is a branch-and-bound solver that computes a lower bound with an LP relaxation obtained through reformulation techniques [12, 18, 20]. As for most MINLP solvers, Couenne uses a procedure to gradually refine the LP relaxation by repeatedly solving the LP relaxation at each node of the branch-and-bound tree, obtaining a solution x^{LP}, and seeking an inequality violated by x^{LP} which strengthens the relaxation.

Generating LTIs amounts to finding $x^\star$ associated with a violated LTI. We omit the details of the separation algorithm, but point out that the procedure finds a violated LTI in $\mathcal{O}(n)$. In these experiments, at each branch-and-bound node Couenne used up to four rounds of cuts to refine the LP relaxation.

Although LTIs can be separated for M_n, Couenne does not generate inequalities for the convex hull of $M_k^\star$ with $k \geq 3$, hence all of our experiments focus on bilinear terms. Products of more than two variables are decomposed into a set of bilinear terms using the *recursive Arithmetic Interval* (rAI) technique [16] outlined in Section 2. Although each auxiliary y_k introduced by rAI has trivial bounds at the beginning, branching rules (which may also be imposed on y_k) and bound reduction techniques may reduce its bounds and thus require separation of LTIs for some, or all, of the bilinear terms generated.

Also, Couenne can generate LTIs for bilinear sets M_2 not necessarily contained in $\mathbb{R}^3_+$ but in any other orthant, i.e., LTIs are generated when the bound interval of each variable does not have 0 as an interior point: if a variable x_i of a bilinear term has $\ell_i < u_i \leq 0$, then a fictitious variable x'_i with inverted bound interval $[-u_i, -\ell_i]$ replaces x_i.

In order to show the utility of LTIs for M_2, we have compared two variants of Couenne, which we call Couenne and CouenneLTI, on a set of MINLP instances. While the first variant only separates, for each bilinear term, inequalities (1), the second variant adds both these and LTIs—recall that there is no dominance relationship between these two families of inequalities.

We have performed tests on 474 instances from multiple online libraries: globallib[1], minlplib[2], and macminlp[3]. Both variants were allowed two hours of CPU time. All experiments have been carried out on the Palmetto cluster of Clemson University, which has machines with different CPUs and amounts of memory. Although a parallel version of Couenne is currently being developed and the cluster allows running parallel jobs, we have used a serial version of the code for our tests. Also, in order to provide a fair comparison, each instance was solved by the two variants on the same machine.

Out of 474 instances, we only report on the 119 instances that took either or both algorithms more than one minute. Table 1 summarizes the comparison by showing, for each variant, the number of instances

- solved before the time limit (*solved*);
- solved in at most 90% of the other variant's time (*best time*);
- solved using at most 90% of the other variant's BB nodes (*best nodes*);
- for which the variant obtained the best lower bound (*best lower*).

[1] http://www.gamsworld.org/global/globallib.htm
[2] http://www.gamsworld.org/minlp/minlplib.htm
[3] http://www.mcs.anl.gov/~leyffer/MacMINLP

Alg	Solved	Best time	Best nodes	Best lower
A1	26	15	7	24
A2	26	8	11	32

Table 1: Summary of the comparison between Couenne (A1) and CouenneLTI (A2).

The first three parameters refer to instances that at least one variant solved before the time limit, whereas the last one refers to the instances that neither algorithm could solve to optimality. It appears that separating LTIs on "easy" instances, i.e., those that can be solved within the time limit, is of limited impact (mainly on the number of BB nodes) and actually may lead to an increase in CPU time. However, when both algorithms take more than two hours, LTIs help obtain a better lower bound.

Table 2 shows in more detail the performance of both variants of Couenne for some of the instances where the difference in performance is significant, regardless of whether Couenne or CouenneLTI obtained a better result. A more complete report can be found in [4]. The better performance is in bold. The parameters reported in the columns are:

- *Name*, *var*, *con*: Name of the instance, number of variables and of constraints;
- *T(lb)*: the CPU time taken to solve the problem to optimality, or, if no solution was found within the time limit, the lower bound in brackets;
- *node*: the number of BB nodes used before proving optimality or the time limit was passed;
- *ub*: the best known upper bound.

Although the results are only sketched here for reasons of space, it is apparent that some instances highly benefit from adding LTIs. Certain instances (`nvs23`, `nvs24`, `st-e35`) can be solved much more quickly, although it appears that for others (`bayes2-10`, `bayes2-30`, `bayes2-50`, `tln5`) LTIs have the opposite effect.

Name	var	con	COUENNE t(lb)	nodes	COUENNELTI t(lb)	nodes	ub
bayes2-10	86	72	**3553**	124k	(0)	67k	2.55e-4
bayes2-30	86	75	**3072**	130k	(0)	1.5m	4.61e-4
bayes2-50	86	76	**6140**	1727	(0)	2057	0.9298
camcge	209	209	(**-4036**)	535	(-6092)	426	-191.74
ex5-2-5	32	19	(-4832)	1.6m	(**-4775**)	2.1m	-3500
ex5-4-4	27	19	(7257)	3.1m	(**7801**)	2.1m	10077.8
hhfair	27	25	252	30k	**168**	23k	-87.159
space-25	893	235	(89.4)	4388	(90.9)	5278	483.811
nvs23	9	9	(-1240)	2.7m	**237**	61k	-1125.2
nvs24	10	10	(-1200)	2.5m	**6054**	1.7m	-1033.2
st-e35	29	33	(42443)	1.1m	**496**	210k	64868
tln5	35	30	**2506**	2.4m	(9.86)	4.5m	10.3
tln7	63	42	(7.73)	123k	(**9.31**)	1.5m	15.6
water4	195	137	(**716.7**)	1.3m	(655.1)	957k	965.47
waterx	70	54	(636.7)	58k	(**652.4**)	106k	973.91

Table 2: Comparison between COUENNE and COUENNELTI on select instances. Under "t(lb)" columns are reported the CPU time or, if more than two hours, the lower bound in brackets; "ub" is the best known upper bound.

5. Concluding Remarks

We have described a family of linear inequalities of the convex hull of a class of nonconvex sets widely used in MINLP. Their efficiency has only been tested on products of two variables, but we expect to implement the more general procedure in the near future and apply it to MINLP problems with products of more than two variables.

REFERENCES

[1] F. A. Al-Khayyal and J. E. Falk. Jointly constrained biconvex programming. *Mathematics of Operations Research*, 8:273–286, 1983.

[2] X. Bao, N. Sahinidis, and M. Tawarmalani. Multi-term polyhedral relaxations for nonconvex, quadratically constrained quadratic programs. *Optimization Methods and Software*, 24(4-5):485–504, 2009.

[3] P. Belotti, J. Lee, L. Liberti, F. Margot, and A. Wächter. Branching and bounds tightening techniques for non-convex MINLP. *Optimization Methods & Software*, 24(4-5):597–634, 2009.

[4] P. Belotti, A. J. Miller, and M. Namazifar. Linear inequalities for multilinear sets. In preparation, 2011.

[5] S. Cafieri, J. Lee, and L. Liberti. On convex relaxations of quadrilinear terms. *Journal of Global Optimization*, 47:661–685, 2010.

[6] A. Caprara and M. Monaci. Bidimensional packing by bilinear programming. *Mathematical Programming*, 118:75–108, 2009.

[7] COUENNE. `http://www.coin-or.org/Couenne`.

[8] M. Jach, D. Michaels, and R. Weismantel. The convex envelope of $(n-1)$-convex functions. *SIAM J. Optim.*, 19:1451–1466, 2008.

[9] A. H. Land and A. G. Doig. An automatic method of solving discrete programming problems. *Econometrica*, 28(3):497–520, 1960.

[10] R. Lougee-Heimer. The Common Optimization INterface for Operations Research. *IBM Journal of Research and Development*, 47(1):57–66, 2004.

[11] J. Luedtke, M. Namazifar, and J. Linderoth. Some results on the strength of relaxations of multilinear functions. *Optimization Online*, 2711, 2010.

[12] G. P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I — Convex underestimating problems. *Mathematical Programming*, 10:146–175, 1976.

[13] C. A. Meyer and C. A. Floudas. Trilinear monomials with mixed sign domains: Facets of the convex and concave envelopes. *Journal of Global Optimization*, 29(2), 2004.

[14] R. Misener, C. E. Gounaris, and C. A. Floudas. Mathematical modeling and global optimization of large-scale extended pooling problems with the EPA complex emissions constraints. *Computers & Chemical Engineering*, 34(9):1432–1456, 2010.

[15] A. D. Rikun. A convex envelope formula for multilinear functions. *Journal of Global Optimization*, 10:425–437, 1997.

[16] H. S. Ryoo and N. V. Sahinidis. Analysis of bounds for multilinear functions. *Journal of Global Optimization*, 19:403–424, 2001.

[17] M. A. Shaik, C. A. Floudas, J. Kallrath, and H.-J. Pitz. Production scheduling of a large-scale industrial continuous plant: Short-term and medium-term scheduling. *Computers & Chemical Engineering*, 33(3):670–686, 2009.

[18] E. M. B. Smith and C. C. Pantelides. A symbolic reformulation/spatial branch-and-bound algorithm for the global optimisation of nonconvex MINLPs. *Computers & Chem. Eng.*, 23:457–478, 1999.

[19] M. Tawarmalani, J.-P. Richard, and K. Chung. Strong valid inequalities for orthogonal disjunctions and bilinear covering sets. *Mathematical Programming*, 124:481–512, 2010.

[20] M. Tawarmalani and N. V. Sahinidis. Global optimization of mixed-integer nonlinear programs: A theoretical and computational study. *Mathematical Programming*, 99(3):563–591, 2004.

CyberInfrastructure for Mixed-Integer Nonlinear Programming

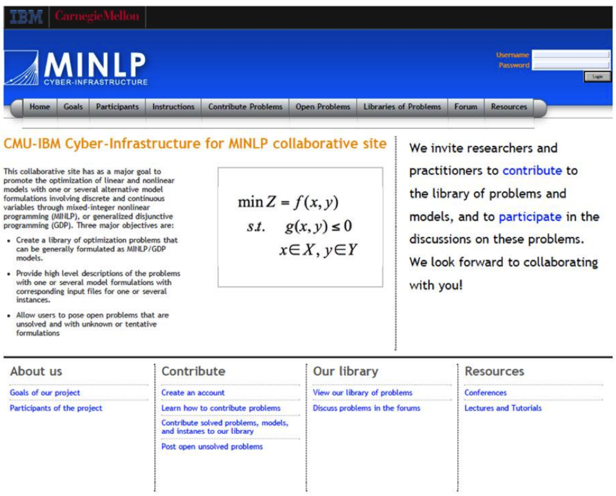
Ignacio E. Grossmann

Carnegie Mellon University (grossmann@cmu.edu).

Jon Lee

IBM T.J. Watson Research Center (jonlee@us.ibm.com).

Carnegie Mellon University and the IBM T. J. Watson Research Center researchers have developed a Collaborative CyberInfrastructure for Mixed-Integer Nonlinear Programming (MINLP): http://www.minlp.org, which is funded by the funded by the National Science Foundation under Grant OCI-0750826: "OpenCyberInfrastructure for Mixed-integer Nonlinear Programming: Collaboration and Deployment via Virtual Environments". The core team consists of: Larry Biegler, Ignacio E. Grossmann, François Margot and Nick Sahinidis of CMU, and Jon Lee and Andreas Wächter of IBM. Additional collaborators include: Pietro Belotti (Clemson University), Pedro Castro (INETI) and Juan Ruiz (CMU). The site was launched in October, 2009. The current homepage is shown below. Over the last 12 months the site has had between 500 and 1000 daily hits, and between 80 and 130 daily visits.



Optimization has been recognized as one of the strategic technologies for cyberinfrastructure computational tools. Many of the challenging optimization models require the use of discrete variables (of-