ON SOME RIGOROUS ASPECTS OF FRAGMENTED CONDENSATION

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ABSTRACT. In this paper we discuss some aspects of finite fragmented condensation from a mathematical perspective. Inspired by techniques of pseudodifferential calculus and semi-classical analysis in Bosonic Quantum Field Theory, we propose a simple way of identifying finite fragmentation, and we analyze the effects of pair interaction on finite fragmented states. In particular, we focus on the persistence of finite fragmented condensation when the gap between the degenerate ground state and the excited states of the corresponding non-interacting system is very large.

1. INTRODUCTION

The phenomenon of fragmented Bose-Einstein condensation has attracted a lot of attention in recent years, both from an experimental and a theoretical point of view [see, *e.g.*, MHUB06, Leg08, and references thereof contained for a detailed bibliography on fragmented condensates]. The physical idea of fragmentation is that in some cases, *e.g.* due to some degeneracy of the low-energy states, the condensed fraction of a bosonic system is distributed among more than one single-particle state. In this paper we analyze two aspects of finite fragmented condensation: one is the identification of a fragmented state in mathematical terms, and the other is persistence of finite fragmentation under time evolution, for mean field systems.

The easiest, and perhaps most widespread, mathematical definition of fragmentation is the following [Leg08]: a system of N bosons with density matrix ϱ_N exhibits *finite fragmented condensation* if there is a finite number bigger or equal than two of eigenvalues of the reduced one-particle density matrix $\gamma_N^{(1)}$ associated to ϱ_N that are of order N, the remaining ones being of order one. If there is convergence, as $N \to \infty$, of the p-particles reduced density matrix $\gamma_N^{(p)}$ in trace class or Hilbert-Schmidt norm, the above definition is equivalent to the fact that the effective one-particle density matrix $\gamma_\infty^{(1)}$ *has a (finite) rank bigger or equal than two* [see LSSY05, for additional details on convergence of reduced density matrices and other mathematical aspects of condensation]. However, despite its simplicity, such mathematical definition of fragmentation is very broad, in the sense that it includes states that are not fragmented *in a physically meaningful way*. In fact, there are statistical mixtures of simply condensed states whose effective one-particle reduced density matrix has rank two or more (we follow again [Leg08] in calling simple condensate a state whose reduced one-particle density matrix has only one macroscopic eigenvalue). Such states represent the physical situation of a system that is simply condensed (with probability one), but where there is only statistical information on which one-particle state it is condensed into. More precisely, the effective one-particle reduced density matrices of the aforementioned statistical mixtures *coincides exactly* with the effective one-particle reduced density matrices of truly fragmented

states. It is therefore desirable to give a different definition/characterization of finite fragmented condensation, able to distinguish truly fragmented states from statistical mixtures of simply condensed states. Since the effective one-particle reduced density matrices may coincide, it is necessary to consider properties of the state that involve two or more particles as well. Based on the comparison of the effective behavior of temperature-zero mean field fragmented states with the one of statistical mixtures with the same effective one-particle reduced density matrix, we propose a simple characterization of finite fragmentation, that distinguishes between fragmented condensation and statistical mixtures of simply condensed states. A precise definition can be found in § 2, however the idea is roughly speaking that a state exhibits fragmented condensation if the rank of $\gamma_\infty^{(1)}$ is two or more, and the rank of $\gamma_\infty^{(p)}$, as a function of p, *is not constant.* Let us remark that in most cases it should be sufficient to check that the rank of $\gamma_\infty^{(2)}$ is different (usually higher) than the rank of $\gamma_\infty^{(1)}$, and it is not necessary to consider at once all the reduced density matrices with an arbitrary number of particles (the collection $\big\{\gamma_\infty^{(p)}\big\}_{p\geq 1}$ determines uniquely the mean-field configuration of the system [see, *e.g.*, AN11, LNR15, AFP16b], but it is often impractical or impossible to compute, either analytically or numerically, all the reduced density matrices of a system with a large number of particles).

Another interesting aspect of finite fragmented condensation is whether it persists under the effect of interactions. It is well-known that simple condensation persists if the particles are interacting among each other in a suitable way. For example, it has been rigorously proved to persist for Bose gases (confined or not) with weak pair interactions, proportional to the inverse of the number of particles (Hartree regime) [see, *e.g.*, GV79a, GV79b, RS09, ESY10, KP10, BPS16], and in dilute Bose gases with intense collisions (Gross-Pitaevskii regime) [see, *e.g.*, ESY10, Pic10, Pic15, BOS15, BS17]. For states exhibiting finite fragmented condensation, the situation is more complicated. In the Hartree regime, Alon, Streltsov, and Cederbaum [ASC07, ASC08] formally derived an effective mean-field evolution for a fragmented condensate, imposing that fragmentation is persisting, with the same number of one-particle states at any time, in the limit of a large number of particles $N \to \infty$. This idea seems close to the so-called Dirac-Frenkel principle used in mathematical physics and numerical analysis [Lub08]. Unfortunately, the error made in imposing that fragmentation with the same number of states holds at any time is in general not converging to zero when $N \to \infty$, as it can be proved applying a series of rigorous results on the mean-field effective evolution in the Hartree regime, for generic many-body states [AN08, AN09, AN11, AN15, AFP16b]. In fact, from these results it follows that finite fragmentation is destroyed by interactions: for almost all times the time-evolved effective one-particle reduced density matrix $\gamma_\infty^{(1)}(t)$ has *infinite rank*. Hence finite fragmented condensation could be rather difficult to detect experimentally in systems where the inter-particle interactions play a role. Nonetheless, we prove that finite fragmented condensation is persisting in interacting systems where there is a very large energy gap between the degenerate ground states and the first excited states of the corresponding non-interacting system. In this case, the effective mean-field evolution is described by equations that are similar to the one introduced in the aforementioned papers [ASC07, ASC08].

In the following, we restrict our analysis to states with simple and finite fragmented condensation at temperature zero (ultra-cold systems), and with weak inter-particle interactions

(Hartree regime). Our results could be extended with suitable modifications to dilute gases in the Gross-Pitaevskii regime. It would be interesting to study fragmented condensation at finite inverse temperature (in the thermodynamic limit), but this would require different mathematical techniques from the ones considered in this paper. We plan to address this question in a future work. We introduce the degeneracy in the ground state of the system, causing finite fragmented condensation, in the most natural way: the degeneracy is caused by spin or pseudo-spin degrees of freedom, that do not affect the Hamiltonian.

The rest of the paper is organized as follows. The characterization of fragmentation that we propose is described in § 2, and in § 3 we analyze the persistence of fragmentation for interacting systems with a large energy gap. In Appendix A we briefly review semiclassical analysis in QFT and its applications to the study of systems of non-relativistic bosons with a very large number of particles. The tools introduced in this appendix are the mathematical backbone of our description of fragmented condensation, and they could be useful to better understand some results used in § 2 and 3. In Appendix B, we provide the mathematical proofs of some important results of § 2 and 3.

2. Characterization of fragmented condensation

In this section we propose a characterization of finite fragmented condensation that is more specialized than the one commonly adopted, with the advantage of being able to distinguish truly fragmented states from statistical mixtures of simply condensed states.

The most common mathematical definition of finite fragmentation, à la Penrose-Onsager [PO56], is the following [Leg08]:

> Let $\gamma_N^{(1)}$ be the one-particle reduced density matrix (1-RDM) of an N-bosons system. Then the system exhibits finite fragmented condensation if there is a finite number bigger or equal than two of eigenvalues of $\gamma_N^{(1)}$ of order N, with the remaining ones being of order one.

It is always possible to take the limit $N \to \infty$ of the 1-RDM trace-class operator in some suitable topology (in particular, it is always possible to take the limit in the weak-* topology); in the explicit cases considered below we have convergence in the relevant trace norm topology, and this ensures that no mass is lost by the RDM in the limit procedure. We denote the resulting effective 1-RDM by $\gamma_\infty^{(1)}$ (and in general the effective p-RDM by $\gamma_\infty^{(p)}$). The advantage of the limit $N \to \infty$ is that one can investigate simple and finite fragmented condensation easily by looking at the rank of the effective p-RDMs $\gamma_\infty^{(p)}$: if the rank is one there is simple condensation, if the rank is (finite and) two or more there is (finite) fragmented condensation.

The above definition seems intuitive, and it indeed captures a property of fragmented condensates: in fact, all fragmented condensates have two or more macroscopic eigenvalues in the relative 1-RDM. The aforementioned property is, however, *not sufficient* to characterize fragmented condensates: it is easy to write statistical mixtures of simply condensed states whose 1-RDMs have two or more macroscopic eigenvalues, and such statistical mixtures describe (albeit incompletely) systems that are simply condensed rather than fragmented.

In addition, the effective 1-RDMs of statistical mixtures of simple condensates of the form

$$(1) \qquad \varrho_{N,\text{stat}} = \sum_{k=1}^{2s+1} \frac{f_k(N)}{N} \big| \underbrace{\varphi_k \otimes \cdots \otimes \varphi_k}_{N} \big\rangle \big\langle \underbrace{\varphi_k \otimes \cdots \otimes \varphi_k}_{N} \big|,$$

coincide with the effective 1-RDMs of fragmented condensates of the form

$$
\begin{aligned}
\varrho_{N,\text{frag}} = \big| \big(\underbrace{\varphi_1 \otimes \cdots \otimes \varphi_1}_{f_1(N)} \big) \vee \ldots \vee \big(\underbrace{\varphi_{2s+1} \otimes \cdots \otimes \varphi_{2s+1}}_{f_{2s+1}(N)} \big) \big\rangle \\
(2) \qquad\qquad\qquad \big\langle \big(\underbrace{\varphi_1 \otimes \cdots \otimes \varphi_1}_{f_1(N)} \big) \vee \ldots \vee \big(\underbrace{\varphi_{2s+1} \otimes \cdots \otimes \varphi_{2s+1}}_{f_{2s+1}(N)} \big) \big|,
\end{aligned}
$$

where $\vee$ stands for the symmetric tensor product, and the φ_k, $k = 1, \ldots, 2s+1$, are mutually orthogonal one-particle states. Here the $f_k(N) \in \mathbb{N}$ are such that $\sum_{k=1}^{2s+1} f_k(N) = N$. The limit numbers $\pi_k = \lim_{N\to\infty} \frac{f_k(N)}{N}$ are: for the state $\varrho_{N,\text{stat}}$ the macroscopic probabilities of the state being simply condensed on the one-particle state φ_k; and for the state $\varrho_{N,\text{frag}}$ the macroscopic fractions of particles in the one-particle states φ_k, within the fragmented condensate. One could think of φ_k, $k = 1, \ldots, 2s+1$, as the $2s+1$ degenerate ground states of a one-particle free Hamiltonian describing a spin or pseudo-spin s boson. We study and compare the p-RDMs of $\varrho_{N,\text{stat}}$ and $\varrho_{N,\text{frag}}$ in detail below. Let us remark that fragmented states analogous to $\varrho_{N,\text{frag}}$ have been considered in [RS16], where the authors prove that, for a system of interacting bosons trapped by a suitably scaled double-well confining potential, the purely factorized state does no longer describe the many-body ground state accurately. There is in fact a transition to a regime in which energy is gained by localizing particles in either of the two wells. It is expected that, in a suitable scaling regime of the double-well, the ground state is well approximated by a state of the form $\varrho_{N,\text{frag}}$.

Since the 1-RDMs of $\varrho_{N,\text{stat}}$ and $\varrho_{N,\text{frag}}$ are equal, a definition of fragmentation that includes $\varrho_{N,\text{frag}}$ but not $\varrho_{N,\text{stat}}$ should involve properties of the system correlating two or more particles. Given a many-body density matrix ϱ_N, its limit $N \to \infty$ may be interpreted as a probability measure μ on the one particle Hilbert space $\mathcal{H}$ of the system (see Appendix A for more details). In addition, the effective p-RDM $\gamma_\infty^{(p)}$ associated to ϱ_N has the following explicit mathematical form:

$$\gamma_\infty^{(p)} = \int_{\mathcal{H}} \big| \underbrace{u \otimes \cdots \otimes u}_{p} \big\rangle \big\langle \underbrace{u \otimes \cdots \otimes u}_{p} \big| \, \mathrm{d}\mu(u).$$

By the Penrose-Onsager definition of condensation it follows that in order to have a simple condensate, the measure μ should be concentrated either on a single point u_0, or be a convex combination of measures concentrated on single points, each one differing from the other only by a phase. In fact, these are the only measures yielding

$$\gamma_{\infty,\text{cond}}^{(1)} = \big| u_0 \big\rangle \big\langle u_0 \big|,$$

for some one-particle wavefunction $u_0 \in \mathcal{H}$. Hence it follows that, for a simple condensate,

$$\gamma_{\infty,\text{cond}}^{(p)} = \big| \underbrace{u_0 \otimes \cdots \otimes u_0}_{p} \big\rangle \big\langle \underbrace{u_0 \otimes \cdots \otimes u_0}_{p} \big|.$$

In addition, the measure ν associated to a statistical mixture of the form

$$\eta_N = \sum_{k=1}^{2s+1} \pi_k(N)\varrho_{N,k}$$

is

$$\nu = \sum_{k=1}^{2s+1} \pi_k(\infty)\mu_k \ ,$$

where $\pi_k(\infty) = \lim_{N\to\infty} \pi_k(N)$ and μ_k is the measure associated to $\varrho_{N,k}$. Therefore the effective p-RDMs of statistical mixtures of simple condensates are of the form

$$\gamma_{\infty,\text{stat}}^{(p)} = \sum_{k=1}^{2s+1} \pi_k \big| \underbrace{u_k \otimes \cdots \otimes u_k}_{p} \big\rangle \big\langle \underbrace{u_k \otimes \cdots \otimes u_k}_{p} \big| \ ,$$

where the probabilities π_k are such that $\sum_{k=1}^{2s+1} \pi_k = 1$. From the above discussion, the following proposition follows.

Proposition 2.1. *The rank $R_{\text{stat}}(p)$ of effective p-RDMs $\gamma_{\infty,\text{stat}}^{(p)}$ corresponding to statistical mixtures of simple condensates is a constant function of $p \geq 1$. In particular, we have that for all $p \geq 1$, $R_{\text{stat}}(p) = \Pi$, with*

$$(3) \qquad \Pi = \text{Card}\Big(\big\{k=1,\ldots,2s+1\,;\,\pi_k \neq 0\big\}\Big) \leq 2s+1 \ .$$

In particular, the effective p-RDMs of the aforementioned mixture $\varrho_{N,\text{stat}}$ are given by the above formula with $u_k = \varphi_k$, and $\pi_k = \lim_{N\to\infty} \frac{f_k(N)}{N}$:

$$\gamma_{\infty,\text{stat}}^{(p)} = \sum_{k=1}^{2s+1} \pi_k \big| \underbrace{\varphi_k \otimes \cdots \otimes \varphi_k}_{p} \big\rangle \big\langle \underbrace{\varphi_k \otimes \cdots \otimes \varphi_k}_{p} \big| \ .$$

Let us now turn our attention to the effective p-RDMs corresponding to the fragmented density matrices $\varrho_{N,\text{frag}}$. The probability measure μ_{frag} corresponding, in the limit $N \to \infty$, to $\varrho_{N,\text{frag}}$ has the following form. Let $P = \big\{k=1,\ldots,2s+1\,;\,\pi_k \neq 0\big\}$ be the set of indices corresponding to non-zero π_ks and let, for any one particle wavefunction $u \in \mathcal{H}$, $\delta_u^{S^1}$ be the following convex combination (average) of delta measures

$$\delta_u^{S^1} = \frac{1}{2\pi} \int_0^{2\pi} \delta_{e^{i\theta}u}\mathrm{d}\theta \ ;$$

then

$$(4) \qquad \mu_{\text{frag}} = \bigotimes_{k\in P} \delta_{\sqrt{\pi_k}\varphi_k}^{S^1} \otimes \delta_0^{\perp} \ ,$$

where $\delta_0^{\perp}$ is the delta in zero acting on the orthogonal complement $\mathcal{H}_1^{\perp}$ of the linear span

$$\mathcal{H}_1 = \text{span}_{\mathbb{C}}\big\{\varphi_k \,,\, k \in P\big\} \ .$$

Hence this measure is not a convex combination of delta measures but rather *a product* of convex combinations of delta measures. Hence the associated p-RDMs

$$\gamma_{\infty,\text{frag}}^{(p)} = \int_{\mathcal{H}} \big| \underbrace{u \otimes \cdots \otimes u}_{p} \big\rangle \big\langle \underbrace{u \otimes \cdots \otimes u}_{p} \big| \, \mathrm{d}\mu_{\text{frag}}(u)$$

should differ from the ones of statistical mixtures of simple condensates such as $\varrho_{N,\text{stat}}$. The explicit form of the p-RDMs is more complicated than for statistical mixtures, and it is given in the following proposition. The corresponding proof can be found in Appendix B.1.

Proposition 2.2. *Let $\varrho_{N,\text{frag}}$ be defined by Eq. (2), and let $\gamma_{\infty,\text{frag}}^{(p)}$, $p \geq 1$, be the associated effective p-RDMs. In addition, define for any $\alpha \geq 1$,*

$$\mathcal{F}_{p,\alpha} := \Big\{ g \in \mathbb{N}^\alpha : \sum_{j=1}^{\alpha} g_j = p \Big\}, \quad c_{p,g} := p! \prod_{\substack{j \in \{1,\dots,2s+1\} \\ g_j \neq 0}} \frac{\pi_j^{g_j}}{g_j!} \, .$$

Then,

$$\gamma_{\infty,\text{frag}}^{(p)} = \sum_{g \in \mathcal{F}_{p,2s+1}} c_{p,g} \, \Big| \big(\underbrace{\varphi_1 \otimes \cdots \otimes \varphi_1}_{g_1} \big) \vee \cdots \vee \big(\underbrace{\varphi_{2s+1} \otimes \cdots \otimes \varphi_{2s+1}}_{g_{2s+1}} \big) \Big\rangle$$
$$\Big\langle \big(\underbrace{\varphi_1 \otimes \cdots \otimes \varphi_1}_{g_1} \big) \vee \cdots \vee \big(\underbrace{\varphi_{2s+1} \otimes \cdots \otimes \varphi_{2s+1}}_{g_{2s+1}} \big) \Big| \, .$$

Corollary 2.3. $\gamma_{\infty,\text{frag}}^{(1)} = \gamma_{\infty,\text{stat}}^{(1)}$

Corollary 2.4. *Let $R_{\text{frag}}(p)$ be the rank of $\gamma_{\infty,\text{frag}}^{(p)}$ as a function of $p \geq 1$, and let Π be defined by Eq. (3). Then*

$$R_{\text{frag}}(p) = \binom{p + \Pi - 1}{p} \, .$$

Hence even if the effective 1-RDMs of $\varrho_{N,\text{stat}}$ and $\varrho_{N,\text{frag}}$ coincide, and have both the same rank Π (with $1 \leq \Pi \leq 2s + 1$), the ≥ 2-RDMs behave quite differently in the two cases for any $\Pi > 1$ (if $\Pi = 1$ both $\varrho_{N,\text{stat}}$ and $\varrho_{N,\text{frag}}$ are describing simple condensates in the same one-particle state). One notable difference is that the rank function $R_{\text{frag}}(p)$ of the fragmented state is a *non-constant function* of $p \geq 1$, while as proved in Proposition 2.1 the rank function $R_{\text{stat}}(p)$ of a statistical mixture of simple condensates is always a *constant function* of p. In particular, R_{frag} is a strictly increasing function (and thus its non-constancy is already verified looking at $p = 1$ and $p = 2$). In our opinion, this feature provides a nice characterization of finite fragmentation, and thus we propose the following modified definition of fragmented condensation:

> Let $\gamma_N^{(p)}$, $p \geq 1$, be the p-RDMs of an N-bosons system. Then the system exhibits finite fragmented condensation if the number of eigenvalues of order N of $\gamma_N^{(p)}$ is a non-constant function $R(p)$ of p, with values in $\mathbb{N}$, and $R(1) \geq 2$.

Let us remark again that the above definition/characterization is reasonably easy to check in concrete examples, and excludes statistical mixtures of simple condensates, since for such states $R(p)$ is constant. In addition, all the examples of physically relevant states with finite fragmented condensation that we know of satisfy the above definition, including, to mention a concrete example, the spin-one fragmented state corresponding to the LPB wavefunction [see LPB98]

$$\big(a_{0,1}^* a_{0,-1}^* + a_{0,-1}^* a_{0,1}^* - a_{0,0}^* a_{0,0}^* \big)^{\frac{N}{2}} |\text{vac}\rangle \, .$$

The LPB state is not of the type $\varrho_{N,\text{frag}}$ previously considered, nonetheless the rank of its effective 2-RDM is different from the rank of its 1-RDM. Therefore we believe that

this definition characterizes finite fragmented condensation better than the usual Penrose-Onsager-like definition, at the same time being only slightly more difficult to verify.

3. Persistence of finite fragmented condensation

In this section we study the behavior of (temperature-zero) finite fragmented condensation under the action of two-body inter-particle interactions of mean-field type (Hartree regime).

Let us consider a many-body system of N bosons with (pseudo-) spin s, whose free one-body energy operator has an energy gap ω between the (degenerate) ground state and the first excited states. We also denote the (re-scaled) inter-particle interaction by V. The Hamiltonian of the system is given by

$$(5) \qquad H_{\omega,N} := \sum_{j=1}^{N} h_{\omega,j} + \frac{1}{N} \sum_{j<k} V(x_j - x_k) \,,$$

acting on the symmetric product space $\mathcal{H}_N := \bigvee_{j=1}^{N} L^2(\mathbb{R}^d, \mathbb{C}^{2s+1})$, $s \in \frac{1}{2}\mathbb{N} \smallsetminus \{0\}$. We require the following mathematical assumptions on the one-particle operator h_ω, acting on $\mathcal{H}_1 = L^2(\mathbb{R}^d, \mathbb{C}^{2s+1})$.

- (Self-adjointness) $h_\omega = \underline{h}_\omega \otimes \mathrm{id}_{\mathbb{C}^{2s+1}}$, $\underline{h}_\omega$ self-adjoint on $L^2(\mathbb{R}^d, \mathbb{C})$;

- (Ground state energy) $\inf \sigma(\underline{h}_\omega) = 0$;

- (Ground state) $\exists \underline{\varphi} \in L^2(\mathbb{R}^d, \mathbb{C})$, $\left\{ \underline{\psi} \in L^2(\mathbb{R}^d, \mathbb{C}), \, \|\underline{\psi}\|_2 = 1, \, \underline{h}_\omega \underline{\psi} = 0 \right\} = \mathrm{span}_{\mathbb{C}} \left\{ \underline{\varphi} \right\}$;

- (Gap condition) $\inf \left(\sigma(\underline{h}_\omega) \smallsetminus \{0\} \right) = \omega \in \mathbb{R}^+ \smallsetminus \{0\}$.

The pair potential $V \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{C}^{2s+1})$ is supposed to be Kato-small, with arbitrarily small bound, as an operator, with respect to h_ω. More precisely, we require that, for every $\varepsilon > 0$, there exists a constant C_ε such that

$$(6) \qquad V^2 \leq \varepsilon h_\omega^2 + C_\varepsilon \,.$$

From the above assumptions, it follows that $H_{\omega,N}$ is self-adjoint on $D\left(\sum_{j=1}^{N} h_{\omega,j} \right)$, and that h_ω has a $2s+1$-fold degenerate ground state, spanned by the orthonormal functions $\varphi_1 = \left(\underline{\varphi}, 0, \ldots, 0 \right), \cdots, \varphi_{2s+1} = \left(0, \ldots, 0, \underline{\varphi} \right)$. The degeneracy is induced by the degrees of freedom due to the particles' (pseudo-) spin. Let us denote

$$(7) \qquad \mathcal{F}_1 := \mathrm{span}_{\mathbb{C}} \left\{ \varphi_1, \ldots, \varphi_{2s+1} \right\} \subset \mathcal{H}_1 \,.$$

Then for any $\jmath = 1, \ldots, 2s+1$, $\psi \in \mathcal{F}_1^\perp$, $\|\psi\|_{\mathcal{H}_1} = 1$:

$$\langle \psi, h_\omega \psi \rangle_{\mathcal{H}_1} - \langle \varphi_k, h_\omega \varphi_k \rangle_{\mathcal{H}_1} \geq \omega \,;$$

in other words, the one-particle Hamiltonian h_ω has also an energy gap of order ω. Given an initial many-body configuration $\Psi_0 \in \mathcal{H}_N$, the time-evolution is given by the Schrödinger equation

$$\begin{cases} i\partial_t \Psi(t) = H_{\omega,N} \Psi(t) \\ \Psi(0) = \Psi_0 \end{cases} \,,$$

whose solution is $\Psi(t) = e^{-itH_{\omega,N}} \Psi_0$. Let us remark that our assumptions are fulfilled by a one-particle hamiltonian with harmonic trap $\underline{h}_\omega = -\Delta + \frac{1}{4}\omega^2 x^2 - \frac{3}{2}\omega$, and by the physically relevant pair interaction with a local Coulomb singularity, i.e. $V(x) \sim |x|^{-1}$.

At initial time, let us consider, as a fragmented condensate, a ground state of the non-interacting system of the following type:

$$\varrho_0 = \varrho_{\text{frag}} = \left| \varphi_1^{\otimes f_1(N)} \vee \cdots \vee \varphi_{2s+1}^{\otimes f_{2s+1}(N)} \right\rangle \left\langle \varphi_1^{\otimes f_1(N)} \vee \cdots \vee \varphi_{2s+1}^{\otimes f_{2s+1}(N)} \right| .$$

As we have already discussed in § 2, the $N \to \infty$ counterpart of ϱ_0 is a probability measure μ_0 on the one-particle space $\mathcal{H}_1$. For all $p \geq 1$,

$$(8) \qquad\qquad \gamma_{\infty,0}^{(p)} = \int_{\mathcal{H}_1} |\psi^{\otimes p}\rangle\langle\psi^{\otimes p}| \, \mathrm{d}\mu_0(\psi) .$$

The measure μ_0 is a $U(1)$-invariant product of convex combinations of delta measures:

$$(9) \qquad\qquad \mu_0 = \bigotimes_{k \in P} \delta_{\sqrt{\pi_k}\varphi_k}^{S_1} \otimes \delta_0^{\perp} ,$$

see the discussion following Eq. (4) for additional details.

Now, suppose that we have prepared the system in the state ϱ_0, and then turn on the inter-particle interaction, letting the system evolve for some time t. It has been proved in [AN15] that for any potential V satisfying the assumptions above, the time-evolved interacting effective p-RDMs, $p \geq 1$, at time t are given by

$$(10) \qquad\qquad \gamma_{\infty,t}^{(p)} = \int_{\mathcal{H}_1} |\psi(t)^{\otimes p}\rangle\langle\psi(t)^{\otimes p}| \, \mathrm{d}\mu_0(\psi) ,$$

where $\psi(t)$ is the unique solution of the effective Hartree Cauchy problem

$$(11) \qquad\qquad \begin{cases} i\partial_t \psi(t) = h_\omega \psi(t) + \big(V * |\psi(t)|\big)\psi(t) \\ \psi(0) = \psi \end{cases} .$$

In other words, the effective probability distribution of single-particle states is pushed forward by the Hartree effective evolution. From this mathematical description it is quite easy to see how the inter-particle interaction destroys finite fragmentation: the nonlinear Hartree evolution destroys, as it pushes forward the effective probability distribution, the factorized structure of the latter, and "spreads" it on the whole Hilbert space of available wavefunctions (in doing this, the relative phases corresponding to the different convex combinations of deltas in the product play a crucial role). As a result, the rank of $\gamma_{\infty,t}^{(p)}$ is, for almost all times $t \in \mathbb{R}$, *infinite*. This does not happen only in the case $\Pi = 1$, *i.e.* when the initial state is actually a simple condensate: in this case the measure is a $U(1)$-invariant convex combination of delta measures, and such structure is preserved by the $U(1)$-invariant nonlinear Hartree evolution; simple condensation is therefore preserved. Let us remark that the error made in approximating the evolved interacting N-particle reduced density matrices with $\gamma_{\infty,t}^{(p)}$ given by Eq. (10) is of order N^{-1} for any time, and this is confirmed by theoretical and numerical analysis [AFP16b].

Hence in general it is not possible to assume that a condensate fragmented into Π states remains Π-fragmented under the action of interactions, unless $\Pi = 1$: the fragmented condensate spreads to have macroscopic occupation on an infinity of one-particle states. Nonetheless, there are suitable situations in which finite fragmented condensation persists under the action of inter-particle interactions, in the sense that the fraction of particles not occupying a specified set of one-particle states is so small to be irrelevant at the macroscopic level. Our aim is to discuss the case of a very large energy gap ω between the degenerate ground state and the first excited states of the aforementioned system of bosons with (pseudo-) spin s is

very large. Intuitively, the explanation is the following: it costs so much energy to transition from the (free) ground state, that inter-particle interactions are not strong enough to cause such transition; thus the particles are effectively constrained to the $2s+1$-dimensional Hilbert space of degenerate ground states of h_ω, and this preserves the finite fragmentation caused by spin degeneracy. We rigorously justify the above intuition, and provide an explicit effective one-particle evolution on the reduced Hilbert space of degenerate ground states that is valid in the limit of a very large energy gap $\omega \to \infty$. Such evolution is conveniently described by the evolution of projections of the one-particle effective wavefunction on each degenerate ground state.

For the sake of simplicity, let us focus on the evolution of the first marginal $\gamma_{\infty,t}^{(1)}(\omega)$ (where the dependence on ω is made explicit to clarify that we are studying the infinite gap limit). By Eqs. (9) and (10), it follows that

$$(12) \qquad \gamma_{\infty,t}^{(1)}(\omega) = \tfrac{1}{(2\pi)^{\Pi}} \int_0^{2\pi} \big| \psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(t) \big\rangle \big\langle \psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(t) \big| \prod_{k \in P} d\theta_k \ ,$$

where $\psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(t)$ is the solution of Eq. (11) with initial condition

$$(13) \qquad \psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(0) = \sum_{k \in P} e^{i\theta_k} \sqrt{\pi_k} \varphi_k \ .$$

Let us decompose now $\psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(t)$ following the Hilbert space decomposition

$$\mathcal{H}_1 = \mathscr{F}_1 \oplus \mathscr{F}_1^\perp \ ,$$

introduced in Eq. (7). Let us stress that here $\mathscr{F}_1$ is spanned by *all* the $2s+1$ degenerate ground states. We obtain

$$(14) \qquad \psi_{\{\theta_\imath, \imath \in P\}}^{(\omega)}(t) = \sum_{k=1}^{2s+1} \kappa_{k,t}(\omega) \varphi_k + \psi_t^\perp(\omega) \ ;$$

where $\kappa_{k,t}(\omega) \in \mathbb{C}$ for any $k = 1, \ldots, 2s+1$, and $\psi_t^\perp(\omega) \in \mathscr{F}_1^\perp$. In the right hand side we have omitted for convenience the dependence on $\{\theta_\imath, \imath \in P\}$, however the presence of such phases plays a crucial role, and should be kept in mind. Finally, let us define for any $\imath < l = 1, \ldots, 2s$ the "averaged" transition amplitudes

$$(15) \qquad \begin{aligned} K_{\imath,t}(\omega) &= \tfrac{1}{(2\pi)^{\Pi}} \int_0^{2\pi} |\kappa_{\imath,t}(\omega)|^2 \prod_{k \in P} d\theta_k \\ K_{\imath\ell,t}(\omega) &= \tfrac{1}{(2\pi)^{\Pi}} \int_0^{2\pi} \overline{\kappa_{\imath,t}(\omega)} \kappa_{\ell,t}(\omega) \prod_{k \in P} d\theta_k \end{aligned} \ ,$$

that thus does not depend on $\{\theta_\imath, \imath \in P\}$ anymore. We also define their limits, which, according to Theorem 3.1, are interpreted as matrix elements of the mean-field 1-RDM in the limit of infinite gap:

$$(16) \qquad \begin{aligned} K_{\imath,t}(\infty) &= \lim_{\omega \to \infty} K_{\imath,t}(\omega) \\ K_{\imath\ell,t}(\infty) &= \lim_{\omega \to \infty} K_{\imath\ell,t}(\omega) \end{aligned} \ .$$

Theorem 3.1.

$$\gamma_{\infty,t}^{(1)}(\infty) := \lim_{N,\omega\to\infty} \gamma_{N,t}^{(1)}(\omega) = \sum_{\imath=1}^{2s+1} K_{\imath,t}(\infty)|\varphi_\imath\rangle\langle\varphi_\imath| + \sum_{\imath<\ell=1}^{2s+1} K_{\imath\ell,t}(\infty)|\varphi_\imath\rangle\langle\varphi_\ell| + \overline{K_{\imath\ell,t}(\infty)}|\varphi_\ell\rangle\langle\varphi_\imath|.$$

$\Pi \le \operatorname{Rank}\gamma_{\infty,t}^{(1)}(\infty) \le 2s+1$ *for all* $t \in \mathbb{R}$, *being equal to* $2s+1$ *for a.e.* t.

Both limits are intended in the norm topology of the space of trace-class operators $\mathfrak{S}^1(\mathcal{H}_1)$, *and the order in which they are taken is* indifferent. *In addition, the evolution of each component* $\kappa_{\jmath,t}(\infty)$ *is described by the ordinary differential equation*

$$(17) \quad \begin{cases} i\partial_t \kappa_{\imath,t}(\infty) = \Big\langle \varphi_\imath \,, V * \Big|\sum_{\ell=1}^{2s+1} \kappa_{\ell,t}(\infty)\varphi_\ell\Big|^2 \Big(\sum_{\ell'=1}^{2s+1} \kappa_{\ell',t}(\infty)\varphi_{\ell'}\Big) \Big\rangle \\ \kappa_{\imath,0}(\infty) = \begin{cases} e^{i\theta_\imath}\sqrt{\pi_\imath} & \text{if } \imath \in P \\ 0 & \text{if } \imath \notin P \end{cases} \end{cases}.$$

Remark. Theorem 3.1 can be extended without effort to any p-RDMs. In fact, we have that

$$\binom{p+\Pi-1}{p} \le \operatorname{Rank}\gamma_{\infty,t}^{(p)}(\infty) \le \binom{p+2s}{p} \text{ for all } t \in \mathbb{R}, \text{ being equal to } \binom{p+2s}{p} \text{ for a.e. } t.$$

The proof of Theorem 3.1 can be found in Appendix B.2. As we discussed previously, Theorem 3.1 rigorously proves the persistence of finite fragmented condensation for interacting systems in the Hartree regime, if the energy gap between the degenerate one-particle ground-state of the non-interacting system causing fragmentation and the relative excited states is very large. In that case, the system remains finitely fragmented on the space of ground states: at almost every time there is a nonzero macroscopic fraction of particles occupying *all* the available degenerate non-interacting ground states, provided that at the initial time at least two of them had macroscopic occupation, and no macroscopic occupation of the orthogonal space of excitations. The effective one-particle reduced density matrix of the system has in fact non-zero action only on the $2s+1$-dimensional subspace $\mathcal{F}_1$ spanned by the degenerate ground states, and its matrix elements can be characterized explicitly. They are the average, over all possible phases, of the coefficients of projection on the ground states of the solution of the Hartree equation on $\mathcal{F}_1$, corresponding to initial data oscillating with the aforementioned phase coefficient. Such evolution shares some similarities with the one given in [ASC07, ASC08].

Appendix A. Systems with many bosons and QFT semiclassical analysis

In order to better understand the main mathematical tools used throughout this paper, let us recast the large N approximation as a semiclassical problem in quantum field theory, and recall some of the main results obtained in the latter concerning semiclassical states. This section may be of independent interest for the reader that is not familiar with semiclassical techniques in bosonic field theories.

Let us start with a simple remark: *it is always possible to see an N-particle bosonic vector* $\psi \in \mathcal{H}_N = \bigvee_{j=1}^{N} \mathcal{H}$, *where* $\mathcal{H}$ *is a separable "one-boson" Hilbert space (in § 1 it was* $L^2(\mathbb{R}^d, \mathbb{C}^{2s+1})$), *as the only non-zero component of a vector* $\Psi = (0,\dots,0,\psi,0,\dots)$ *in the symmetric Fock space* $\Gamma_{\mathrm{s}}(\mathcal{H}) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}_n$. Therefore it is possible to interpret any N-particle bosonic density matrix, with N fixed, as a density matrix with an arbitrary number

of identical particles and probability one of having exactly N particles. It is also possible to interpret an N-body Hamiltonian H_N (with pair interactions) as an Hamiltonian H on the Fock space that agrees with H_N *on the N-particles sector* and that commutes with the number operator: let

$$H_N = \sum_{j=1}^{N} h_j + \frac{1}{N} \sum_{j<k} V(x_j - x_k)$$

be the self-adjoint Hamiltonian defined by Eq. (5) (where we have omitted the ω dependence for simplicity); then the Hamiltonian H on $\Gamma_{\mathrm{s}}(\mathcal{H})$ defined by

$$H = \int h(x,y) a^*(x) a(y) \mathrm{d}x \mathrm{d}y + \frac{1}{2N} \int V(x-y) a^*(x) a^*(y) a(x) a(y) \mathrm{d}x \mathrm{d}y \, ,$$

where $h(x,y)$ is the integral kernel of the self-adjoint operator h, and a^*, a are the bosonic creation and annihilation operator-valued distributions, is self-adjoint [see, *e.g.*, GV70, Fal15] and agrees with H_N when restricted to $\mathcal{H}_N$ (so in particular $H\Psi\big|_{\mathcal{H}_N} = H_N\psi$). So for any vector $\psi \in \mathcal{H}_N$, $e^{-itH_N}\psi = e^{-itH}\Psi\big|_{\mathcal{H}_N}$, the other components of the latter being zero. In other words, we can study the evolution of an N-body system with pair interactions directly in the Fock space setting (restricting to vectors whose only non-zero component is in the N-particle sector).

Let us now show that the limit $N \to \infty$ can be seen as a classical limit in quantum field theory (whose limit "classical" field is the one-particle mean-field wavefunction). First of all, let us define the semiclassical parameter $\varepsilon = N^{-1}$. Therefore H depends on ε, as well as, in general, the vector $\Psi = \Psi_\varepsilon$. In order to make the semiclassical nature of the problem more explicit, let us define new creation and annihilation operator-valued distributions $a_\varepsilon^* = \sqrt{\varepsilon} a^*$ and $a_\varepsilon = \sqrt{\varepsilon} a$ satisfying

$$[a_\varepsilon(x), a_\varepsilon^*(y)] = \varepsilon \delta(x - y) \, .$$

If we rewrite H in terms of a_ε^* and a_ε, we obtain $H = \frac{1}{\varepsilon} H_\varepsilon$, where

$$H_\varepsilon = \int h(x,y) a_\varepsilon^*(x) a_\varepsilon^*(y) \mathrm{d}x \mathrm{d}y + \frac{1}{2} \int V(x-y) a_\varepsilon^*(x) a_\varepsilon^*(y) a_\varepsilon(x) a_\varepsilon(y) \mathrm{d}x \mathrm{d}y$$

is the energy per particle ($H_\varepsilon = \frac{H}{N}$). Therefore, the evolution of the system is described by $e^{-i\frac{t}{\varepsilon}H_\varepsilon}\Psi_\varepsilon$, and the creation and annihilation operators corresponding to the canonical field observables of the system satisfy "semiclassical" ε-dependent commutation relations. In other words, the parameter ε for this (non-relativistic) bosonic quantum field theory is perfectly analogous to $\hbar$ in ordinary quantum mechanics, and the system admits therefore a semiclassical description.

Semiclassical analysis for bosonic quantum field theories has been studied rigorously [see, *e.g.*, AN08, Fal13, AF14, AN15, Fal16, ABN17, AF17, Fal18, CF18, CFO18], and share some similarities with the better known quantum mechanical version (that can in fact be recovered as a special case). There are, however, some significant differences due to the fact that the classical fields' phase space is infinite-dimensional. Let us outline the basic ideas. Fields (both at the classical and quantum level) are described mathematically as distributions, making sense when smeared by suitably regular test functions. Given a real vector space (X, ς) of test functions with a symplectic form (that due to its symplectic nature is sometimes called the phase space), it is possible to construct the algebra of quantum observables $\mathbb{W}_\varepsilon(X, \varsigma)$ satisfying ε-dependent canonical commutation relations (bosonic algebra of observables), and

quantum states as continuous linear functionals of norm one on such algebra. In the limit $\varepsilon \to 0$, one would like to interpret observables and states as classical objects, *i.e.* functions and states of classical fields. In particular, since the algebra of bosonic observables can be represented in the symmetric Fock space $\Gamma_\mathrm{s}(\mathcal{H})$ whenever (X, ς) originates from a complex pre-Hilbert space $\mathfrak{H}$ ($X = \mathfrak{H}_\mathbb{R}$ and $\varsigma = \mathrm{Im}\langle \,\cdot\, , \,\cdot\, \rangle_\mathfrak{H}$) such that $\mathcal{H} = \overline{\mathfrak{H}}$, one would like to have at least such a description for polynomial field observables and density matrices in the Fock space.

For quantum states, it is actually possible to give a very general semiclassical description. Given a state ω_ε on $\mathbb{W}_\varepsilon(X, \varsigma)$, there is always a (generalized) subsequence $\omega_{\varepsilon_\beta}$ that converges to some classical state M (in a suitable topology) that is a cylindrical probability on the space of classical fields X_X^*, the latter being the algebraic dual of X endowed with the weak $\sigma(X^*, X)$ topology. M is called the cylindrical Wigner measure associated to the state $\omega_{\varepsilon_\beta}$, and the subsequence extraction is a mathematical requirement that does not have physical relevance. Such description agrees with the physical intuition: the classical counterpart of a quantum state is a "probability measure" in a space of classical fields acting on the test functions as (non-regular) distributions. In addition it can be shown that this description is unique up to isomorphisms, and "optimal" in the sense that given any cylindrical probability on X_X^* there is at least one quantum state that converges to it semiclassically. There are two features that are not completely satisfactory: M is in general only a cylindrical probability and not a true probability measure [for an introduction to cylindrical probabilities, see Sch73, VTC87], and the classical fields in X_X^* can be very singular, hence difficult to study mathematically (the space of classical fields is in some sense "too large"). Cylindrical measures are physically relevant, since they are the classical counterpart of suitable thermodynamic states (*e.g.* suitable grand-canonical Gibbs states). The space of classical fields is too large because the quantum states considered in this abstract setting are too general, and one should restrict to physical quantum states. It is possible to formulate sufficient conditions on states such that all the corresponding Wigner measures are concentrated as true probability measures in more reasonable spaces of classical fields. In our non-relativistic many bosons system, it can be proved that *all Wigner measures corresponding, in the limit $N \to \infty$, to N-particle density matrices are Radon probability measures, concentrated on the unit ball of $\mathcal{H}$, that are invariant under* $\mathrm{U}(1)$ *symmetry transformations on $\mathcal{H}$.* The situation may be more complicated in relativistic quantum mechanics since the number of particles is not conserved by the evolution. Let us remark that the action of linear symplectic transformations on test functions induces on one hand a quantum transformation on states (and also observables), and on the other hand a classical transformation on classical fields. The transformed quantum state converges semiclassically to the corresponding Wigner measure pushed forward by the classical fields' transformation.

The semiclassical analysis of quantum field observables is more difficult in general, but it is possible for a wide class of observables in the Fock representation that are polynomial in the fields, and normal ordered. For example, the second quantized operator H_ε can be seen as the Wick quantization of the following densely defined symbol on $\mathcal{H}$:

$$\mathcal{E}(u) = \int h(x, y)\bar{u}(x)u(y)\mathrm{d}x\mathrm{d}y + \frac{1}{2}\int V(x - y)|u(x)|^2|u(y)|^2\mathrm{d}x\mathrm{d}y \,,$$

defined and real-valued for any $u \in D(h) \subseteq \mathcal{H}$. The function u is the classical counterpart of the annihilation operator-valued distribution, and it is interpreted as the one-particle effective wavefunction in the mean-field limit. It can be proved that for any density matrix ϱ_ε such that $\mathrm{Tr}(\varrho_\varepsilon H_\varepsilon) \leq C$, the associated Wigner measure μ is concentrated on $D(h)$, and

$$\lim_{\varepsilon \to 0} \mathrm{Tr}(\varrho_\varepsilon H_\varepsilon) = \int_{D(h)} \mathcal{E}(u)\mathrm{d}\mu(u) \ .$$

In particular, it follows that the expectation on an N-body state of the energy per-particle converges as $N \to \infty$ to the average of the effective one-particle energy with respect to the mean-field probability distribution of one-particle wavefunctions. A similar description can be given for other physically relevant observables of the system. The p-RDM $\gamma_\varepsilon^{(p)}$, associated to a density matrix ϱ_ε converging to the Wigner measure μ, also converges:

$$\lim_{\varepsilon \to 0} \gamma_\varepsilon^{(p)} = \int_{\mathcal{H}} \big| \underbrace{u \otimes \cdots \otimes u}_{p} \big\rangle \big\langle \underbrace{u \otimes \cdots \otimes u}_{p} \big| \, \mathrm{d}\mu(u) \ ,$$

and the convergence always holds in the weak-* topology of trace class operators, and for suitable density matrices also in trace norm topology.

It is also possible to characterize explicitly the effective evolution of the mean-field particle in the limit. Let $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon} \varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon}$ be the many-body evolution of the density matrix ϱ_ε. Then, given an observable A_ε that is the Wick quantization of a (suitably nice) polynomial symbol $\mathcal{A}(u)$, we get

$$\lim_{\varepsilon \to 0} \mathrm{Tr}(\varrho_\varepsilon(t)A_\varepsilon) = \int_{\mathcal{H}} \mathcal{A}(u_t)\mathrm{d}\mu(u) \ ,$$

where u_t is the solution of the Hartree equation $i\partial_t u_t = hu_t + V * |u_t|^2 u_t$ with initial datum $u_0 = u$. In other words, the mean-field counterpart of an evolved N-body state is a probability pushed forward by the nonlinear Hartree flow governing the effective evolution.

Let us conclude this section remarking that it is possible to compute explicitly the Wigner measure, for physically relevant quantum states. This is done with the aid of the "non-commutative Fourier transform" of a state, *i.e.* testing the convergence with Weyl operators (exponential of the field smeared on test functions). Such average converges to the Fourier transform of the Wigner measure, that identifies it uniquely. For example, an N-particle density matrix of the form $\big| u^{\otimes N} \big\rangle \big\langle u^{\otimes N} \big|$ converges to the measure $\int_0^{2\pi} \delta(z - e^{i\theta}u)\frac{\mathrm{d}\theta}{2\pi}$ (that is U(1)-invariant as expected), while the squeezed coherent state $C_\varepsilon(u)$ of minimal uncertainty converges to the delta measure $\delta(z - u)$ (that is not U(1) invariant since the coherent state has non-zero components on any fixed-particle sector of the Fock space). The Wigner measures corresponding to fragmented density matrices have already been introduced in § 2. For other explicit examples, refer, *e.g.*, to [AN08, AFP16a].

Appendix B. Mathematical proofs

In this appendix we collect the mathematical proofs of the results discussed in § 2 and 3.

B.1. **Proof of Proposition 2.2.** To prove Proposition 2.2 it is important to better understand the combinatorial factors appearing in $\varrho_{N,\mathrm{frag}}$ and $\gamma_{\infty,\mathrm{frag}}^{(p)}$. Recall that the wavefunction

$\varphi_1^{\otimes f_1(N)} \vee \cdots \vee \varphi_{2s+1}^{\otimes f_{2s+1}(N)}$ is defined as follows:

$$
\varphi_1^{\otimes f_1(N)} \vee \cdots \vee \varphi_{2s+1}^{\otimes f_{2s+1}(N)} (x_1, \ldots x_N) := C_{f(N),N} \sum_{\sigma \in \mathscr{S}_N} \prod_{n=1}^{N} \sum_{j=1}^{2s+1} \chi_{F_j}(n) \varphi_j(x_{\sigma(n)})
$$

(18)

$$
= C_{f(N),N} \sum_{\sigma \in \mathscr{S}_N} \prod_{n=1}^{N} \sum_{j=1}^{2s+1} \chi_{\sigma(F_j)}(n) \varphi_j(x_n)
$$

with $C_{f,N} := \left(N! \prod_{j=1}^{2s+1} f_j! \right)^{-\frac{1}{2}}$, and $F_j := \left(\sum_{l=1}^{j-1} f_l(N), \sum_{l=1}^{j} f_l(N) \right] \cap \mathbb{N}$ for any $j \in \{1, \ldots, 2s+1\}$. It follows from the orthogonality of the vectors φ_j that cancellations occur in the partial trace of $\varrho_{N,\text{frag}}$; nonetheless the number of vectors φ_j is the same on each side of the projection. More precisely,

(19)

$$
[Tr_{N-p}\varrho_{N,\text{frag}}] (x_1, \ldots, x_p; y_1, \ldots, y_p) = C_{f(N),N}^2 \sum_{\sigma,\tau \in \mathscr{S}_N} \left(\prod_{n=1}^{p} \sum_{j=1}^{2s+1} \chi_{\sigma(F_j) \cap \{1,\ldots,k\}}(n) \varphi_j(x_n) \right)
$$

$$
\cdot \left(\prod_{\nu=1}^{p} \sum_{\gamma=1}^{2s+1} \chi_{\tau(F_\gamma) \cap \{1,\ldots,k\}}(\nu) \overline{\varphi_\gamma(y_\nu)} \right) \left(\prod_{m=k+1}^{N} \sum_{l=1}^{2s+1} \chi_{\sigma(F_l) \cap \tau(F_l) \cap \{k+1,\ldots,N\}}(m) \right).
$$

The last term in the product naturally translate on a condition on the family of possible permutations that we can choose from. Let us identify, for any permutation $\sigma \in \mathscr{S}_N$, the set $I[\sigma] = \sigma^{-1}(\{1, \ldots, p\})$. The permutation σ can be seen as a permutation from $I[\sigma]$ to p elements, and another from the complement of $I[\sigma]$ to (the other) $N-p$ elements. This idea can be carried further, and it is easy to realize that every permutation $\sigma \in \mathscr{S}_N$ can be thought of as a triple (σ_1, σ_2, I) with $\sigma_1 \in \mathscr{S}_p$, $\sigma_2 \in \mathscr{S}_{N-p}$ and $I \subseteq \{1, \ldots, N\}$ with $\#I = p$.

Now it is important to understand how the choice of k particles affects the total distribution of particles. In this sense, one should first notice that $f(N)$ (as a frequency distribution) is identified uniquely by the partition of connected subsets $F = \{F_j, \ 1 \le j \le 2s+1\}$. Both the family $F \cap I[\sigma]$ and $F \cap (\{1, \ldots, N\} \smallsetminus I[\sigma])$ are partitions of connected subsets, respectively of $I[\sigma]$ and of its complement. To these two families we can now associate two different frequency distributions g and h, respectively in the sets $\mathcal{F}_{p,2s+1}^N$ and $\mathcal{F}_{N-p,2s+1}^N$, where

(20)
$$
\mathcal{F}_{p,\alpha}^N := \left\{ g : \{1, \ldots, \alpha\} \to \mathbb{N} : \ g_j \le f_j(N), \ \sum_{j=1}^{\alpha} g_j = k \right\}.
$$

Denote $F^g := F \cap I[\sigma]$, and $F^h := F \cap (\{1, \ldots, N\} \smallsetminus I[\sigma])$. These two families of sets (and the corresponding distributions g and h) are identified, in a non-unique fashion, by the set $I[\sigma]$. It is hence possible to rewrite Eq. (19) as

(21) $[Tr_{N-p}\varrho_{N,\text{frag}}] (x_1, \ldots, x_p; y_1, \ldots, y_p) =$

$$
\sum_{\substack{\sigma_1,\tau_1 \in \mathscr{S}_p, \ \sigma_2,\tau_2 \in \mathscr{S}_{N-p} \\ I,J \subseteq \{1,\ldots,N\}, \#I = \#J = k \\ \sigma_2(F_l^{h(I)}) = \tau_2(F_l^{h(J)}), 1 \le l \le 2s+1}} C_{f(N),N}^2 \left(\prod_{n=1}^{p} \sum_{l=1}^{2s+1} \chi_{\sigma_1(F_l^{g(I)})}(n) \varphi_l(x_n) \right) \left(\prod_{\nu=1}^{p} \sum_{\gamma=1}^{2s+1} \chi_{\tau_1(F_\gamma^{g(J)})}(\nu) \overline{\varphi_\gamma(y_\nu)} \right).
$$

To get rid of σ_2 and τ_2 it is useful to compute the number of permutations that respect the constraint in the sum, that is

$$(22) \quad \#\left\{\sigma_2, \tau_2 \in \mathscr{S}_{N-p}: \; \sigma_2\left(F_l^{h(I)}\right) = \sigma_2\left(F_l^{h(J)}\right), \; 1 \le l \le 2s+1\right\}$$

$$= \begin{cases} (N-p)! \prod_{l=1}^{2s+1} [h(I)_l]! \equiv C_{h(I),N-p}^{-2} & \text{if } g(I) = g(J); \\ 0 & \text{else.} \end{cases}$$

Moreover, since the terms of the summation do not depend directly on I and J, we would like to substitute them with the possible choices of g. To do so, it is necessary to understand how many choices of I correspond to the same distribution function g. Equivalently, we should know in how many ways it is possible to choose g_j elements in $f_j(N)$. Clearly, the answer is given by a binomial coefficient. Therefore, it follows that

$$(23)$$
$$[Tr_{N-p}\varrho_{N,\text{frag}}]\,(x_1,\ldots,x_p;y_1,\ldots,y_p) =$$

$$\sum_{\substack{\sigma,\tau \in \mathscr{S}_k \\ g \in \mathcal{F}_{p,2s+1}^N}} \frac{C_{f(N),N}^2}{C_{f(N)-g,N-p}^2} \left[\prod_{j=1}^{2s+1}\binom{f_j(N)}{g_j}\right]^2 \left(\prod_{n=1}^{p}\sum_{l=1}^{2s+1}\chi_{\sigma(F_l^g)}(n)\varphi_l(x_n)\right) \left(\prod_{\nu=1}^{p}\sum_{\gamma=1}^{2s+1}\chi_{\tau(F_\gamma^g)}(\nu)\overline{\varphi_\gamma(y_\nu)}\right).$$

It is not difficult to see the operator corresponding to that kernel is a combination of projectors:

$$(24)$$
$$Tr_{N-p}\varrho_{N,\text{frag}} = \sum_{g \in \mathcal{F}_{p,2s+1}^N} \binom{N}{k}^{-1}\left[\prod_{j=1}^{2s+1}\binom{f_j(N)}{g_j}\right]\left|\varphi_1^{\otimes g_1}\vee\cdots\vee\varphi_{2s+1}^{\otimes g_{2s+1}}\right\rangle\left\langle\varphi_1^{\otimes g_1}\vee\cdots\vee\varphi_{2s+1}^{\otimes g_{2s+1}}\right|.$$

To compute the limit $N \to \infty$ it is convenient to calculate the limit of each coefficient in the sum. Recall that $f_j = \pi_j N + o(N)$. In particular this implies that for any strictly positive integer λ smaller than f_j, the binomial coefficient $\binom{f_j}{\lambda} = \frac{1}{\lambda!}N^\lambda\left(\pi_j^\lambda + o(1)\right)$. Hence we get

$$(25)$$
$$\lim_{N\to+\infty}\binom{N}{k}^{-1}\prod_{j=1}^{2s+1}\binom{f_j(N)}{g_j} = \lim_{N\to+\infty}\binom{N}{k}^{-1}\prod_{\substack{j\in\{1,\ldots,2s+1\}\\ j:\, g_j\neq 0}}\binom{f_j(N)}{g_j} = k!\prod_{\substack{j\in\{1,\ldots,2s+1\}\\ j:\, g_j\neq 0}}\frac{\pi_j^{g_j}}{g_j!}.$$

From Eq. (25) it follows that

$$\gamma_{\infty,\text{frag}}^{(p)} := \text{w}^*{-}\lim_{N\to\infty} Tr_{N-p}\,\varrho_{N,\text{frag}}$$

$$(26)$$
$$= \sum_{g \in \mathcal{F}_{p,2s+1}} k! \left[\prod_{\substack{j\in\{1,\ldots,2s+1\}\\ j:\, g_j\neq 0}}\frac{\pi_j^{g_j}}{g_j!}\right]\left|\varphi_1^{\otimes g_1}\vee\cdots\vee\varphi_{2s+1}^{\otimes g_{2s+1}}\right\rangle\left\langle\varphi_1^{\otimes g_1}\vee\cdots\vee\varphi_{2s+1}^{\otimes g_{2s+1}}\right|.$$

The rank of such density matrix is the following:

$$(27) \qquad \text{Rank } \gamma_{\infty,\text{frag}}^{(p)} = \#\left\{g \in \mathcal{F}_p: \; \pi_j = 0 \Rightarrow g_j = 0, 1 \le j \le s\right\} = \binom{p+\Pi-1}{p}.$$

B.2. **Proof of Theorem 3.1.** We prove Theorem 3.1 in two steps. Firstly, in Appendix B.2.1 we prove the desired limit taking before $N \to \infty$, and then $\omega \to \infty$.

Then in Appendix B.2.2 we take the limits in the reverse order $\omega \to \infty$ and $N \to \infty$, obtaining the same result.

B.2.1. *The limit $N \to \infty$, $\omega \to \infty$.* We consider directly the effective problem, where the limit $N \to \infty$ has already been performed [see AN15, AFP16b, for a detailed discussion of the limit $N \to \infty$ for fragmented and more general states]. We prove that, at every time $t \geq 0$, the component of $\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t)$ that is orthogonal to all the φ_k is L^2-small whenever ω is big. Then, we show that the coefficients $\{\kappa_{\imath,t}(\omega)\}$ converge to the solution $\{\kappa_{\imath,t}(\infty)\}$ of (17) (the existence and uniqueness of solutions to this system of ODE does not present difficulties). The above implies that $\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t)$ has a limit for every $\theta_\imath \in [0, 2\pi]$ and for every $t \geq 0$, as ω becomes large. Then we show how this implies the convergence of the 1-RDM. This concludes the proof of Theorem 3.1 if the limits are taken in the order $N \to \infty$, and then $\omega \to \infty$. The converse order is considered in the next section.

The first step is to prove the following proposition.

Proposition B.1. *Consider the decomposition*

$$\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t) = \sum_{k=1}^{2s+1} \kappa_{k,t}(\omega) \varphi_k + \psi_t^\perp(\omega)$$

of the solution to the Hartree equation (11) with initial datum

$$\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(0) = \sum_{k \in P} e^{i\theta_k} \sqrt{\pi_k} \varphi_k .$$

Then there exists $C > 0$ such that for every $\theta_\imath \in [0, 2\pi]$, $\imath \in P$,

$$\tag{28} \|\psi_t^\perp(\omega)\|_2 \leq \frac{C}{\omega^{1/2}} .$$

In particular,

$$\tag{29} \lim_{\omega \to \infty} \|\psi_t^\perp(\omega)\|_2 = 0 .$$

Proof. The Hartree energy

$$\tag{30} \mathcal{E}[\psi] = \langle \psi, h_\omega \psi \rangle + \langle \psi, V * |\psi|^2 \psi \rangle$$

is conserved along the Hamiltonian flow of (11). It is then sufficient to show that $\mathcal{E}[\psi^{(\omega)}(t)]$ controls the norm $\|\psi_t^\perp(\omega)\|_2$.

By the relative boundedness assumption on V, *i.e.* $V(x - y) \leq \varepsilon h_x + C_\varepsilon$, for every $\psi \in \mathcal{H}_1$ with $\|\psi\| = 1$ we have

$$\tag{31} \left| \langle \psi, V * |\psi|^2 \psi \rangle \right| \leq \langle \psi \otimes \psi, \left[\varepsilon h_{\omega,x} + C_\varepsilon \right] \psi \otimes \psi \rangle \leq \varepsilon \langle \psi, h_\omega \psi \rangle + C_\varepsilon ,$$

for every $\varepsilon > 0$. This immediately implies

$$(1 - \varepsilon) \langle \psi, h_\omega \psi \rangle \leq \mathcal{E}[\psi] + C_\varepsilon .$$

Now since $\mathcal{E}[\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t)] = \mathcal{E}[\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(0)]$, we deduce that

$$\tag{32} (1 - \varepsilon) \langle \psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t), h_\omega \psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t) \rangle \leq \mathcal{E}[\psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(0)] + C_\varepsilon .$$

Moreover, by the hypotheses on h_ω we have

$$\tag{33} \langle \psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t), h_\omega \psi^{(\omega)}_{\{\theta_\imath, \imath \in P\}}(t) \rangle \geq \omega \|\psi_t^\perp(\omega)\|_2^2 ,$$

and, by comparing (32) and (33), we deduce that for any $0 < \varepsilon < 1$,

$$(34) \qquad \|\psi_t^\perp(\omega)\|_2^2 \le \frac{\mathcal{E}[\psi_{\{\theta_i, i \in P\}}^{(\omega)}(0)] + C_\varepsilon}{(1-\varepsilon)\omega} \ .$$

This concludes the proof, since by (31) we have that

$$\left| \mathcal{E}[\psi_{\{\theta_i, i \in P\}}^{(\omega)}(0)] \right| = \left| \langle \psi_{\{\theta_i, i \in P\}}^{(\omega)}(0), V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(0)|^2 \psi_{\{\theta_i, i \in P\}}^{(\omega)}(0) \rangle \right|$$

is bounded by a constant independent of ω. ⊣

The next step is to show that $\kappa_{i,t}(\omega)$ converges to $\kappa_{i,t}(\infty)$.

Proposition B.2. *For every $1 \le i \le 2s + 1$ and for every $t \ge 0$ we have*

$$(35) \qquad |\kappa_{i,t}(\omega) - \kappa_{i,t}(\infty)|^2 \le \frac{1}{\omega} e^{C|t|} ,$$

for some positive C, independent on t or ω.

Proof. By direct computation one finds that the coefficients $\{\kappa_{i,t}(\omega)\}$ satisfy the system of ODE

$$(36) \qquad \begin{cases} i\partial_t \kappa_{i,t}(\omega) = \left\langle \varphi_i , \ V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 (\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)) \right\rangle \\ \kappa_{i,0}(\omega) = \begin{cases} e^{i\theta_i}\sqrt{\pi_i} & \text{if } i \in P \\ 0 & \text{if } i \notin P \end{cases} \end{cases} .$$

Let us compute

$$\partial_t \left| \kappa_{i,t}(\omega) - \kappa_{i,t}(\infty) \right|^2 = \Im\left[\left(\overline{\kappa_{i,t}(\omega)} - \overline{\kappa_{i,t}(\infty)} \right) \left(\left\langle \varphi_i , \ V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 (\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)) \right\rangle \right.\right.$$

$$\left.\left. - \left\langle \varphi_i , \ V * \left| \sum_{\ell=1}^{2s+1} \kappa_{\ell,t}(\infty)\varphi_\ell \right|^2 \left(\sum_{\ell'=1}^{2s+1} \kappa_{\ell',t}(\infty)\varphi_{\ell'} \right) \right\rangle \right) \right]$$

$$= \Im\left[\left(\overline{\kappa_{i,t}(\omega)} - \overline{\kappa_{i,t}(\infty)} \right) \left\langle \varphi_i , \ V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 \left(\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t) - \sum_{\ell'=1}^{2s+1} \kappa_{\ell',t}(\infty)\varphi_{\ell'} \right) \right\rangle \right]$$

$$+ \Im\left[\left(\overline{\kappa_{i,t}(\omega)} - \overline{\kappa_{i,t}(\infty)} \right) \left\langle \varphi_i , \ V * \left(\left| \sum_{\ell=1}^{2s+1} \kappa_{\ell,t}(\infty)\varphi_\ell \right|^2 - |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 \right) \left(\sum_{\ell'=1}^{2s+1} \kappa_{\ell',t}(\infty)\varphi_{\ell'} \right) \right\rangle \right]$$

$$=: I_i + II_i .$$

We treat the two terms separately.

$$|I_i| \le \left| \kappa_{i,t}(\omega) - \kappa_{i,t}(\infty) \right| \left\| V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 \varphi_i \right\|_2 \left\| \psi_{\{\theta_i, i \in P\}}^{(\omega)}(t) - \sum_{\ell'=1}^{2s+1} \kappa_{\ell',t}(\infty)\varphi_{\ell'} \right\|_2 .$$

By the assumptions on V one finds

$$\left\| V * |\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t)|^2 \varphi_i \right\|_2 \le \varepsilon \langle \psi_{\{\theta_i, i \in P\}}^{(\omega)}(t), h_\omega \psi_{\{\theta_i, i \in P\}}^{(\omega)}(t) \rangle + C_\varepsilon \le C ,$$

for some constant C that does not depend on ω. Moreover,

$$\psi_{\{\theta_i, i \in P\}}^{(\omega)}(t) - \sum_{\ell=1}^{2s+1} \kappa_{\ell,t}(\infty)\varphi_\ell = \sum_{\ell=1}^{2s+1} \left(\kappa_{\ell,t}(\omega) - \kappa_{\ell,t}(\infty) \right)\varphi_\ell + \psi_t^\perp(\omega).$$

Hence,

$$
\begin{aligned}
|I_i| \ &\le\ C\left|\kappa_{i,t}(\omega) - \kappa_{i,t}(\infty)\right| \left[\sum_{\ell'=1}^{2s+1}\left|\kappa_{\ell',t}(\omega) - \kappa_{\ell',t}(\infty)\right| + \|\psi_t^\perp(\omega)\|_2\right] \\
&\le\ C\left|\kappa_{i,t}(\omega) - \kappa_{i,t}(\infty)\right| \left[\sum_{\ell'=1}^{2s+1}\left|\kappa_{\ell',t}(\omega) - \kappa_{\ell',t}(\infty)\right| + \frac{C}{\omega^{1/2}}\right],
\end{aligned}
$$

(37)

having used Proposition B.1 in the last inequality.

The estimate of II_i is analogous, and we get

$$
|II_i| \ \le\ C\left|\kappa_{i,t}(\omega) - \kappa_{i,t}(\infty)\right| \left[\sum_{\ell=1}^{2s+1}\left|\kappa_{\ell,t}(\omega) - \kappa_{\ell,t}(\infty)\right| + \frac{C}{\omega^{1/2}}\right].
$$

Denoting $A(t) := \sum_{k=1}^{2s+1} \left|\kappa_{k,t}(\omega) - \kappa_{k,t}(\infty)\right|^2$, we have thus

$$
\dot{A}(t) \ \le\ C\big(A(t) + \omega^{-1}\big),
$$

that yields, by Grönwall's inequality,

(38)
$$
A(t) \ \le\ \frac{1}{\omega}e^{Ct},
$$

since $A(0) = 0$. ⊣

We are now able to conclude the proof of Theorem 3.1 for the limits taken in the order $N \to \infty$, $\omega \to \infty$. Recall that, from (12),

$$
\gamma_{\infty,t}^{(1)}(\omega) = \frac{1}{(2\pi)^\Pi}\int_0^{2\pi} \left|\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right\rangle\left\langle\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right| \prod_{k\in P} \mathrm{d}\theta_k.
$$

We will show that the difference

(39) $D(\omega,t) := \gamma_{\infty,t}^{(1)}(\omega) - \sum_{i=1}^{2s+1} K_{i,t}(\infty)|\varphi_i\rangle\langle\varphi_i| - \sum_{i<\ell=1}^{2s+1} K_{i\ell,t}(\infty)|\varphi_i\rangle\langle\varphi_\ell| - \overline{K_{i\ell,t}(\infty)}|\varphi_\ell\rangle\langle\varphi_i|$

converges to zero in trace norm, as $\omega \to \infty$. This will prove the claim. We use the identity

$$
\begin{aligned}
\sum_{i=1}^{2s+1} K_{i,t}(\infty)|\varphi_i\rangle\langle\varphi_i| &+ \sum_{i<\ell=1}^{2s+1} K_{i\ell,t}(\infty)|\varphi_i\rangle\langle\varphi_\ell| + \overline{K_{i\ell,t}(\infty)}|\varphi_\ell\rangle\langle\varphi_i| \\
&= \frac{1}{(2\pi)^\Pi}\int_0^{2\pi}\left|\sum_{\ell=1}^{2s+1}\kappa_{\ell,t}(\infty)\varphi_\ell\right\rangle\left\langle\sum_{\ell'=1}^{2s+1}\kappa_{\ell',t}(\infty)\varphi_{\ell'}\right| \prod_{k\in P}\mathrm{d}\theta_k.
\end{aligned}
$$

Hence, it is possible to write

$$
\begin{aligned}
\|D(\omega,t)\|_{\mathfrak{S}^1} &= \left\|\frac{1}{(2\pi)^\Pi}\int_0^{2\pi}\left[\left|\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right\rangle\left\langle\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right| - \left|\sum_{\ell=1}^{2s+1}\kappa_{\ell,t}(\infty)\varphi_\ell\right\rangle\left\langle\sum_{\ell'=1}^{2s+1}\kappa_{\ell',t}(\infty)\varphi_{\ell'}\right|\right]\prod_{k\in P}\mathrm{d}\theta_k\right\|_{\mathfrak{S}^1} \\
&\le \frac{1}{(2\pi)^\Pi}\int_0^{2\pi}\left\|\left|\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right\rangle\left\langle\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t)\right| - \left|\sum_{\ell=1}^{2s+1}\kappa_{\ell,t}(\infty)\varphi_J\right\rangle\left\langle\sum_{\ell'=1}^{2s+1}\kappa_{\ell',t}(\infty)\varphi_{\ell'}\right|\right\|_{\mathfrak{S}^1}\prod_{k\in P}\mathrm{d}\theta_k \\
&\le \frac{2}{(2\pi)^\Pi}\int_0^{2\pi}\left\|\psi_{\{\theta_i,i\in P\}}^{(\omega)}(t) - \sum_{\ell=1}^{2s+1}\kappa_{\ell,t}(\infty)\varphi_\ell\right\|_2\prod_{k\in P}\mathrm{d}\theta_k;
\end{aligned}
$$

in the last inequality we have used that $\|p_{\psi_1} - p_{\psi_2}\|_{\mathfrak{S}^1} \le 2\|\psi_1 - \psi_2\|_{L^2}$, since p_{ψ_i} is the rank-one projection onto the linear span of ψ_i, and $\|\psi_i\|_2 = 1$ for $i = 1, 2$. By Propositions B.1

and B.2,

$$\left\| \psi_{\{\theta_i, i \in P\}}^{(\omega)}(t) - \sum_{\ell=1}^{2s+1} \kappa_{\ell,t}(\infty)\varphi_\ell \right\|_{L^2} \leq \frac{Ce^{K|t|}}{\omega^{1/2}} \,,$$

where $C, K > 0$ and are independent of ω, t.

B.2.2. *The limit $\omega \to \infty$, $N \to \infty$.* We define the orthogonal projection P_N as the one whose range is spanned by many-body vectors in which all particles occupy one of the φ_k, with $k = 1, \ldots, 2s + 1$. In other words,

$$(40) \qquad \mathcal{H}_{P_N} := \operatorname{ran} P_N = \left(\operatorname{span}\{\varphi_1, \ldots, \varphi_{2s+1}\} \right)^{\vee N} \cong \mathbb{C}^D \,,$$

for a suitable D (that depends both on N and $2s + 1$). In the following, it should be kept in mind that the vector $\Psi(t)$ depends on ω, even if we may avoid to mention it explicitly. Let us define

$$(41) \qquad \Phi_\omega(t) := P_N \Psi(t) \equiv P_N e^{-itH_{\omega,N}} \Psi_0$$

and

$$(42) \qquad \Phi_\infty(t) := e^{-itP_N V_N P_N} \Psi_0 \,,$$

having used the notation

$$V_N := \frac{1}{N} \sum_{j<k} V(x_j - x_k) \,.$$

If no confusion arises, we may omit the subscript of P_N. Firstly, we prove that $\Phi_\infty(t)$ is the limit of $\Psi(t)$ as $\omega \to \infty$. It is sufficient to prove that $(1 - P)\Psi(t)$ vanishes in the norm of $\mathcal{H}_N$ as $\omega \to \infty$, and that $\Phi_\omega(t)$ converges to $\Phi_\infty(t)$.

Proposition B.3. *Let $\Psi(t) = e^{-itH_{\omega,N}} \Psi_0$, with*

$$\Psi_0 = \varphi_1^{\otimes f_1(N)} \vee \cdots \vee \varphi_{2s+1}^{\otimes f_{2s+1}(N)}.$$

Then, for any $t \in \mathbb{R}$, there exists $C > 0$, independent of ω and t, such that

$$(43) \qquad \left\| P\Psi(t) - \Psi(t) \right\|_{\mathcal{H}_N}^2 \leq \frac{CN}{\omega} \,.$$

In particular,

$$\lim_{\omega \to \infty} \left\| P\Psi(t) - \Psi(t) \right\|_{\mathcal{H}_N} = 0 \,.$$

Proof. By conservation of the many-body energy, we have

$$(44) \qquad \langle \Psi(t), H_{\omega,N} \Psi(t) \rangle = \langle \Psi_0, H_{\omega,N} \Psi_0 \rangle = \frac{N-1}{2} \langle \Psi_0, V(x_1 - x_2)\Psi_0 \rangle \,.$$

On the other hand, by the relative boundedness of V with respect to h_ω, for every $\varepsilon > 0$ we have

$$\langle \Psi(t), H_{\omega,N} \Psi(t) \rangle \geq (1 - \varepsilon) \sum_{j=1}^{N} \langle \Psi(t), h_{\omega,j} \Psi(t) \rangle - C_\varepsilon \,;$$

hence, using the gap assumption on h_ω,

$$\langle \Psi(t), H_{\omega,N} \Psi(t) \rangle \geq (1 - \varepsilon) \sum_{j=1}^{N} \langle (1 - P)\Psi(t), h_{\omega,j}(1 - P)\Psi(t) \rangle - C_\varepsilon$$

$$\geq \omega(1 - \varepsilon)\|(1 - P)\Psi(t)\|_{\mathcal{H}_N}^2 - C_\varepsilon \,.$$

Using Eq. (44) on the left hand side, the proof is concluded.

$\dashv$

Proposition B.4. *For every $t \in \mathbb{R}$,*

$$(45) \qquad \lim_{\omega \to \infty} \left\| \Phi_\omega(t) - \Phi_\infty(t) \right\|_{\mathcal{H}_P} = 0 .$$

Proof. Using the fact that $PH_{\omega,N}(1-P) = PV_N(1-P)$, we deduce

$$i\partial_t \Phi_\omega(t) = PV_N P \Phi_\omega(t) + PV_N(1-P)\Psi(t) ,$$

that is equivalent to the integral equation

$$\Phi_\omega(t) = e^{-itPV_N P}\Psi_0 + \int_0^t \mathrm{d}s \, e^{-i(t-s)PV_N P} PV_N(1-P)\Psi(s) .$$

Comparing with the definition of $\Phi_\infty(t)$, we obtain, using the Kato-smallness of V, for any $\varepsilon > 0$:

$$\left\| \Phi_\omega(t) - \Phi_\infty(t) \right\|_{\mathcal{H}_P} \leq \varepsilon |t| \sup_{s \in [0,|t|]} \left\| \sum_{j=1}^N h_{\omega,j}(1-P)\Psi(s) \right\|_{\mathcal{H}_N} + C_{\varepsilon,N}|t| \sup_{s \in [0,|t|]} \left\| (1-P)\Psi(s) \right\|_{\mathcal{H}_N} .$$

Now, since both $(1-P)\Psi(s)$ and $\sum_{j=1}^N h_{\omega,j}(1-P)\Psi(s)$ are strongly continuous with respect to $s \in [0,|t|]$, and since $\sum_{j=1}^N h_{\omega,j}$ is a closed operator, the result follows from Proposition B.3.

$\dashv$

Propositions B.3 and B.4 yield

$$(46) \qquad \lim_{\omega \to \infty} \left\| e^{-itH_{\omega,N}}\Psi_0 - e^{-itPV_N P}\Psi_0 \right\|_{\mathcal{H}_N} = 0 .$$

It remains to prove that the reduced density matrix $\gamma_{N,t}^{(1)}(\infty)$ associated to $\Phi_\infty(t)$ converges to $\gamma_{\infty,t}^{(1)}(\infty)$ in trace norm as $N \to \infty$. This is in fact a problem of finite-dimensional semiclassical analysis (with N^{-1} as semiclassical parameter). Consider the operator

$$W_N := \frac{1}{N} \bigoplus_{n \in \mathbb{N}} P_n \sum_{1=j<k}^n V(x_j - x_k) P_n$$

on $\Gamma_s(\mathbb{C}^{2s+1}) \cong L^2(\mathbb{R}^{2s+1})$ (where the isomorphism is intended between unitarily equivalent irreducible representations of the C*-algebra of canonical commutation relations). On one hand it agrees with $P_N V_N P_N$ when restricted to the sector with $n = N$, and on the other hand it is the Wick quantization of a symbol $\sigma(\zeta, \bar\zeta)$ on $\mathbb{C}^{2s+1}$. Such symbol σ is, if we make the identifications $\zeta = (\kappa_1, \ldots, \kappa_{2s+1})$, and $u_\zeta = \sum_{j=1}^{2s+1} \kappa_j \varphi_j \in \mathcal{H}_1$,

$$(47) \qquad \sigma(\zeta, \bar\zeta) = \frac{1}{2} \int_{\mathbb{R}^d} V(x-y)\bar{u}_\zeta(x)\bar{u}_\zeta(y)u_\zeta(x)u_\zeta(y)\mathrm{d}x\mathrm{d}y .$$

Eq. (47) defines precisely the energy of the Hamilton-Jacobi equations (17). The trace-norm convergence of reduced density matrices is well-known in this finite-dimensional context, and yields the sought result. This concludes the proof of Theorem 3.1.

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