ON A FOURTH-ORDER EQUATION OF MOORE-GIBSON-THOMPSON TYPE

FILIPPO DELL'ORO AND VITTORINO PATA

Abstract. An abstract version of the fourth-order equation

$$\partial_{tttt} u + \alpha \partial_{ttt} u + \beta \partial_{tt} u - \gamma \Delta \partial_{tt} u - \delta \Delta \partial_t u - \varrho \Delta u = 0$$

subject to the homogeneous Dirichlet boundary condition is analyzed. Such a model encompasses the Moore-Gibson-Thompson equation with memory in presence of an exponential kernel. The stability properties of the related solution semigroup are investigated. In particular, a necessary and sufficient condition for exponential stability is established, in terms of the values of certain stability numbers depending on the strictly positive parameters $\alpha, \beta, \gamma, \delta, \varrho$.

1. Introduction

1.1. **Motivations.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider for every $t > 0$ the Moore-Gibson-Thompson (MGT) equation with memory treated in [3, 7, 8]

$$\partial_{ttt} u + a \partial_{tt} u - b \Delta \partial_t u - c \Delta u + \int_0^t g(s) \Delta u(t-s) \mathrm{d}s = 0 \tag{1.1}$$

in the unknown variable $u = u(\boldsymbol{x}, t) : \Omega \times [0, \infty) \to \mathbb{R}$ subject to the homogeneous Dirichlet boundary condition

$$u(\boldsymbol{x}, t)_{|\boldsymbol{x} \in \partial\Omega} = 0.$$

Here a, b, c are strictly positive constants complying with the structural constraint

$$\mu := b - \frac{c}{a} \geq 0,$$

while the so-called memory kernel $g \in W^{1,1}(\mathbb{R}^+)$ is a (nonnegative) nonincreasing absolutely continuous convex function on $\mathbb{R}^+ = (0, \infty)$ of total mass

$$\int_0^\infty g(s) \mathrm{d}s < c, \tag{1.2}$$

satisfying for some $\ell > 0$ the relation

$$g'(s) + \ell g(s) \leq 0, \quad \forall s \in \mathbb{R}^+. \tag{1.3}$$

The equation is supplemented with the initial conditions assigned at time $t = 0$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \partial_t u(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}), \qquad \partial_{tt} u(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}),$$

2000 *Mathematics Subject Classification.* 35B35, 35G05, 47D06, 74D05.

Key words and phrases. Moore-Gibson-Thompson equation with memory, fourth-order equation, solution semigroup, exponential stability, growth bound.

being $u_0, v_0, w_0 : \Omega \to \mathbb{R}$ prescribed data.

• When $g \equiv 0$, the model boils down to the MGT equation arising in acoustics

$$\partial_{ttt} u + a\partial_{tt} u - b\Delta\partial_t u - c\Delta u = 0, \tag{1.4}$$

accounting for the second sound effects and the associated thermal relaxation in viscous fluids [5, 10, 11, 13, 14]. The asymptotic properties of the solutions to (1.4) have been studied by several authors (see e.g. [4, 6, 9]). Summarizing the earlier literature, such an equation generates a semigroup on the natural weak energy space which is exponentially stable if and only if $\mu > 0$, whereas the energy is conserved when $\mu = 0$. It is worth mentioning that the semigroup exists also if $\mu < 0$, but in that case the energy blows up as time goes to infinity.

• When $g \not\equiv 0$, the memory term introduces further dissipation. Hence, as expected, for $\mu > 0$, the energy $\mathsf{Q}(t)$ associated to (1.1) and defined as

$$\begin{aligned}\mathsf{Q}(t) = \|\nabla u(t)\|^2_{L^2(\Omega)} + \|\nabla\partial_t u(t)\|^2_{L^2(\Omega)} + \|\partial_{tt} u(t)\|^2_{L^2(\Omega)} \\ - \int_0^t g'(s)\|\nabla(u(t) - u(t-s))\|^2_{L^2(\Omega)} \mathrm{d}s\end{aligned}$$

decays exponentially to zero as well (see [7, 8]), that is,

$$\mathsf{Q}(t) \le C\mathsf{Q}(0)\mathrm{e}^{-\omega t},$$

for some $\omega > 0$ and $C \ge 1$. Nevertheless, since the dissipation mechanism of (1.1) is stronger than the one of (1.4), one might think that exponential stability occurs also in the case $\mu = 0$. On the contrary, as shown in the recent paper [3], when $\mu = 0$ the memory contribution is capable to drive the system to zero, i.e. for every fixed initial energy $\mathsf{Q}(0)$ it is always true that

$$\lim_{t\to\infty} \mathsf{Q}(t) = 0,$$

but the decay is not exponential.

1.2. **The fourth-order equation.** From the physical viewpoint, the most relevant case in connection with (1.1) is the one of the (negative) exponential kernel

$$g(s) = d\mathrm{e}^{-\ell s},$$

where the strictly positive constants d, ℓ fulfill the relation

$$\frac{d}{\ell} < c,$$

in compliance with (1.2)-(1.3). In this situation, equation (1.1) reads

$$\partial_{ttt} u + a\partial_{tt} u - b\Delta\partial_t u - c\Delta u + d\int_0^t \mathrm{e}^{-\ell s}\Delta u(t-s)\mathrm{d}s = 0. \tag{1.5}$$

Then, taking the sum

$$\partial_t(1.5) + \ell(1.5),$$

we obtain

$$\partial_{tttt} u + (a+\ell)\partial_{ttt} u + a\ell\partial_{tt} u - b\Delta\partial_{tt} u - (c+b\ell)\Delta\partial_t u - (c\ell - d)\Delta u = 0. \tag{1.6}$$

Motivated by the discussion above, we consider in more generality the boundary value problem

$$\begin{cases} \partial_{tttt}u + \alpha\partial_{ttt}u + \beta\partial_{tt}u - \gamma\Delta\partial_{tt}u - \delta\Delta\partial_t u - \varrho\Delta u = 0, \\ u_{|\partial\Omega} = 0, \end{cases} \tag{1.7}$$

for some parameters

$$\alpha, \beta, \gamma, \delta, \varrho > 0.$$

As we will see, problem (1.7) is well-posed in the natural weak energy space

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega)$$

for *any* choice of the (positive) structural parameters. At the same time, the associated solution semigroup $S(t)$ exhibits an extremely rich dynamics, which turns out to be dramatically influenced by the two *stability numbers*

$$\varkappa = \gamma - \frac{\delta}{\alpha} \qquad \text{and} \qquad \varpi = \beta - \frac{\varrho\alpha}{\delta}, \tag{1.8}$$

and by the first eigenvalue $\lambda_1 > 0$ of the Laplace-Dirichlet operator $-\Delta$. In this work, we are interested in establishing necessary and sufficient conditions on $\varkappa$ and ϖ in order to ensure stability of the solutions to (1.7). More precisely, we focus on the growth bound of $S(t)$. We recall the definition.

Definition 1.1. The *growth bound* of the semigroup $S(t)$ is the number $\omega_* \in [-\infty, \infty)$ defined as

$$\omega_* = \inf\left\{\omega \in \mathbb{R} : \|S(t)\| \leq C\mathrm{e}^{\omega t}\right\}$$

for some $C = C(\omega)$ and every $t \geq 0$. Here, the norm $\|S(t)\|$ is understood in the space of bounded linear operators on $\mathcal{H}$.

The most interesting case is when $\omega_* < 0$, yielding exponential stability. In this situation, there exist $\omega > 0$ and $C \geq 1$ such that

$$\|S(t)\| \leq C\mathrm{e}^{-\omega t}.$$

Instead, when $\omega_* > 0$, there are solutions with energy growing exponentially fast. The limit case $\omega_* = 0$ corresponds either to a bounded semigroup, possibly stable but not exponentially stable, or to a semigroup whose energy blows up at infinity, but slower than any exponential (e.g. polynomially). The aim of the present paper is to give a complete characterization of (the sign of) ω_* in terms of the stability numbers $\varkappa$ and ϖ. In particular, we will show that $S(t)$ is exponentially stable if and only if

$$\varkappa > 0 \qquad \text{and} \qquad \varpi > -\lambda_1\varkappa.$$

1.3. **On the comparison between (1.5) and (1.6).** Writing $\varkappa$ and ϖ in terms of the numbers a, b, c, d, ℓ of the "concrete" fourth-order equation (1.6), we find the relations

$$\varkappa = \frac{a\mu}{a+\ell} \qquad \text{and} \qquad \varpi = \frac{a\mu\ell^2 + d(a+\ell)}{c+b\ell}.$$

Hence, when $\mu > 0$, both $\varkappa$ and ϖ are positive, meaning that (1.6) is exponentially stable. In this situation, the energy $\mathsf{Q}(t)$ associated with the Volterra-type equation (1.5) decays exponentially as well. On the other hand, when $\mu = 0$, the semigroup $S(t)$ is not exponentially stable, nor is $\mathsf{Q}(t)$ as shown in [3]. Still, the latter conclusion cannot be drawn directly from the knowledge that $S(t)$ is not exponentially stable. Indeed, let us consider the generalization of (1.5) to the case of infinite memory, i.e.

$$\partial_{ttt}u + a\partial_{tt}u - b\Delta\partial_t u - c\Delta u + d\int_0^\infty \mathrm{e}^{-\ell s}\Delta u(t-s)\mathrm{d}s = 0, \tag{1.9}$$

where $u(t)$ is understood to be an assigned datum for $t \leq 0$. Within the so-called history space framework of Dafermos [2], equation (1.9) can be shown to generate a solution semigroup $\Sigma(t)$ acting on a suitable Hilbert space $\mathcal{V}$. It is also possible to prove that such a $\Sigma(t)$ is exponentially stable on $\mathcal{V}$ if and only if the same is true for the semigroup $S(t)$ on $\mathcal{H}$. On the other hand, since (1.5) is just a particular instance of (1.9) corresponding to null initial past histories of u, the lack of exponential stability of $\Sigma(t)$ does not imply that $\mathsf{Q}(t)$ is not exponentially stable (see [1] for an example in this direction).

1.4. **Plan of the paper.** In the forthcoming Sections 2 and 3 we reformulate the problem in an abstract setting, and we establish the existence of the solution semigroup $S(t)$. In Section 4, we provide a detailed description of the spectrum of its infinitesimal generator and its relation with the growth bound ω_*. The subsequent Sections 5-6 are devoted to the main results on the asymptotic properties of $S(t)$.

2. The Abstract Formulation

In order to reformulate (1.7) in an abstract framework, let $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let

$$A : \mathfrak{D}(A) \subset \mathrm{H} \to \mathrm{H}$$

be a strictly positive unbounded linear operator of domain $\mathfrak{D}(A)$ densely (but not necessarily compactly) embedded into H. For $t > 0$, we consider the fourth-order equation in the unknown variable $u = u(t)$

$$\partial_{tttt}u + \alpha\partial_{ttt}u + \beta\partial_{tt}u + \gamma A\partial_{tt}u + \delta A\partial_t u + \varrho A u = 0. \tag{2.1}$$

It is apparent to see that (1.7) is just a particular realization of (2.1) corresponding to the choice $\mathrm{H} = L^2(\Omega)$ and

$$A = -\Delta \qquad \text{with} \qquad \mathfrak{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

2.1. **Functional setting.** For $r \in \mathbb{R}$, we introduce the hierarchy of nested Hilbert spaces (the subscript r will be always omitted whenever zero)

$$\mathrm{H}_r = \mathfrak{D}(A^{\frac{r}{2}}), \qquad \langle u,v\rangle_r = \langle A^{\frac{r}{2}}u, A^{\frac{r}{2}}v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}}u\|.$$

The symbol $\langle\cdot,\cdot\rangle$ will also stand for duality product between H_r and its dual space H_{-r}. Moreover, we have the Poincaré inequality

$$\lambda_1\|u\|^2 \leq \|u\|_1^2, \quad \forall u \in \mathrm{H}_1, \tag{2.2}$$

where

$$\lambda_1 = \min\{\lambda : \lambda \in \sigma(A)\} > 0,$$

being $\sigma(A)$ the spectrum of A. The phase space of our problem is

$$\mathcal{H} = \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H},$$

endowed with the standard Euclidean product norm

$$\|(u,v,w,z)\|_{\mathcal{H}}^2 = \|u\|_1^2 + \|v\|_1^2 + \|w\|_1^2 + \|z\|^2.$$

2.2. **The operator $\mathbb{A}$.** Introducing the state vector

$$\boldsymbol{U}(t) = (u(t), v(t), w(t), z(t)),$$

we view (2.1) as the ODE in the space $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(t) = \mathbb{A}\boldsymbol{U}(t) \tag{2.3}$$

where $\mathbb{A}$ is the (closed) linear operator defined as

$$\mathbb{A}\begin{pmatrix} u \\ v \\ w \\ z \end{pmatrix} = \begin{pmatrix} v \\ w \\ z \\ -\alpha z - \beta w - A(\gamma w + \delta v + \varrho u) \end{pmatrix}$$

with dense domain

$$\mathfrak{D}(\mathbb{A}) = \left\{ (u,v,w,z) \in \mathcal{H} \,\middle|\, \begin{array}{c} z \in \mathrm{H}_1 \\ \gamma w + \delta v + \varrho u \in \mathrm{H}_2 \end{array} \right\}.$$

The equation is complemented with the initial condition $\boldsymbol{U}(0) = \boldsymbol{U}_0 \in \mathcal{H}$.

3. The Solution Semigroup

The first step is an existence and uniqueness result for (2.1).

Theorem 3.1. *The operator $\mathbb{A}$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup*

$$S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.$$

Such a semigroup is ω-contractive with respect to an equivalent norm $|\cdot|_{\mathcal{H}}$, namely, there exists $\omega \geq 0$ such that

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \leq \mathrm{e}^{\omega t}|\boldsymbol{U}_0|_{\mathcal{H}} \tag{3.1}$$

for every $\boldsymbol{U}_0 \in \mathcal{H}$ and every $t \geq 0$.

The proof of Theorem 3.1 can be carried out by exploiting one's favorite method, e.g. the Hille-Yosida or the Lumer-Phillips Theorems (see [12]), or via direct energy estimates within a Galerkin scheme. However, no matter which is the proof one could have in mind, the natural product norm $\|\cdot\|_{\mathcal{H}}$ of $\mathcal{H}$ does not seem to be appropriate for the analysis of (2.1). Reason why we now consider a different, albeit equivalent, norm that is tailored to the structure of the problem. To this end, we introduce the (nonnegative) number m as follows:

$$\begin{cases} m > -\dfrac{\alpha\varkappa}{\gamma} & \text{if } \ \varkappa \leq 0, \\ m = 0 & \text{if } \ \varkappa > 0, \end{cases}$$

where $\varkappa$ is the stability number defined in (1.8), together with the strictly positive constants

$$\alpha_m := \alpha + m \qquad \text{and} \qquad \varkappa_m := \gamma - \frac{\delta}{\alpha_m}.$$

For all $\boldsymbol{U} = (u, v, w, z)$ and $\tilde{\boldsymbol{U}} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{z})$ in $\mathcal{H}$, the function

$$
\begin{aligned}
(\boldsymbol{U}, \tilde{\boldsymbol{U}})_{\mathcal{H}} = {} & \frac{\delta}{\alpha_m}\langle w + \alpha_m v + \frac{\alpha_m \varrho}{\delta} u, \tilde{w} + \alpha_m \tilde{v} + \frac{\alpha_m \varrho}{\delta}\tilde{u}\rangle_1 + \frac{\varkappa_m \alpha_m \varrho}{\delta}\langle v, \tilde{v}\rangle_1 \\
& + \varkappa_m \langle w, \tilde{w}\rangle_1 + \langle z + \alpha_m w + \frac{\alpha_m \varrho}{\delta} v, \tilde{z} + \alpha_m \tilde{w} + \frac{\alpha_m \varrho}{\delta}\tilde{v}\rangle
\end{aligned}
$$

is an inner product on $\mathcal{H}$, with induced norm

$$
\begin{aligned}
|\boldsymbol{U}|^2_{\mathcal{H}} = {} & \frac{\delta}{\alpha_m}\|w + \alpha_m v + \frac{\alpha_m \varrho}{\delta} u\|^2_1 + \frac{\varkappa_m \alpha_m \varrho}{\delta}\|v\|^2_1 \\
& + \varkappa_m \|w\|^2_1 + \|z + \alpha_m w + \frac{\alpha_m \varrho}{\delta} v\|^2.
\end{aligned}
$$

This is a direct consequence of the next result.

Lemma 3.2. *There exists a structural constant $\mathfrak{c} > 0$, depending also of the choice of m, such that*

$$
\mathfrak{c}\|\boldsymbol{U}\|^2_{\mathcal{H}} \leq |\boldsymbol{U}|^2_{\mathcal{H}} \leq \frac{1}{\mathfrak{c}}\|\boldsymbol{U}\|^2_{\mathcal{H}}, \quad \forall \boldsymbol{U} \in \mathcal{H}.
$$

Proof. We limit ourselves to prove the estimate from below (the other one is much simpler and left to the reader). In what follows, the Poincaré inequality (2.2) will be used several times without explicit mention. For all $0 < \varepsilon < 1$, an application of the Young inequality yields the control

$$
\begin{aligned}
|\boldsymbol{U}|^2_{\mathcal{H}} \geq {} & \frac{\varepsilon \alpha_m \varrho^2}{\delta}\|u\|^2_1 + \frac{\varkappa_m \alpha_m \varrho}{\delta}\|v\|^2_1 + \varkappa_m \|w\|^2_1 + \varepsilon\|z\|^2 \\
& - \frac{\varepsilon \delta}{\alpha_m(1-\varepsilon)}\|w + \alpha_m v\|^2_1 - \frac{\varepsilon \alpha_m^2}{\lambda_1(1-\varepsilon)}\|w + \frac{\varrho}{\delta} v\|^2_1 \\
\geq {} & \frac{\varepsilon \alpha_m \varrho^2}{\delta}\|u\|^2_1 + \Big[\frac{\varkappa_m \alpha_m \varrho}{\delta} - \frac{2\varepsilon}{1-\varepsilon}\Big(\delta \alpha_m + \frac{\alpha_m^2 \varrho^2}{\delta^2 \lambda_1}\Big)\Big]\|v\|^2_1 \\
& + \Big[\varkappa_m - \frac{2\varepsilon}{1-\varepsilon}\Big(\frac{\delta}{\alpha_m} + \frac{\alpha_m^2}{\lambda_1}\Big)\Big]\|w\|^2_1 + \varepsilon\|z\|^2.
\end{aligned}
$$

Being $\varkappa_m > 0$ it is apparent to see that, fixing $\varepsilon > 0$ small enough, there exists $\mathfrak{c} > 0$ such that the right-hand side of the inequality above is greater then or equal to $\mathfrak{c}\|\boldsymbol{U}\|^2_{\mathcal{H}}$. ☐

Remark 3.3. When $\varkappa = 0$, we can choose m to be any positive real number. Later on, we will be interested in taking m suitably small and, at the same time, in knowing the behavior of the corresponding constant $\mathfrak{c} = \mathfrak{c}(m)$ as $m \to 0$. From the proof above one can see that $\mathfrak{c}(m)$ goes to zero, but not faster than m. More precisely, choosing $\varepsilon = \varepsilon(m)$ in a proper way, the relation

$$
\limsup_{m \to 0^+} \frac{m}{\mathfrak{c}(m)} < \infty \tag{3.2}
$$

holds. The details are left to the reader.

General agreement. From now on, if not explicitly stated, the space $\mathcal{H}$ is understood to be endowed with the equivalent inner product $(\cdot,\cdot)_{\mathcal{H}}$ and norm $|\cdot|_{\mathcal{H}}$.

Proof of Theorem 3.1. For a fixed $T>0$, we consider a regular solution to (2.1)

$$\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t), \partial_{ttt} u(t)) \in \mathcal{C}([0,T],\mathcal{H})$$

on the time interval $[0,T]$, and we set

$$\begin{aligned}
\mathsf{E}(t) &= \frac{1}{2}|\boldsymbol{U}(t)|^2_{\mathcal{H}} \\
&= \frac{1}{2}\Big[\frac{\delta}{\alpha_m}\|\partial_{tt}u(t) + \alpha_m\partial_t u(t) + \frac{\alpha_m\varrho}{\delta}u(t)\|_1^2 + \frac{\varkappa_m\alpha_m\varrho}{\delta}\|\partial_t u(t)\|_1^2 \\
&\quad + \varkappa_m\|\partial_{tt}u(t)\|_1^2 + \|\partial_{ttt}u(t) + \alpha_m\partial_{tt}u(t) + \frac{\alpha_m\varrho}{\delta}\partial_t u(t)\|^2\Big].
\end{aligned}$$

Our aim is to provide the basic energy estimate

$$\mathsf{E}(t) \leq \mathsf{E}(0)\mathrm{e}^{2\omega t} \tag{3.3}$$

for some $\omega \geq 0$. Being the problem linear, this suffices for well-posedness (see [4] for details). In particular, (3.1) holds true. In order to prove (3.3), we take the inner product in $\mathcal{H}$ of (2.3) with $\boldsymbol{U}(t)$. By means of direct calculations, we obtain the *energy identity*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E} + \alpha_m\varkappa_m\|\partial_{tt}u\|_1^2 &= m\langle\partial_{ttt}u, \partial_{ttt}u + \alpha_m\partial_{tt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle \\
&\quad + \Big(\frac{m\varrho}{\delta} - \varpi\Big)\langle\partial_{tt}u, \partial_{ttt}u + \alpha_m\partial_{tt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle.
\end{aligned}\tag{3.4}$$

Exploiting Lemma 3.2, together with the Young and the Poincaré inequalities, there exists $\omega = \omega(\alpha,\beta,\gamma,\delta,\varrho,\lambda_1,m) \geq 0$ such that the right-hand side is less than or equal to $2\omega\mathsf{E}$. Recalling that $\varkappa_m > 0$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E} \leq 2\omega\mathsf{E}.$$

Finally, an application of the Gronwall lemma yields (3.3).

4. The Spectrum of $\mathbb{A}$

Before stating our main results on the asymptotic properties of $S(t)$, we provide a complete characterization of the spectrum of (the complexification of) the operator $\mathbb{A}$. Such a description will play a crucial role in the sequel. To this end, for every fixed $\lambda > 0$, we consider the fourth-order polynomial in the variable $\zeta \in \mathbb{C}$ defined as

$$P_\lambda(\zeta) = \zeta^4 + \alpha\zeta^3 + (\beta + \gamma\lambda)\zeta^2 + \delta\lambda\zeta + \varrho\lambda.$$

Theorem 4.1. *The spectrum of* $\mathbb{A}$ *is given by*

$$\sigma(\mathbb{A}) = \bigcup_{\lambda\in\sigma(A)} \big\{\zeta\in\mathbb{C} : P_\lambda(\zeta) = 0\big\} \cup \big\{\zeta_1, \zeta_2\big\},$$

where ζ_1, ζ_2 *are the two complex solutions of the second-order equation*

$$\gamma\zeta^2 + \delta\zeta + \varrho = 0.$$

Proof. Let $\boldsymbol{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ be arbitrarily fixed. We look for a unique solution $\boldsymbol{U} = (u, v, w, z) \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation

$$\zeta \boldsymbol{U} - \mathbb{A}\boldsymbol{U} = \boldsymbol{F}$$

which, written componentwise, reads

$$\begin{cases} \zeta u - v = f_1, \\ \zeta v - w = f_2, \\ \zeta w - z = f_3, \\ \zeta z + \alpha z + \beta w + A(\gamma w + \delta v + \varrho u) = f_4. \end{cases} \tag{4.1}$$

From the system above, we infer that

$$\zeta^4 u + \alpha\zeta^3 u + \beta\zeta^2 u + \gamma\zeta^2 Au + \delta\zeta Au + \varrho Au = Af,$$

having set

$$f = A^{-1}\big[(\zeta^3 + \alpha\zeta^2 + \beta\zeta)f_1 + (\zeta^2 + \alpha\zeta + \beta)f_2 + (\zeta + \alpha)f_3 + f_4\big] + (\gamma\zeta + \delta)f_1 + \gamma f_2.$$

It is apparent to see that $f \in \mathrm{H}_1$. Moreover, by the functional calculus

$$u = \int_{\sigma(A)} \frac{\lambda}{P_\lambda(\zeta)}\, \mathrm{d}E_A(\lambda)\, f,$$

where E_A is the spectral measure of A. Therefore, $u \in \mathrm{H}_1$ for every given $f \in \mathrm{H}_1$ if and only if

$$\sup_{\lambda\in\sigma(A)} \left| \frac{\lambda}{P_\lambda(\zeta)} \right| < \infty.$$

Being $\sigma(A) \subset \mathbb{R}$ a closed set, the latter occurs if and only if

$$\gamma\zeta^2 + \delta\zeta + \varrho \neq 0 \qquad \text{and} \qquad P_\lambda(\zeta) \neq 0, \quad \forall \lambda \in \sigma(A).$$

In which case, from system (4.1), one concludes that $\boldsymbol{U} \in \mathfrak{D}(\mathbb{A})$ is the unique solution to the resolvent equation. In turn, ζ belongs to the resolvent set $\rho(\mathbb{A})$. □

Definition 4.2. The *spectral bound* of $\mathbb{A}$ is the number σ_* defined as

$$\sigma_* = \sup\{\mathfrak{Re}\,\zeta : \zeta \in \sigma(\mathbb{A})\}.$$

Being $\mathbb{A}$ the infinitesimal generator of $S(t)$, it is well known from the general theory of linear semigroups that σ_* does not exceed the growth bound ω_*, namely,

$$\sigma_* \leq \omega_*. \tag{4.2}$$

In particular, if $S(t)$ is exponentially stable it follows that $\sigma_* < 0$.

We conclude with a technical result needed in the course of the investigation of the decay properties of (2.1). To this end, for every fixed $\lambda \in \sigma(A)$, we introduce the (real) function

$$R_\lambda(x) = x^4 + \alpha x^3 + (\beta+\gamma\lambda)x^2 + \delta\lambda x + \varrho\lambda + \Big(\frac{4x^3+3\alpha x^2+2(\beta+\gamma\lambda)x+\delta\lambda}{\alpha+4x}\Big)^2 - \frac{(4x^3+3\alpha x^2+2(\beta+\gamma\lambda)x+\delta\lambda)(6x^2+3\alpha x+\beta+\gamma\lambda)}{\alpha+4x}. \tag{4.3}$$

Lemma 4.3. *Let $\lambda \in \sigma(A)$ be fixed. Assume there exist $p \neq -\alpha/4$ and $q > 0$ such that*

$$R_\lambda(p) = 0 \qquad \textit{and} \qquad q^2 = \frac{4p^3+3\alpha p^2+2(\beta+\gamma\lambda)p+\delta\lambda}{\alpha+4p}.$$

Then the numbers

$$\zeta_\pm = p \pm \mathrm{i}q$$

belong to $\sigma(\mathbb{A})$.

Proof. By direct calculations,

$$\mathfrak{Re}\big[P_\lambda(\zeta_\pm)\big] = p^4 + \alpha p^3 + (\beta+\gamma\lambda)p^2 + \delta\lambda p + \varrho\lambda + q^2(q^2 - 6p^2 - 3\alpha p - (\beta+\gamma\lambda))$$

and

$$\mathfrak{Im}\big[P_\lambda(\zeta_\pm)\big] = \pm\Big[4(p^2-q^2)pq + \alpha(3p^2-q^2)q + 2(\beta+\gamma\lambda)pq + \delta\lambda q\Big].$$

Substituting the expression of q^2 into the formulae above, we get

$$\mathfrak{Im}\big[P_\lambda(\zeta_\pm)\big] = 0 \qquad \text{and} \qquad \mathfrak{Re}\big[P_\lambda(\zeta_\pm)\big] = R_\lambda(p) = 0.$$

The claim follows from Theorem 4.1.

5. Exponential Stability

Our main result provides a necessary and sufficient condition for the exponential stability of $S(t)$ in terms of the values of the stability numbers $\varkappa$ and ϖ defined in (1.8).

Theorem 5.1. *The semigroup $S(t)$ is exponentially stable if and only if*

$$\varkappa > 0 \qquad \textit{and} \qquad \varpi > -\lambda_1\varkappa.$$

The remaining of the section is devoted to the proof of Theorem 5.1.

5.1. **Proof of Theorem 5.1 (Sufficiency).** As customary, we agree to work with (regular) solutions

$$\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt}u(t), \partial_{ttt}u(t)) = S(t)\boldsymbol{U}_0$$

arising from initial data $\boldsymbol{U}_0$ belonging to the domain of the infinitesimal generator $\mathbb{A}$. In what follows, $C > 0$ will denote a *generic* positive constant depending only on the structural quantities of the problem, but independent of $\boldsymbol{U}_0$. Besides, the Hölder, Young and Poincaré inequalities will be used several times, often without explicit mention.

Due to the assumption $\varkappa > 0$, we have

$$m = 0, \qquad \alpha_m = \alpha, \qquad \varkappa_m = \varkappa.$$

As a consequence, the energy identity (3.4) takes the simpler form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E} + \alpha\varkappa\|\partial_{tt}u\|_1^2 + \varpi\langle\partial_{tt}u, \partial_{ttt}u + \alpha\partial_{tt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle = 0,$$

where now

$$\begin{aligned}\mathsf{E}(t) = \frac{1}{2}\Big[&\frac{\delta}{\alpha}\|\partial_{tt}u(t) + \alpha\partial_t u(t) + \frac{\alpha\varrho}{\delta}u(t)\|_1^2 + \frac{\varkappa\alpha\varrho}{\delta}\|\partial_t u(t)\|_1^2\\ &+ \varkappa\|\partial_{tt}u(t)\|_1^2 + \|\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \frac{\alpha\varrho}{\delta}\partial_t u(t)\|^2\Big].\end{aligned}$$

In addition, since

$$\varpi\langle\partial_{tt}u, \partial_{ttt}u + \alpha\partial_{tt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle = \frac{\varpi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|\partial_{tt}u\|^2 + \frac{\alpha\varrho}{\delta}\|\partial_t u\|^2\Big] + \alpha\varpi\|\partial_{tt}u\|^2,$$

we arrive at the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F} + \alpha\varkappa\|\partial_{tt}u\|_1^2 + \alpha\varpi\|\partial_{tt}u\|^2 = 0$$

having set

$$\mathsf{F}(t) = \mathsf{E}(t) + \frac{\varpi}{2}\Big[\|\partial_{tt}u(t)\|^2 + \frac{\alpha\varrho}{\delta}\|\partial_t u(t)\|^2\Big].$$

Being $\varpi > -\lambda_1\varkappa$, there exists $\nu > 0$ such that

$$\varpi = \nu - \lambda_1\varkappa.$$

Hence, estimating

$$\varkappa\|\partial_{tt}u\|_1^2 + \varpi\|\partial_{tt}u\|^2 \geq \theta\|\partial_{tt}u\|_1^2$$

with

$$\theta = \min\Big\{\varkappa, \frac{\nu}{\lambda_1}\Big\} > 0,$$

we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{F} + \alpha\theta\|\partial_{tt}u\|_1^2 \leq 0. \tag{5.1}$$

By the same token, we achieve the control

$$\begin{aligned}\mathsf{F}(t) \geq \frac{1}{2}\Big[&\frac{\delta}{\alpha}\|\partial_{tt}u(t) + \alpha\partial_t u(t) + \frac{\alpha\varrho}{\delta}u(t)\|_1^2 + \frac{\theta\alpha\varrho}{\delta}\|\partial_t u(t)\|_1^2\\ &+ \theta\|\partial_{tt}u(t)\|_1^2 + \|\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \frac{\alpha\varrho}{\delta}\partial_t u(t)\|^2\Big].\end{aligned}$$

As a consequence, arguing as in the proof of Lemma 3.2 (with θ in place of $\varkappa_m$), we infer that

$$\frac{1}{C}\|\boldsymbol{U}(t)\|_{\mathcal{H}}^2 \leq \mathsf{F}(t) \leq C\|\boldsymbol{U}(t)\|_{\mathcal{H}}^2. \tag{5.2}$$

Our next aim is to reconstruct the term $\|\boldsymbol{U}\|_{\mathcal{H}}^2$ on the left-hand side of (5.1). To this end, we introduce three auxiliary functionals

$$
\begin{aligned}
\Phi(t) &= \frac{1}{2}\|\partial_{ttt}u(t)\|^2 + \frac{\gamma}{2}\|\partial_{tt}u(t)\|_1^2 + \langle \delta\partial_t u(t) + \varrho u(t), \partial_{tt}u(t)\rangle_1,\\
\Psi(t) &= \langle \partial_{ttt}u(t) + \alpha\partial_{tt}u(t), \partial_t u(t)\rangle + \frac{\varrho}{2}\|u(t)\|_1^2,\\
\Upsilon(t) &= \langle \partial_{ttt}u(t), u(t)\rangle.
\end{aligned}
$$

By direct calculations, the functional Φ fulfills the identity

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \alpha\|\partial_{ttt}u\|^2 = \delta\|\partial_{tt}u\|_1^2 - \beta\langle\partial_{tt}u, \partial_{ttt}u\rangle + \varrho\langle\partial_t u, \partial_{tt}u\rangle_1.
$$

It is also immediate to see that

$$
\delta\|\partial_{tt}u\|_1^2 - \beta\langle\partial_{tt}u, \partial_{ttt}u\rangle + \varrho\langle\partial_t u, \partial_{tt}u\rangle_1 \leq \frac{\alpha}{2}\|\partial_{ttt}u\|^2 + \frac{\delta}{4}\|\partial_t u\|_1^2 + C\|\partial_{tt}u\|_1^2.
$$

Thus, we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Phi + \frac{\alpha}{2}\|\partial_{ttt}u\|^2 \leq \frac{\delta}{4}\|\partial_t u\|_1^2 + C\|\partial_{tt}u\|_1^2. \tag{5.3}
$$

Concerning the functional Ψ, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \delta\|\partial_t u\|_1^2 &= \alpha\|\partial_{tt}u\|^2 + \langle\partial_{ttt}u, \partial_{tt}u\rangle - \beta\langle\partial_{tt}u, \partial_t u\rangle - \gamma\langle\partial_{tt}u, \partial_t u\rangle_1\\
&\leq \frac{\alpha}{4}\|\partial_{ttt}u\|^2 + \frac{\delta}{2}\|\partial_t u\|_1^2 + C\|\partial_{tt}u\|_1^2.
\end{aligned}
$$

Hence, we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Psi + \frac{\delta}{2}\|\partial_t u\|_1^2 \leq \frac{\alpha}{4}\|\partial_{ttt}u\|^2 + C\|\partial_{tt}u\|_1^2. \tag{5.4}
$$

Finally, the functional Υ satisfies the equality

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon + \varrho\|u\|_1^2 = \langle\partial_{ttt}u, \partial_t u\rangle - \langle\alpha\partial_{ttt}u + \beta\partial_{tt}u, u\rangle - \langle\gamma\partial_{tt}u + \delta\partial_t u, u\rangle_1.
$$

Estimating the right-hand side as

$$
\begin{aligned}
&\langle\partial_{ttt}u, \partial_t u\rangle - \langle\alpha\partial_{ttt}u + \beta\partial_{tt}u, u\rangle - \langle\gamma\partial_{tt}u + \delta\partial_t u, u\rangle_1\\
&\leq \frac{\varrho}{2}\|u\|_1^2 + C\|\partial_{ttt}u\|^2 + C\|\partial_{tt}u\|_1^2 + C\|\partial_t u\|_1^2,
\end{aligned}
$$

we conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon + \frac{\varrho}{2}\|u\|_1^2 \leq C\|\partial_{ttt}u\|^2 + C\|\partial_{tt}u\|_1^2 + C\|\partial_t u\|_1^2. \tag{5.5}
$$

At this point, for every $\varepsilon > 0$, we set

$$
\Lambda_\varepsilon(t) = \mathsf{F}(t) + \varepsilon[\Phi(t) + \Psi(t)] + \varepsilon^2\Upsilon(t).
$$

In the light of (5.1) and (5.3)-(5.5), the functional Λ_ε fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon\Big(\frac{\alpha}{4} - C\varepsilon\Big)\|\partial_{ttt}u\|^2 + (\alpha\theta - C\varepsilon - C\varepsilon^2)\|\partial_{tt}u\|_1^2 + \varepsilon\Big(\frac{\delta}{4} - C\varepsilon\Big)\|\partial_t u\|_1^2 + \frac{\varepsilon^2\varrho}{2}\|u\|_1^2 \le 0.$$

Hence, for all $\varepsilon > 0$ sufficiently small, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + \varepsilon^3\|\boldsymbol{U}\|_{\mathcal{H}}^2 \le 0.$$

Exploiting now (5.2) it is readily seen that, possibly reducing $\varepsilon > 0$,

$$\frac{1}{C}\|\boldsymbol{U}(t)\|_{\mathcal{H}}^2 \le \Lambda_\varepsilon(t) \le C\|\boldsymbol{U}(t)\|_{\mathcal{H}}^2. \tag{5.6}$$

In conclusion, fixing $\varepsilon > 0$ small enough, from the differential inequality above we learn that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_\varepsilon + 2\omega\Lambda_\varepsilon \le 0$$

for some $\omega = \omega(\varepsilon) > 0$. An application of the Gronwall lemma, together with (5.6), yields the desired exponential decay.

5.2. **Proof of Theorem 5.1 (Necessity).** Due to (4.2), in order to prove the lack of exponential stability it is enough showing that $\sigma_* \ge 0$. We shall treat separately four cases. In what follows, the function R_λ is given by (4.3).

◇ **Case 1: $\boldsymbol{\varpi < -\lambda_1\varkappa}$.** It is immediate to see that

$$R_{\lambda_1}(0) = -\frac{\delta\lambda_1}{\alpha}(\varpi + \lambda_1\varkappa) > 0 \qquad \text{and} \qquad \lim_{x\to\infty} R_{\lambda_1}(x) = -\infty.$$

Being R_{λ_1} continuous on $[0,\infty)$, there exists a real root $p > 0$. In the light of Lemma 4.3, the complex numbers

$$\zeta_\pm = p \pm \mathrm{i}\sqrt{\frac{4p^3 + 3\alpha p^2 + 2(\beta + \gamma\lambda_1)p + \delta\lambda_1}{\alpha + 4p}}$$

belong to $\sigma(\mathbb{A})$, and thus $\sigma_* > 0$.

◇ **Case 2: $\boldsymbol{\varpi = -\lambda_1\varkappa}$.** We have

$$R_{\lambda_1}(0) = 0.$$

According to Lemma 4.3, the spectrum of $\mathbb{A}$ contains the two purely imaginary numbers

$$\zeta_\pm = \pm\mathrm{i}\sqrt{\frac{\delta\lambda_1}{\alpha}},$$

and therefore $\sigma_* \ge 0$.

◇ **Case 3: $\boldsymbol{\varkappa < 0}$.** For any given $\lambda \in \sigma(A)$,

$$R_\lambda(0) = -\frac{\delta\lambda}{\alpha}(\varpi + \lambda\varkappa) \qquad \text{and} \qquad \lim_{x\to\infty} R_\lambda(x) = -\infty.$$

Being A an unbounded operator, λ can be taken arbitrarily large. Since

$$R_\lambda(0) \sim -\frac{\delta\lambda^2\varkappa}{\alpha} > 0 \quad \text{as } \lambda \to \infty,$$

the value $R_\lambda(0)$ eventually becomes positive. Arguing as in Case 1, we conclude that $\sigma_* > 0$.

◇ **Case 4: $\varkappa = 0$.** It is sufficient to analyze only the situation when $\varpi > 0$, since the case $\varpi \leq \varkappa = 0$ has been already addressed. Choose $\lambda_n \in \sigma(A)$ such that $\lambda_n \to \infty$ (this is possible since A is unbounded). In particular, for any fixed n, we have

$$R_{\lambda_n}(0) = -\frac{\delta\lambda_n\varpi}{\alpha} < 0.$$

Moreover, taking

$$M > \frac{\alpha\varpi}{2\gamma}$$

and exploiting the assumption $\varkappa = 0$, by direct computations we obtain

$$\lim_{n\to\infty} \frac{1}{\lambda_n} R_{\lambda_n}\Big(-\frac{M}{\lambda_n}\Big) = \frac{\delta(2\gamma M - \alpha\varpi)}{\alpha^2} > 0.$$

Therefore, for all n sufficiently large, the equation $R_{\lambda_n}(p) = 0$ admits a real solution

$$p_n = -\frac{M_n}{\lambda_n} < 0 \qquad \text{with} \qquad 0 < M_n < M.$$

In particular, $p_n \to 0$ as $n \to \infty$. By the same token,

$$q_n^2 := \frac{4p_n^3 + 3\alpha p_n^2 + 2(\beta + \gamma\lambda_n)p_n + \delta\lambda_n}{\alpha + 4p_n} \sim \frac{\delta\lambda_n}{\alpha} \to \infty.$$

Hence, for every n large enough, the hypotheses of Lemma 4.3 are satisfied, meaning that

$$p_n \pm \mathrm{i}q_n \in \sigma(\mathbb{A}).$$

Since $p_n \to 0$, we conclude that $\sigma_* \geq 0$. □

Remark 5.2. From the proof of the necessity part of Theorem 5.1 we learn that, if Case 1 or Case 3 hold, the spectral bound σ_* is strictly positive. Accordingly, the semigroup $S(t)$ has solutions with energy growing exponentially fast.

6. The Remaining Cases

From the discussion above, ω_* is greater than or equal to zero if

(i) $\varkappa > 0$ and $\varpi = -\lambda_1\varkappa$, or
(ii) $\varkappa = 0$ and $\varpi \geq 0$.

Actually, in these situations, ω_* turns out to be exactly zero. To see that, for an arbitrarily fixed $\varepsilon > 0$, we consider the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{U}(t) = \mathbb{A}\boldsymbol{U}(t) - \varepsilon\boldsymbol{U}(t). \tag{6.1}$$

It is well known that the operator $\mathbb{A} - \varepsilon\mathbb{I}$ generates the $\mathcal{C}_0$-semigroup

$$S_\varepsilon(t) = \mathrm{e}^{-\varepsilon t}S(t)$$

on the space $\mathcal{H}$ (see e.g. [12]). In particular,

$$\|S_\varepsilon(t)\| = \mathrm{e}^{-\varepsilon t}\|S(t)\|.$$

Lemma 6.1. *Assume that* (i) *holds. Then, for every fixed* $\varepsilon > 0$*, the semigroup* $S_\varepsilon(t)$ *is bounded.*

Proof. As usual, we work with (regular) solutions to (6.1)

$$\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t), \partial_{ttt} u(t)) = S_\varepsilon(t)\boldsymbol{U}_0$$

arising from initial data $\boldsymbol{U}_0$ belonging to the domain of $\mathbb{A}$. First, we take the inner product in $\mathcal{H}$ of (6.1) with $\boldsymbol{U}(t)$. Recalling that the constant m equals zero, we obtain the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{U}|^2_{\mathcal{H}} + \varepsilon|\boldsymbol{U}|^2_{\mathcal{H}} + \alpha\varkappa\|\partial_{tt}u\|^2_1 - \alpha\lambda_1\varkappa\|\partial_{tt}u\|^2 - \lambda_1\varkappa\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle = 0.$$

Applying the Poincaré inequality (2.2), we infer that

$$\alpha\varkappa\|\partial_{tt}u\|^2_1 - \alpha\lambda_1\varkappa\|\partial_{tt}u\|^2 \geq 0.$$

Thus, writing

$$\begin{aligned} &- \lambda_1\varkappa\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle \\ &= \frac{(\nu - \lambda_1\varkappa)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\partial_{tt}u\|^2 + \frac{\alpha\varrho}{\delta}\|\partial_t u\|^2\big] - \nu\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle \end{aligned}$$

for some $0 < \nu < \lambda_1\varkappa$ to be fixed later, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{G} + \varepsilon|\boldsymbol{U}|^2_{\mathcal{H}} \leq \nu\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle,$$

having set

$$\mathsf{G}(t) = \frac{1}{2}\Big[|\boldsymbol{U}(t)|^2_{\mathcal{H}} + (\nu - \lambda_1\varkappa)\|\partial_{tt}u(t)\|^2 + (\nu - \lambda_1\varkappa)\frac{\alpha\varrho}{\delta}\|\partial_t u(t)\|^2\Big].$$

Exploiting Lemma 3.2, it is immediate to see that the right-hand side is controlled by

$$\nu\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha\varrho}{\delta}\partial_t u\rangle \leq \nu C|\boldsymbol{U}|^2_{\mathcal{H}},$$

where $C > 0$ is a structural constant depending on the parameters of the problem but independent of ν. Accordingly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{G} + (\varepsilon - \nu C)|\boldsymbol{U}|^2_{\mathcal{H}} \leq 0.$$

Due to (2.2), the functional G fulfills

$$\begin{aligned} \mathsf{G}(t) \geq \frac{1}{2}\Big[&\frac{\delta}{\alpha}\|\partial_{tt}u(t) + \alpha\partial_t u(t) + \frac{\alpha\varrho}{\delta}u(t)\|^2_1 + \frac{\nu\alpha\varrho}{\delta\lambda_1}\|\partial_t u(t)\|^2_1 \\ &+ \frac{\nu}{\lambda_1}\|\partial_{tt}u(t)\|^2_1 + \|\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \frac{\alpha\varrho}{\delta}\partial_t u(t)\|^2\Big]. \end{aligned}$$

Hence, making use of the inequality above and arguing as in the proof of Lemma 3.2 (with ν/λ_1 in place of $\varkappa_m$), we find

$$c_\nu|\boldsymbol{U}(t)|^2_{\mathcal{H}} \leq G(t) \leq \frac{1}{2}|\boldsymbol{U}(t)|^2_{\mathcal{H}},$$

for some $c_\nu > 0$ depending on ν. At this point, fixing

$$\nu = \nu(\varepsilon) = \min\Big\{\frac{\varepsilon}{2C}, \frac{\lambda_1\varkappa}{2}\Big\},$$

we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{G} + 2\omega\mathsf{G} \leq 0,$$

for some $\omega = \omega(\varepsilon) > 0$. The Gronwall lemma completes the argument.

Lemma 6.2. *Assume that* (ii) *holds. Then, for every fixed* $\varepsilon > 0$, *the semigroup* $S_\varepsilon(t)$ *is bounded.*

Proof. Let $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$ be any (regular) initial datum. Along the proof, $C_m > 0$ will denote a *generic* positive constant depending on $m > 0$ (besides the other structural quantities of the problem) but independent of $\boldsymbol{U}_0$. A multiplication in $\mathcal{H}$ of (6.1) with $\boldsymbol{U}(t)$ leads to the equality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{U}|^2_{\mathcal{H}} + \varepsilon|\boldsymbol{U}|^2_{\mathcal{H}} + \alpha_m\varkappa_m\|\partial_{tt}u\|^2_1 + \alpha_m\varpi\|\partial_{tt}u\|^2 + \varpi\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle \\ &= m\langle\partial_{ttt}u + \frac{\varrho}{\delta}\partial_{tt}u, \partial_{ttt}u + \alpha_m\partial_{tt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle.\end{aligned}$$

Writing

$$\varpi\langle\partial_{tt}u, \partial_{ttt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle = \frac{\varpi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\partial_{tt}u\|^2 + \frac{\alpha_m\varrho}{\delta}\|\partial_t u\|^2\big],$$

and recalling that $\varpi \geq 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{L} + \varepsilon|\boldsymbol{U}|^2_{\mathcal{H}} \leq m\langle\partial_{ttt}u + \frac{\varrho}{\delta}\partial_{tt}u, \partial_{ttt}u + \alpha_m\partial_{tt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle, \tag{6.2}$$

having set

$$\mathsf{L}(t) = \frac{1}{2}\Big[|\boldsymbol{U}(t)|^2_{\mathcal{H}} + \varpi\|\partial_{tt}u(t)\|^2 + \frac{\varpi\alpha_m\varrho}{\delta}\|\partial_t u(t)\|^2\Big].$$

The differential inequality (6.2) above holds true for $m = 0$ as well. In this situation, since $\alpha_m = \alpha$ and $\varkappa_m = \varkappa = 0$, it turns into

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\mathcal{E} + \frac{\varpi}{2}\|\partial_{tt}u\|^2 + \frac{\varpi\alpha\varrho}{2\delta}\|\partial_t u\|^2\Big] + 2\varepsilon\mathcal{E} \leq 0,$$

where $\mathcal{E}(t)$ denotes the *pseudoenergy*

$$\mathcal{E}(t) = \frac{1}{2}\Big[\frac{\delta}{\alpha}\|\partial_{tt}u(t) + \alpha\partial_t u(t) + \frac{\alpha\varrho}{\delta}u(t)\|^2_1 + \|\partial_{ttt}u(t) + \alpha\partial_{tt}u(t) + \frac{\alpha\varrho}{\delta}\partial_t u(t)\|^2\Big].$$

Being $\mathcal{E}(t) \geq 0$, an integration over $[0, t]$, together with (2.2) and Lemma 3.2, gives

$$\mathcal{E}(t) \leq \mathcal{E}(t) + \frac{\varpi}{2}\|\partial_{tt}u(t)\|^2 + \frac{\varpi\alpha\varrho}{2\delta}\|\partial_t u(t)\|^2 \leq C_m|\boldsymbol{U}_0|^2_{\mathcal{H}}, \tag{6.3}$$

where the dependence on m of the constant C_m comes from the inequality of Lemma 3.2. Coming back to (6.2), the right-hand side is estimated as

$$m\langle\partial_{ttt}u + \frac{\varrho}{\delta}\partial_{tt}u, \partial_{ttt}u + \alpha_m\partial_{tt}u + \frac{\alpha_m\varrho}{\delta}\partial_t u\rangle \leq c_1 m\sqrt{\mathcal{E}}\|\boldsymbol{U}\|_{\mathcal{H}} + c_1 m^2\|\boldsymbol{U}\|^2_{\mathcal{H}},$$

where $c_1 > 0$ is a structural constant depending of $\alpha, \beta, \gamma, \delta, \varrho, \lambda_1$, but independent of m and $\boldsymbol{U}_0$. Invoking (6.3), we get

$$c_1 m\sqrt{\mathcal{E}}\|\boldsymbol{U}\|_{\mathcal{H}} + c_1 m^2\|\boldsymbol{U}\|_{\mathcal{H}}^2 \leq 2c_1 m^2\|\boldsymbol{U}\|_{\mathcal{H}}^2 + \frac{c_1}{4}\mathcal{E} \leq 2c_1 m^2\|\boldsymbol{U}\|_{\mathcal{H}}^2 + C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2.$$

Therefore, inequality (6.2) takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{L} + \varepsilon|\boldsymbol{U}|_{\mathcal{H}}^2 \leq 2c_1 m^2\|\boldsymbol{U}\|_{\mathcal{H}}^2 + C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2.$$

At this point, we have

$$2c_1 m^2\|\boldsymbol{U}\|_{\mathcal{H}}^2 \leq \frac{2c_1 m^2}{\mathfrak{c}}|\boldsymbol{U}|_{\mathcal{H}}^2$$

where $\mathfrak{c} = \mathfrak{c}(m) > 0$ is the constant of Lemma 3.2. Recalling (3.2), there exists $c_2 > 0$ independent of m and $\boldsymbol{U}_0$ such that

$$\frac{2c_1 m^2}{\mathfrak{c}}|\boldsymbol{U}|_{\mathcal{H}}^2 \leq c_2 m|\boldsymbol{U}|_{\mathcal{H}}^2,$$

provided that $m > 0$ is sufficiently small. Accordingly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{L} + (\varepsilon - c_2 m)|\boldsymbol{U}|_{\mathcal{H}}^2 \leq C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2.$$

Exploiting once more the assumption $\varpi \geq 0$, together with Lemma 3.2, we also find the controls

$$\frac{1}{2}|\boldsymbol{U}(t)|_{\mathcal{H}}^2 \leq \mathsf{L}(t) \leq C_m|\boldsymbol{U}(t)|_{\mathcal{H}}^2. \tag{6.4}$$

In conclusion, fixing $m = m(\varepsilon) > 0$ small enough, there exists $\omega = \omega(\varepsilon) > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{L} + 2\omega\mathsf{L} \leq C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2.$$

Owing to the Gronwall lemma and (6.4), we arrive at

$$|\boldsymbol{U}(t)|_{\mathcal{H}}^2 \leq 2\mathsf{L}(t) \leq 2\mathsf{L}(0)\mathrm{e}^{-2\omega t} + \frac{C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2}{\omega} \leq C_m|\boldsymbol{U}_0|_{\mathcal{H}}^2.$$

By density, the control above holds true for every initial datum $\boldsymbol{U}_0 \in \mathcal{H}$ as well, yielding the sought boundedness of $S_\varepsilon(t)$. □

We are now in a position to show that $\omega_* = 0$. Being $S_\varepsilon(t)$ bounded, for all $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$\|S(t)\| = \mathrm{e}^{\varepsilon t}\|S_\varepsilon(t)\| \leq C\mathrm{e}^{\varepsilon t}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, the conclusion follows.

We subsume all the results obtained so far in a theorem.

Theorem 6.3. *The following hold.*

- $\omega_* < 0$ *whenever* $\varkappa > 0$ *and* $\varpi > -\lambda_1\varkappa$.
- $\omega_* > 0$ *whenever* $\varkappa < 0$, *or whenever* $\varpi < -\lambda_1\varkappa$.
- $\omega_* = 0$ *whenever* $\varkappa > 0$ *and* $\varpi = -\lambda_1\varkappa$, *or whenever* $\varkappa = 0$ *and* $\varpi \geq 0$.

We conclude by analyzing in more detail the case $\omega_* = 0$, namely, the borderline situation between exponential stability and blow up of solutions. In several equations of physical interest, in cases of this kind the semigroup is stable, namely, every trajectory goes to zero. As a byproduct, it turns out to be bounded. Here, we will show that stability does not occur when both $\varkappa$ and ϖ equal zero. Nonetheless, we cannot exclude that the semigroup $S(t)$ is bounded.

Theorem 6.4. *Assume that $\varkappa = \varpi = 0$. Then $S(t)$ is not stable, namely, there exists an initial datum $\boldsymbol{U}_0 \in \mathcal{H}$ such that*

$$S(t)\boldsymbol{U}_0 \not\to 0.$$

In order to prove Theorem 6.4, a preliminary result is needed.

Lemma 6.5. *Let $\varkappa = \varpi = 0$, and let $u(t)$ be the solution to equation (2.1) corresponding to the initial datum $\boldsymbol{U}_0 = (u_0, v_0, w_0, z_0) \in \mathcal{H}$. Then, the following mutually disjoint*[1] *situations occur.*

- *if $\alpha^2 > 4\beta$, then $u(t)$ solves the Cauchy problem*

$$\begin{cases} \partial_{tt} u(t) + \gamma A u(t) = \dfrac{(\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t})(z_0 + \gamma A v_0) - (r_2 \mathrm{e}^{r_1 t} - r_1 \mathrm{e}^{r_2 t})(w_0 + \gamma A u_0)}{r_1 - r_2}, \\ u(0) = u_0, \\ \partial_t u(0) = v_0, \end{cases} \tag{6.5}$$

where

$$r_1 = \frac{-\alpha + \sqrt{\alpha^2 - 4\beta}}{2} < 0 \qquad \text{and} \qquad r_2 = \frac{-\alpha - \sqrt{\alpha^2 - 4\beta}}{2} < 0.$$

- *if $\alpha^2 = 4\beta$, then $u(t)$ solves the Cauchy problem*

$$\begin{cases} \partial_{tt} u(t) + \gamma A u(t) = t\mathrm{e}^{-\frac{\alpha t}{2}}(z_0 + \gamma A v_0) + \dfrac{\mathrm{e}^{-\frac{\alpha t}{2}}(2 + \alpha t)(w_0 + \gamma A u_0)}{2}, \\ u(0) = u_0, \\ \partial_t u(0) = v_0, \end{cases} \tag{6.6}$$

- *if $\alpha^2 < 4\beta$, then $u(t)$ solves the Cauchy problem*

$$\begin{cases} \partial_{tt} u(t) + \gamma A u(t) = \mathrm{e}^{-\frac{\alpha t}{2}} f(t)(z_0 + \gamma A v_0) + \mathrm{e}^{-\frac{\alpha t}{2}} g(t)(w_0 + \gamma A u_0), \\ u(0) = u_0, \\ \partial_t u(0) = v_0, \end{cases} \tag{6.7}$$

where

$$\begin{aligned} f(t) &= \frac{2}{\sqrt{4\beta - \alpha^2}} \sin\Big(\frac{\sqrt{4\beta - \alpha^2}}{2} t\Big), \\ g(t) &= \cos\Big(\frac{\sqrt{4\beta - \alpha^2}}{2} t\Big) + \frac{\alpha}{\sqrt{4\beta - \alpha^2}} \sin\Big(\frac{\sqrt{4\beta - \alpha^2}}{2} t\Big). \end{aligned}$$

[1] It is apparent to see that all the three cases are possible.

Proof. When $\varkappa = \varpi = 0$, equation (2.1) can be rewritten in the simpler form

$$\partial_{tt}\phi + \alpha\partial_t\phi + \beta\phi = 0,$$

having set

$$\phi(t) = \partial_{tt}u(t) + \gamma Au(t).$$

Solving the differential equation above and imposing the initial conditions we conclude that, in dependence of the value of the discriminant $\alpha^2 - 4\beta$ to the associated characteristic equation, the function ϕ is a solution to (6.5), (6.6) or (6.7).

Proof of Theorem 6.4. Let $u_0, v_0 \in \mathrm{H}_2$ be arbitrarily fixed. In the light of Lemma 6.5, the solution to (2.1) corresponding to initial data of the form

$$\boldsymbol{U}_0 = (u_0, v_0, -\gamma Au_0, -\gamma Av_0) \in \mathcal{H}$$

solves the conservative wave equation

$$\begin{cases} \partial_{tt}u(t) + \gamma Au(t) = 0, \\ u(0) = u_0, \\ \partial_t u(0) = v_0. \end{cases}$$

In particular, for all $t \geq 0$, the equality

$$\|\partial_t u(t)\|^2 + \gamma\|u(t)\|_1^2 = \|v_0\|^2 + \gamma\|u_0\|_1^2$$

holds. Hence, exploiting Lemma 3.2, we conclude that

$$\begin{aligned} |S(t)\boldsymbol{U}_0|_{\mathcal{H}}^2 &\geq \mathfrak{c}\min\{1/\gamma, \lambda_1\}\big(\|\partial_t u(t)\|^2 + \gamma\|u(t)\|_1^2\big) \\ &= \mathfrak{c}\min\{1/\gamma, \lambda_1\}\big(\|v_0\|^2 + \gamma\|u_0\|_1^2\big). \end{aligned}$$

The proof is finished.

References

[1] M. Conti, S. Gatti and V. Pata, *Uniform decay properties of linear Volterra integro-differential equations*, Math. Models Methods Appl. Sci. **18** (2008), 21-45.

[2] C.M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 297-308.

[3] F. Dell'Oro, I. Lasiecka and V. Pata, *The Moore-Gibson-Thompson equation with memory in the critical case*, J. Differential Equations **261** (2016), 4188–4222.

[4] F. Dell'Oro and V. Pata, *On the Moore-Gibson-Thompson equation and its relation to linear viscoelasticity*, Appl. Math. Optim. (to appear).

[5] P. Jordan, *Second-sound phenomena in inviscid, thermally relaxing gases*, Discrete Contin. Dyn. Syst. Ser. B **19** (2014), 2189–2205.

[6] B. Kaltenbacher, I. Lasiecka and R. Marchand, *Wellposedness and exponential decay rates for the Moore-Gibson-Thompson equation arising in high intensity ultrasound*, Control Cybernet. **40** (2011), 971–988.

[7] I. Lasiecka and X. Wang, *Moore-Gibson-Thompson equation with memory, part I: Exponential decay of energy*, Z. Angew. Math. Phys. **67** (2016), n.17.

[8] I. Lasiecka and X. Wang, *Moore-Gibson-Thompson equation with memory, part II: General decay of energy*, J. Differential Equations **259** (2015), 7610–7635.

[9] R. Marchand, T. McDevitt and R. Triggiani, *An abstract semigroup approach to the third-order Moore-Gibson-Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability*, Math. Methods Appl. Sci. **35** (2012), 1896–1929.

[10] F.K. Moore and W.E. Gibson, *Propagation of weak disturbances in a gas subject to relaxation effects*, J. Aero/Space Sci. **27** (1960), 117–127.

[11] K. Naugolnykh and L. Ostrovsky, *Nonlinear wave processes in acoustics*, Cambridge University Press, Cambridge, 1998.

[12] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[13] Professor Stokes, *An examination of the possible effect of the radiation of heat on the propagation of sound*, Philos. Mag. Series 4 **1** (1851), 305–317.

[14] P.A. Thompson, *Compressible-fluid dynamics*, McGraw-Hill, New York, 1972.

Politecnico di Milano - Dipartimento di Matematica
Via Bonardi 9, 20133 Milano, Italy
E-mail address: `filippo.delloro@polimi.it` (F. Dell'Oro)
E-mail address: `vittorino.pata@polimi.it` (V. Pata)