A "right" path to cyclic polygons

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Abstract

It is well known that Heron's theorem provides an explicit formula for the area of a triangle, as a symmetric function of the lengths of its sides. It has been extended by Brahmagupta to quadrilaterals inscribed in a circle (cyclic quadrilaterals). A natural problem is trying to further generalize the result to cyclic polygons with a larger number of edges, which, surprisingly, has revealed to be far from simple. In this paper we investigate such a problem by following a new and elementary approach. We start from the simple observation that the incircle of a right triangle touches its hypothenuse in a point that splits it into two segments, the product of whose lengths equals the area of the triangle. From this curious fact we derive in a few lines: an unusual proof of the Pythagoras' theorem, Heron's theorem for right triangles, Heron's theorem for general triangles, and Brahmagupta's theorem for cyclic quadrangles. This suggests that cutting the edges of a cyclic polygon by means of suitable points should be the "right" working method. Indeed, following this idea, we obtain an explicit formula for the area of any convex cyclic polygon, as a symmetric function of the segments split on its edges by the incircles of a triangulation. We also show that such a symmetry can be rediscovered in Heron's and Brahmagupta's results, which consequently represent special cases of the general provided formula.

MSC: 52A10;52A38

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1 Introduction

A natural and largely considered question in convex geometry is the determination of the area A of a convex polygon as a function of the lengths of its sides. The problem goes back to Heron of Alexandria, that was able to solve the problem in the case of a triangle. Later, in the seventh century, Brahmagupta extended the result to cyclic quadrilaterals, namely to quadrilaterals inscribed in a circle (see for instance [1]). Several results concerning the geometry of cyclic polygons have been obtained in different areas of research (see [2, 3, 6, 4, 11]), which points out a general interest for such geometric objects. It is therefore natural trying to further extend to cyclic polygons with a larger number of edges the nice and ancient formulae by Heron and Brahmagupta. Surprisingly, this has revealed to be far from simple. In [8, 9] an algebraic formulation of the problem led D.P. Robbins to find formulae for cyclic pentagons and cyclic hexagons. It was observed that Heron and Brahmagupta's formulae can be restated in a form where $16A^2$ represents a monic polynomial whose coefficients are symmetric polynomials in the squares of the sides. This generalizes to cyclic pentagons and hexagons, where the polynomial have degree 7, but the formulae, even if holding also in the non convex case, do not provide an easy explicit form for the area (see also [7] for interesting comments and remarks). Formulae of the same kind have been conjectured [8], and later proved [5], even for heptagons and octagons, also illuminating some mysterious features of Robbins' formulas for the areas of cyclic pentagons and hexagons (see also [10] for further detail on Robbin's conjectures). The resulting formulae are interesting, but are very complex and do not seem to

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provide a general picture that could be easily generalizable to polygons with an arbitrary large number of edges.

In this article we follow a different approach, which leads to a complete solution of the considered problem. The leitmotif of our paper is to point out that the role played by the edges in Heron's and Brahmagupta's theorems must be replaced by the segments cut on the edges of a polygon by the incircles of the triangles of a triangulation of the polygon. First of all, we show that Pytagoras', Heron's and Brahmagupta's theorems can be linked together thanks to a simple result concerning the area of a right triangle. Then, we generalize Heron's and Brahmagupta's results to a symmetric coordinate free formula that holds true for any cyclic polygon.

2 Heron's formula

Let ABC be a right triangle and let I be its incenter (see Fig.1). Since $\hat{B}$ is a right angle and since the incircle is tangent perpendicularly to the three sides of ABC, we have $r = IJ = IK = IH = BJ = BK$, where r is the inradius. The internal bisectors of ABC are concurrent in I and this implies $AJ = AH = s$ and $CH = CK = t$.

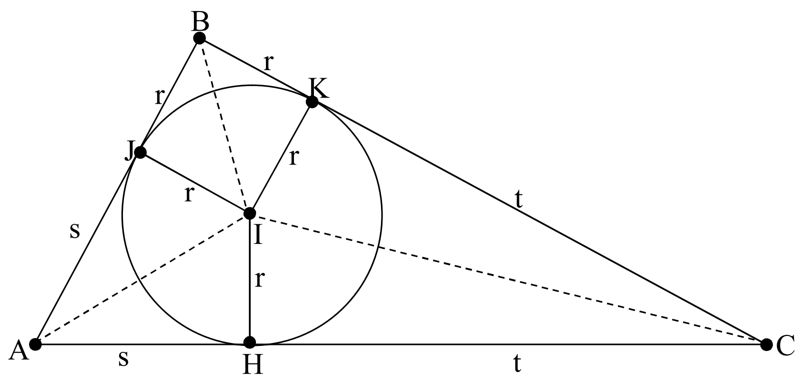

Figure 1: A right triangle ABC.

Denoting the area of a triangle with vertical bars we have

$$|ABC| = |AIB| + |BIC| + |CIA|.$$

The half-perimeter of ABC is $p = r + s + t$ while all the three triangles on the r.h.s. of the above equality have altitude r with respect to their sides AB, BC, and AC. Therefore we have

$$|ABC| = r(r + s + t). \tag{1}$$

Remark 1. In case ABC is not a right triangle, Formula (1) generalizes to

$$|ABC| = R(r + s + t), \tag{2}$$

where R is the incircle of ABC, meaning that $|ABC|$ is a symmetric function in r, s, t.

Theorem 2. The area of a right triangle ABC is equal to the area of the rectangle of sides $s = AH$ and $t = CH$, where H is the point where the incircle is tangent to the hypotenuse AC.

Proof Clearly $|ABC| = \frac{1}{2}(s + r)(t + r)$, and by (1) we get

$$st + rs + rt + r^2 = 2(r^2 + rs + rt)$$

which we simplify into $st = r^2 + rs + rt = r(r + s + t) = |ABC|$. ✓

Theorem 3. [Heron's formula for right triangles] The area of a right triangle is equal to $\sqrt{p(p-a)(p-b)(p-c)}$, where a, b, c are the lengths of its sides (taken in any order) and $p = (a+b+c)/2$ is its half perimeter.

Proof Using the same notation (as in Fig.1) we need to show that

$$|ABC|^2 = str(r+s+t),$$

but this is immediate, being $|ABC| = st$ by Theorem 2, and also $|ABC| = r(r+s+t)$ by (1). ✓
Theorem 3 shows that, in any right angle triangles, Heron's formula can be rediscovered by starting from the symmetric formula provided by Theorem 2. We wish now to extend such a result to any triangle.

Theorem 4. [Pythagoras] In a right triangle the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares whose sides are the two legs.

Proof We have $r(r+s+t) = st$. Multiplying by 2 and adding s^2+t^2 on both sides we get $s^2+t^2+2r^2+2rs+2rt = s^2+t^2+2st$, i.e.

$$(s+r)^2 + (r+t)^2 = (s+t)^2. \quad ✓$$

Thanks to Pythagoras' Theorem we can easily extend Heron's Theorem to any triangle. To this, let ABC be a generic triangle, and let CH be the altitude on its edge AB (see Figure 2), and assume H is between A and B (in any triangle surely exists an altitude with this property).

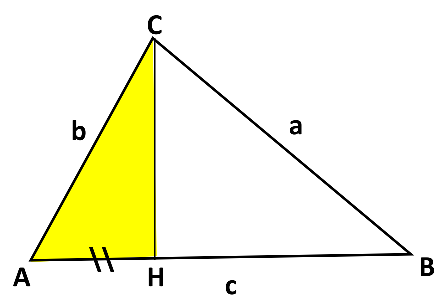

Figure 2: A generic triangle ABC.

Let $p = r+s+t$ be the semiperimeter of ABC, and let $\varphi = \sqrt{rst(r+s+t)} = \sqrt{p(p-a)(p-b)(p-c)}$. By Pythagoras' Theorem in AHC and CHB, we have

$$\begin{aligned} c^2 &= (AH+HB)^2 = AH^2 + BH^2 + 2(AH)(HB) = \\ &= a^2+b^2-2CH^2+2(AH)(HB) = a^2+b^2-2CH^2+2\sqrt{(a^2-CH^2)(b^2-CH^2)} \end{aligned}$$

and, solving for CH we get

$$CH = \frac{\sqrt{4a^2b^2-(c^2-a^2-b^2)^2}}{2c} = \frac{2\sqrt{p(p-a)(p-b)(p-c)}}{c} = \frac{2}{c}\varphi.$$

Therefore, from $2|ABC| = cCH$, Heron's theorem for ABC follows. ✓

Remark 5. By (2) and Heron's Theorem written in the form $|abc| = \sqrt{rst(r+s+t)}$ we can obtain the incircle R of any triangle as a symmetric function of r, s, t as follows (see Figure 3)

$$R = \sqrt{\frac{rst}{p}}. \tag{3}$$

being $p = r+s+t$ the half perimeter of ABC.

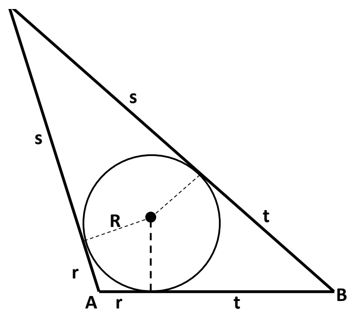

Figure 3: Incircle of a generic triangle ABC.

3 Brahmagupta's formula

We wish now to show how Brahmagupta's formula can be obtained by exploiting the same idea of symmetry considered in the previous section. with respect to considered a symmetry , we prove the following result.

Theorem 6. Let ABC, ADC be two triangles inscribed in a same circumference. If s_1, t_1 and s_2, t_2 are the lengths of the two segments split on the common edge AC by the respective incircles, then

$$|ABC||ACD| = s_1 s_2 t_1 t_2.$$

Proof. In the cyclic quadrangle $ABCD$ the halves of $A\hat{B}C$ and $A\hat{D}C$ are complementary angles. Therefore the shaded right triangles in Figure 4 are similar, and consequently $\frac{R_1}{r_1} = \frac{r_2}{R_2}$, that is $R_1 R_2 = r_1 r_2$.

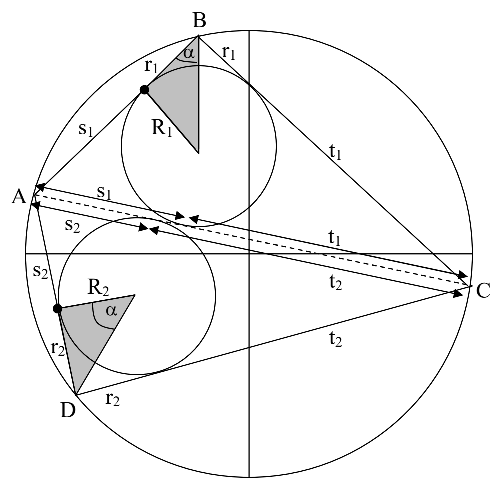

Figure 4: A general cyclic quadrangle $ABCD$.

By (3) we have $p_1 R_1^2 = r_1 s_1 t_1$ and $p_2 R_2^2 = r_2 s_2 t_2$, being p_1, p_2 the half perimeter of ABC and ADC respectively, so that

$$(p_1 R_1 p_2 R_2)^2 = p_1(r_1 s_1 t_1) p_2(r_2 s_2 t_2) = p_1 p_2 (r_1 r_2)(s_1 t_1 s_2 t_2) = p_1 p_2 (R_1 R_2)(s_1 t_1 s_2 t_2),$$

and consequently $|ABC||ACD| = p_1 R_1 p_2 R_2 = s_1 t_1 s_2 t_2$. ✓

Remark 7. Since $R_1 R_2 = r_1 r_2$ we have also $|ABC||ACD| = p_1 R_1 p_2 R_2 = p_1 p_2 r_1 r_2$.

Theorem 8. [Brahmagupta's formula for cyclic quadrangles] The area of a convex quadrangle that can be inscribed in a circle (a cyclic quadrangle) is equal to $\sqrt{(p-a)(p-b)(p-c)(p-d)}$, where a, b, c, d are the lengths of its sides (taken in any order) and $p = (a+b+c+d)/2$ is its half perimeter.

Proof. Let $ABCD$ be split into ABC and ACD, as in Figure 4, and assume $a = s_1+r_1 = AB, b = t_1+r_1 = BC, c = t_2+r_2 = CD, d = s_2+r_2 = AD$, so that $p = r_1+r_2+(s_1+t_1) = r_1+r_2+(s_2+t_2)$. Starting from $|ABCD|^2 = (|ABC|+|ACD|)^2 = |ABC|^2+|ACD|^2+2|ABC||ACD|$, we use the previous theorem, and the Remark, to write $2|ABC||ACD| = s_1t_1s_2t_2 + p_1p_2r_1r_2$, where p_1, p_2 are the half perimeters of ABC and ACD, respectively. Moreover, by Heron's formula, $|ABC|^2 = (r_1+s_1+t_1)r_1s_1t_1$, and $|ACD|^2 = (r_2+s_2+t_2)r_2s_2t_2$. Then, also using $s_1+t_1 = s_2+t_2$, we get

$$\begin{aligned}
|ABCD|^2 &= \boxed{(r_1+s_1+t_1)r_1s_1t_1} + \boxed{\boxed{(r_2+s_2+t_2)r_2s_2t_2}} + \\
&\quad + \boxed{\boxed{s_1t_1s_2t_2}} + \boxed{r_1r_2(r_1+s_1+t_1)(r_2+s_2+t_2)} \\
&= \boxed{r_1(r_1+s_1+t_1)(r_2(r_2+s_2+t_2)+s_1t_1)} \\
&\quad + \boxed{\boxed{s_2t_2(r_2(r_2+s_2+t_2)+s_1t_1)}} \\
&= (r_2(r_2+\boxed{s_2+t_2})+s_1t_1)(r_1(r_1+\boxed{\boxed{s_1+t_1}})+s_2t_2) \\
&= (r_2(r_2+\boxed{\boxed{s_1+t_1}})+s_1t_1)(r_1(r_1+\boxed{s_2+t_2})+s_2t_2) \\
&= (r_2+t_1)(r_2+s_1)(r_1+s_2)(r_1+t_2) \\
&= (p-a)(p-b)(p-c)(p-d). \checkmark
\end{aligned}$$

4 The area of a circular polygon having an arbitrary number of edges

In this section we generalize the previous results to a cyclic polygon P_n, having $n+2$ edges for any $n \geq 1$. Let us observe that Heron's formula has been extended to Brahmagupta's formula by considering a cyclic quadrilateral Q as the union of two triangles, $Q = T_1 \cup T_2$, and then focusing on the segments r_1, s_1, t_1 and r_2, s_2, t_2 determined, respectively, on the edges of T_1 and T_2 by the tangent points of the corresponding incircles. This provides the square of the area of Q as a polynomial function, symmetric under the exchange of r_1, s_1, t_1 with r_2, s_2, t_2.

As a consequence we are inspired to investigate the square of the area $A(n)$ of a generic cyclic polygon P_n by looking at the partitions of the edges of P_n determined by the tangent points of the incircles of some triangulatio. To this, let us first observe that P_n can always be assumed as a union of n consecutive triangles $T_1, T_2, ..., T_n$, all having a common vertex. For $i = 1, ..., n-1$, denote by $L_{i,i+1}$ the common edge between the two consecutive triangles T_i, T_{i+1}. Let p_j, A_j, R_j be, respectively, the semiperimeter, the area and the radius of the incircle of T_j, $j = 1, ..., n$. Also, let r_j, s_j, t_j be the segments cut on the edges of T_j by its incircle, where $L_{i,i+1} = s_i + t_i = s_{i+1} + r_{i+1}$, $i = 1, ..., n-1$ (see Figure 5).

For the sake of brevity, and in order to avoid heavy notations, in the following theorems we assume to be equal to 1 all the meaningless products.

Theorem 9. Let P_n be a cyclic polygon consisting of $n+2$ edges, $n \geq 1$. Then it results

$$A_hA_k = s_ht_hs_kr_k \prod_{i=h+1}^{k-1} \frac{s_i}{p_i} = p_hr_hp_kt_k \prod_{i=h+1}^{k-1} \frac{p_i}{s_i}, \qquad \text{for } 1 \leq h < k \leq n. \tag{4}$$

Proof. In order to prove the first equality in (4) we apply Theorem 6 iteratively, so that

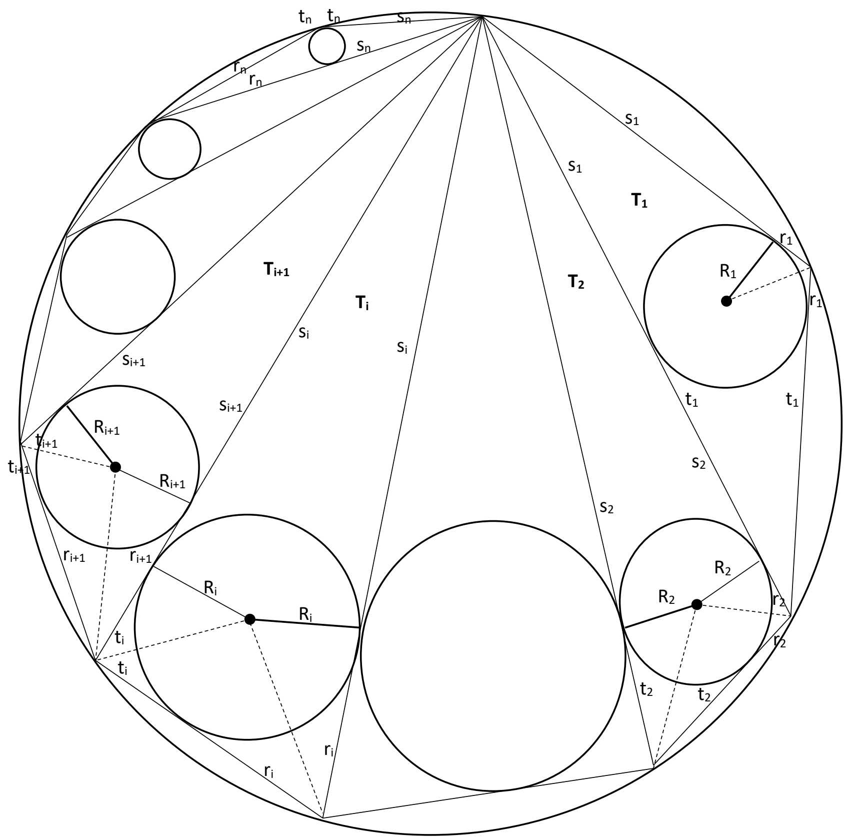

Figure 5: Triangulation of a cyclic polygons.

$$\begin{aligned}
&A_h A_{h+1} = s_h t_h s_{h+1} r_{h+1}\\
&A_{h+1} A_{h+2} = s_{h+1} t_{h+1} s_{h+2} r_{h+2}\\
&...\\
&A_{k-1} A_k = s_{k-1} t_{k-1} s_k r_k.
\end{aligned}$$

By multiplying on both sides we get

$$A_h (A_{h+1} A_{h+2} ... A_{k-1})^2 A_k = s_h t_h \left(\prod_{i=h+1}^{k-1} r_i s_i^2 t_i \right) s_k r_k.$$

By Heron's Theorem applied to $T_{h+1}, T_{h+2}, ..., T_{k-1}$ we have

$$A_h A_k = \frac{s_h t_h \left(\prod_{i=h+1}^{k-1} r_i s_i^2 t_i \right) s_k r_k}{\prod_{i=h+1}^{k-1} r_i s_i t_i p_i} = s_h t_h s_k r_k \prod_{i=h+1}^{k-1} \frac{s_i}{p_i},$$

and the first equality in (4) is obtained. For the proof of the second equality, let us observe that, by similitude, it results $\dfrac{R_i}{r_i} = \dfrac{t_{i+1}}{R_{i+1}}$, for all $i = 1, ..., n-1$. Therefore we get

$$\begin{aligned}
&R_h R_{h+1} = r_h t_{h+1}\\
&R_{h+1} R_{h+2} = r_{h+1} t_{h+2}\\
&...\\
&R_{k-1} R_k = r_{k-1} t_k.
\end{aligned}$$

By multiplying on both sides, it results

$$R_h(R_{h+1}R_{h+2}...R_{k-1})^2R_k = R_hR_k\prod_{i=h+1}^{k-1}R_i^2 = r_ht_k\prod_{i=h+1}^{k-1}r_it_i,$$

and applying (3) to all R_i^2 we have

$$R_hR_k = \frac{r_ht_k\prod_{i=h+1}^{k-1}r_it_i}{\prod_{i=h+1}^{k-1}\frac{r_is_it_i}{p_i}},$$

and consequently

$$R_hR_k = r_ht_k\prod_{i=h+1}^{k-1}\frac{p_i}{s_i} \quad \text{for } 1 \le h < k \le n. \tag{5}$$

then The second equality in (4) follows immediately from (5), being $A_hA_k = p_hR_hp_kR_k$.✓

Assuming $P = T_1 \cup T_2 \cup ... \cup T_n$ as in Figure 5, and using the same notations as above we can now prove a general formula for the area of a cyclic polygon with any number of edges.

Theorem 10. Let P_n be a cyclic polygon consisting of $n+2$ edges, $n \ge 1$, and let A be its area. Then it results

$$A(n)^2 = \left(p_1r_1 + \sum_{q=2}^{n}r_qs_q\prod_{m=2}^{q-1}\frac{s_m}{p_m}\right)\left(s_1t_1 + \sum_{q=2}^{n}p_qt_q\prod_{m=2}^{q-1}\frac{p_m}{s_m}\right).$$

Proof. Since P_n is the union of $T_1, ..., T_n$, by Heron's Theorem, and using both equalities in (4), we can writewe have

$$\begin{aligned}
A^2 &= (A_1 + A_2 + ... + A_n)^2 = \\
&= \sum_{j=1}^{n}A_j^2 + 2\sum_{1\le h<k\le n}A_hA_k = \\
&= \sum_{j=1}^{n}p_jr_js_jt_j + \sum_{1\le h<k\le n}s_ht_hs_kr_k\prod_{i=h+1}^{k-1}\frac{s_i}{p_i} + \sum_{1\le h<k\le n}p_hr_hp_kt_k\prod_{i=h+1}^{k-1}\frac{p_i}{s_i}.
\end{aligned}$$

Let's collect as follows (where each one of the three terms appearing in each bracket comes from the corresponding sum)

$$\begin{aligned}
A^2 &= p_1r_1\left(s_1t_1 + 0 + \sum_{k>1}p_kt_k\prod_{i=2}^{k-1}\frac{p_i}{s_i}\right) + \\
&+ r_2s_2\left(p_2t_2 + s_1t_1 + \frac{p_2}{s_2}\sum_{k>2}p_kt_k\prod_{i=3}^{k-1}\frac{p_i}{s_i}\right) + \\
&+ r_3s_3\frac{s_2}{p_2}\left(\frac{p_2}{s_2}p_3t_3 + (s_1t_1 + p_2t_2) + \frac{p_2}{s_2}\frac{p_3}{s_3}\sum_{k>3}p_kt_k\prod_{i=4}^{k-1}\frac{p_i}{s_i}\right) + \\
&+ r_4s_4\frac{s_2}{p_2}\frac{s_3}{p_3}\left(\frac{p_2}{s_2}\frac{p_3}{s_3}p_4t_4 + (s_1t_1 + p_2t_2 + \frac{p_2}{s_2}p_3t_3) + \frac{p_2}{s_2}\frac{p_3}{s_3}\frac{p_4}{s_4}\sum_{k>4}p_kt_k\prod_{i=5}^{k-1}\frac{p_i}{s_i}\right) + \\
&+ ... + \\
&+ r_ns_n\frac{s_2}{p_2}\frac{s_3}{p_3}...\frac{s_{n-1}}{p_{n-1}}\left(\frac{p_2}{s_2}\frac{p_3}{s_3}...\frac{p_{n-1}}{s_{n-1}}p_nt_n + \left(s_1t_1 + p_2t_2 + p_3t_3\frac{p_2}{s_2} + ... + p_{n-1}t_{n-1}\frac{p_2}{s_2}\frac{p_3}{s_3}...\frac{p_{n-2}}{s_{n-2}}\right) + 0\right) = \\
&= \left(p_1r_1 + \sum_{q=2}^{n}r_qs_q\prod_{m=2}^{q-1}\frac{s_m}{p_m}\right)\left(s_1t_1 + \sum_{q=2}^{n}p_qt_q\prod_{m=2}^{q-1}\frac{p_m}{s_m}\right).
\end{aligned}$$

✓

Remark 11. We emphasize that the formula obtained for $A(n)^2$ is symmetric under mutually exchanging r_j with t_j, and s_j with p_j, for all $j \in \{1, ..., n\}$. We also note that only the terms concerning the partitions of the edges of the polygon P_n are in fact necessary. Indeed, $p_1 = r_1+s_1+t_1$ can be immediately computed once we know the terms r_1, s_1, t_1 determined on the edges of P_n by the incircle of T_1. For $1 < q < n-1$, due to consecutiveness, we have $s_q = s_{q-1} + t_{q-1} - r_q$, so that, by recursion, we get

$$s_q = s_1 + \sum_{h=1}^{q-1} t_h - \sum_{k=2}^{q} r_k, \tag{6}$$

$$p_q = r_q + s_q + t_q = \begin{cases} s_1 + t_1 + t_2 & \text{if } q = 2 \\ s_1 + \sum_{h=1}^{q} t_h - \sum_{k=2}^{q-1} r_k & \text{if } 2 < q < n-1. \end{cases} \tag{7}$$

Consequently, all terms appearing in $A(n)^2$ can be computed once s_1, s_n, r_j, t_j are known.

Examples
We can easily rediscover Heron's and Brahmagupta's results from the provided symmetric function.

• $n = 1$

$$A(1)^2 = (p_1 r_1 + 0)(s_1 t_1 + 0) = p_1 r_1 s_1 t_1 \quad \textbf{Heron's theorem}.$$

• $n = 2$

$$\begin{aligned} A(2)^2 &= (p_1 r_1 + r_2 s_2)(s_1 t_1 + p_2 t_2) = \\ &= ((r_1 + s_1 + t_1) r_1 + r_2 s_2)(s_1 t_1 + (r_2 + s_2 + t_2) t_2 = \\ &= ((r_1 + s_2 + r_2) r_1 + r_2 s_2)(s_1 t_1 + (s_1 + t_1 + t_2) t_2 = \\ &= (r_1 + s_2)(r_1 + r_2)(t_2 + s_1)(t_2 + t_1). \end{aligned}$$

Being $s_1 + t_1 = s_2 + r_2$, it is $p = r_1 + r_2 + s_1 + t_1 = r_1 + r_2 + t_2 + s_2$, so that, if we denote, respectively, by a, b, c, d the edges $s_2 + t_2$, $r_2 + t_2$, $r_1 + s_1$ and $r_1 + t_1$, then

$$A(2)^2 = (p - a)(p - b)(p - c)(p - d) \quad \textbf{Brahmagupta's theorem}.$$

• $n = 3$ **generalization of Brahmagupta's theorem to cyclic pentagons**

$$A(3)^2 = \left(p_1 r_1 + r_2 s_2 + r_3 s_3 \frac{s_2}{p_2}\right)\left(s_1 t_1 + p_2 t_2 + p_3 t_3 \frac{p_2}{s_2}\right).$$

• $n = 4$ **generalization of Brahmagupta's theorem to cyclic hexagons**

$$A(4)^2 = \left(p_1 r_1 + r_2 s_2 + r_3 s_3 \frac{s_2}{p_2} + r_4 s_4 \frac{s_2}{p_2}\frac{s_3}{p_3}\right)\left(s_1 t_1 + p_2 t_2 + p_3 t_3 \frac{p_2}{s_2} + p_4 t_4 \frac{p_2}{s_2}\frac{p_3}{s_3}\right).$$

We can even extend the formula to $n = 0$ by assuming $A(0) = 0$, where P_0 is a polygon degenerated in a segment, which can be obtained by progressively removing an edge from a starting polygon P_n having $n + 2$ edges

5 Conclusion and remarks

We have shown that Heron's and Brahmagupta's theorems can be extended to a formula that provides the square of the area of a convex cyclic polygon as a symmetric polynomial of the segments determined on the edges by the incircles of a suitable triangulation. We remark that the formula is coordinate-free as one should expect from the intrinsic geometric nature of the problem. Differently, using for instance Green's theorem, it would be quite easy to provide a coordinate dependent result.

In our opinion the obtained formula is the natural generalization of what happens for triangle and cyclic quadrilaterals, where the lengths of the edges explicitly appear in the computation of the area.

This is just because the number of involved edges is small, so that the segments determined by the edge partitions induced by the incircles can be easily related to the original lengths of the edges of the polygon. We also remark that the incircles can be elementary constructed, so that the provided formula also determines an elementary computation of the square of the area of any convex cyclic polygon

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