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Identification conditions in simultaneous systems of cointegrating equations with integrated variables of higher order

Rocco Mosconi [a,1], Paolo Paruolo [b,*]

[a] Politecnico di Milano, Piazza L. da Vinci 32, 20133 Milano, Italy

[b] European Commission, Joint Research Centre (JRC), Via E.Fermi 2749, I-21027 Ispra (VA), Italy

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ABSTRACT

This paper discusses identification of systems of simultaneous cointegrating equations with integrated variables of order two or higher, under constraints on the cointegration parameters. Rank and order conditions for identification are provided for general linear constraints, covering both cross-equation and equation-by-equation restrictions.

1. Introduction

The identification problem of system of simultaneous equations (SSE) lies at the heart of classical econometrics, see e.g. Koopmans (1949). Rank (and order) conditions for identification of these systems are well summarized in Fisher (1966) or Sargan (1988).

Simultaneous systems of cointegrating (CI) equations have revived interest on SSE over the last three decades, especially for variables integrated of order 1, I(1), see Engle and Granger (1987). When identifying restrictions are placed only on the CI parameters, the rank and order conditions for identification for I(1) simultaneous systems of CI equations, here indicated as I(1) SSE, coincide with the classical ones for SSE, see e.g. Saikkonen (1993), Davidson (1994) and Johansen (1995). The present paper discusses identification for SSE with integrated variables of order higher than 1, when restrictions are only placed on the CI parameters, and shows that the rank and order conditions have relevant differences in this higher order case.

CI SSE with variables integrated of order 2, or I(2) SSE, have been used to accommodate models of stock and flow variables, of inventories, and of consumption, income and wealth, see Klein (1950), Hendry and von Ungern-Sternberg (1981) and Granger and Lee (1989). A different rationale for I(2) SSE is provided by the literature on integral control mechanisms in economics initiated by Phillips (1954, 1956, 1957) in relation to the Error Correction Mechanism, EC, see Haldrup and Salmon (1998).

In I(2) systems, CI equations may involve both stocks and flows; these equations are called 'integral control' in the EC literature, or 'multi-cointegrating' relations (multi-CI), see Granger and Lee (1989). They are also a special case of 'polynomial-cointegration' relations, as introduced by Engle and Yoo (1991). A different type of CI equations consists of linear combinations of flow variables only; they represent balancing equations for flows, and they called 'proportional control' relations in the EC literature.

Identification of I(2) SSE has been addressed mostly through 'normalization' schemes, both in the parametric case, see Johansen (1997), and in the semi-parametric approach of Stock and Watson (1993), where the short-run dynamics are not estimated parametrically.

The purpose of the present paper is to discuss the identification problem in the I(d) SSE, with $d = 2, 3, \ldots,$ allowing for the possibility of over-identification, giving rank and order conditions.

* Corresponding author. Fax: +39 0332 785733.
E-mail addresses: rocco.mosconi@polimi.it (R. Mosconi), paolo.paruolo@jrc.ec.europa.eu (P. Paruolo).

1 Fax: +39 02700423151.

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These conditions generalize the ones valid for I(1) SSE, to which they reduce setting $d = 1$.

The rest of the paper is organized as follows: Section 2 gives motivation via a simple model of inventories; Section 3 defines I(2) SSE and discusses observational equivalence; Section 4 presents rank and order conditions. Section 5 discusses identification for higher order systems; Section 6 concludes. Proofs are placed in an Online Appendix.

In the following $a := b$ and $b =: a$ indicate that a is defined by b; $(a : b)$ indicates the matrix obtained by horizontally concatenating a and b. For any full column rank matrix H, $\text{col}(H)$ is the linear span of the columns of H, $\bar{H}$ indicates $H(H'H)^{-1}$ and $H_\perp$ indicates a basis of the orthogonal complement of the space spanned by the columns of H. Moreover $P_H := H(H'H)^{-1}H'$ indicates the orthogonal projection matrix on the columns of H, and $P_{H_\perp} = I - P_H$ denotes the orthogonal projection matrix on its orthogonal complement. vec is the column stacking operator, $\otimes$ is the Kronecker product, $\text{diag}(A_1, \dots, A_n)$ is a matrix with $A_1, \dots, A_n$ as diagonal (non necessarily square) blocks and zeros elsewhere.

The vector process X_t is said to be integrated of order d (with integer d), I(d), when $\Delta^d X_t - m_t = F(L)\varepsilon_t$ is a stationary linear process, m_t is a deterministic process, L is the lag operator, $\Delta := 1 - L$ and $F(z) = \sum_{i=0}^t F_i z^i$ is convergent in the disk $U_a := \{z : |z| < 1 + a\}$, $a > 0$. Here it is assumed that $F(z)$ is of full rank over U_a with the possible exception of $z = 1$, where the MA impact matrix $F(1)$ is assumed to be non-zero, see Johansen (1996). When $F(1)$ is of full rank, the process is said to be 'non cointegrating I(d)', indicated as ncI(d).

2. Motivating example

This section reports a model of inventories taken from Granger and Lee (1989), that motivates the derivations in the paper. Let y_t and w_t represent sales and production of a (possibly composite) good. Sales y_t are market-driven and trending; in particular Granger and Lee assume that they are I(1). Production w_t is chosen to meet demand y_t, i.e. y_t and w_t have the same trend. Hence $z_t := w_t - y_t$, the change in inventory, is stationary. This corresponds to a proportional control relationship among the flow variables $\Delta X_t := (w_t : y_t)'$; in other words ΔX_t is CI with cointegrating vector $(1 : -1)'$, i.e.

$$(1 : -1)\Delta X_t = u_{1t}, \tag{2.1}$$

where u_{1t} is a stationary process. The stock of inventories $Z_t = \sum_{i=1}^t z_i + Z_0$ can be expressed in terms of the cumulated production, $W_t = \sum_{i=1}^t w_i + W_0$ and cumulated sales $Y_t = \sum_{i=1}^t s_i + Y_0$, as $Z_t = W_t - Y_t$. Because w_t and y_t are assumed to be I(1), W_t and Y_t are I(2).

The principle of inventory proportionality anchors the inventory stock Z_t to a fraction of sales y_t, i.e. it implies the multi-CI relationship $Z_t = ay_t + u_{0t}$, with u_{0t} stationary, which may be written as

$$(1 : -1 : 0 : -a)\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = u_{0t}. \tag{2.2}$$

Observe that the CI relations (2.1) and (2.2) form an SSE of two equations,

$$\begin{pmatrix} 1 & -1 & 0 & -a \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = u_t, \tag{2.3}$$

where $u_t := (u_{0t} : u_{1t})'$ is a stationary error term.

The present paper investigates the following question: is the multi-CI vector in (2.2) *unique* (with or without the 0 restriction in the third entry)? Pre-multiplication of (2.3) by the following 2×2 matrix

$$Q = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{2.4}$$

with generic b, gives a system of equations with $(1 : -1 : b : -(a+b))'$ in place of $(1 : -1 : 0 : -a)'$ as the first equation (the multi-CI relation). One may expect that, when the third entry of the multi-CI vector is restricted to 0, the first equation (as well as the system) is identified.

This example, which motivates the derivations in the paper, is deliberately very simple, with only one proportional control relationship associated with the differenced (single) multi-CI relation. In the general case, discussed in the following section, there may be additional proportional control relationships.

3. I(2) simultaneous system of equations

This section introduces the I(2) SSE, discusses Observational Equivalence (OE) and the class of Q transformations on the CI parameters that induces OE. Let X_t be a $p \times 1$ vector of I(2) variables. The multi-CI relations involving X_t are of the type

$$\beta' X_t + \upsilon' \Delta X_t = \left(\beta' \mid \upsilon' \right)\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_{0t} + u_{0t} \tag{3.1}$$

where β and υ are $p \times r$ and β is of full column rank r, $r < p$, and u_{0t} is stationary. Here μ_{0t} denotes a deterministic vector.

The first difference of Eq. (3.1), $\beta'\Delta X_t + \upsilon'\Delta^2 X_t$, is also stationary; this implies that $\beta'\Delta X_t$ is stationary, given that $\Delta^2 X_t$ is stationary, because X_t is I(2). Moreover, other CI relations involving only ΔX_t can be present in the form $\gamma'\Delta X_t$ where γ is $p \times s$, of full column rank and linearly independent from β, with $s < p - r$. Taken together, the proportional control relations are given by

$$\begin{pmatrix} \gamma' \\ \hline \beta' \end{pmatrix}\Delta X_t = \left(\begin{array}{c|c} 0 & \gamma' \\ \hline 0 & \beta' \end{array}\right)\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_{1t} + u_{1t}, \tag{3.2}$$

where $u_{1t} := (u'_{1\gamma,t} : u'_{1\beta,t})'$ is a stationary process, $u_{1\beta,t} := \Delta u_{0t} - \upsilon'\Delta^2 X_t$, with $(u'_{0,t} : u'_{1\gamma,t})'$ an ncI(0) process, and μ_{1t} denotes a deterministic vector.

Collecting (3.1) and (3.2), the following system of $k := 2r + s$ stationary SSE results

$$\zeta'\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \left(\begin{array}{c|c} \beta' & \upsilon' \\ \hline 0 & \gamma' \\ 0 & \beta' \end{array}\right)\begin{pmatrix} X_t \\ \Delta X_t \end{pmatrix} = \mu_t + u_t, \tag{3.3}$$

where ζ' indicates the matrix of CI SSE coefficients, $\mu_t := (\mu'_{0t} : \mu'_{1t})'$ and $u_t := (u'_{0t} : u'_{1t})'$ is stationary.

Eq. (3.3) is the relevant SSE for the discussion of identification in I(2) cointegrated system. Note that ζ' contains 0 entries in the lower left corner and presents cross-equation restrictions, given by the presence of β' in the first and third block of rows.[2]

3.1. The identification problem

This subsection describes the Q transformation that gives rise to the identification problem in the I(2) SSE. Consider pre-multiplying ζ' in Eq. (3.3) by Q with

$$Q := \begin{pmatrix} \underset{r\times r}{Q_{00}} & Q_{0\gamma} & Q_{0\beta} \\ 0 & \underset{s\times s}{Q_{\gamma\gamma}} & Q_{\gamma\beta} \\ 0 & 0 & Q_{00} \end{pmatrix} \tag{3.4}$$

[2] Note that the CI SSE could be also written as $\zeta'(X'_t : \Delta X'_{t-n})'$ for $n = 1, 2, \dots$, because $\zeta'(X'_t : \Delta X'_{t-n})' - \zeta'(X'_t : \Delta X'_t)' = (\upsilon : \gamma : \beta)'(1 + L + \dots + L^{n-1})\Delta^2 X_t$ is stationary.

where Q_{00} and $Q_{\gamma\gamma}$ are non-singular square matrices of order r and s; the number of generically non-zero elements of Q is given by $q := r^2+(r+s)^2$. Pre-multiplying ζ' by Q gives rise to an equivalent I(2) SSE; in fact observe that

$$Q\zeta' = \begin{pmatrix} Q_{00}\beta' & (Q_{00}\upsilon' + Q_{0\gamma}\gamma' + Q_{0\beta}\beta') \\ 0 & (Q_{\gamma\gamma}\gamma' + Q_{\gamma\beta}\beta') \\ 0 & Q_{00}\beta' \end{pmatrix} =: \begin{pmatrix} \beta^{\circ\prime} & \upsilon^{\circ\prime} \\ 0 & \gamma^{\circ\prime} \\ 0 & \beta^{\circ\prime} \end{pmatrix} =: \zeta^{\circ\prime} \tag{3.5}$$

where $\beta^{\circ\prime} := Q_{00}\beta'$, $\gamma^{\circ\prime} := Q_{\gamma\gamma}\gamma' + Q_{\gamma\beta}\beta'$, $\upsilon^{\circ\prime} = Q_{00}\upsilon' + Q_{0\gamma}\gamma' + Q_{0\beta}\beta'$. Notice that the number of integral-control relations (r) and of proportional-control relations $(r+s)$ is unaffected by the Q transformation, and that $\zeta^{\circ\prime} := Q\zeta'$ has the same zero-restrictions and cross-equation constraints as ζ' in (3.3). This is the identification problem in SSE with I(2) variables.

This identification problem differs from the one encountered in I(1) system where OE is associated with pre-multiplication by any non-singular matrix Q, see Saikkonen (1993), Davidson (1994) and Johansen (1995).

3.2. Observational equivalence

This subsection shows that (a) the Q transformation defines OE values of the parameters in terms of the likelihood and (b) a similar OE applies to the representation of Stock and Watson (1993), which is used in semi-parametric models. This implies that the rank and order conditions derived in the next subsection apply both in the parametric and in the semi-parametric settings.

Consider first the EC representation of a VAR $A(L)X_t = \varepsilon_t$ under the conditions of the I(2) representation theorem of Johansen (1992). Here $A(L) = I - \sum_{i=1}^{h} A_i L^i$ is the AR polynomial, and it is assumed that $A(z)$ is convergent in the disk U_a and that $A(z)$ is of full rank over U_a with the possible exception of $z = 1$.

Johansen (1992) derived conditions under which a VAR process satisfying these assumptions is I(2), see also Johansen (1996) Chapter 4. These conditions are: (i) $A(1) = -\alpha\beta'$ of reduced rank $r < p$ and (ii) $P_{\alpha_\perp}\dot{A}P_{\beta_\perp} = \alpha_1\beta_1'$ of reduced rank $s < p - r$, and (iii) $P_{(\alpha:\alpha_1)_\perp}(\frac{1}{2}\ddot{A}+\dot{A}\bar{\beta}\bar{\alpha}'\dot{A})P_{(\beta:\beta_1)_\perp}$ of full rank $p-r-s$, where $\dot{A}$ and $\ddot{A}$ are the first and second derivative of $A(z)$ with respect to z, evaluated at $z = 1$.

Under conditions (i) and (ii), it can be shown that the VAR model can be parametrized as the following EC[3]

$$\Delta^2 X_t = \eta\zeta' \begin{pmatrix} X_{t-1} \\ \Delta X_{t-1} \end{pmatrix} + \Upsilon \begin{pmatrix} \Delta^2 X_{t-1} \\ \vdots \\ \Delta^2 X_{t-h+2} \end{pmatrix} + \varepsilon_t, \qquad \zeta := \begin{pmatrix} \beta & 0 & 0 \\ \upsilon & \gamma & \beta \end{pmatrix}. \tag{3.6}$$

The parameters of the model are the unrestricted adjustment matrix η, the CI matrix ζ, the short run dynamics matrix Υ and Ω, the variance–covariance matrix of ε_t. The Gaussian log-likelihood ℓ associated with observations $X_1, \ldots, X_T$ and parameters $\xi := (\eta, \zeta, \Upsilon, \Omega)$ is proportional to $\ell(\xi) := -\frac{1}{2}(T \log\det\Omega + \sum_{t=1}^{T} \varepsilon_t'\Omega^{-1}\varepsilon_t)$, when ε_t is taken to be iid $N(0, \Omega)$. The parameter space for $\xi := (\eta, \zeta, \Upsilon, \Omega)$ is unrestricted, except for the requirements on ζ to have the structure in (3.3) and on Ω to be positive definite.

Consider next the square invertible matrix Q in (3.4) and insert $Q^{-1}Q$ between η and ζ' in (3.6); it is simple to observe that $\xi := (\eta, \zeta, \Upsilon, \Omega)$ and $\xi^\circ := (\eta Q^{-1}, \zeta Q', \Upsilon, \Omega)$ produce the same likelihood, $\ell(\xi) = \ell(\xi^\circ)$, i.e. that ξ is OE to ξ°. This shows that the class of transformations Q creates OE in terms of the likelihood of the EC (3.6).

The Q transformation in (3.4) has a similar effect on the representation in Stock and Watson (1993). Let X_t be I(2) with MA representation $\Delta^2 X_t = F(L)\varepsilon_t$; under the condition that $F(z)^{-1}$ has a pole of order 2 at $z = 1$, there exists some ncI(0) process $H(L)\varepsilon_t$ and some square and nonsingular matrix $B := (b_2 : b_1 : b_0)$ of order p, with b_i of dimension $p \times r_i$, $r_i \geq 0$, such that one can define $y_t := B'X_t = (y_t^{2\prime} : y_t^{1\prime} : y_t^{0\prime})'$, with $y_t^i = b_i'X_t$, and

$$\begin{pmatrix} \Delta^2 I_{r_2} & 0 & 0 \\ -\theta_{1,2}^1\Delta & \Delta I_{r_1} & 0 \\ -\theta_{0,2}^1\Delta - \theta_{0,2}^0 & -\theta_{0,1}^0 & I_{r_0} \end{pmatrix} \begin{pmatrix} y_t^2 \\ y_t^1 \\ y_t^0 \end{pmatrix} = H(L)\varepsilon_t, \tag{3.7}$$

see Stock and Watson (1993) eq. (3.2).[4] Collecting terms with equal order of differencing, one can write (3.7) as

$$\begin{pmatrix} b_2'\Delta^2 \\ \gamma'\Delta \\ \beta' + \upsilon'\Delta \end{pmatrix} X_t = H(L)\varepsilon_t \tag{3.8}$$

where $u_t := H(L)\varepsilon_t$ is ncI(0) and

$$\begin{pmatrix} b_2' \\ \gamma' \\ \beta' \end{pmatrix} := \begin{pmatrix} I_{r_2} & 0 & 0 \\ -\theta_{1,2}^1 & I_{r_1} & 0 \\ -\theta_{0,2}^0 & -\theta_{0,1}^0 & I_{r_0} \end{pmatrix} \begin{pmatrix} b_2' \\ b_1' \\ b_0' \end{pmatrix}, \qquad \upsilon' := -\begin{pmatrix} \theta_{0,2}^1 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_2' \\ b_1' \\ b_0' \end{pmatrix}. \tag{3.9}$$

Stock and Watson (1993) take $B := (b_2 : b_1 : b_0)$ to be a permutation matrix of order p[5]; this restriction is not necessary, and one can take B to be any appropriate nonsingular matrix, where in particular b_i and b_j need not be orthogonal, $i, j = 0, 1, 2$; see also Boswijk (2000). Note here that $(b_2 : \gamma : \beta)$ is square and nonsingular, being the product of B, which is nonsingular, and the block triangular matrix in (3.9) with identities on the main diagonal, which is also nonsingular. Representation (3.8) is called SW in the following.

The next theorem shows in what sense SW is Q-invariant.

Theorem 1 (Q-*invariance of SW*). *Let X_t be $I(2)$ with MA representation $\Delta^2 X_t = F(L)\varepsilon_t$ where $F(L)\varepsilon_t$ is $I(0)$ and let (3.7) be its SW, where $H(L)\varepsilon_t$ is ncI (0). Then X_t also satisfies the SW*

$$\begin{pmatrix} b_2'\Delta^2 \\ \gamma^{\circ\prime}\Delta \\ \beta^{\circ\prime} + \upsilon^{\circ\prime}\Delta \end{pmatrix} X_t = H^\circ(L)\varepsilon_t \tag{3.10}$$

[3] Let $\Delta^2 X_t = \Pi X_{t-1} + \Gamma\Delta X_{t-1} + \sum_{i=1}^{k-2} \Upsilon_i \Delta^2 X_{t-i} + \varepsilon_t$, where condition (i) implies $\Pi = \alpha\beta'$. Next define $\tau := (\beta : \gamma)$ where γ is any matrix that satisfies $\text{col}(\beta : \gamma) = \text{col}(\beta : \beta_1)$ and consider $\Gamma = \Gamma P_\tau + \Gamma P_{\tau_\perp} = \lambda_\star\tau' + \alpha\delta\tau_\perp'$ where $\lambda_\star := \Gamma\bar{\tau}$, $\delta := \bar{\alpha}'\Gamma\bar{\tau}_\perp$ because $\alpha_\perp'\Gamma\tau_\perp = 0$ by condition (ii). Adding and subtracting $\alpha c\tau'$ one obtains $\Gamma = \lambda\tau' + \alpha\upsilon'$ where $\lambda := \lambda_\star - \alpha c$ and $\upsilon' := \delta\tau_\perp' + c\tau' = (c : \delta)(\tau : \tau_\perp)'$. Here $\eta := (\alpha : \lambda)$. Because $(\tau : \tau_\perp)$ is square and nonsingular and no restrictions are placed on $(c : \delta)$, this shows that υ' is not restricted to lie in any specific subspace.

[4] The notation y_t^{d+1-i} (respectively $\theta_{d+1-s,d+1-j}^{i-1}$) here corresponds to y_t^i (respectively $\theta_{s,j}^{d-i}$) in Stock and Watson (1993). Moreover, $r_0 = r$, $r_1 = s$, $r_2 = p - r - s$ here correspond to k_2, k_1, k_0 there.

[5] I.e. a matrix obtained by rearranging the rows or columns of the identity matrix of order p.

where $H^\circ(L)\varepsilon_t$ is ncI(0), $\beta^{\circ\prime} := Q_{00}\beta'$, $\gamma^{\circ\prime} := Q_{\gamma\gamma}\gamma' + Q_{\gamma\beta}\beta'$, $\upsilon^{\circ\prime} = Q_{00}\upsilon' + Q_{0\gamma}\gamma' + Q_{0\beta}\beta'$ are the elements of $\zeta^{\circ\prime} = Q\zeta'$ in (3.5) and Q_{ij} are the blocks of the Q matrix in (3.4), $i, j = 0, \gamma, \beta$.

4. I(2) identification conditions

This section considers the I(2) SSE (3.3) under general linear restrictions on ζ. Consider the following linear restrictions

$$\underset{m_\star \times f_\star}{R'_\star}\ \theta = c_\star, \qquad \underset{f_\star \times 1}{\theta} := \begin{pmatrix} \operatorname{vec}\begin{pmatrix}\beta\\ \upsilon\end{pmatrix} \\ \operatorname{vec}\ \gamma \end{pmatrix} \tag{4.1}$$

where $f_\star := p(2r + s)$. The next theorem gives rank and order conditions for (4.1) to identify ζ.

Theorem 2 (*Rank and Order Conditions for I(2) SSE*)**.** *A necessary and sufficient condition (rank condition) for the restrictions (4.1) to identify ζ in the I(2) SSE (3.3) is that the matrix*

$$R'_\star \operatorname{diag}\left(I_r \otimes \zeta, I_s \otimes (\gamma : \beta)\right) \tag{4.2}$$

is of full column rank q, where $q = r^2 + (r + s)^2$. A necessary but not sufficient condition (order condition) for (4.2) to have full column rank is that its number of rows is greater than or equal to its number of columns, that is

$$m_\star \geq q. \tag{4.3}$$

A few remarks are in order.

Remark 3 (*No Integral Control Relations, $r = 0$*)**.** If $r = 0$, then $I_r \otimes \zeta$ and β are dropped from (4.2) and the rank condition reduces to $\operatorname{rank} R'_\star (I_s \otimes \gamma) = s^2$. This is the usual rank condition for identification in a standard I(1) SSE, see Johansen (1995), due to the fact that the I(2) SSE simplifies into an I(1) SSE in first differences.

Remark 4 (*No Additional Proportional Control Relations, $s = 0$*)**.** If $s = 0$, then γ is dropped from ζ, which simplifies into

$$\zeta = \begin{pmatrix} \beta & 0 \\ \upsilon & \beta \end{pmatrix}, \tag{4.4}$$

and $I_s \otimes (\gamma : \beta)$ is dropped from (4.2). The rank condition becomes $\operatorname{rank} R'_\star (I_r \otimes \zeta) = 2r^2$. Note that this is *not* the usual rank condition for identification in a standard I(1) SSE. The I(2) SSE (4.4) in fact still involves υ and the β block is repeated in the upper left and lower right corners.

Remark 5 (*Practical Implementation of the Rank Condition*)**.** In order to check the rank condition on matrix (4.2), consider the restrictions (4.1) in explicit form, i.e. $\theta = H_\star\varphi + h_\star$, where $H_\star = R_{\star\perp}$ and $c_\star = R'_\star h_\star$; here φ contains the unrestricted parameters in θ. For given value of φ, φ° say, one can form θ° as $\theta^\circ = H_\star\varphi^\circ + h_\star$, and hence ζ° and $(\gamma^\circ : \beta^\circ)$ using the definition of θ. One can then numerically find the rank of $R'_\star \operatorname{diag}(I_r \otimes \zeta^\circ, I_s \otimes (\gamma^\circ : \beta^\circ))$ e.g. by computing its singular values.

One way to choose φ° can be for instance to generate this as a random draw from some distribution with Lebesgue density, such as the Gaussian. If the rank condition is satisfied outside a set of Lebesgue measure zero, then the probability of drawing an element from this set is zero, see Boswijk and Doornik (2004).

Remark 6 (*Role of the Order Condition*)**.** The order condition can be used – as in classical SSE – as a preliminary check to control that the number of restrictions in (4.1) is at least equal to q.

Next consider the rank and order conditions for equation-by-equation constraints, where ith column of ζ is indicated as ζ_i and the ith column of γ as γ_i. These constraints can be formulated as follows

$$\begin{aligned} \underset{m_i \times 2p}{R'_i}\ \zeta_i &= c_i, \qquad i = 1, \ldots, r, \\ \underset{m_i \times p}{R'_i}\ \gamma_{i-r} &= c_i, \qquad i = r + 1, \ldots, r + s. \end{aligned} \tag{4.5}$$

These restrictions are a special case of (4.1), with

$$R_\star = \operatorname{diag}(\mathcal{R}_1, \mathcal{R}_2),$$

where $\mathcal{R}_1 = \operatorname{diag}(R_1, \ldots, R_r)$ collects the first r equations and $\mathcal{R}_2 = \operatorname{diag}(R_{r+1}, \ldots, R_{r+s})$ the next s equations (concerning γ). The following corollary specializes the rank and order conditions to the case of equation-by-equation restrictions.

Corollary 7 (*Identification, Equation-by-equation Restrictions*)**.** *Let the restrictions be given as in (4.5); then the ith column of ζ, $i = 1, \ldots, r$, is identified if and only if*

$$\operatorname{rank}\left(R'_i\zeta\right) = 2r + s, \qquad i = 1, \ldots, r; \tag{4.6}$$

column number $i - r$ in γ for $i = r + 1, \ldots, r + s$ is identified if and only if:

$$\operatorname{rank}\left(R'_i(\gamma : \beta)\right) = r + s, \qquad i = r + 1, \ldots, r + s. \tag{4.7}$$

The joint validity of rank conditions (4.6) for $i = 1, \ldots, r$ and (4.7) for $i = r + 1, \ldots, r + s$ is equivalent to the full column rank of (4.2), which can also be expressed equivalently as follows

$$\begin{aligned} &\operatorname{rank}\left(\mathcal{R}'_1 (I_r \otimes \zeta)\right) = r\,(2r + s) \quad \textit{and} \\ &\operatorname{rank}\left(\mathcal{R}'_2 (I_s \otimes (\gamma : \beta))\right) = s\,(r + s)\,. \end{aligned} \tag{4.8}$$

A necessary but not sufficient condition (order condition) for (4.6) is

$$m_i \geq 2r + s, \qquad i = 1, \ldots, r. \tag{4.9}$$

Similarly, a necessary but not sufficient condition (order condition) for (4.7) is

$$m_i \geq r + s, \qquad i = r + 1, \ldots, r + s. \tag{4.10}$$

5. Systems of equations with integrated variables of higher order

This section discusses the rank and order conditions for the I(d) SSE, $d = 1, 2, 3, \ldots$. In this section r and s of the previous sections are indicated as r_0 and r_1 respectively, while β, γ and υ are indicated here as γ_0, γ_1 and υ_{01}. As shown in the Online Appendix, the relevant I(d) SSE is given by

$$\begin{aligned} \underset{k \times pd}{\zeta'} \begin{pmatrix} X_t \\ \Delta X_t \\ \vdots \\ \Delta^{d-1}X_t \end{pmatrix} &= \begin{pmatrix} \varphi'_0 & \upsilon'_{01} & \upsilon'_{02} & \cdots & \upsilon'_{0d-1} \\ 0 & \varphi'_1 & \upsilon'_{12} & & \upsilon'_{1d-1} \\ 0 & 0 & \varphi'_2 & & \upsilon'_{2d-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & \varphi'_{d-1} \end{pmatrix} \begin{pmatrix} X_t \\ \Delta X_t \\ \vdots \\ \Delta^{d-1}X_t \end{pmatrix} \\ &= \mu_t + u_t \end{aligned} \tag{5.1}$$

where u_t is I(0), $\varphi_i := (\gamma_i : \gamma_{i-1} : \ldots : \gamma_1 : \gamma_0) = (\gamma_i : \varphi_{i-1})$, of dimension $p \times k_i$ with $k_i := \sum_{j=0}^{i} r_j$, $k := \sum_{i=0}^{d-1} k_i$, and $r_i \geq 0$ is the number of columns in γ_i, $i = 0, \ldots, d - 1$; see Online Appendix for details on the definition of other matrices. System (5.1) is henceforth referred to as an I(d) SSE.

Restrictions on the SSE (5.1) are expressed as follows:

$$\underset{m \times f}{R'}\operatorname{vec}\ \zeta = c. \tag{5.2}$$

Here $f := kpd$. As for the I(2) case, let θ be the $f_\star \times 1$ vector containing the generically nonzero elements of ζ; the linear restrictions (5.2) can be equivalently expressed as

$$\underset{m_\star \times f_\star}{R'_\star} \theta = c_\star. \tag{5.3}$$

Without loss of generality, one can assume that $f - f_\star$ of the restrictions in (5.2) ensure that ζ' has a block-triangular structure with cross-equation restrictions,[6] while $m_\star := m - f + f_\star$ are possibly (over-)identifying restrictions on θ. Because ζ and θ contain the same parameter matrices, there exists some non-singular matrix A with entries equal to 0 or 1 that satisfies $\operatorname{vec} \zeta = A\theta$.

The Q transformation that induces lack of identification in (5.1) is of the form

$$Q = \begin{pmatrix} Q_{00} & Q_{01} & \cdots & Q_{0,d-1} \\ 0 & Q_{11} & \cdots & Q_{1,d-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & Q_{d-1,d-1} \end{pmatrix}, \qquad \underset{k_j \times k_j}{Q_{jj}} = \begin{pmatrix} Q_{j,\gamma_j} & Q_{j,\varphi_{j-1}} \\ 0 & Q_{j-1,j-1} \end{pmatrix}, \quad j = 1, \ldots, d-1, \tag{5.4}$$

where Q_{00}, Q_{j,γ_j} for $j = 1, \ldots, d-1$ are square and nonsingular; hence also Q is square and nonsingular. The number of generically non-zero elements of Q is still indicated by q, g denotes the $q \times 1$ vector containing the generically non-zero elements of Q in (5.4) and N indicates the (unique, 0–1) matrix that maps g into $\operatorname{vec}(Q')$, i.e. such that $\operatorname{vec}(Q') = Ng$.

One can now state the rank and order conditions for identification of ζ in the I(d) SSE.

Theorem 8 (*Rank and Order Conditions for the I(d) SSE*)**.** *A necessary and sufficient condition (rank condition) for the restrictions (5.2) to identify ζ in an I(d) SSE (5.1) is that the matrix*

$$R'(I_k \otimes \zeta)N \tag{5.5}$$

is of full column rank q. A necessary but not sufficient condition (order condition) for the rank condition to hold is

$$m_\star \geq q. \tag{5.6}$$

For the case $d = 2$, the condition of full rank of (5.5) is equivalent to the requirement of full rank of (4.2) and the order condition (5.6) is equivalent to (4.3).

Remark 9 (*Differences with the Rank Condition for Standard SSE*)**.** The rank condition in (5.5) can be compared with the one obtained for standard SSE, see e.g. Sargan (1988), Chapter 3, Theorem 1. The matrix $R'(I_k \otimes \zeta)N$ in the rank condition here is very similar to the matrix $R'(I_k \otimes \zeta)$ in the standard case, the only difference being the additional multiplicative factor N here. This is due to fact that the class of matrices Q in (5.4) is different from the of square and nonsingular matrices, which is the one that induces OE in standard SSE.

Remark 10 (*The I(1) Case is Covered*)**.** In the I(1) case, one has that $Q = Q_{00}$, $N = I$ and the rank condition (5.5) reduces to the standard one. Hence Theorem 8 covers also the case $d = 1$, and it is hence an extension of it.

[6] These concern the fact that φ_{i-1} appears as one component in $\varphi_i = (\gamma_i : \varphi_{i-1})$ for $i = 1, \ldots, d-1$.

6. Conclusions

This paper provides rank and order conditions for identification in I(d) systems, $d = 1, 2, \ldots$ under general linear hypotheses on the cointegrating vectors. The advantage of the present algebraic approach in the discussion of identification of the I(d) SSE is that it works for all approaches for which the Q transformation induces observational equivalence, which includes parametric and semiparametric specifications.

These results are relevant also when using sequential identification schemes. In fact, one could consider procedure that first aims at identifying β through affine restrictions of the type $R'_\circ \operatorname{vec} \beta = c_\circ$, and subsequently consider identification of υ, γ, with or without β fixed. The first-stage identification of β is standard, see Johansen et al. (2010), and the associated rank condition is $\operatorname{rank} R'_\circ(I_r \otimes \beta) = r^2$. When β is identified, the identification problem for υ, γ is still associated with a Q transformation of the type (3.4) with $Q_{00} = I_r$; hence the present discussion of identification is relevant also for this sequential procedure.

In particular, the identification analysis of the coefficients in υ, i.e. the coefficients involving the flow variables in the multi-CI equations, requires a joint non-standard analysis of υ, γ and β, since each column of υ could be replaced with a linear combination of columns in υ, γ and β.

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Appendix A. Proofs

Proofs of Theorems are reported in the supplementary material related to this article, which can be found online at http://dx.doi.org/10.1016/j.jeconom.2017.01.007.

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