A Doubly Blended Model for Multiscale Atmospheric Dynamics

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ABSTRACT

The compressible flow equations for a moist, multicomponent fluid constitute the most comprehensive description of atmospheric dynamics used in meteorological practice. Yet, compressibility effects are often considered weak and acoustic waves outright unimportant in the atmosphere, except possibly for Lamb waves on very large scales. This has led to the development of ''soundproof'' models, which suppress sound waves entirely and provide good approximations for small-scale to mesoscale motions. Most global flow models are based instead on the hydrostatic primitive equations that only suppress vertically propagating acoustic modes and are applicable to relatively large-scale motions. Generalized models have been proposed that combine the advantages of the hydrostatic primitive and the soundproof equation sets. In this note, the authors reveal close relationships between the compressible, pseudoincompressible (soundproof), hydrostatic primitive, and the Arakawa and Konor unified model equations by introducing a continuous two-parameter (i.e., ''doubly blended'') family of models that defaults to either of these limiting cases for particular parameter constellations.

1. Introduction

The full compressible (FC) flow equations rank as the most comprehensive model for representing atmospheric fluid flows. They support sound waves, yet acoustic effects are often considered to be of little relevance for weather and climate—although this is not entirely obvious for very-large-scale modes. Atmospheric motions at small-scale and mesoscale are therefore frequently modeled by approximate ''soundproof'' equations that suppress elastic effects and are justified by low-Mach-number scalings involving small density or pressure perturbations around a background state (Ogura and Phillips 1962; Lipps and Hemler 1982; Durran 1989). In contrast, global numerical weather prediction codes have largely relied upon the hydrostatic (HY) primitive equations, which remove vertically propagating sound waves only and are justified by scaling arguments involving small vertical-to-horizontal-scale aspect ratios (e.g., White et al. 2005).

Recent efforts focused on bridging these small- and large-scale models in unified soundproof analytical formulations that have shown competitive behavior with respect to established approaches (Durran 2008; Arakawa and Konor 2009; Konor 2014; Dubos and Voitus 2014). While the general operational viability of reduced analytical models has been questioned on the grounds of inferior performance in normal-mode analyses (Davies et al. 2003; Dukowicz 2013), other studies found that numerical errors incurred with different discretizations applied to a single set of equations may outweigh analytical model-to-model errors (Smolarkiewicz and Dörnbrack 2008). In an effort to facilitate like-to-like comparison of soundproof and compressible formulations and controlled treatment of acoustics triggered by unbalanced initial data, Benacchio et al. (2014) devised a continuously blended multimodel discretization where thermodynamically consistent pseudoincompressible (PI) and fully compressible dynamics are accessed by simple switching within a single numerical framework [see also Klein et al. (2014); Benacchio (2014); Benacchio et al. (2015)].

We note that Gatti-Bono and Colella (2006) and Smolarkiewicz et al. (2014) propose related approaches to designing unified computational frameworks for

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different sets of governing equations. While these authors implement different model equations based on the same fundamental numerical operators and thus achieve like-to-like comparability as outlined above, they do not pursue a continuous blending of models at the analytical or coding level as proposed here.

In this short note, we broaden the scope of the blending approach by Benacchio (2014) and Benacchio et al. (2015) to include larger-scale dynamics. We devise a two-parameter family of models that accesses pseudoincompressible, hydrostatic primitive, unified anelastic and quasi-hydrostatic, as well as fully compressible dynamics, depending on the binary choice of switches. For simplicity and brevity, we have excluded the Coriolis term in the sequel, as its inclusion would call for a third parameter controlling geostrophic balance.

We first revise in section 2 the blending between the FC and PI models suggested by Benacchio et al. (2014). The blending switch now features as a parameter in the equation of state instead of appearing explicitly in the pressure equation. Energy conservation of the resulting blended model family is discussed. Section 3 describes a blended full compressible–hydrostatic model and its energy conservation. The suppression of vertical acoustics is achieved again through a switch in the equation of state, whereas control of hydrostasy requires a second switching component in the vertical momentum balance. Suppressing vertical acoustics by the same mechanism but allowing for nonhydrostatic motions leads to a model that is conceptually very close to the unified anelastic and quasi-hydrostatic system of equations (AK) of Arakawa and Konor (2009). Section 4 presents this model variant and demonstrates that it is—in fact—equivalent to the AK model. Section 5 presents a new two-parameter blended model family that combines the FC, PI, HY, and AK models in one and the same formulation. Conclusions are drawn in section 6.

2. FC–PI: Governing equations blended via the state equation

a. FC–PI blended equations

The dimensionless, dry, inviscid, full compressible equations in Cartesian coordinates read

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1a}$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0, \tag{1b}$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{P}{\Gamma}\nabla \pi = -\rho g \mathbf{k}, \tag{1c}$$

$$P = \pi^{1/(\gamma\Gamma)}, \tag{1d}$$

where ρ, π, $\boldsymbol{v} = (u, v, w) \equiv (\mathbf{u}, w)$ are density, Exner pressure, and velocity—the latter decomposed into the horizontal and vertical velocities, $\mathbf{u}$ and w, respectively; $\gamma = c_p/c_v$ is the isentropic exponent (c_p and c_v the specific heat at constant pressure and constant volume, respectively) $\Gamma = (\gamma - 1)/\gamma$, g is the gravitational acceleration, ∇ is the gradient operator, subscripts indicate partial derivatives, and $\mathbf{k}$ is the vertical unit vector. For simplicity, we restrict our analysis to adiabatic flows. In (1), pressure and density are nondimensionalized by standard values p_{ref} and ρ_{ref}, the velocity is nondimensionalized by $\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}$, and the mass-weighted potential temperature, $\rho\theta \equiv P$, is nondimensionalized by $\rho_{\text{ref}} T_{\text{ref}}$. As in Klein (2009) and Benacchio et al. (2014), P is used here to formulate the internal energy equation in conservation form in (1b). We note in passing that (1a) and (1b) imply that the potential temperature

$$\theta = \frac{P}{\rho} \quad \text{satisfies} \quad \frac{D\theta}{Dt} \equiv \theta_t + \boldsymbol{v} \cdot \nabla\theta = 0. \tag{2}$$

Benacchio et al. (2014) introduced a continuous blending between the compressible equations in (1) and the pseudoincompressible model by introducing a "pseudoincompressible switch" as a factor $\alpha \in [0, 1]$ multiplying P_t in (1b). For $\alpha = 0$, this leads to the velocity divergence constraint of the pseudoincompressible model, $\nabla \cdot (P\boldsymbol{v}) = 0$, and it implies that $P \equiv P_0(z)$ remains fixed in time and equal to the mean hydrostatic distribution. Here we deviate from this approach by introducing the background hydrostatic pressure into a blended variant of the state equation in (1d) while leaving the internal energy equation in (1b) untouched. Thus, we consider

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{3a}$$

$$(P_\alpha)_t + \nabla \cdot (P_\alpha \boldsymbol{v}) = 0, \tag{3b}$$

$$(\rho \boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{P_\alpha}{\Gamma}\nabla(\pi_\alpha + \pi') = -\rho g \mathbf{k}, \tag{3c}$$

where

$$\pi_\alpha = \alpha\pi + (1-\alpha)\pi_0, \tag{4a}$$

$$\pi' = (1-\alpha)\frac{\Gamma}{P_0}(\pi^{1/\Gamma} - \pi_0^{1/\Gamma}), \tag{4b}$$

$$P_\alpha = \left(\frac{\alpha}{P} + \frac{1-\alpha}{P_0}\right)^{-1}, \tag{4c}$$

$$P = \pi^{1/(\gamma\Gamma)}, \quad P_0 = \pi_0^{1/(\gamma\Gamma)}, \quad \theta = P_\alpha/\rho, \tag{4d}$$

and $\pi_0(z)$ is a hydrostatic background state Exner pressure distribution that defines a mean potential temperature stratification $\theta_0(z)$ through hydrostatic balance

$$\frac{d\pi_0}{dz} = -\frac{\Gamma g}{\theta_0}, \quad \pi_0(0) = 1. \tag{5}$$

For $\alpha = 1$, the blended FC–PI system in (3) and (4) reduces to the compressible Euler equations from (1) and to the pseudoincompressible model (Durran 1989) for $\alpha = 0$. As confirmed by a characteristic analysis (not shown for conciseness), for small perturbations of P around P_0, the quantity $\sqrt{\alpha}$ in (4) plays the role of the Mach number in a low-Mach-number system. In the appendix we demonstrate that for $\alpha \in (0, 1)$ system (3) with (4) is equivalent to a blended system that, as in Benacchio et al. (2014), involves the pressure–density form of the momentum equation and a thermodynamic consistency correction following Klein and Pauluis (2012). The analogy with the formulation of the pseudoincompressible limit in Benacchio et al. (2014) becomes transparent when we note that

$$(P_\alpha)_t = \frac{\partial P_\alpha/\partial p}{\partial \pi/\partial p}\pi_t = \alpha\frac{P_\alpha^2}{\gamma\Gamma p}\pi_t \tag{6}$$

so that the factor α appears in front of the (Exner) pressure time derivative.

b. Energy conservation for the FC–PI blended equations

To identify a total energy conservation law for the blended system (3), it is convenient to adopt its pressure-density formulation (see appendix). Thus (3c) is replaced with

$$\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{1}{\rho}\nabla p = -g\mathbf{k}\left(1 + \frac{p'}{\rho_* c_0^2}\right), \tag{7}$$

where

$$p = \pi^{1/\Gamma}, \quad p_0 = \pi_0^{1/\Gamma}, \quad \rho_* = \frac{P_\alpha}{\theta_0},$$
$$c_0^2 = \frac{\gamma p_0}{\rho_0} \equiv \gamma\theta_0\pi_0, \quad p' = (1-\alpha)(p - p_0). \tag{8}$$

Multiplying (7) by $\rho\boldsymbol{v}$ and using the equation once in advective form and once in conservative form exploiting (3a) so that $\rho(\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v}) = (\rho\boldsymbol{v})_t + \nabla\cdot(\rho\boldsymbol{v}\circ\boldsymbol{v})$, dividing by 2 and rearranging terms we obtain the kinetic and potential energy balance,

$$(\rho E_M)_t + \nabla\cdot[\boldsymbol{v}(\rho E_M + p)] = p\nabla\cdot\boldsymbol{v} - gw\rho\frac{p'}{\rho_* c_0^2}, \tag{9}$$

where

$$E_M = \frac{\mathbf{u}^2 + w^2}{2} + gz \tag{10}$$

is the specific mechanical energy of the system. Next we define the specific internal energy as

$$U = \frac{\theta\pi_\alpha}{\gamma - 1}, \tag{11}$$

with the blended Exner pressure π_α from (4a). Starting from the P equation in (3b), and using the blended state equation in (4c) and the definition of the pseudodensity ρ_* in (8), we obtain

$$(\rho U)_t + \nabla\cdot(\boldsymbol{v}\rho U) = -p\nabla\cdot\boldsymbol{v} + gw\rho\frac{p'}{\rho_* c_0^2}. \tag{12}$$

Adding (9) and (12) we find a conservation law for the total energy:

$$(\rho E_T)_t + \nabla\cdot[\boldsymbol{v}(\rho E_T + p)] = 0, \quad \text{where} \quad E_T = U + E_M. \tag{13}$$

3. FC–HY: Blending the full compressible and hydrostatic equations

a. FC–HY blended equations

Let $\beta \in [0, 1]$ be a hydrostatic switch in analogy with the pseudoincompressible switch α of section 2. For $\beta = 0$ we wish to see a variant of the hydrostatic primitive equations and for $\beta = 1$ the full nonhydrostatic equations. To this end, we subject the full compressible equations in (1) to a transformation of variables that is compatible with the asymptotic scalings of space, time, and velocity used in deriving the hydrostatic primitive equations through standard scale analysis [see, e.g., section 2.7 of Klein (2010), where ϵ^{α_x} corresponds to the present $\sqrt{\beta}$]. Therefore, for a vertical-to-horizontal-space-scale anisotropy $\beta \ll 1$ we consider rescaled variables

$$\hat{t} = \sqrt{\beta}t, \quad \hat{\mathbf{x}} = \sqrt{\beta}\mathbf{x}, \quad \hat{z} = z, \quad \hat{\mathbf{u}} = \mathbf{u}, \quad \hat{w} = w/\sqrt{\beta}. \tag{14}$$

In addition, pressure and density are written as

$$\pi = \pi_h + \beta\pi', \quad \rho = \rho_h + \beta\rho', \tag{15}$$

with the hydrostatic relation between π_h and $\theta = P/\rho$—that is,

$$\pi_{hz} = -g\Gamma/\theta, \tag{16}$$

and with a bottom boundary condition for π_h yet to be determined. Once π_h is defined, (15) defines the nonhydrostatic pressure and density components π' and ρ', given π and ρ, namely

$$\pi' = (\pi - \pi_h)/\beta, \quad \rho' = (\rho - \rho_h)/\beta, \quad P = \pi^{1/(\gamma\Gamma)},$$
$$P_h = \pi_h^{1/(\gamma\Gamma)}, \quad \rho_h = P_h/\theta. \tag{17}$$

With these scalings and definitions, and dropping the circumflexes, the FC–HY blended equations read

$$\rho_t + \nabla_\parallel \cdot (\rho \mathbf{u}) + (\rho w)_z = 0, \tag{18a}$$
$$P_t + \nabla_\parallel \cdot (P\mathbf{u}) + (Pw)_z = 0, \tag{18b}$$
$$(\rho\mathbf{u})_t + \nabla_\parallel \cdot (\rho\mathbf{u}\circ\mathbf{u}) + (\rho\mathbf{u}w)_z + \frac{P}{\Gamma}\nabla_\parallel(\pi_h + \beta\pi') = 0, \tag{18c}$$
$$\beta[(\rho w)_t + \nabla_\parallel \cdot (\rho w\mathbf{u}) + (\rho w^2)_z] + \frac{P}{\Gamma}(\pi_h + \beta\pi')_z = -g\rho. \tag{18d}$$

Here $\nabla_\parallel$ is the horizontal gradient operator. At leading order, the vertical momentum balance from (18d) yields the hydrostatic balance

$$\frac{P_h}{\Gamma}\pi_{hz} = -g\rho_h. \tag{19}$$

b. The hydrostatic limit for $\beta = 0$

For $\beta = 0$, the equations in (15)–(18) yield the hydrostatic primitive equations in Cartesian coordinates [see, e.g., section 2.7 of Klein (2010)]:

$$\rho_t + \nabla_\parallel \cdot (\rho \mathbf{u}) + (\rho w)_z = 0, \tag{20a}$$
$$P_{ht} + \nabla_\parallel \cdot (P_h\mathbf{u}) + (P_h w)_z = 0, \tag{20b}$$
$$(\rho\mathbf{u})_t + \nabla_\parallel \cdot (\rho\mathbf{u}\circ\mathbf{u}) + (\rho\mathbf{u}w)_z + \frac{P_h}{\Gamma}\nabla_\parallel\pi_h = 0, \tag{20c}$$
$$\frac{P_h}{\Gamma}\pi_{hz} = -g\rho_h, \tag{20d}$$
$$P = \pi_h^{1/(\gamma\Gamma)}. \tag{20e}$$

Together with appropriate boundary conditions for the integration of (20d), (20) constitutes a closed system. The vertical velocity w is determined from (20b), which becomes a velocity divergence constraint once π_h is determined.

c. Energy conservation for the blended compressible–hydrostatic system

The blended FC–HY system in (18) is equivalent to the full compressible Euler equations in (1) for any $\beta \in (0, 1]$ as they emerge from the latter as the result of an invertible transformation of variables. Therefore, they inherit the total energy conservation law (13) with

$$E_T = \frac{1}{\gamma - 1}\frac{p_h + \beta p'}{\rho_h + \beta\rho'} + \frac{\mathbf{u}^2 + \beta w^2}{2} + gz \tag{21}$$

in the rescaled variables. The formal limit for $\beta = 0$ of this expression is

$$E_T|_{\beta=0} = \frac{p_h/\rho_h}{\gamma - 1} + \frac{\mathbf{u}^2}{2} + gz, \tag{22}$$

and this quantity satisfies the conservation law (13) when p is replaced with p_h. The kinetic energy of the vertical motion $w^2/2$ is missing from (22). This is compatible with the standard scale analysis leading to the hydrostatic primitive equations, where the vertical velocity is smaller than the horizontal velocity by a factor of the aspect ratio of the flow domain [see, e.g., section 5.4 of White (2002)]. The mechanical energy balance is obtained by scalar multiplication of (20c) by $\mathbf{u}$ and multiplication of (20d) by w,

$$(\rho E_M)_t + \nabla \cdot [(\rho E_M + p_h)\boldsymbol{v}] = -p_h \nabla \cdot \boldsymbol{v}, \tag{23}$$

where

$$E_M = \frac{\mathbf{u}^2}{2} + gz. \tag{24}$$

The balance for the internal energy is obtained from (20b) using the equation of state in (20e),

$$(\rho U)_t + \nabla_\parallel \cdot (\rho U \boldsymbol{v}) = p_h \nabla \cdot \boldsymbol{v}, \tag{25}$$

where

$$U = \frac{p_h/\rho}{\gamma - 1}. \tag{26}$$

Summing (23) and (25), we have

$$(\rho E_T)_t + \nabla \cdot [(\rho E_T + p_h)\boldsymbol{v}] = 0. \tag{27}$$

4. FC–AK: Blending the full compressible and the Arakawa and Konor (2009) unified model

a. FC–AK blended equations

Reconsidering the blended FC–HY formulation in (18), we observe that the blending parameter β induces two different transitions at the same time. On the one hand, since for $\beta \to 0$,

$$P = (\pi_h + \beta\pi')^{1/(\gamma\Gamma)} \to P_h = \pi_h^{1/(\gamma\Gamma)}, \tag{28}$$

it restricts compressibility to the effects of hydrostatic pressure variations only. This is analogous to how the FC–PI blending parameter, α, suppresses compressibility altogether by replacing P with $P_0(z)$ in the P equation (see section 2). On the other hand, the FC–HY blending parameter β controls the influence of the perturbation pressure π' on horizontal momentum in (18c). For $\beta = 0$ only the horizontal gradient of the hydrostatic pressure remains in the horizontal momentum equation, which thus decouples from the vertical momentum balance.

Adopting only the compressibility constraint in (28) while maintaining fully three-dimensional coupling of the momentum equations through the perturbation pressure, we find

$$\rho_t + \nabla_\parallel \cdot (\rho \mathbf{u}) + (\rho w)_z = 0, \tag{29a}$$

$$P_t + \nabla_\parallel \cdot (P \mathbf{u}) + (P w)_z = 0, \tag{29b}$$

$$(\rho \mathbf{u})_t + \nabla_\parallel \cdot (\rho \mathbf{u} \circ \mathbf{u}) + (\rho \mathbf{u} w)_z + \frac{P}{\Gamma} \nabla_\parallel (\pi_h + \pi') = 0, \tag{29c}$$

$$(\rho w)_t + \nabla_\parallel \cdot (\rho w \mathbf{u}) + (\rho w^2)_z + \frac{P}{\Gamma} (\pi_h + \pi')_z = -\rho g, \tag{29d}$$

$$\frac{P_h}{\Gamma} \pi_{hz} = -\rho_h g, \tag{29e}$$

with

$$P = (\pi_h + \beta_c \pi')^{1/(\gamma\Gamma)}, \quad P_h = \pi_h^{1/(\gamma\Gamma)}, \quad \rho_h = \frac{P_h}{\theta}, \quad (\beta_c \in [0, 1]). \tag{30}$$

We have used β_c instead of β here as the notation for the blending parameter to indicate that here the parameter only constrains compressibility but does not imply hydrostasy. In fact, the parameter β_c appears in the equation of state only as was the case with the FC–PI switch α in (4).

b. *The limiting case $\beta_c = 0$ and comparison with Arakawa and Konor (2009)*

In the limiting case of $\beta_c = 0$, system (29) reduces to

$$\rho_t + \nabla_\parallel \cdot (\rho \mathbf{u}) + (\rho w)_z = 0, \tag{31a}$$

$$P_{ht} + \nabla_\parallel \cdot (P_h \mathbf{u}) + (P_h w)_z = 0, \tag{31b}$$

$$(\rho \mathbf{u})_t + \nabla_\parallel \cdot (\rho \mathbf{u} \circ \mathbf{u}) + (\rho \mathbf{u} w)_z + \frac{P_h}{\Gamma} \nabla_\parallel (\pi_h + \pi') = 0, \tag{31c}$$

$$(\rho w)_t + \nabla_\parallel \cdot (\rho w \mathbf{u}) + (\rho w^2)_z + \frac{P_h}{\Gamma} (\pi_h + \pi')_z = -\rho g, \tag{31d}$$

$$\frac{P_h}{\Gamma} \pi_{hz} = -\rho g, \tag{31e}$$

$$P_h = \pi_h^{1/(\gamma\Gamma)}. \tag{31f}$$

System (31) is the unified anelastic and quasi-hydrostatic model proposed by Arakawa and Konor (2009), provided we identify the density variable $\rho = P_h/\theta \equiv \rho_h$ in (31) with their quasi-hydrostatic density ρ_{qs} [see their (1.5)]. In addition, expressions (31a) and (31b) imply that the potential temperature $\theta = P_h/\rho$ satisfies the advection equation

$$\theta_t + \boldsymbol{v} \cdot \nabla \theta = 0, \tag{32}$$

which is Arakawa and Konor's (2009) (2.8) in the present adiabatic setting. Incidentally, (31) is also (20) with a full vertical momentum equation, as expected. Inclusion of a diabatic source term would be straightforward, and it would imply a related source term in (31b).

Dividing (31c) and (31d) by ρ, using (31a), and employing (31e), yields the momentum equation in advective form:

$$\boldsymbol{v}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{\theta}{\Gamma} (\nabla_\parallel \pi_h + \nabla \pi') = 0, \tag{33}$$

which is Arakawa and Konor's (2009) (3.1) without the Coriolis and general forcing terms. Including these terms would again be straightforward.

In this derivation we have identified the present π_h with Arakawa and Konor's (2009) quasi-hydrostatic pressure π_{qs}. In our system, this variable satisfies the hydrostatic equation in (31e), which we rewrite using $\theta = P_h/\rho$ and $\Gamma = c_p/R$, so that in our nondimensional notation we have

$$\pi_{hz} = -\frac{\Gamma g}{\theta}, \tag{34}$$

which is Arakawa and Konor's (2009) (2.2).

The equation of state in (31f) is equivalent to Arakawa and Konor's (2009) (2.6) combined with (2.4). In fact, translated to the present dimensionless notation, we have

$$P_h = \frac{\rho_{qs} R \theta}{p_{00}} = \pi_{qs}^{1/\kappa - 1} = \pi_h^{1/(\gamma\Gamma)}, \tag{35}$$

since $\kappa = \Gamma = (\gamma - 1)/\gamma$ in the notation of Arakawa and Konor (2009).

Therefore, except for nondimensionalization and momentum and energy source terms, system (31) coincides with Arakawa and Konor's (2009) unified anelastic and quasi-hydrostatic model. From the perspective of the present derivation, the label "unified pseudoincompressible and quasi-hydrostatic model" would seem more appropriate because the suppression of compressibility effects in constructing the model is the same as that underlying the derivation of the pseudoincompressible model by Durran (1989) (see section 2).

The reader is referred to Arakawa and Konor (2009) for an analysis of the model's energy budget. They show that there is no *local* conservation law for total energy but that *global* conservation can be achieved through a suitable adjustment of the domain-averaged perturbation pressure.

c. The hydrostatic pressure field

Arakawa and Konor (2009) suggest a particular formulation for determining the bottom boundary condition for the quasi-hydrostatic Exner pressure π_h that is derived combining mass conservation, hydrostatic balance for π_h, and the equations of state for the quasi-hydrostatic relations (ρ, π_h, θ). Using instead the equation for $P_h = \rho\theta$ from (31b), we suggest a different approach. Integrating (34) vertically in $[0, z]$, we find

$$\pi_h(t, \mathbf{x}, z) = \pi_{h0}(t, \mathbf{x}) - \Gamma g \int_0^z \frac{1}{\theta(t, \mathbf{x}, \zeta)} d\zeta, \tag{36}$$

where the surface Exner pressure π_{h0} remains to be determined. To this end, we integrate (31b) in $z \in [0, H]$, assuming for simplicity a flat bottom and top, so $(Pw)_{z=0} = (Pw)_{z=H} \equiv 0$. Letting

$$\langle X \rangle \equiv \int_0^H X\, dz \tag{37}$$

and using that $P_h = \pi_h^{1/(\gamma\Gamma)}$, we have

$$\langle P_h \rangle_t \equiv \frac{1}{\gamma\Gamma}\left\langle \frac{P_h}{\pi_h} \right\rangle \pi_{h0,t} - \frac{g}{\gamma}\left\langle \frac{P_h}{\pi_h} \int_0^z \left(\frac{1}{\theta}\right)_t d\zeta \right\rangle$$
$$= -\langle \nabla_\parallel \cdot (P_h \mathbf{u}) \rangle. \tag{38}$$

While (38) is not a closed equation for π_{h0} owing to the presence of θ, $(\theta^{-1})_t$, and $\mathbf{u}$, casting it into an implicit equation for an update $\delta\pi_{h0} = \pi_{h0}^{n+1} - \pi_{h0}^n$ over a time step in a numerical discretization should be straightforward within a semi-implicit framework [see also related discussion in Marshall et al. (1997); Smolarkiewicz et al. (2001)].

5. Two-parameter blended formulation

The three blended models FC–PI, FC–HY, and FC–AK dependent on α, β, and β_c can be cast jointly in the following two-parameter family, where $\xi, \eta \in [0, 1]$:

TABLE 1. Switches and limiting models for the two-parameter family of models [see (39)].

	ξ	
η	0	1
0	PI	HY
1	AK	FC

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{39a}$$

$$(P_B)_t + \nabla \cdot (P_B \boldsymbol{v}) = 0, \tag{39b}$$

$$(\rho\boldsymbol{v})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{v}) + \frac{P_B}{\Gamma}\nabla\pi_B = -\rho g \mathbf{k}, \tag{39c}$$

with the definitions

$$P_B = \xi\eta P + (\xi + \eta - 2\xi\eta)P_h + (1-\xi)(1-\eta)P_0 \tag{40}$$

and

$$\pi_B = P_B^{\gamma\Gamma} + (1-\xi)\pi', \quad \rho = \rho_h + (1-\xi)\rho'. \tag{41}$$

The four choices $\xi, \eta = \{0, 1\}$ enable access to the four models considered in this paper (Table 1). From the pseudoincompressible system, $(\xi, \eta) = (0, 0)$, moving "diagonally" to $(\xi, \eta) = (1, 1)$ reproduces the gradual introduction of compressibility, $\alpha = 0 \to 1$, in the FC–PI blended model (3) and (4), as analyzed in Benacchio et al. (2014). From the full compressible system, setting $\eta = 1$ and $\xi \in [0, 1]$ tunes the relative importance of P_h and P in P_B, and the effects of the perturbation pressure π' within the blended FC–AK model (29) and (30). Finally, setting $\xi = 1$ and $\eta = [0, 1]$ yields the blended FC–HY model (18). Letting $\eta \to 0$ suppresses vertically propagating sound waves and recovers hydrostasy within the limiting model (20).

6. Conclusions

In this paper we have introduced a new two-parameter family of dynamical models for atmospheric flows. The family encompasses the pseudoincompressible, hydrostatic primitive, unified Arakawa and Konor (2009), and full compressible models and allows access to reduced soundproof dynamics by straightforward switching. As a byproduct of the derivation, we have found that, in the context of our blended formulation, the suppression of compressibility in the model by Arakawa and Konor (2009) mirrors the same process in the pseudoincompressible model. A normal-mode analysis, not reported here for conciseness, corroborates the equivalence of system (31) with the model by Arakawa and Konor (2009).

The two-way blended formulation (39)–(41) naturally lends itself to a continuously tunable numerical discretization along the lines of Benacchio et al. (2014), enabling an effective treatment of fast acoustic and gravity waves within an analytical framework that covers small-, meso-, synoptic-, and planetary-scale motions. In this approach, unbalanced modes are filtered (e.g., from assimilated data) by running the related balanced model for a few time steps and then tuning back smoothly to the full compressible model over a few more steps. An open issue in this context concerns the conservation properties of the blended model family for intermediate values $\eta, \xi \in (0, 1)$.

The development presented in this paper becomes all the more attractive in light not only of the current efforts to show viability of soundproof models at synoptic and planetary scales (Smolarkiewicz et al. 2014; Kurowski et al. 2015) but also of the imminent necessity to compare performance of hydrostatic and nonhydrostatic codes at global operational resolutions finer than 10 km afforded by next-generation exascale supercomputers (Wedi et al. 2012; Smolarkiewicz et al. 2015).

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APPENDIX

Pressure–Density Formulation of the FC–PI System

Here we demonstrate that the FC–PI system from (3) and (4) can equivalently be written with the momentum equation in a pressure–density form as in the $(\text{FC–PI})^{\text{tc}}$ system of Benacchio et al. (2014). Then, maintaining (3a) and (3b), we replace (3c) with (Benacchio et al. 2014)

$$\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{1}{\rho}\nabla p = -g\mathbf{k}\left(1 + \frac{p'}{\rho_* c_0^2}\right) \tag{A1}$$

and, in the sequel, let

$$\pi = p^\Gamma, \quad \pi_0 = p_0^\Gamma \quad \pi' = \frac{\Gamma p'}{P_0} \equiv (1-\alpha)\frac{\Gamma(p-p_0)}{P_0}. \tag{A2}$$

Now, comparing (A1) with (3c), we need to verify that $(1/\rho)\nabla p + g\mathbf{k}[p'/(\rho_* c_0^2)] = (\theta/\Gamma)\nabla(\pi_\alpha + \pi')$:

$$\underline{\underline{\frac{1}{\rho}\nabla p}} = \frac{\theta}{P_\alpha}\nabla p = \theta\nabla\left(\frac{p}{P_\alpha}\right) - \theta p\nabla\left(\frac{1}{P_\alpha}\right) \tag{A3a}$$

$$= \theta\nabla\{p[\alpha p^{-1/\gamma} + (1-\alpha)p_0^{-1/\gamma}]\} - \theta p\nabla[\alpha p^{-1/\gamma} + (1-\alpha)p_0^{-1/\gamma}] \tag{A3b}$$

$$= \alpha\theta\nabla p^\Gamma + (1-\alpha)\theta\nabla(pp_0^{-1/\gamma}) + \alpha\frac{\theta p}{\gamma}\frac{p^{-1/\gamma}}{p}\nabla p - (1-\alpha)\theta p\nabla p_0^{-1/\gamma} \tag{A3c}$$

$$= \alpha\theta\left[1 + \frac{1}{\gamma\Gamma}\right]\nabla p^\Gamma + (1-\alpha)\theta\left[\nabla\left(\frac{p-p_0}{p_0^{1/\gamma}}\right) - (p-p_0)\nabla p_0^{-1/\gamma}\right] \tag{A3d}$$

$$+ (1-\alpha)\theta(\nabla p_0^{1-1/\gamma} - p_0\nabla p_0^{-1/\gamma}) \tag{A3e}$$

$$= \frac{\theta}{\Gamma}\nabla\left[\alpha\pi + (1-\alpha)\pi_0 + (1-\alpha)\frac{\Gamma(p-p_0)}{P_0}\right] + (1-\alpha)\theta\frac{p-p_0}{\gamma p_0^{1+1/\gamma}}\frac{dp_0}{dz}\mathbf{k} \tag{A3f}$$

$$= \frac{\theta}{\Gamma}\nabla(\pi_\alpha + \pi') - g\mathbf{k}(1-\alpha)\frac{\theta}{P_0}\frac{p-p_0}{\gamma p_0/\rho_0} = \underline{\underline{\frac{\theta}{\Gamma}\nabla(\pi_\alpha + \pi') - g\mathbf{k}\frac{p'}{\rho_* c_0^2}}}\,. \tag{A3g}$$

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