On the Moore–Gibson–Thompson Equation and Its Relation to Linear Viscoelasticity

Filippo Dell'Oro[1] · **Vittorino Pata**[1]

Abstract We discuss the parallel between the third-order Moore–Gibson–Thompson equation

$$\partial_{ttt} u + \alpha \partial_{tt} u - \beta \Delta \partial_t u - \gamma \Delta u = 0$$

depending on the parameters $\alpha, \beta, \gamma > 0$, and the equation of linear viscoelasticity

$$\partial_{tt} u(t) - \kappa(0) \Delta u(t) - \int_0^\infty \kappa'(s) \Delta u(t-s)\, \mathrm{d}s = 0$$

for the particular choice of the exponential kernel

$$\kappa(s) = a\mathrm{e}^{-bs} + c$$

with $a, b, c > 0$. In particular, the latter model is shown to exhibit a preservation of regularity for a certain class of initial data, which is unexpected in presence of a general memory kernel κ.

Keywords Moore–Gibson–Thompson equation · Exponential decay · Linear viscoelasticity

Mathematics Subject Classification 35B35 · 35G05 · 47D06 · 74D05

✉ Vittorino Pata
vittorino.pata@polimi.it

Filippo Dell'Oro
filippo.delloro@polimi.it

1 Dipartimento di Matematica, Politecnico di Milano, Via Bonardi 9, 20133 Milan, Italy

This is a post-peer-review, pre-copyedit version of an article published in Appl Math Optim (2017) 76:641–655. The final authenticated version is available online at: https://doi.org/10.1007/s00245-016-9365-1

1 Introduction

1.1 The MGT Equation

Given a smooth bounded domain $\Omega \subset \mathbb{R}^3$, we consider for $t > 0$ the third-order Moore–Gibson–Thompson (MGT) equation

$$\partial_{ttt} u + \alpha \partial_{tt} u - \beta \Delta \partial_t u - \gamma \Delta u = 0 \tag{1.1}$$

in the unknown variable $u = u(\boldsymbol{x}, t) : \Omega \times [0, \infty) \to \mathbb{R}$, where $\alpha, \beta, \gamma > 0$ are fixed constants, and

$$-\Delta \qquad \text{with domain} \qquad \mathfrak{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$$

is the Laplace–Dirichlet operator on the Hilbert space $L^2(\Omega)$. The equation is supplemented with the initial conditions assigned at time $t = 0$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \qquad \partial_t u(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}), \qquad \partial_{tt} u(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}),$$

being $u_0, v_0, w_0 : \Omega \to \mathbb{R}$ prescribed initial data. From the physical viewpoint, the MGT equation (1.1) arises in the modeling of wave propagation in viscous thermally relaxing fluids [9,15], although its first derivation seems to be due to Stokes [14]. Such an equation is known to generate a (linear) solution semigroup $S(t)$ acting on the natural weak energy space

$$\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

However, although the structural constants α, β, γ are all strictly positive, the asymptotic properties of $S(t)$ turn out to depend on the *stability number*

$$\varkappa = \beta - \frac{\gamma}{\alpha}.$$

Indeed, the following hold [5,8].

- If $\varkappa > 0$ then $S(t)$ is exponentially stable.
- If $\varkappa = 0$ then $S(t)$ is marginally stable.
- If $\varkappa < 0$ then $S(t)$ is unstable.

The reason of such a different behavior is that, in order for the system to exhibit exponential stability of the trajectories, the ratio between the sound speed γ and the sound diffusivity β has to be small with respect to the natural damping coefficient α (see [5]). Here, we want to give an alternative interpretation. To this end, let us first examine another (apparently unrelated) problem.

1.2 The Equation of Viscoelasticity

We consider for $t > 0$ the integro-differential equation[1]

$$\partial_{tt}u(t) - \kappa(0)\Delta u(t) - \int_0^\infty \kappa'(s)\Delta u(t-s)\,\mathrm{d}s = 0, \tag{1.2}$$

which rules the evolution of the relative displacement field $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ in a linearly viscoelastic solid occupying a volume Ω at rest (see e.g. [13]). The variable u is understood to be an assigned datum for negative times $t \leq 0$, while the function $\kappa \in \mathcal{C}^2(\mathbb{R}^+)$, usually called *memory kernel*, is supposed to be nonnegative, nonincreasing, convex, and such that

$$\kappa(0) > \kappa(\infty) > 0.$$

In his famous article [2], C.M. Dafermos proposed to introduce the auxiliary variable

$$\eta^t(s) = u(t) - u(t-s), \quad t \geq 0,\ s > 0,$$

containing all the information on the past history of u, which allows to rewrite (1.2) in the form

$$\begin{cases} \partial_{tt}u(t) - \kappa(\infty)\Delta u(t) + \displaystyle\int_0^\infty \kappa'(s)\Delta\eta^t(s)\,\mathrm{d}s = 0, \\ \partial_t\eta^t(s) = -\partial_s\eta^t(s) + \partial_t u(t). \end{cases} \tag{1.3}$$

The latter system generates a contraction semigroup $\Sigma(t)$ of solutions, acting on a suitable Hilbert space $\mathcal{V}$ accounting for the presence of the past history component η. This semigroup is long known to be stable, that is, all its trajectories vanish as t goes to infinity [2]. On the contrary, the problem of the uniform (exponential) stability has been completely understood only quite recently, although sufficient conditions can be found e.g. in [4,7,10]. Indeed, we have the following result from [12].

Theorem 1.1 *The semigroup $\Sigma(t)$ on $\mathcal{V}$ is exponentially stable if and only if there exist two constants $C \geq 1$ and $\delta > 0$ such that, for every $t \geq 0$ and every $s > 0$,*

$$\kappa'(t+s) \geq C\mathrm{e}^{-\delta t}\kappa'(s). \tag{1.4}$$

1.3 The Exponential Kernel

From the physical side, the most interesting (and most relevant) case for the equation of viscoelasticity (1.2) is the exponential kernel

$$\kappa(s) = a\mathrm{e}^{-bs} + c,$$

[1] Here and in what follows, the *prime* denotes the derivative with respect to the variable $s > 0$.

for some strictly positive constants a, b, c, which naturally arises in the description of viscoelastic solids via rheological models (see e.g. [3]). In which case, (1.2) becomes

$$\partial_{tt}u(t) - (a+c)\Delta u(t) + ab\int_0^\infty e^{-bs}\Delta u(t-s)\,ds = 0. \tag{1.5}$$

Observe that this particular κ complies with (1.4), which is actually an equality for $C = 1$ and $\delta = b$. Accordingly, the corresponding semigroup $\Sigma(t)$ is exponentially stable.

Differentiating (1.5) with respect to time, we have

$$\partial_{ttt}u(t) - (a+c)\Delta\partial_t u(t) + ab\Delta u(t) - ab^2\int_0^\infty e^{-bs}\Delta u(t-s)\,ds = 0. \tag{1.6}$$

Then, taking the sum $b(1.5) + (1.6)$, we obtain

$$\partial_{ttt}u + b\partial_{tt}u - (a+c)\Delta\partial_t u - bc\Delta u = 0, \tag{1.7}$$

which is nothing but the MGT equation with

$$\alpha = b, \qquad \beta = a + c, \qquad \gamma = bc.$$

An interpretation of (1.7) as a model of linear viscoelasticity has been also proposed in [1]. In particular, the stability number reads

$$\varkappa = a > 0,$$

telling that the semigroup $S(t)$ generated by (1.7) is exponentially stable as well. It is also clear that (1.7) can always be written in the form (1.1) by choosing

$$a = \varkappa, \qquad b = \alpha, \qquad c = \frac{\gamma}{\alpha},$$

provided that $\varkappa = \beta - \gamma/\alpha > 0$.

In summary: the MGT equation (1.1) is exponentially stable if and only if it models linear viscoelasticity, which is the case if and only if $\varkappa > 0$.

Note that, in the limit situation where $a = 0$, equation (1.7) becomes

$$\partial_t(\partial_{tt}u - c\Delta u) + b(\partial_{tt}u - c\Delta u) = 0,$$

namely, the sum of a conservative wave equation with its time derivative.

2 The Abstract Formulation

We will actually analyze an abstract generalization of (1.1). We begin with some notation.

2.1 Functional Setting

Let $(\mathrm{H}, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a separable real Hilbert space, and let

$$A : \mathfrak{D}(A) \subset \mathrm{H} \to \mathrm{H}$$

be a strictly positive selfadjoint unbounded linear operator of domain $\mathfrak{D}(A)$ densely (but not necessarily compactly) embedded into H. For $r \in \mathbb{R}$, we introduce the family of nested Hilbert spaces (the subscript r will be always omitted whenever zero)

$$\mathrm{H}_r = \mathfrak{D}(A^{\frac{r}{2}}), \qquad \langle u, v\rangle_r = \langle A^{\frac{r}{2}} u, A^{\frac{r}{2}} v\rangle, \qquad \|u\|_r = \|A^{\frac{r}{2}} u\|.$$

Then we define the product Hilbert space

$$\mathcal{H} = \mathrm{H}_1 \times \mathrm{H}_1 \times \mathrm{H},$$

endowed with the standard Euclidean product norm

$$\|(u, v, w)\|^2_{\mathcal{H}} = \|u\|^2_1 + \|v\|^2_1 + \|w\|^2.$$

2.2 The Abstract Equation

For $t > 0$, we consider the third-order evolution equation in the unknown variable $u = u(t)$

$$\partial_{ttt} u + \alpha \partial_{tt} u + \beta A \partial_t u + \gamma A u = 0, \tag{2.1}$$

of which (1.1) is just the particular instance obtained by the choice $\mathrm{H} = L^2(\Omega)$ and $A = -\Delta$ with the Dirichlet boundary condition. Introducing the state vector

$$\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t)),$$

we view (2.1) as the ordinary differential equation in $\mathcal{H}$

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{U}(t) = \mathbb{A} \boldsymbol{U}(t)$$

where $\mathbb{A}$ is the (closed) linear operator acting as

$$\mathbb{A}(u, v, w) = (v, w, -\alpha w - A\left[\beta v + \gamma u\right])$$

of dense domain

$$\mathfrak{D}(\mathbb{A}) = \left\{(u, v, w) \in \mathcal{H} : \ w \in \mathrm{H}_1, \ \beta v + \gamma u \in \mathrm{H}_2\right\} \subset \mathcal{H}.$$

The equation is supplemented with the initial condition $\boldsymbol{U}(0) = \boldsymbol{U}_0 \in \mathcal{H}$.

2.3 The Spectrum of $\mathbb{A}$

For further use, we are interested to describe the spectrum of (the complexification of) the operator $\mathbb{A}$. To this end, we define for every $\lambda > 0$ the third-order complex polynomial

$$P_\lambda(\mu) = \mu^3 + \alpha\mu^2 + \lambda\beta\mu + \lambda\gamma.$$

Proposition 2.1 *The spectrum of $\mathbb{A}$ is given by*

$$\sigma(\mathbb{A}) = \bigcup_{\lambda\in\sigma(A)} \big\{\mu \in \mathbb{C} : P_\lambda(\mu) = 0\big\} \cup \Big\{-\frac{\gamma}{\beta}\Big\}.$$

Proof Given $\boldsymbol{F} = (f, g, h) \in \mathcal{H}$, let us look for a unique solution $\boldsymbol{U} = (u, v, w) \in \mathfrak{D}(\mathbb{A})$ to the resolvent equation

$$\mu\boldsymbol{U} - \mathbb{A}\boldsymbol{U} = \boldsymbol{F}.$$

In components, this reads

$$\begin{cases} \mu u - v = f, \\ \mu v - w = g, \\ \mu w + \alpha w + A(\beta v + \gamma u) = h. \end{cases}$$

Writing v and w in terms of u, we obtain

$$\mu^3 u + \alpha\mu^2 u + \mu\beta A u + \gamma A u = A\psi,$$

where

$$\psi = A^{-1}\big[h + (\mu f + g)(\mu + \alpha)\big] + \beta f \in \mathrm{H}_1.$$

By the functional calculus,

$$u = \int_{\sigma(A)} \frac{\lambda}{P_\lambda(\mu)}\, \mathrm{d}E_A(\lambda)\, \psi,$$

being E_A the spectral measure of A. It is then clear that $u \in \mathrm{H}_1$ for any given $\psi \in \mathrm{H}_1$ if and only if

$$\sup_{\lambda\in\sigma(A)} \left|\frac{\lambda}{P_\lambda(\mu)}\right| < \infty.$$

Since $\sigma(A)$ is a closed subset of the real line, this occurs if and only if

$$\mu \neq -\frac{\gamma}{\beta} \qquad \text{and} \qquad P_\lambda(\mu) \neq 0, \ \forall \lambda \in \sigma(A).$$

In which case, we learn from the system above that

$$
\begin{aligned}
v &= \mu u - f \in \mathrm{H}_1, \\
w &= \mu v - g \in \mathrm{H}_1, \\
\beta v + \gamma u &= A^{-1}\big[h - (\mu + \alpha) w\big] \in \mathrm{H}_2,
\end{aligned}
$$

meaning that $\boldsymbol{U} \in \mathfrak{D}(\mathbb{A})$ is the unique solution to the resolvent equation. Accordingly, μ belongs to the resolvent set $\rho(\mathbb{A})$. □

Remark When A has compact resolvent, a detailed description of $\sigma(\mathbb{A})$ has been given in [8]. In that work, three branches of eigenvalues have been explicitly displayed. Among them, there is one branch of negative eigenvalues in a sharp finite interval, monotonically converging to $-\frac{\gamma}{\beta}$ from the left, the limit point $-\frac{\gamma}{\beta}$ being in the continuous spectrum of $\mathbb{A}$. Moreover, it is proved that the whole state space is the direct sum of the spans of the eigenvectors corresponding to each branch of eigenvalues, and the infinitesimal generator is a normal operator when restricted to each summand. Since the set of eigenvectors forms a Riesz basis, there exists a bounded invertible operator that transforms the original operator into a normal one. In particular, the transformed semigroup is normal (and contractive when $\varkappa > 0$). See [8] for more details.

3 The Solution Semigroup

The existence and uniqueness result for (2.1) is stated in the next theorem.

Theorem 3.1 *The operator $\mathbb{A}$ is the infinitesimal generator of a $\mathcal{C}_0$-semigroup*

$$
S(t) = \mathrm{e}^{t\mathbb{A}} : \mathcal{H} \to \mathcal{H}.
$$

Theorem 3.1 is proved in [5,8] by means of linear semigroups techniques. Here we propose a simple argument of PDE flavor, which is also applicable to nonlinear generalizations of the equation.

Proof We choose $m \geq 0$ large enough that

$$
\varkappa_m = \beta - \frac{\gamma}{\alpha + m} > 0.
$$

Calling for simplicity $\alpha_m = \alpha + m$, we consider the (equivalent) norm on $\mathcal{H}$

$$
|(u, v, w)|_{\mathcal{H}}^2 = \frac{\gamma}{\alpha_m} \|v + \alpha_m u\|_1^2 + \|w + \alpha_m v\|^2 + \varkappa_m \|v\|_1^2.
$$

With this position, for an arbitrarily fixed time $T > 0$, let

$$
\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t)) \in \mathcal{C}([0, T], \mathcal{H})
$$

be a regular solution to (2.1) on $[0, T]$. Introducing the natural energy

$$E(t) = \frac{1}{2}\left[\frac{\gamma}{\alpha_m}\|\partial_t u(t) + \alpha_m u(t)\|_1^2 + \|\partial_{tt} u(t) + \alpha_m \partial_t u(t)\|^2 + \varkappa_m \|\partial_t u(t)\|_1^2\right],$$

we take the product in H of (2.1) and $\partial_{tt} u + \alpha_m \partial_t u$. Exploiting the Young and the Poincaré inequalities, we easily get

$$\frac{\mathrm{d}}{\mathrm{d}t} E + \alpha_m \varkappa_m \|\partial_t u\|_1^2 = m\big(\|\partial_{tt} u\|^2 + \alpha_m \langle \partial_{tt} u, \partial_t u\rangle\big) \leq 2\omega E,$$

for some $\omega \geq 0$ depending only on m and on the structural constants of the problem. Then from the Gronwall lemma we draw the estimate

$$E(t) \leq E(0)\mathrm{e}^{2\omega t}. \tag{3.1}$$

For any fixed $\boldsymbol{U}_0 \in \mathcal{H}$, this gives the required uniform bound in $L^\infty(0, T; \mathcal{H})$ of any sequence $\boldsymbol{U}^n$ of Galerkin approximants with initial data $\boldsymbol{U}_0^n \to \boldsymbol{U}_0$ in $\mathcal{H}$, implying the weak-$*$ convergence (up to a subsequence)

$$\boldsymbol{U}^n \rightharpoonup \boldsymbol{U} \quad \text{in } L^\infty(0, T; \mathcal{H}),$$

for some $\boldsymbol{U}$ that solves the equation in the weak sense. At the same time, since the equation is linear, (3.1) holds for the difference $\boldsymbol{U}^n - \boldsymbol{U}^k$ of two approximants, yielding the convergence of the entire sequence $\boldsymbol{U}^n$ to its limit $\boldsymbol{U}$ in the topology of $\mathcal{C}([0, T], \mathcal{H})$. By the same token, the energy of the difference of two solutions satisfies (3.1) as well, providing the continuous dependence estimate

$$|S(t)\boldsymbol{U}_1 - S(t)\boldsymbol{U}_2|_{\mathcal{H}} \leq \mathrm{e}^{\omega t}|\boldsymbol{U}_1 - \boldsymbol{U}_2|_{\mathcal{H}}$$

for every pair of initial data $\boldsymbol{U}_1, \boldsymbol{U}_2 \in \mathcal{H}$.

In particular, we proved that the semigroup $S(t)$ is ω-contractive with respect to the equivalent norm $|\cdot|_{\mathcal{H}}$, i.e.

$$|S(t)\boldsymbol{U}_0|_{\mathcal{H}} \leq \mathrm{e}^{\omega t}|\boldsymbol{U}_0|_{\mathcal{H}}, \quad \forall \boldsymbol{U}_0 \in \mathcal{H}.$$

Note that when $\varkappa > 0$ we can take $m = 0$ (hence $\alpha_m = \alpha$ and $\varkappa_m = \varkappa$). The basic energy identity, now valid for all initial data $\boldsymbol{U}_0 \in \mathfrak{D}(\mathbb{A})$, becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} E + \alpha\varkappa \|\partial_t u\|_1^2 = 0,$$

telling that $S(t)$ is a contraction semigroup with respect to $|\cdot|_{\mathcal{H}}$.

Remark In the "noncritical" case $\varkappa > 0$, the contractivity of the semigroup $S(t)$ is already contained in [6] (see the proof of Theorem 2.2), where the authors use a different norm, equivalent to $|\cdot|_{\mathcal{H}}$.

Remark Actually, in the case $\varkappa > 0$, by recasting the calculations above using A^{1+r} in place of the operator A, the contractivity of $S(t)$ can be proven also in the space

$$\mathcal{H}_r = \mathrm{H}_{1+r} \times \mathrm{H}_{1+r} \times \mathrm{H}_r, \quad \forall r \in \mathbb{R}.$$

4 Asymptotic Behavior

- If $\varkappa > 0$, then $S(t)$ is exponentially stable. Namely, there exist $\varepsilon > 0$ and $C \geq 1$ such that the operator norm of $S(t)$ fulfills

$$\|S(t)\|_{\mathfrak{L}(\mathcal{H})} \leq C\mathrm{e}^{-\varepsilon t}. \tag{4.1}$$

  This result is proved in [5] (by means of the renowned Datko theorem) and in [8]. In particular, in the latter paper, the generation of a strongly continuous *group* has been shown, and uniform decay has been obtained in several state settings, with sharp explicit rates.
- The case $\varkappa = 0$ is much different. Indeed, as shown in [5,8], the *pseudoenergy*

$$\mathcal{E}(t) = \frac{1}{2}\Big[\beta\|\partial_t u(t) + \alpha u(t)\|_1^2 + \|\partial_{tt} u(t) + \alpha\partial_t u(t)\|^2\Big]$$

  is conserved. In fact, setting $\phi = \partial_{tt}\, u + \beta A u$, equation (2.1) reads

$$\partial_t \phi + \alpha\phi = 0.$$

  As a consequence,

$$\phi(t) = \mathrm{e}^{-\alpha t}\phi(0),$$

  meaning that u solves the Cauchy problem

$$\begin{cases} \partial_{tt} u(t) + \beta A u(t) = \mathrm{e}^{-\alpha t}(w_0 + \beta A u_0), \\ u(0) = u_0, \\ \partial_t u(0) = v_0. \end{cases}$$

  This, together with the conservation of the pseudoenergy, imply that $S(t)$ is a bounded semigroup. In particular, choosing $w_0 = -\beta A u_0$ with $u_0 \in \mathrm{H}_2$, we have the conservative wave equation.
- We finally discuss the case $\varkappa < 0$. The instability of $S(t)$ has been shown via numerical simulations in [5], whereas when A has compact resolvent a rigorous analysis has been made in [8]. In what follows, we agree to work with the complex-ification of the operator $\mathbb{A}$, as well as with the complexification of the semigroup $S(t)$. With standard notation, let

$$\omega_* = \lim_{t\to\infty} \frac{1}{t} \log \|S(t)\|_{\mathfrak{L}(\mathcal{H})}$$

be the *growth bound* of $S(t)$. It is well-known that

$$\sigma_* = \sup\{\Re\mu : \mu \in \sigma(\mathbb{A})\} \leq \omega_*.$$

Lemma 4.1 *Assume that $\varkappa < 0$. Then, for any fixed $\lambda \in \sigma(A)$, the complex polynomial*

$$P_\lambda(\mu) = \mu^3 + \alpha\mu^2 + \lambda\beta\mu + \lambda\gamma$$

has always three distinct roots: a real root $\mu_1(\lambda) < 0$, and a pair of complex conjugate ones $\mu_2(\lambda)$, $\mu_3(\lambda)$ with positive real part.

In the light of Proposition 2.1, we have an immediate corollary.

Corollary 4.2 *If $\varkappa < 0$ it follows that $\sigma_* > 0$, implying in turn $\omega_* > 0$. Therefore the semigroup $S(t)$ has solutions with energy growing exponentially fast.*

Proof of Lemma 4.1. Let $\lambda \in \sigma(A)$ be fixed. Since

$$P_\lambda(0) = \lambda\gamma > 0,$$

it is clear that there exists a real root $\mu_1(\lambda) < 0$. We prove that $P_\lambda(\mu)$ admits also two complex conjugate roots with positive real part. To this end, writing $\mu = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$, from the equation $P_\lambda(\mu) = 0$ we obtain the system

$$\begin{cases} (x^2 - y^2)(x + \alpha) - 2xy^2 + x\lambda\beta + \lambda\gamma = 0, \\ (x^2 - y^2)y + 2xy(x + \alpha) + y\lambda\beta = 0. \end{cases}$$

Assuming $y \neq 0$, and substituting the second equation into the first one, we are led to

$$Q_\lambda(x) = 8x^3 + 8x^2\alpha + 2x(\alpha^2 + \lambda\beta) + \lambda\alpha\varkappa = 0.$$

Since

$$Q_\lambda(0) = \lambda\alpha\varkappa < 0,$$

the cubic polynomial $Q_\lambda(x)$ has a real root $\xi > 0$. In turn, the two complex conjugate numbers

$$\mu_{2,3}(\lambda) = \xi \pm \mathrm{i}\sqrt{3\xi^2 + 2\xi\alpha + \lambda\beta}$$

are roots of $P_\lambda(\mu)$.

Remark An elementary visual argument provides the lower bound

$$\omega_* \geq \sigma_* = -\frac{\alpha\varkappa}{2\beta}.$$

Indeed, it is enough noting that the number ξ is the abscissa of the intersection between the cubic $8x^3 + 8x^2\alpha$ and the line $-2x(\alpha^2 + \lambda\beta) - \lambda\alpha\varkappa$, and since A is unbounded we can let $\lambda \to +\infty$.

5 Rigorous Comparison with the Equation of Viscoelasticity

In this final section, we discuss the link between the abstract form of system (1.3) with the exponential kernel

$$\kappa(s) = a\mathrm{e}^{-bs} + c,$$

and (2.1) in the case when $\varkappa > 0$, which can be more conveniently rewritten as

$$\partial_{ttt}u + b\partial_{tt}u + (a + c)A\partial_t u + bcAu = 0. \tag{5.1}$$

To this end, let $\mathcal{M}$ be the space of square summable functions $\eta : \mathbb{R}^+ \to \mathrm{H}_1$ with respect to the measure $ab\mathrm{e}^{-bs}\mathrm{d}s$, endowed with the norm

$$\|\eta\|^2_{\mathcal{M}} = ab \int_0^\infty \mathrm{e}^{-bs} \|\eta(s)\|_1^2 \,\mathrm{d}s,$$

and define the product Hilbert space

$$\mathcal{V} = \mathrm{H}_1 \times \mathrm{H} \times \mathcal{M},$$

with the usual Euclidean product norm. Then, introducing the infinitesimal generator of the right-translation semigroup on $\mathcal{M}$, i.e. the linear operator T given by

$$\mathsf{T}\eta = -\eta' \qquad \text{with domain} \qquad \mathfrak{D}(\mathsf{T}) = \big\{\eta \in \mathcal{M} : \eta' \in \mathcal{M},\ \eta(0) = 0\big\},$$

the abstract version of (1.3) with the exponential kernel reads

$$\begin{cases} \partial_{tt}u(t) + cAu(t) + ab \displaystyle\int_0^\infty \mathrm{e}^{-bs} A\eta^t(s)\,\mathrm{d}s = 0, \\ \partial_t \eta = \mathsf{T}\eta^t + \partial_t u(t). \end{cases} \tag{5.2}$$

Introducing the three-component vector $\boldsymbol{V}(t) = (u(t), \partial_t u(t), \eta^t)$, we view system (5.2) as the ordinary differential equation in $\mathcal{V}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{V}(t) = \mathbb{L}\boldsymbol{V}(t),$$

where $\mathbb{L}$ is the linear operator on $\mathcal{V}$ defined as

$$\mathbb{L}(u, v, \eta) = \bigg(v, -A\Big[cu + ab\int_0^\infty \mathrm{e}^{-bs}\eta(s)\,\mathrm{d}s\Big], \mathsf{T}\eta + v\bigg),$$

with domain

$$\mathfrak{D}(\mathbb{L}) = \left\{ (u, v, \eta) \in \mathcal{V} : \ v \in \mathrm{H}_1, \ cu + ab \int_0^\infty \mathrm{e}^{-bs} \eta(s)\,\mathrm{d}s \in \mathrm{H}_2, \ \eta \in \mathfrak{D}(\mathsf{T}) \right\}.$$

This equation generates an exponentially stable linear contraction semigroup

$$\Sigma(t) = \mathrm{e}^{t\mathbb{L}} : \mathcal{V} \to \mathcal{V}.$$

Moreover, for any initial datum $\boldsymbol{V}_0 = (u_0, v_0, \eta_0) \in \mathcal{V}$, the third component of the solution $\Sigma(t)\boldsymbol{V}_0 = (u(t), \partial_t u(t), \eta^t)$ fulfills the representation formula (see e.g. [11])

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ \eta_0(s-t) + u(t) - u_0 & s > t. \end{cases} \tag{5.3}$$

The rigorous comparison between (5.1) and (5.2) is established the next theorem.

Theorem 5.1 *The following hold.*

(i) *Let* $\boldsymbol{V}(t) = (u(t), \partial_t u(t), \eta^t)$ *be the solution to* (5.2) *corresponding to any initial datum*

$$\boldsymbol{V}_0 = (u_0, v_0, \eta_0) \in \mathcal{V}$$

satisfying the regularity conditions

$$v_0 \in \mathrm{H}_1 \quad \text{and} \quad cu_0 + ab \int_0^\infty \mathrm{e}^{-bs} \eta_0(s)\,\mathrm{d}s \in \mathrm{H}_2. \tag{5.4}$$

Then $\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t))$ *is the solution to* (5.1) *with initial datum*

$$\boldsymbol{U}_0 = (u_0, v_0, w_0) \in \mathcal{H},$$

where we set

$$w_0 = -A\Big[cu_0 + ab \int_0^\infty \mathrm{e}^{-bs} \eta_0(s)\,\mathrm{d}s\Big]. \tag{5.5}$$

(ii) *Conversely, let* $\boldsymbol{U}(t) = (u(t), \partial_t u(t), \partial_{tt} u(t))$ *be the solution to* (5.1) *corresponding to any initial datum*

$$\boldsymbol{U}_0 = (u_0, v_0, w_0) \in \mathcal{H}.$$

Then, for every $\eta_0 \in \mathcal{M}$ *satisfying the relation*

$$ab \int_0^\infty \mathrm{e}^{-bs} \eta_0(s) = -cu_0 - A^{-1} w_0, \tag{5.6}$$

the function $\boldsymbol{V}(t) = (u(t), \partial_t u(t), \eta^t)$, with η^t given by (5.3), is the solution to (5.2) with initial datum

$$\boldsymbol{V}_0 = (u_0, v_0, \eta_0) \in \mathcal{V}.$$

Remark If $\boldsymbol{V}_0 \in \mathcal{V}$ satisfies (5.4), it does not necessarily mean that $\boldsymbol{V}_0 \in \mathfrak{D}(\mathbb{L})$. Also note that the function η_0 satisfying (5.6) is not uniquely determined: one possibility is, for instance,

$$\eta_0 = -\frac{1}{a}\big[A^{-1}w_0 + cu_0\big],$$

which is actually constant in s, but there are infinitely many other possible choices.

Proof We will limit ourselves to prove (i), since repeating the argument backwards the proof of (ii) follows. Let then

$$\boldsymbol{V}(t) = (u(t), \partial_t u(t), \eta^t)$$

be the solution to (5.2) corresponding to an initial datum $\boldsymbol{V}_0 = (u_0, v_0, \eta_0)$ complying with (5.4). Substituting the representation formula[2] (5.3) into the first equation of (5.2), we obtain the identity

$$\partial_{tt}u(t) + (a+c)Au(t) - ab\mathrm{e}^{-bt}\int_0^t \mathrm{e}^{bs}Au(s)\,\mathrm{d}s = -\mathrm{e}^{-bt}Aq_0, \tag{5.7}$$

having set

$$q_0 = ab\int_0^\infty \mathrm{e}^{-bs}\eta_0(s)\,\mathrm{d}s - au_0 \in \mathrm{H}_1.$$

In particular, due to (5.4),

$$\partial_{tt}u(0) = -A\big[(a+c)u_0 + q_0\big] = -A\left[cu_0 + ab\int_0^\infty \mathrm{e}^{-bs}\eta_0(s)\,\mathrm{d}s\right] \in \mathrm{H}.$$

Next, multiplying equality (5.7) by e^{bt}, and taking the derivative with respect to time, we have

$$\mathrm{e}^{bt}(\partial_{ttt}u + b\partial_{tt}u) + (a+c)\mathrm{e}^{bt}(A\partial_t u + bAu) - ab\mathrm{e}^{bt}Au = 0,$$

and a final multiplication by e^{-bt} yields (5.1).

[2] The representation formula (5.3) is actually completely equivalent to the second equation of (5.2), once the initial conditions are fixed.

As a byproduct, system (5.2) exhibits a preservation of regularity for a certain class of initial data, which does not generally occur for different types of memory kernels κ.

Corollary 5.2 *Let $\boldsymbol{V}_0 = (u_0, v_0, \eta_0) \in \mathcal{V}$ satisfy the further regularity conditions* (5.4). *Then, the solution $\Sigma(t)\boldsymbol{V}_0 = (u(t), \partial_t u(t), \eta^t)$ fulfills*

$$u \in \mathcal{C}^1([0,\infty), \mathrm{H}_1) \cap \mathcal{C}^2([0,\infty), \mathrm{H}).$$

Moreover, there exists $\varepsilon > 0$ and $C \geq 1$ such that

$$\|\Sigma(t)\boldsymbol{V}_0\|_{\mathcal{V}} + \|\partial_t u(t)\|_1 + \|\partial_{tt} u(t)\| \leq C\big[\|u_0\|_1 + \|v_0\|_1 + \|w_0\| + \|\eta_0\|_{\mathcal{M}}\big]\mathrm{e}^{-\varepsilon t},$$

with w_0 given by (5.5).

Proof We know from Theorem 5.1 that the function $u(t)$ solves (5.1). Therefore, u has the claimed regularity. Since $S(t)$ and $\Sigma(t)$ are both exponentially stable (in the respective spaces), there exist $\varepsilon > 0$ and $C \geq 1$ such that

$$\|\partial_t u(t)\|_1 + \|\partial_{tt} u(t)\| \leq C\big[\|u_0\|_1 + \|v_0\|_1 + \|w_0\|\big]\mathrm{e}^{-\varepsilon t},$$

and

$$\|\Sigma(t)\boldsymbol{V}_0\|_{\mathcal{V}} \leq C\big[\|u_0\|_1 + \|v_0\| + \|\eta_0\|_{\mathcal{M}}\big]\mathrm{e}^{-\varepsilon t}.$$

Adding the relations, and estimating $\|v_0\|$ with the Poincaré inequality, we are done.

Acknowledgements We thank the referees for careful reading and very useful comments.

References

1. D'Acunto, B., D'Anna, A., Renno, P.: On the motion of a viscoelastic solid in presence of a rigid wall. Z. Angew. Math. Phys. **34**, 421–438 (1983)
2. Dafermos, C.M.: Asymptotic stability in viscoelasticity. Arch. Ration. Mech. Anal. **37**, 297–308 (1970)
3. Drozdov, A.D., Kolmanovskii, V.B.: Stability in Viscoelasticity. North-Holland, Amsterdam (1994)
4. Fabrizio, M., Lazzari, B.: On the existence and asymptotic stability of solutions for linear viscoelastic solids. Arch. Ration. Mech. Anal. **116**, 139–152 (1991)
5. Kaltenbacher, B., Lasiecka, I., Marchand, R.: Wellposedness and exponential decay rates for the Moore–Gibson–Thompson equation arising in high intensity ultrasound. Control Cybern. **40**, 971–988 (2011)
6. Lasiecka, I., Wang, X.: Moore–Gibson–Thompson equation with memory, part I: exponential decay of energy. Z. Angew. Math. Phys. **67**, 17 (2016)
7. Liu, Z., Zheng, S.: On the exponential stability of linear viscoelasticity and thermoviscoelasticity. Q. Appl. Math. **54**, 21–31 (1996)
8. Marchand, R., McDevitt, T., Triggiani, R.: An abstract semigroup approach to the third-order Moore–Gibson–Thompson partial differential equation arising in high-intensity ultrasound: structural decomposition, spectral analysis, exponential stability. Math. Methods Appl. Sci. **35**, 1896–1929 (2012)
9. Moore, F.K., Gibson, W.E.: Propagation of weak disturbances in a gas subject to relaxation effects. J. Aerosp. Sci. **27**, 117–127 (1960)

10. Muñoz Rivera, J.E.: Asymptotic behaviour in linear viscoelasticity. Q. Appl. Math. **52**, 629–648 (1994)
11. Pata, V.: Stability and exponential stability in linear viscoelasticity. Milan J. Math. **77**, 333–360 (2009)
12. Pata, V.: Exponential stability in linear viscoelasticity with almost flat memory kernels. Commun. Pure Appl. Anal. **9**, 721–730 (2010)
13. Renardy, M., Hrusa, W.J., Nohel, J.A.: Mathematical Problems in Viscoelasticity. Longman Scientific & Technical, New York (1987)
14. Stokes, G.G.: An examination of the possible effect of the radiation of heat on the propagation of sound. Philos. Mag. Ser. **4**(1), 305–317 (1851)
15. Thompson, P.A.: Compressible-Fluid Dynamics. McGraw-Hill, New York (1972)