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BOHR'S CORRESPONDENCE PRINCIPLE FOR THE RENORMALIZED NELSON MODEL*

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Abstract. This paper studies the derivation of the nonlinear system of Schrödinger–Klein–Gordon (S-KG) equations, coupled by a Yukawa-type interaction, from a microscopic quantum field model of nonrelativistic particles interacting with a relativistic scalar field introduced by Edward Nelson in the mid 1960s. In particular, we prove that the quantum states evolved by the microscopic dynamics converge, in the classical limit, to their Wigner measures pushed forward by the S-KG flow. To define the microscopic dynamics it is not sufficient to quantize the classical energy, since the system requires a self-energy renormalization; it is therefore noteworthy, as well as one of the main technical difficulties of the analysis, that the classical limit is not affected by such renormalization. This last fact is proved with the aid of a classical dressing transformation.

Key words. Nelson model, Schrödinger–Klein–Gordon system, Yukawa interaction, classical limit

AMS subject classifications. 35Q40, 81T16, 81Q20

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1. Introduction. Modern theoretical physics explains how matter interacts with radiation and proposes phenomenological models of quantum field theory that in principle describe such fundamental interaction. Giving firm mathematical foundations to these models is known to be a difficult task related to renormalization theory [21, 55, 72, 73, 81, 94]. Since the 1950s there have been spectacular advances in these problems culminating with the perturbative renormalization of quantum electrodynamics, the birth of the renormalization group method, and the renormalizability of gauge field theories. Nevertheless, conceptual mathematical difficulties remain as well as outstanding open problems; see [85, 103]. The purpose of the present article is to study the quantum-classical correspondence for a simple renormalized model of particles interacting with a scalar field: the Nelson model. We believe that the study of the relationship between classical and quantum nonlinear field theories sheds light on the mathematical foundation of renormalization theory. In particular, in the case considered here the renormalization procedure turns out to be related to a normal form implemented by nonlinear symplectic transformations on the classical phase-space. The interested reader may find a formal discussion concerning the possibility of a different point of view on renormalization in an extended version of this article [6].

The so-called Nelson model is a system of quantum field theory that has been widely studied from a mathematical standpoint; see, e.g., [1, 14, 15, 16, 17, 23, 43, 53, 58, 62, 63, 64, 96, 100, 110]. It consists of nonrelativistic spin zero particles interacting with a scalar boson field and can be used to model various systems of physical interest, such as nucleons interacting with a meson field. In the mid 1960s

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Edward Nelson rigorously constructed a quantum dynamics for this model free of ultraviolet (high energy) cutoffs in the particle-field coupling; see [97]. This is done by means of a renormalization procedure: roughly speaking, we need to subtract a divergent quantity from the Hamiltonian so the latter can be defined as a self-adjoint operator in the limit of the ultraviolet cutoff being removed. The quantum dynamics is rather singular in this case (renormalization is necessary); and the resulting generator has no explicit form as an operator, though it is unitarily equivalent to an explicit one. Since the work of Gross [79] and Nelson [97] it has been believed, but never proved, that the renormalized dynamics is generated by a canonical quantization of the Schrödinger–Klein–Gordon (S-KG) system with Yukawa coupling. In other words, the quantum fluctuations of the particle-field system are centered around the classical trajectories of the S-KG system at a certain scale, and the renormalization procedure preserves the suitable quantum-classical correspondence as well as being necessary to define the quantum dynamics. We give a mathematical formulation of such a result in Theorem 1.1 in the form of a Bohr correspondence principle. Consequently, our result justifies in some sense the use of the S-KG system as a model of nucleon-meson interaction; see, e.g., [18, 19, 40, 59, 70, 99].

Recently, the authors of this paper have studied the classical limit of the Nelson model in its regularized version [5, 51]. We have proved that the quantum dynamic converges when an effective semiclassical parameter $\varepsilon \to 0$, toward a nonlinear Hamiltonian flow on a classical phase space. This flow is governed by an S-KG system with a regularized Yukawa-type coupling. To extend the classical-quantum correspondence to the system without ultraviolet cutoff, we partially rely on the recent techniques elaborated on in the mean-field approximation of many-body Schrödinger dynamics in [9, 10, 11, 12], as well as on the result with cutoff [5]. As a matter of fact, the renormalization procedure, implemented by a dressing transform, generates a many-body Schrödinger dynamics in a mean-field scaling; see, e.g., [7, 20, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 45, 46, 47, 48, 49, 50, 75, 76, 77, 78, 83, 86, 87, 88, 90, 95, 101, 107, 108, 109] for an overview on rigorous derivations of mean field dynamics for bosonic systems. So it has been convenient for us that the mean-field approximation has already been derived with the same general techniques that equally allow study of the classical approximation of quantum field theory models. The result is further discussed in subsection 1.2, and all the details and proofs are provided in section 4.

For the sake of presentation, we collected the notation and basic definitions used throughout the paper in the subsection 1.1 below. In subsection 1.2 we present our main result on the classical-quantum correspondence principle. The rest of the paper is organized as follows: in section 2 we review the basic properties of the quantum system and the usual procedure of renormalization with some crucial uniform estimates; in section 3 we analyze the classical S-KG dynamics and the classical dressing transformation; in section 4 we study in detail the classical limit of the renormalized Nelson model and prove our main theorem, Theorem 1.1.

1.1. Notation and general definitions.

* We fix once and for all $\bar{\varepsilon}, m_0, M > 0$. We also define the function $\omega(k) = \sqrt{k^2 + m_0^2}$.
* The effective (semiclassical) parameter will be denoted by $\varepsilon \in (0, \bar{\varepsilon})$.
* Let $\mathscr{Z}$ be a Hilbert space; then we denote by $\Gamma_s(\mathscr{Z})$ the symmetric Fock space

over $\mathcal{Z}$. We have that

$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \mathcal{Z}^{\otimes_s n} \quad \text{with} \quad \mathcal{Z}^{\otimes_s 0} = \mathbb{C} \,,$$

where $\mathcal{Z}^{\otimes_s n}$ is the n-fold symmetrized tensor product.

* Let X be an operator on a Hilbert space $\mathcal{Z}$. We will usually denote by $D(X) \subset \mathcal{Z}$ its domain of definition and by $Q(X) \subset \mathcal{Z}$ the domain of definition of the corresponding quadratic form.
* Let $S : \mathcal{Z} \supseteq D(S) \to \mathcal{Z}$ be a densely defined self-adjoint operator on $\mathcal{Z}$. Its second quantization $d\Gamma(S)$ is the self-adjoint operator on $\Gamma_s(\mathcal{Z})$ defined by

$$d\Gamma(S)\big|_{D(S)^{\otimes_s^{alg} n}} = \varepsilon \sum_{k=1}^{n} 1 \otimes \cdots \otimes \underbrace{S}_{k} \otimes \cdots \otimes 1 \,,$$

where $D(S)^{\otimes_s^{alg} n}$ denotes the algebraic tensor product. In particular, the operator $d\Gamma(1)$ is the scaled number operator which we simply denote by N without stressing the ε-dependence.
* We denote by $\mathcal{C}_0^\infty(N)$ the subspace of finite particle vectors:

$$\mathcal{C}_0^\infty(N) = \{\Psi \in \Gamma_s(\mathcal{Z}) \,;\, \exists \bar{n} \in \mathbb{N}, \Psi\big|_{\mathcal{Z}^{\otimes_s n}} = 0 \;\forall n > \bar{n}\} \,.$$

* Let U be a unitary operator on $\mathcal{Z}$. We define $\Gamma(U)$ to be the unitary operator on $\Gamma_s(\mathcal{Z})$ given by

$$\Gamma(U)|_{\mathcal{Z}^{\otimes_s n}} = \bigotimes_{k=1}^{n} U \,.$$

It then follows that for any one-parameter group $U = e^{itS}$ of unitary operators on $\mathcal{Z}$, its second quantization satisfies the following identity: $\Gamma(e^{itS}) = e^{i\frac{t}{\varepsilon} d\Gamma(S)}$.
* On $\Gamma_s(\mathcal{Z})$, we define the annihilation/creation operators $a^{\#}(g)$, $g \in \mathcal{Z}$, by their action on $f^{\otimes n} \in \mathcal{Z}^{\otimes_s n}$ (with $a(g) f_0 = 0$ for any $f_0 \in \mathcal{Z}^{\otimes_s 0} = \mathbb{C}$):

$$\begin{aligned} a(g) f^{\otimes n} &= \sqrt{\varepsilon n}\, \langle g, f\rangle_{\mathcal{Z}}\, f^{\otimes(n-1)} \,, \\ a^*(g) f^{\otimes n} &= \sqrt{\varepsilon(n+1)}\, g \otimes_s f^{\otimes n} \,. \end{aligned}$$

They satisfy the canonical commutation relations (CCR), $[a(f), a^*(g)] = \varepsilon \langle f, g\rangle_{\mathcal{Z}}$.
If $\mathcal{Z} = L^2(\mathbb{R}^d)$, it is useful to introduce the operator valued distributions $a^{\#}(x)$ defined by

$$a(g) = \int_{\mathbb{R}^d} \bar{g}(x) a(x) dx, \quad a^*(g) = \int_{\mathbb{R}^d} g(x) a^*(x) dx.$$

* $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)) \simeq \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(L^2(\mathbb{R}^3))$. We denote by $\psi^{\#}(x)$ and N_1 the annihilation/creation and number operators corresponding to the nucleons (conventionally taken to be the first Fock space) and by $a^{\#}(k)$ and N_2 the annihilation/creation and number operators corresponding to the

meson scalar field (second Fock space). In particular, we will always use the following ε-dependent representation of the CCR if not specified otherwise:

$$[\psi(x), \psi^*(x')] = \varepsilon\delta(x - x'), \quad [a(k), a^*(k')] = \varepsilon\delta(k - k').$$

* We will sometimes use the following decomposition:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \text{ with } \mathcal{H}_n = \big(L^2(\mathbb{R}^3)\big)^{\otimes_s n} \otimes \Gamma_s(L^2(\mathbb{R}^3)) .$$

  We denote by $T^{(n)} := T\big|_{\mathcal{H}_n}$ the restriction to $\mathcal{H}_n$ of any operator T on $\mathcal{H}$.

* On $\mathcal{H}$, the Segal field operator is given for $\xi = \xi_1 \oplus \xi_2 \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ by

$$R(\xi) = \big(\psi^*(\xi_1) + \psi(\xi_1) + a^*(\xi_2) + a(\xi_2)\big)/\sqrt{2}\ ,$$

  and therefore the Weyl operator becomes

$$W(\xi) = e^{\frac{i}{\sqrt{2}}\left(\psi^*(\xi_1)+\psi(\xi_1)\right)}\, e^{\frac{i}{\sqrt{2}}\left(a^*(\xi_2)+a(\xi_2)\right)} .$$

* Given a Hilbert space $\mathcal{Z}$, we denote by $\mathcal{L}(\mathcal{Z})$ the C^*-algebra of bounded operators; by $\mathcal{K}(\mathcal{Z}) \subset \mathcal{L}(\mathcal{Z})$ the C^*-algebra of compact operators; and by $\mathcal{L}^1(\mathcal{Z}) \subset \mathcal{K}(\mathcal{Z})$ the trace-class ideal.
* We denote classical Hamiltonian flows by boldface capital letters (e.g., $\mathbf{E}(\cdot)$) and their corresponding energy functional by script capital letters (e.g., $\mathscr{E}$).
* Let $f \in \mathscr{S}'(\mathbb{R}^d)$ be a tempered distribution. We denote by $\mathcal{F}(f)(k)$ its Fourier transform

$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ik\cdot x} dx .$$

* We denote by $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions of compact support. We denote by $H^s(\mathbb{R}^d)$ the nonhomogeneous Sobolev space

$$H^s(\mathbb{R}^d) = \Big\{ f \in \mathscr{S}'(\mathbb{R}^d)\ ,\ \int_{\mathbb{R}^d} (1 + |k|^2)^s |\mathcal{F}(f)(k)|^2 dk < +\infty \Big\}$$

  and denote its "Fourier transform" by

$$\mathcal{F}H^s(\mathbb{R}^d) = \Big\{ f \in \mathscr{S}'(\mathbb{R}^d)\, ,\, \mathcal{F}^{-1} f \in H^s(\mathbb{R}^d) \Big\} .$$

* Let $\mathcal{Z}$ be a Hilbert space. We denote by $\mathfrak{P}(\mathcal{Z})$ the set of Borel probability measures on $\mathcal{Z}$.

1.2. The classical limit of the renormalized Nelson model. The S-KG equations with Yukawa-like coupling are a widely studied system of nonlinear PDEs in three dimensions; see, e.g., [18, 19, 40, 59, 60, 61, 70, 99]. This system is usually written as

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M} u + Vu + Au, \\ (\Box + m_0^2)A = -|u|^2, \end{cases}$$

where $m_0, M > 0$ are real parameters and V is a nonnegative potential that is confining or equal to zero. For our purposes, it is more useful to rewrite it in an equivalent form using the complex field α as a dynamical variable instead of $(A, \dot{A})$ (see (47) of section 3 for the details):

$$\text{(S-KG)} \qquad \begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + A(\alpha)u, \\ i\partial_t \alpha = \omega\alpha + \dfrac{1}{\sqrt{2\omega}}\mathcal{F}\big(|u|^2\big), \end{cases}$$

where $\omega(k) = \sqrt{k^2 + m_0^2}$, and

$$A(\alpha)(t,x) = \tfrac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \tfrac{1}{\sqrt{2\omega(k)}} \big(\bar{\alpha}(t,k)e^{-ik\cdot x} + \alpha(t,k)e^{ik\cdot x}\big)dk\ .$$

The aforementioned system of equations can be seen as a Hamiltonian system corresponding to the following energy functional, densely defined on[1] $L^2 \oplus L^2$:

$$\begin{aligned} \mathscr{E}(u,\alpha) :=& \Big\langle u, \Big(-\tfrac{\Delta}{2M} + V\Big)u\Big\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 \\ &+ \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}} \Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dxdk\ . \end{aligned}$$

With suitable assumptions on the external potential V, one proves the global existence of the associated flow $\mathbf{E}(t)$. A more detailed discussion of global well-posedness can be found in subsection 3.3, where sufficient conditions on V are given (assumption (A-V)). In other words, there exist a Hilbert space $\mathcal{D}$, densely embedded in $L^2 \oplus L^2$, such that $\mathbf{E} : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ associates to a given point (u, α) on $\mathcal{D}$, and to a given time t, the solution at time t of the Cauchy problem associated S-KG equation (S-KG) above with initial datum (u, α).

A question of significant interest, both mathematically and physically, is whether it is possible to quantize the S-KG dynamics with Yukawa coupling as a consistent theory that describes quantum-mechanically the particle-field interaction. As mentioned previously, Nelson rigorously constructed a self-adjoint operator satisfying in some sense the above requirement [97]. Afterward the model was proved to satisfy some of the main properties that are familiar in the axiomatic approach to quantum fields; see [24]. Furthermore, asymptotic completeness was proved to be true in [4]. The problem of quantization of such infinite dimensional nonlinear dynamics is related to constructive quantum field theory. The general framework is as follows.

Let $\mathscr{Z}$ be a complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. We define the associated symplectic structure $\Sigma(\mathscr{Z})$ as the pair $\{\mathcal{Y}, B(\,\cdot\,,\,\cdot\,)\}$ where $\mathcal{Y}$ is $\mathscr{Z}$ considered as a real Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle_r = \mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle$, and $B(\,\cdot\,,\,\cdot\,)$ is the symplectic form defined by $B(\,\cdot\,,\,\cdot\,) = \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle$. Following [106], we define a (bosonic) quantization of the structure $\Sigma(\mathscr{Z})$ to be any linear map $R(\cdot)$ from $\mathcal{Y}$ to self-adjoint operators on a complex Hilbert space such that

* the Weyl operator $W(z) = e^{iR(z)}$ is weakly continuous when restricted to any finite dimensional subspace of $\mathcal{Y}$;
* $W(z_1)W(z_2) = e^{-\frac{i}{2}B(z_1,z_2)}W(z_1 + z_2)$ for any $z_1, z_2 \in \mathcal{Y}$ (Weyl's relations).

[1] Sometimes the shorthand notation $L^2 \oplus L^2$ is used instead of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$ if no confusion arises.

When the dimension of $\mathscr{Z}$ is not finite, there are uncountably many irreducible unitarily inequivalent Segal quantizations of $\Sigma(\mathscr{Z})$ (or representations of Weyl's relations). A representation of particular relevance in physics is the so-called Fock representation [42, 54] on the symmetric Fock space $\Gamma_s(\mathscr{Z})$. Once this representation is considered, there is a natural way to quantize polynomial functionals on $\mathscr{Z}$ into quadratic forms on $\Gamma_s(\mathscr{Z})$ according to the Wick or normal order (we briefly outline the essential features of Wick quantization in subsection 4.3; the reader may refer to [9, 22, 44] for a more detailed presentation).

Applying these rules, the formal quantization of the classical energy $\mathscr{E}$ yields a quadratic form h on the Fock space $\Gamma_s(L^2 \oplus L^2)$ which plays the role of a quantum energy. The difficulty now lies in the fact that the quadratic form h does not define straightforwardly a dynamical system (i.e., h may not define a self-adjoint operator). Nevertheless, according to the work of Nelson, it is possible to define a so-called renormalized self-adjoint operator H^{ren} associated in some specific sense to h. Let us briefly outline how (the reader can find a detailed derivation in section 2). Since the quadratic form h is ill-behaved for high momenta of the scalar field, it is customary to introduce a (smooth) ultraviolet cutoff χ_σ that cuts all fields' momenta of magnitude bigger than $\sigma \in \mathbb{R}_+$ off the interaction. The resulting quadratic form h_σ now takes the form $h_\sigma(\,\cdot\,,\,\cdot\,) = \langle\,\cdot\,, H_\sigma\,\cdot\,\rangle$, where H_σ is the self-adjoint and bounded from below operator on $\Gamma_s(L^2 \oplus L^2)$ defined by

$$H_\sigma = d\Gamma(-\tfrac{\Delta}{2M} + V) + d\Gamma(\omega) + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi^*(x)\Big(a^*\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big) + a\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big)\Big)\psi(x)dx.$$

In order to define the dynamics, one should find a way to let $\sigma \to \infty$ and still obtain a self-adjoint operator. This is done by means of a unitary transformation U_σ, $\sigma \le \infty$, called dressing transformation (see subsection 2.1) and subtracting a divergent constant from the Hamiltonian (energy renormalization). The action of the dressing transformation on H_σ singles out both the "natural" domain of the Hamiltonian and the divergent constant. In fact, define the operator $\hat{H}_\sigma = U_\sigma H_\sigma U_\sigma^* - \varepsilon N_1 E_\sigma$, where E_σ is the so-called particle's self-energy that diverges as $\sigma \to \infty$. The number $\varepsilon > 0$ is the semiclassical parameter, which will be introduced shortly, and which can be seen as a constant in this quantum setting. The dressing U_σ and therefore $\hat{H}_\sigma$ depend on the function $(1 - \chi_{\sigma_0})$, $\sigma_0 < \sigma$ arbitrary. Hence $(1 - \chi_{\sigma_0})$ acts as an effective cutoff from below on the momenta. The possibility of choosing σ_0 big enough plays a crucial role in proving self-adjointness in the limit $\sigma \to \infty$. How big σ_0 should be depends, however, on the number of nonrelativistic particles in a given state. Hence it is useful to exploit the fact that $\hat{H}_\sigma$ (and also H_σ) commute with the particle's number operator N_1, and could therefore be written

$$\hat{H}_\sigma = \bigoplus_{n\in\mathbb{N}} \hat{H}_\sigma^{(n)}\ ,$$

where each $\hat{H}_\sigma^{(n)}$ is a self-adjoint operator on the n-particle sector $\mathcal{H}_n$.

It can be proven (see Theorem 2.11), that for any $n \in \mathbb{N}$ there exists a $\sigma_0(n,\varepsilon)$ big enough[2] such that the quadratic form $\hat{h}_\sigma^{(n)}$, associated to $\hat{H}_\sigma^{(n)}$, is closed and bounded from below for any $\sigma \le \infty$ (in particular for $\sigma = \infty$) on the same dense form domain $D\big((H_0^{(n)})^{1/2}\big)$, and that $\hat{H}_\sigma^{(n)}$ converges to $\hat{H}_\infty^{(n)}$ in the norm resolvent

[2]Again the ε-dependence is not important for the present discussion, but we should keep track of it since it plays a prominent role in the classical limit.

sense as $\sigma \to \infty$. The Hamiltonian $\hat{H}_\infty^{(n)}$ is the dressed renormalized Hamiltonian with n fixed nonrelativistic particles. There are several possible ways to extend the Hamiltonian, and the dynamics generated by it, to the whole Fock space $\mathcal{H} = \Gamma_s(L^2 \oplus L^2)$. Motivated by the type of quantum states that we want to study in the classical limit, we choose to define the dynamics as $e^{-i\frac{t}{\varepsilon}\hat{H}_\infty^{(n)}}$ only up to a maximal number of nonrelativistic particles $\mathfrak{N}$, and for more particles we fix it to be trivial (i.e., generated by zero). Such $\mathfrak{N}$ is defined by inversion of the function $n \mapsto \sigma_0(n, \varepsilon)$: for any $\sigma_0 \in \mathbb{R}_+$, there exists an $\mathfrak{N}(\varepsilon, \sigma_0)$ such that $\hat{h}_\sigma^{(n)}$ is closed and bounded from below for any $n \leq \mathfrak{N}(\varepsilon, \sigma_0)$. The renormalized dressed Hamiltonian $\hat{H}_\varepsilon^{\mathrm{ren}}$ is then defined as

$$\hat{H}_\varepsilon^{\mathrm{ren}}\big|_{\mathcal{H}_n} = \begin{cases} \hat{H}_\infty^{(n)} & \text{if } n \leq \mathfrak{N}(\varepsilon, \sigma_0), \\ 0 & \text{if } n > \mathfrak{N}(\varepsilon, \sigma_0), \end{cases}$$

and it generates a nontrivial dynamics for any n-particle sector up to $\mathfrak{N}(\varepsilon, \sigma_0)$. The undressed renormalized Hamiltonian is accordingly defined as

$$H_\varepsilon^{\mathrm{ren}} = U_\infty^* \hat{H}_\varepsilon^{\mathrm{ren}}(\varepsilon) U_\infty \ .$$

The ε-dependence of the Hamiltonians has been emphasized for later convenience.

It should be apparent that the renormalization procedure above substantially obscures the relationship between the classical and the quantum theory. In particular it is unclear, even at the formal level, whether the quantum dynamics generated by $H_\varepsilon^{\mathrm{ren}}$ is still related to the original S-KG equation (S-KG) or not. Therefore, we believe that it is mathematically interesting to study Bohr's correspondence principle in the renormalized Nelson model.

> *Bohr's principle*: "The quantum system should reproduce, in the limit of large quantum numbers, the classical behavior."

This principle may be reformulated as follows. We make the quantization procedure dependent on the effective semiclassical parameter ε (already introduced above), which would converge to zero in the limit. The physical interpretation is that ε is a quantity of the same order of magnitude as the Planck constant, which becomes negligible when large energies and orbits are considered. In the Fock representation, we introduce the ε-dependence in the annihilation and creation operator valued distributions $\psi^\#(x)$ and $a^\#(k)$, whose commutation relations then become $[\psi(x), \psi^*(x')] = \varepsilon\delta(x - x')$ and $[a(k), a^*(k')] = \varepsilon\delta(k - k')$. If in the limit $\varepsilon \to 0$ the quantum unitary dynamics converges toward the Hamiltonian flow generated by the S-KG equation with Yukawa interaction, Bohr's principle is satisfied.

If the phase space $\mathcal{Z}$ is finite dimensional, the quantum-classical correspondence has been proved in the context of semiclassical or microlocal analysis, with the aid of pseudodifferential calculus, Wigner measures, or coherent states; see, e.g., [2, 39, 41, 65, 66, 80, 82, 84, 91, 93, 104]. If $\mathcal{Z}$ is infinite dimensional, the situation is more complicated, and there are fewer results for systems with an unconserved number of particles [5, 9, 13, 56, 57, 67]. The approach we adopt here makes use of the infinite dimensional Wigner measures introduced by [9, 10, 11, 12]. Note that Wigner measures are related to phase-space analysis and are in general an effective tool for the study of the classical limit. Let us consider a family of quantum states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ that are normal with respect to the Fock representation; i.e., each ϱ_ε is a positive and trace class operator on the Fock space, with trace one. Given such a family, we say that a Borel probability measure μ on $\mathcal{Z}$ is a Wigner measure associated to it if there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar{\varepsilon})$ such that $\varepsilon_k \to 0$ and[3]

$$\lim_{k\to\infty} \mathrm{Tr}[\varrho_{\varepsilon_k} W(\xi)] = \int_{\mathcal{Z}} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle_{\mathcal{Z}}} d\mu(z) \ \ \forall \xi \in \mathcal{Z} \,. \tag{1}$$

We denote by $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)$ the set of Wigner measures associated to $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$. Let $e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}$ be the quantum dynamics on $\Gamma_s(\mathcal{Z})$, $\mathcal{Z} = L^2 \oplus L^2$; then the time-evolved quantum states can be written as $(e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}})_{\varepsilon\in(0,\bar{\varepsilon})}$. Bohr's principle is satisfied if the Wigner measures of the time-evolved quantum states are exactly the push-forward, by the classical flow $\mathbf{E}(t)$, of the initial Wigner measures at time $t = 0$; i.e.,

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}, \varepsilon \in (0,\bar{\varepsilon})\Big) = \Big\{\mathbf{E}(t)_{\#}\mu \,,\, \mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon}))\Big\} \,. \tag{2}$$

To ensure that $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)$ is not empty, it is sufficient to assume that there exist $\delta > 0$ and $C > 0$ such that, for any $\varepsilon \in (0,\bar{\varepsilon})$, $\mathrm{Tr}[\varrho_\varepsilon N^\delta] < C$, where N is the number operator of the Fock space $\Gamma_s(\mathcal{Z})$ with $\mathcal{Z} = L^2 \oplus L^2$. Actually, we make the following more restrictive assumptions: Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\Gamma_s(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3))$; then

$$\exists \mathfrak{C} > 0 \,,\, \forall \varepsilon \in (0,\bar{\varepsilon}) \,,\, \forall k \in \mathbb{N} \,,\, \mathrm{Tr}[\varrho_\varepsilon N_1^k] \leq \mathfrak{C}^k \,, \tag{A-n}$$

$$\exists C > 0 \,,\, \forall \varepsilon \in (0,\bar{\varepsilon}) \,,\, \mathrm{Tr}[\varrho_\varepsilon (N + U_\infty^* H_0 U_\infty)] \leq C \,, \tag{A-h}$$

where N_1 is the nucleonic number operator, $N = N_1 + N_2$ is the total number operator, H_0 is the free Hamiltonian defined by (5), and U_∞ is the unitary quantum dressing defined in Lemma 2.3. Therefore, under these assumptions, the set of Wigner measures associated to $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is not empty, as proved in Lemma 4.12. As a matter of fact, it could even be possible to remove assumption (A-n), but it has an important role in connection with the parameter σ_0 related to the renormalization procedure. This condition restricts the considered states ϱ_ε to be at most with $[\mathfrak{C}/\varepsilon]$ nucleons.

We are now in a position to state precisely our result: *the Bohr's correspondence principle holds between the renormalized quantum dynamics of the Nelson model generated by $H_\varepsilon^{\mathrm{ren}}$ and the S-KG classical flow generated by $\mathscr{E}$.* The operator $H_\varepsilon^{\mathrm{ren}}$ is constructed in subsection 2.3 according to Definition 2.14. The Hilbert space $\mathcal{D}$ of global well-posedness for (S-KG) is, explicitly, $\mathcal{D} = Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$, with the form domain endowed with the graph norm.

THEOREM 1.1. *Let $\mathbf{E} : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ be the S-KG flow provided by Theorem* 3.15 *and solving the equation* (S-KG) *with a potential V satisfying assumption* (A-V). *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states in $\Gamma_s\big(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\big)$ that satisfies assumptions* (A-n) *and* (A-h). *Then, the following hold:*

(i) *There exists a $\sigma_0 \in \mathbb{R}_+$ such that the dynamics $e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}$ is nontrivial on the states ϱ_ε.*

(ii) $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big) \neq \emptyset$.

(iii) *For any $t \in \mathbb{R}$,*

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}, \varepsilon \in (0,\bar{\varepsilon})\Big) = \Big\{\mathbf{E}(t)_{\#}\mu \,,\, \mu \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)\Big\} \,. \tag{3}$$

[3] $W(\xi)$ is the ε_k-dependent Weyl operator explicitly defined by (65).

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar{\varepsilon})$ *be a sequence such that* $\lim_{k\to\infty}\varepsilon_k = 0$ *and* $\mathcal{M}\big(\varrho_{\varepsilon_k}, k\in\mathbb{N}\big) = \{\mu\}$*; i.e., for any* $\xi \in L^2\oplus L^2$,

$$\lim_{k\to\infty} \mathrm{Tr}[\varrho_{\varepsilon_k} W(\xi)] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle} d\mu(z);$$

then for any $t\in\mathbb{R}$, $\mathcal{M}\big(e^{-i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}, k\in\mathbb{N}\big) = \{\mathbf{E}(t)_{\#}\mu\}$*; i.e., for any* $\xi\in L^2\oplus L^2$,

$$\lim_{k\to\infty} \mathrm{Tr}\Big[e^{-i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}} W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle} d(\mathbf{E}(t)_{\#}\mu)(z)\,. \tag{4}$$

Remark 1.2.

* The choice of σ_0 is related to our Definition 2.14 of the renormalized dynamics and the localization of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfying assumption (A-n) (see Lemma 4.2). Actually, one can take any $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ where $K>0$ is a constant given in Theorem 2.11.
* We remark that every Wigner measure $\mu\in\mathcal{M}(\varrho_\varepsilon, \varepsilon\in(0,\bar{\varepsilon}))$, with $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfying assumption (A-h), is a Borel probability measure on $\mathcal{D}$ equipped with its graph norm; hence the push-forward by means of the classical flow $\mathbf{E}$ is well defined (see subsection 4.4).
* Adopting a shorthand notation, the last assertion of the above theorem can be written as

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t\in\mathbb{R}\,,\ e^{-i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}} \to \mathbf{E}(t)_{\#}\mu\right).$$

2. The quantum system: Nelson Hamiltonian. In this section we define the quantum system of "nucleons" interacting with a meson field and briefly review the standard renormalization procedure due to [97]. Since we are interested in the classical limit and our original and dressed Hamiltonians depend on an effective parameter $\varepsilon\in(0,\bar{\varepsilon})$, it is necessary to check that several known estimates of the quantum theory are uniform with respect to ε. This step is crucial and motivates this brief revisitation of the Nelson renormalization procedure.

On $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3))\otimes\Gamma_s(L^2(\mathbb{R}^3))$ we define the following free Hamiltonian as the positive self-adjoint operator given by

$$\begin{aligned} H_0 &= \int_{\mathbb{R}^3} \psi^*(x)\Big(-\tfrac{\Delta}{2M}+V(x)\Big)\psi(x)dx + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k)dk \\ &= d\Gamma(-\tfrac{\Delta}{2M}+V) + d\Gamma(\omega), \end{aligned} \tag{5}$$

where $M>0$, $V\in L^2_{loc}(\mathbb{R}^3,\mathbb{R}_+)$, and $\omega(k)=\sqrt{k^2+m_0^2}$, $m_0>0$. We denote its domain of self-adjointness by $D(H_0)$. We denote by $d\Gamma$ the second quantization acting either on the first or second Fock space, when no confusion arises.

Now let $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^3)$, $0\leq\chi\leq 1$, and $\chi\equiv 1$ if $|k|\leq 1$, and $\chi\equiv 0$ if $|k|\geq 2$. Then, for all $\sigma>0$ define $\chi_\sigma(k)=\chi(k/\sigma)$; it will play the role of an ultraviolet cutoff in the interaction. The Nelson Hamiltonian with cutoff thus has the form

$$H_\sigma = H_0 + \tfrac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\psi^*(x)\Big(a^*\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big) + a\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big)\Big)\psi(x)dx\,. \tag{6}$$

We will denote the interaction part by $H_I(\sigma) = H_\sigma - H_0$.

Remark 2.1. There is no loss of generality in the choice of χ as a radial function; see [4, Proposition 3.9].

The following proposition shows the self-adjointness of H_σ; see, e.g., [5, Proposition 2.5] or [52].

PROPOSITION 2.2. *For any $\sigma > 0$, H_σ is essentially self-adjoint on $D(H_0) \cap \mathcal{C}_0^\infty(N)$.*

To obtain a meaningful limit when $\sigma \to \infty$, we use a dressing transformation, introduced in the physics literature by [74] following the work of van Hove [112, 113]. The dressing and the renormalization procedures are described in subsections 2.1 and 2.2, respectively. In subsection 2.3 we discuss a possible extension of the renormalized Hamiltonian on $\mathcal{H}_n$ to the whole Fock space $\mathcal{H}$. The extension we choose is not the only possible one; however, the choice is motivated by two facts: other extensions should provide the same classical limit, and our choice $\hat{H}_\varepsilon^{\mathrm{ren}}$ is, in our opinion, more consistent with the quantization procedure of the classical energy functional.

2.1. Dressing. The dressing transform was introduced as an alternative way of doing renormalization in the Hamiltonian formalism and has been utilized in a rigorous fashion in various situations; see, e.g., [68, 71, 81, 97]. For the Nelson Hamiltonian, it consists of a unitary transformation that singles out the singular self-energy.

From now on, let $0 < \sigma_0 < \sigma$, with σ_0 fixed. Then define

$$g_\sigma(k) = -\frac{i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \frac{\chi_\sigma(k) - \chi_{\sigma_0}(k)}{\frac{k^2}{2M} + \omega(k)} \,, \tag{7}$$

$$E_\sigma = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\omega(k)} \frac{(\chi_\sigma - \chi_{\sigma_0})^2(k)}{\frac{k^2}{2M} + \omega(k)} dk - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\chi_\sigma(k)}{\omega(k)} \frac{(\chi_\sigma - \chi_{\sigma_0})(k)}{\frac{k^2}{2M} + \omega(k)} dk \,. \tag{8}$$

The dressing transformation is the unitary operator generated by (the dependence on σ_0 will be usually omitted)

$$T_\sigma = \int_{\mathbb{R}^3} \psi^*(x) \Big(a^*(g_\sigma e^{-ik\cdot x}) + a(g_\sigma e^{-ik\cdot x}) \Big) \psi(x) dx \,. \tag{9}$$

The function $g_\sigma \in L^2(\mathbb{R}^3)$ for all $\sigma \leq \infty$; therefore, it is possible to prove the following lemma, e.g., utilizing the criterion of [52].

LEMMA 2.3. *For any $\sigma \leq \infty$, T_σ is essentially self-adjoint on $\mathcal{C}_0^\infty(N)$. We denote by $U_\sigma(\theta)$ the corresponding one-parameter unitary group $U_\sigma(\theta) = e^{-i\frac{\theta}{\varepsilon}T_\sigma}$.*

For the sake of brevity, we will write $U_\sigma := U_\sigma(-1)$. We remark that T_σ and H_σ preserve the number of "nucleons"; i.e., for any $\sigma \leq \infty$, $\sigma' < \infty$,

$$[T_\sigma, N_1] = 0 = [H_{\sigma'}, N_1] \,. \tag{10}$$

The above operators also commute in the resolvent sense. We are now in a position to define the dressed Hamiltonian

$$\hat{H}_\sigma := U_\sigma \big(H_\sigma - \varepsilon N_1 E_\sigma \big) U_\sigma^* \,. \tag{11}$$

The operator $\hat{H}_\sigma$ is self-adjoint for any $\sigma < \infty$, since H_σ and N_1 are commuting self-adjoint operators and U_σ is unitary. The purpose is to show that the quadratic form associated with $\hat{H}_\sigma\big|_{\mathcal{H}_n}$ satisfies the hypotheses of the Kato–Lax–Milgram–Nelson theorem (KLMN theorem), even when $\sigma = \infty$, so it is possible to define uniquely a self-adjoint operator $\hat{H}_\infty$. In order to do that, we need to study in detail the form associated with $\hat{H}_\sigma^{(n)}$. For the sake of completeness, let us recall the aforementioned KLMN theorem; see also [102, Theorem X.17].

THEOREM 2.4 (KLMN). *Let T_0 be a positive self-adjoint operator on a Hilbert space $\mathscr{K}$ and $q_0 : Q(T_0) \to \mathbb{R}_+$ be its associated quadratic form, defined on the dense domain $Q(T_0) \subset \mathscr{K}$. Let $q : Q(T_0) \to \mathbb{R}$ be a symmetric quadratic form such that there exist $0 \leq a < 1$ and $b \in \mathbb{R}_+$ such that*

$$\forall \psi \in Q(T_0)\ ,\ |q(\psi)| \leq a\, q_0(\psi) + b\, \langle \psi, \psi \rangle_{\mathscr{K}}\ .$$

Then there exist a unique self-adjoint operator T with domain $D(T) \subset Q(T_0)$, associated quadratic form q_T, and form domain $Q(T) = Q(T_0)$, such that

$$\forall \psi \in Q(T_0)\ ,\ q_T(\psi) = q_0(\psi) + q(\psi)\ ,$$

T is bounded from below by $-b$, and any domain of essential self-adjointness of T_0 is a core for q_T.

By (11), it follows immediately that

$$\hat{H}_\sigma^{(n)} = \varepsilon U_\sigma^{(n)} \left(\frac{H_\sigma^{(n)}}{\varepsilon} - (\varepsilon n) E_\sigma \right) (U_\sigma^{(n)})^*\ . \tag{12}$$

A suitable calculation yields (see, e.g., [4, 97, equations (15)–(20) of the second reference]

$$\begin{aligned} \hat{H}_\sigma^{(n)} = H_{\sigma_0}^{(n)} + \varepsilon^2 \sum_{i<j} V_\sigma(x_i - x_j) + \frac{\varepsilon}{2M} \sum_{j=1}^{n} \Bigg(\Big(a^*(r_\sigma e^{-ik\cdot x_j})^2 + a(r_\sigma e^{-ik\cdot x_j})^2 \Big) \\ + 2a^*(r_\sigma e^{-ik\cdot x_j}) a(r_\sigma e^{-ik\cdot x_j}) - 2\Big(D_{x_j} a(r_\sigma e^{-ik\cdot x_j}) + a^*(r_\sigma e^{-ik\cdot x_j}) D_{x_j} \Big) \Bigg), \end{aligned} \tag{13}$$

where $D_{x_j} = -i\nabla_{x_j}$ and

$$r_\sigma(k) = -ikg_\sigma(k)\,,$$

$$V_\sigma(x) = 2\mathrm{Re} \int_{\mathbb{R}^3} \omega(k) |g_\sigma(k)|^2 e^{-ik\cdot x} dk - 4\mathrm{Im} \int_{\mathbb{R}^3} \frac{\bar{g}_\sigma(k)}{(2\pi)^{3/2}} \frac{\chi_\sigma(k)}{\sqrt{2\omega(k)}} e^{-ik\cdot x} dk\ . \tag{14}$$

It is also possible to write $\hat{H}_\sigma$ in its second quantized form as

$$\hat{H}_\sigma = H_0 + \hat{H}_I(\sigma)\ , \tag{15}$$

$$\begin{aligned} \hat{H}_I(\sigma) = H_I(\sigma_0) + \frac{1}{2} \int_{\mathbb{R}^6} \psi^*(x)\psi^*(y) V_\sigma(x-y) \psi(x)\psi(y) dx dy \\ + \tfrac{1}{2M} \int_{\mathbb{R}^3} \psi^*(x) \Bigg(\Big(a^*(r_\sigma e^{-ik\cdot x})^2 + a(r_\sigma e^{-ik\cdot x})^2 \Big) + 2a^*(r_\sigma e^{-ik\cdot x}) a(r_\sigma e^{-ik\cdot x}) \\ -2\Big(D_x a(r_\sigma e^{-ik\cdot x}) + a^*(r_\sigma e^{-ik\cdot x}) D_x \Big) \Bigg) \psi(x) dx\ . \end{aligned} \tag{16}$$

Remark 2.5. The dressed interaction Hamiltonian $\hat{H}_I(\sigma)$ contains a first term analogous to the undressed interaction with cutoff, a second term of two-body interaction between nucleons, and a more singular term that can be only defined as a form when $\sigma = \infty$.

2.2. Renormalization. We will now define the renormalized self-adjoint operator $\hat{H}_\infty^{(n)}$. A simple calculation shows that $E_\sigma \to -\infty$ when $\sigma \to +\infty$; hence the subtraction of the self-energy in the definition (11) of $\hat{H}_\sigma$ is necessary. It is actually the only renormalization necessary for this system. We prove that the quadratic form associated with $\hat{H}_\sigma^{(n)}$ of (13) has meaning for any $\sigma \leq \infty$, and the KLMN theorem can be applied, with a suitable choice of σ_0, and bounds that are uniform with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Let us start with some preparatory lemmas.

LEMMA 2.6. *For any* $0 \leq \sigma \leq \infty$, *the symmetric function* V_σ *satisfies*

(i) $V_\sigma (1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^3))$,

(ii) $(1-\Delta)^{-1/2} V_\sigma (1-\Delta)^{-1/2} \in \mathcal{K}(L^2(\mathbb{R}^3))$.

In particular, $V_\sigma \in L^s(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)$ *for any* $s \in [2, +\infty[$.

Proof. It is sufficient to show [12, Corollary D.6] that $V_\sigma \in L^{3,\infty}(\mathbb{R}^3)$ (weak-L^p spaces). Write $V_\sigma = V_\sigma^{(1)} + V_\sigma^{(2)}$,

$$V_\sigma^{(1)}(x) = 2\mathrm{Re} \int_{\mathbb{R}^3} \omega(k) |g_\sigma(k)|^2 e^{-ik\cdot x} dk = 2(2\pi)^{3/2} \mathrm{Re} \mathcal{F}\Big(\omega |g_\sigma|^2\Big)(x) \,, \tag{17}$$

$$V_\sigma^{(2)}(x) = -2\sqrt{2} \mathrm{Im} \int_{\mathbb{R}^3} \frac{\bar{g}_\sigma(k)}{(2\pi)^{3/2}} \frac{\chi_\sigma(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x} dk = -2\sqrt{2} \mathrm{Im} \mathcal{F}\Big(\bar{g}_\sigma \frac{\chi_\sigma}{\sqrt{\omega}}\Big)(x) \,. \tag{18}$$

- $\big[V_\sigma^{(1)}\big]$. For any $\sigma \leq \infty$, $\omega |g_\sigma|^2 \in L^{s'}(\mathbb{R}^3)$, $1 \leq s' \leq 2$. Then $V_\sigma^{(1)} \in L^s(\mathbb{R}^3)$ for any $s \in [2, +\infty]$; furthermore, $V_\sigma^{(1)} \in \mathcal{C}_0(\mathbb{R}^3)$ (the space of continuous functions converging to zero at infinity). Hence $V_\sigma^{(1)} \in L^{3,\infty}(\mathbb{R}^3)$.
- $\big[V_\sigma^{(2)}\big]$. For any $\sigma \leq \infty$, $\bar{g}_\sigma \frac{\chi_\sigma}{\sqrt{\omega}} \in L^{s'}(\mathbb{R}^3)$, $1 < s' \leq 2$. Therefore, $V_\sigma^{(2)} \in L^s(\mathbb{R}^3)$ for any $s \in [2, +\infty[$. It remains to show that $V_\sigma^{(2)} \in L^{3,\infty}(\mathbb{R}^3)$. Define $f(k) \in L^2(\mathbb{R}^3)$:

$$f(k) := \frac{\chi_\sigma(k)}{\omega(k)} \frac{\big(\chi_\sigma - \chi_{\sigma_0}\big)(k)}{\frac{k^2}{2M} + \omega(k)} \,. \tag{19}$$

Then there is a constant $c > 0$ such that $|V_\sigma^{(2)}(x)| \leq c |\mathcal{F}(f)(x)|$, where the Fourier transform is intended to be on $L^2(\mathbb{R}^3)$. The function f is radial, so we introduce the spherical coordinates $(r, \theta, \phi) \equiv k \in \mathbb{R}^3$, such that the z-axis coincides with the vector x. We then obtain

$$\lim_{R\to+\infty} \int_{B(0,R)} f(k) e^{-ik\cdot x} dk = \lim_{R\to+\infty} \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \, r^2 f(r),$$

$$e^{-ir|x|\cos\theta} \sin\theta = 2\pi \lim_{R\to+\infty} \int_0^R dr \int_{-1}^1 dy \, r^2 f(r) e^{-ir|x|y}$$

$$= \frac{4\pi}{|x|} \lim_{R\to+\infty} \int_0^R f(r) r \sin(r|x|) dr \,.$$

Since for any $\sigma \leq +\infty$, $f(r)r \in L^1(\mathbb{R})$, we can take the limit $R \to +\infty$ and conclude that

$$\mathcal{F}(f)(x) = \frac{4\pi}{|x|} \int_0^{+\infty} f(r) r \sin(r|x|) dr \,. \tag{20}$$

Therefore, for any $x \in \mathbb{R}^3 \setminus \{0\}$, there exists a $0 < \tilde{c} \leq 4\pi c \|f(r)r\|_{L^1(\mathbb{R})}$ such that

$$|V_\sigma^{(2)}(x)| \leq \frac{\tilde{c}}{|x|} \,. \tag{21}$$

Let λ be the Lebesgue measure in $\mathbb{R}^3$. Since $\{x : |V_\sigma^{(2)}| > t\} \subset \{x : \frac{\tilde{c}}{|x|} > t\}$, there is a positive C such that

$$\lambda\Big\{x : |V_\sigma^{(2)}(x)| > t\Big\} \leq \lambda\Big\{x : \frac{\tilde{c}}{|x|} > t\Big\} \leq \frac{C}{t^3} \,. \tag{22}$$

Finally, (22) implies $V_\sigma^{(2)} \in L^{3,\infty}(\mathbb{R}^3)$. ∎

LEMMA 2.7. *There exists $c > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma \leq +\infty$,*

$$\Big\| \big[(H_0+1)^{-1/2} D_{x_j} a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq \frac{c}{\sqrt{n\varepsilon}} \|\omega^{-1/2} r_\sigma\|_2 \,, \tag{23}$$

$$\Big\| \big[(H_0+1)^{-1/2} a^*(r_\sigma e^{-ik\cdot x_j}) D_{x_j}(H_0+1)^{-1/2}\big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq \frac{c}{\sqrt{n\varepsilon}} \|\omega^{-1/2} r_\sigma\|_2 \,. \tag{24}$$

Moreover, (23) *holds if we replace the left H_0 by $d\Gamma(-\frac{\Delta}{2M} + V)$ and the right H_0 by $d\Gamma(\omega)$, and similarly* (24) *holds if we replace the left H_0 by $d\Gamma(\omega)$ and the right H_0 by $d\Gamma(-\frac{\Delta}{2M} + V)$.*

Proof. Let $S_n \equiv S_n \otimes 1$ be the symmetrizer on $\mathcal{H}_n$ (acting only on the $\{x_1, \ldots, x_n\}$ variables) and $\Psi_n \in \mathcal{H}_n$ with $n > 0$. Then

$$\langle \Psi_n, d\Gamma(-\Delta)\Psi_n \rangle = \langle \Psi_n, (n\varepsilon) S_n (D_{x_1})^2 \otimes 1^{n-1} \Psi_n \rangle = (n\varepsilon) \langle \Psi_n, (D_{x_j})^2 \Psi_n \rangle \,.$$

Hence $(n\varepsilon)\|D_{x_j}\Psi_n\|^2 \leq \big\|\big(d\Gamma(-\Delta)+1\big)^{1/2}\Psi_n\big\|^2$. It follows that

$$\begin{aligned} &\Big\| \big[D_{x_j}\big(d\Gamma(-\Delta)+1\big)^{-1/2}\big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq \frac{1}{\sqrt{n\varepsilon}} \,, \\ &\Big\| \big[\big(d\Gamma(-\Delta)+1\big)^{-1/2} D_{x_j}\big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq \frac{1}{\sqrt{n\varepsilon}} \,. \end{aligned} \tag{25}$$

Using (25), we obtain for any $\Psi_n \in \mathcal{H}_n$, with $\|\Psi_n\| = 1$,

$$\begin{aligned} &\Big\| (H_0+1)^{-1/2} D_{x_j} a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\Psi_n \Big\| \\ &\qquad \leq \frac{c}{\sqrt{n\varepsilon}} \Big\| a(r_\sigma e^{-ik\cdot x_j})\big(d\Gamma(\omega)+1\big)^{-1/2}\Psi_n \Big\| \\ &\qquad \leq \frac{c}{\sqrt{n\varepsilon}} \|\omega^{-1/2} r_\sigma\|_2 \,, \end{aligned}$$

where the last inequality follows from standard estimates on the Fock space; see [5, Lemma 2.1]. The bound (24) is obtained by adjunction. ∎

LEMMA 2.8. *There exists $c > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma \leq +\infty$,*

$$\Big\| \big[(H_0+1)^{-1/2} a^*(r_\sigma e^{-ik\cdot x_j}) a(r_\sigma e^{-ik\cdot x_j}) (H_0+1)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c \|\omega^{-1/2} r_\sigma\|_2^2 \,, \tag{26}$$

$$\Big\| \big[(H_0+1)^{-1/2} \big(a^*(r_\sigma e^{-ik\cdot x_j}) \big)^2 (H_0+1)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c \|\omega^{-1/4} r_\sigma\|_2^2 \,, \tag{27}$$

$$\Big\| \big[(H_0+1)^{-1/2} \big(a(r_\sigma e^{-ik\cdot x_j}) \big)^2 (H_0+1)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c \|\omega^{-1/4} r_\sigma\|_2^2 \,. \tag{28}$$

The same bounds hold if H_0 is replaced by $d\Gamma(\omega)$.

Proof. First, observe that, since $m_0 > 0$, there exists $c > 0$ such that, uniformly in $\varepsilon \in (0, \bar{\varepsilon})$,

$$\Big\| (H_0+1)^{-1/2} \big(d\Gamma(\omega) + 1 \big)^{1/2} \Big\|_{\mathcal{L}(\mathcal{H})} \leq c \,, \; \Big\| (H_0+1)^{-1/2} (N_2+1)^{1/2} \Big\|_{\mathcal{L}(\mathcal{H})} \leq c \,.$$

Inequality (26) is easy to prove:

$$\begin{aligned} &\Big\| \big[(H_0+1)^{-1/2} a^*(r_\sigma e^{-ik\cdot x_j}) a(r_\sigma e^{-ik\cdot x_j}) (H_0+1)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \\ &\qquad \leq c \Big\| \big[\big(d\Gamma(\omega)+1 \big)^{-1/2} a^*(r_\sigma e^{-ik\cdot x_j}) \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \\ &\qquad\quad \cdot \Big\| \big[a(r_\sigma e^{-ik\cdot x_j}) \big(d\Gamma(\omega)+1 \big)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c \|\omega^{-1/2} r_\sigma\|_2^2 \,. \end{aligned}$$

For the proof of (27) the reader may refer to [4, Lemma 3.3 (iv)]. Finally, (28) follows from (27) by adjunction. ∎

LEMMA 2.9. *There exists $c(\sigma_0) > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and $\lambda \geq 1$,*

$$\Big\| \big[\big(H_0+\lambda \big)^{-1/2} H_I(\sigma_0) \big(H_0+\lambda \big)^{-1/2} \big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c(\sigma_0) \lambda^{-1/2} (n\varepsilon) \,, \tag{29}$$

$$\Big\| \Big[\big(H_0+\lambda \big)^{-1/2} \varepsilon^2 \sum_{i<j} V_\sigma(x_i - x_j) \big(H_0+\lambda \big)^{-1/2} \Big]^{(n)} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \leq c(\sigma_0) \lambda^{-1/2} \sqrt{n\varepsilon(1+n\varepsilon)} \,. \tag{30}$$

Proof. The inequality (29) can be proved by a standard argument on the Fock space; see, e.g., [51, Proposition IV.1].

To prove (30) we proceed as follows. First, by means of (i) of Lemma 2.6, we can write

$$\begin{aligned} &\Big\| \big(-\Delta_{x_i} + \lambda \big)^{-1/2} V_\sigma(x_i - x_j) \big(-\Delta_{x_i} + \lambda \big)^{-1/2} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \\ &\qquad \leq \lambda^{-1/2} \Big\| V_\sigma(x_i) \big(-\Delta_{x_i} + \lambda \big)^{-1/2} \Big\|_{\mathcal{L}(\mathcal{H}_n)} \\ &\qquad \leq c(\sigma_0) \lambda^{-1/2} \,. \end{aligned}$$

Therefore, $V_\sigma(x_i - x_j) \leq c(\sigma_0) \lambda^{-1/2} \big(-\Delta_{x_i} + \lambda \big)$. Let $\Psi_n \in \mathcal{H}_n$; using its symmetry and some algebraic manipulations, we can write

$$\begin{aligned}\Big\langle \Psi_n, \varepsilon^2 \sum_{i<j} V_\sigma(x_i - x_j)\Psi_n\Big\rangle &\leq c(\sigma_0)(n\varepsilon)^2\big\langle \Psi_n, \big(\lambda^{-1/2}(D_{x_1})^2 + \lambda^{1/2}\big)\Psi_n\big\rangle \\ &= c(\sigma_0)\big\langle \Psi_n, N_1\big(\lambda^{-1/2} d\Gamma(D_x^2) + \lambda^{1/2}N_1\big)\Psi_n\big\rangle \\ \leq c(\sigma_0)\lambda^{-1/2}\Big[\Big\| N_1^{1/2}\big(d\Gamma(D_x^2)+\lambda\big)^{1/2}\Psi_n\Big\|^2 &+ \Big\| N_1\big(d\Gamma(D_x^2)+\lambda\big)^{1/2}\Psi_n\Big\|^2\Big] \\ &\leq c(\sigma_0)\lambda^{-1/2}\big\langle \Psi_n, \big(N_1 + N_1^2\big)\big(d\Gamma(D_x^2)+\lambda\big)\Psi_n\big\rangle \ ,\end{aligned}$$

where the constant $c(\sigma_0)$ is redefined in each inequality. The result follows since N_1 commutes with $d\Gamma(D_x^2)$. ∎

Combining Lemmas 2.7 to 2.9 together, we can prove easily the following proposition.

PROPOSITION 2.10. *There exist $c > 0$ and $c(\sigma_0) > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\lambda \geq 1$, $\sigma_0 < \sigma \leq +\infty$, and for any $\Psi \in D(N_1)$,*

$$\begin{aligned}\Big\| \big(H_0+\lambda\big)^{-1/2} \hat{H}_I(\sigma)\big(H_0+\lambda\big)^{-1/2}\Psi\Big\| \leq \Big[c\big(\|\omega^{-1/2}r_\sigma\|_2^2 \\ + \|\omega^{-1/4}r_\sigma\|_2^2 + \|\omega^{-1/2}r_\sigma\|_2\big) + c(\sigma_0)\lambda^{-1/2}\Big] \Big\|(N_1+1)\Psi\Big\| \ .\end{aligned} \tag{31}$$

Consider now $\hat{H}_I(\sigma)^{(n)}$. It follows easily from (31) above that for any $\sigma_0 < \sigma \leq +\infty$ and $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$,

$$\begin{aligned}\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq \Big[c(n\varepsilon+1)\big(\|\omega^{-1/2}r_\sigma\|_2^2 + \|\omega^{-1/4}r_\sigma\|_2^2 + \|\omega^{-1/2}r_\sigma\|_2\big) \\ + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle \\ +\lambda\Big[c(n\varepsilon+1)\big(\|\omega^{-1/2}r_\sigma\|_2^2 + \|\omega^{-1/4}r_\sigma\|_2^2 + \|\omega^{-1/2}r_\sigma\|_2\big) \\ + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, \Psi_n\big\rangle \ .\end{aligned} \tag{32}$$

Consider now the term $\big(\|\omega^{-1/2}r_\sigma\|_2^2 + \|\omega^{-1/4}r_\sigma\|_2^2 + \|\omega^{-1/2}r_\sigma\|_2\big)$; by definition of r_σ, there exists $c > 0$ such that, uniformly in $\sigma \leq +\infty$,

$$\|\omega^{-1/2}r_\sigma\|_2^2 + \|\omega^{-1/4}r_\sigma\|_2^2 + \|\omega^{-1/2}r_\sigma\|_2 \leq c\big(\sigma_0^{-2} + \sigma_0^{-1}\big) \ . \tag{33}$$

Hence for any $\sigma_0 \geq 1$ there exist $K > 0$ ($K = 2c$), $c(\sigma_0) > 0$, and $C(n, \varepsilon, \lambda, \sigma_0) > 0$ such that (32) becomes

$$\begin{aligned}\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq \Big[\tfrac{K(n\varepsilon+1)}{\sigma_0} + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle \\ + C(n,\varepsilon,\lambda,\sigma_0)\big\langle \Psi_n, \Psi_n\big\rangle \ .\end{aligned} \tag{34}$$

Therefore, choosing

$$\sigma_0 > 2K(n\varepsilon+1) \tag{35}$$

and then $\lambda > \big(2c(\sigma_0)(n\varepsilon+1)\big)^2$, we obtain the following bound for any $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$, with $a < 1$, $b > 0$, and uniformly in $\sigma_0 < \sigma \leq +\infty$:

$$\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq a\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle + b\big\langle \Psi_n, \Psi_n\big\rangle \ . \tag{36}$$

Applying the KLMN theorem, (36) proves the following result; see, e.g., [4, 97] for additional details.

THEOREM 2.11. *There exists $K > 0$ such that, for any $n \in \mathbb{N}$ and $\varepsilon \in (0, \bar{\varepsilon})$ the following statements hold:*

(i) *For any $\big(2K(n\varepsilon + 1)\big) < \sigma_0 < \sigma \leq +\infty$, there exists a unique self-adjoint operator $\hat{H}_\sigma^{(n)}$ with domain $\hat{D}_\sigma^{(n)} \subset D\big((H_0^{(n)})^{1/2}\big) \subset \mathcal{H}_n$ associated to the symmetric form $\hat{h}_\sigma^{(n)}(\cdot,\cdot)$, defined for any $\Psi, \Phi \in D\big((H_0^{(n)})^{1/2}\big)$ as*

$$\hat{h}_\sigma^{(n)}(\Psi, \Phi) = \big\langle \Psi, H_0^{(n)} \Phi \big\rangle + \big\langle \Psi, \hat{H}_I(\sigma)^{(n)} \Phi \big\rangle \ . \tag{37}$$

The operator $\hat{H}_\sigma^{(n)}$ is bounded from below, with bound $-b_{\sigma_0}(\sigma)$ (where $|b_{\sigma_0}(\sigma)|$ is a bounded increasing function of σ).

(ii) *The following convergence holds in the norm topology of $\mathcal{L}(\mathcal{H}_n)$:*

$$\lim_{\sigma \to +\infty} (z - \hat{H}_\sigma^{(n)})^{-1} = (z - \hat{H}_\infty^{(n)})^{-1} \quad \forall\, z \in \mathbb{C} \smallsetminus \mathbb{R}. \tag{38}$$

(iii) *For any $t \in \mathbb{R}$, the following convergence holds in the strong topology of $\mathcal{L}(\mathcal{H}_n)$:*

$$s - \lim_{\sigma \to +\infty} e^{-i\frac{t}{\varepsilon}\hat{H}_\sigma^{(n)}} = e^{-i\frac{t}{\varepsilon}\hat{H}_\infty^{(n)}} \ . \tag{39}$$

Remark 2.12. The operator $\hat{H}_\infty^{(n)}$ can be decomposed only in the sense of forms, i.e.,

$$\hat{H}_\infty^{(n)} = H_0^{(n)} \dotplus \hat{H}_I^{(n)}(\infty) \ , \tag{40}$$

where $\dotplus$ has to be intended as the form sum.

2.3. Extension of $\hat{H}_\infty^{(n)}$ to $\mathcal{H}$. We have defined the self-adjoint operator $\hat{H}_\infty^{(n)}$ which depends on σ_0 for each $n \in \mathbb{N}$. Now we are interested in extending it to the whole space $\mathcal{H}$. This can be done in at least two different ways. However, we choose the one that is more suitable for interpreting $\hat{H}_\infty$ as the Wick quantization of a classical symbol.

Let K be defined by Theorem 2.11. Then define $\mathfrak{N}(\varepsilon, \sigma_0) \in \mathbb{N}$ by

$$\mathfrak{N}(\varepsilon, \sigma_0) = \Big[\frac{\sigma_0 - 2K}{2K\varepsilon} - 1\Big] \ , \tag{41}$$

where the square brackets mean that we take the integer part if the number within is positive, and zero otherwise.

DEFINITION 2.13 ($\hat{H}_\varepsilon^{\mathrm{ren}}$). *Let $0 \leq \sigma_0 < +\infty$ be fixed. Then we define $\hat{H}_\varepsilon^{\mathrm{ren}}$ on $\mathcal{H}$ by*

$$\hat{H}_\varepsilon^{\mathrm{ren}}\big|_{\mathcal{H}_n} = \begin{cases} \hat{H}_\infty^{(n)} & \text{if } n \leq \mathfrak{N}(\varepsilon, \sigma_0), \\ 0 & \text{if } n > \mathfrak{N}(\varepsilon, \sigma_0), \end{cases} \tag{42}$$

where $\mathfrak{N}(\varepsilon, \sigma_0)$ is defined by (41). We may also write $\hat{H}_\varepsilon^{\mathrm{ren}} = H_0 \dotplus \hat{H}_I^{\mathrm{ren}}$ as a sum of quadratic forms.

The operator $\hat{H}_\varepsilon^{\mathrm{ren}}$ is self-adjoint on $\mathcal{H}$, with a domain of self adjointness:

$$\hat{D}_{\mathrm{ren}}(\varepsilon,\sigma_0)=\Big\{\Psi\in\mathcal{H}\ ,\ \Psi\big|_{\mathcal{H}_n}\in\hat{D}_\infty^{(n)}\text{ for any }n\leq\mathfrak{N}(\varepsilon,\sigma_0)\Big\}\ . \tag{43}$$

Acting with the dressing operator U_∞ defined in Lemma 2.3 (with the same fixed σ_0 as for $\hat{H}_\varepsilon^{\mathrm{ren}}$), we can also define the undressed extension $H_\varepsilon^{\mathrm{ren}}$.

DEFINITION 2.14 ($H_\varepsilon^{\mathrm{ren}}$). *Let $0\leq\sigma_0<+\infty$ be fixed. Then we define the following operator on $\mathcal{H}$:*

$$H_\varepsilon^{\mathrm{ren}}=U_\infty^*\,\hat{H}_\varepsilon^{\mathrm{ren}}\,U_\infty\ . \tag{44}$$

The operator $H_\varepsilon^{\mathrm{ren}}$ is self-adjoint on $\mathcal{H}$, with a domain of self adjointness:

$$D_{\mathrm{ren}}(\varepsilon,\sigma_0)=\Big\{\Psi\in\mathcal{H}\ ,\ \Psi\big|_{\mathcal{H}_n}\in e^{-\frac{i}{\varepsilon}T_\infty^{(n)}}\hat{D}_\infty^{(n)}\text{ for any }n\leq\mathfrak{N}(\varepsilon,\sigma_0)\Big\}\ . \tag{45}$$

Remark 2.15. Let $\sigma_0\geq 0$ be fixed. Then the $\hat{H}_\sigma$ given by (11) defines, in the limit $\sigma\to\infty$, a symmetric quadratic form $\hat{h}_\infty$ on $D(H_0^{1/2})\subset\mathcal{H}$. Also $\hat{H}_\varepsilon^{\mathrm{ren}}$ defines a quadratic form $\hat{h}_\varepsilon^{\mathrm{ren}}$. We have[4]

$$\hat{h}_\infty(\mathbb{1}_{[0,\mathfrak{N}]}(N_1)\,\cdot\,,\,\cdot\,)=\hat{h}_\varepsilon^{\mathrm{ren}}(\mathbb{1}_{[0,\mathfrak{N}]}(N_1)\,\cdot\,,\,\cdot\,)\ . \tag{46}$$

However, we are not able to prove that there is a self-adjoint operator on $\mathcal{H}$ associated to $\hat{h}_\infty$, and it is possible that there is none.

3. The classical system: S-KG equations. In this section we define the Schrödinger–Klein–Gordon (S-KG) system, with initial data in a suitable dense subset of $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, that describes the classical dynamics of a particle-field interaction. Then we introduce the classical dressing transformation (viewed itself as a dynamical system) and then study the transformation it induces on the Hamiltonian functional. Finally, we discuss the global existence of unique solutions of the classical equations, both in their original and dressed forms.

The Yukawa coupling. The S-KG[Y] system (Schrödinger–Klein–Gordon with Yukawa interaction), or undressed classical equations, is defined by

$$\begin{cases} i\partial_t u=-\dfrac{\Delta}{2M}u+Vu+Au,\\ (\Box+m_0^2)A=-|u|^2,\end{cases} \tag{S-KG[Y]}$$

where $V:\mathbb{R}^3\to\mathbb{R}$ is an external potential. The Klein–Gordon equation can be rewritten as a system of two first-order equations with respect to time, considering A and its time derivative $\dot{A}=\partial_t A$ as independent variables. In our context, it is even more useful to introduce the complex field α, defined by

$$A(x)=\tfrac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\tfrac{1}{\sqrt{2\omega(k)}}\big(\bar{\alpha}(k)e^{-ik\cdot x}+\alpha(k)e^{ik\cdot x}\big)dk\ , \tag{47}$$

$$\dot{A}(x)=-\tfrac{i}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\sqrt{\tfrac{\omega(k)}{2}}\big(\alpha(k)e^{ik\cdot x}-\bar{\alpha}(k)e^{-ik\cdot x}\big)dk\ . \tag{48}$$

[4] $\mathbb{1}_{[0,\mathfrak{N}]}(N_1)$ is the orthogonal projector on $\bigoplus_{n=0}^{\mathfrak{N}}\mathcal{H}_n$.

Then it is possible to rewrite (S-KG[Y]) as the equivalent system[5]

$$\text{(S-KG)}\qquad \begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + Au, \\ i\partial_t \alpha = \omega\alpha + \dfrac{1}{\sqrt{2\omega}}\mathcal{F}\big(|u|^2\big). \end{cases}$$

The "dressed" coupling. The system that arises from the dressed interaction is quite complicated. We will denote it by S-KG[D], and it has the following form:[6]
(S-KG[D])

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + (W * |u|^2)u + [(\varphi * A) + (\xi * \partial_t A)]u \\ \qquad\qquad + \displaystyle\sum_{i=1}^{3}[(\rho^{(i)} * A)\partial_{(i)} + (\zeta^{(i)} * A)^2]u, \\ (\Box + m_0^2)A = -\varphi * |u|^2 + i\displaystyle\sum_{i=1}^{3}\rho^{(i)} * [(u\partial_{(i)}u) - \sqrt{2M}(\zeta^{(i)} * \partial_t A)], \end{cases}$$

where $V, W, \varphi : \mathbb{R}^3 \to \mathbb{R}$ with W, φ even; $\xi : \mathbb{R}^3 \to \mathbb{C}$, even; $\rho : (\mathbb{R}^3)^3 \to \mathbb{C}$, odd; and $\zeta : (\mathbb{R}^3)^3 \to \mathbb{R}$, odd. Obviously also (S-KG[D]) can be written as an equivalent system with unknowns u and α (omitted here). As discussed in detail in subsection 3.3, with a suitable choice of W, φ, ξ, ρ, and ζ the global well-posedness of (S-KG[D]) follows directly from the global well-posedness of (S-KG[Y]).

3.1. Dressing. We look for a classical correspondent of the dressing transformation $U_\infty(\theta)$. Since $U_\infty(\theta)$ is a one-parameter group of unitary transformations on $\mathcal{H}$, the classical counterpart of its generator is expected to induce a nonlinear evolution on the phase-space $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, using the quantum-classical correspondence principle for systems with infinite degrees of freedom; see, e.g., [9, 69, 82]. The resulting "classical dressing" $D_{g_\infty}(\theta)$ plays a crucial role in proving our results: on one hand it is necessary to link the S-KG classical dynamics with the quantum dressed one; on the other it is at the heart of the "classical" renormalization procedure.

Let $g \in L^2(\mathbb{R}^3)$; define the functional $\mathscr{D}_g : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \mathbb{R}$,

$$\mathscr{D}_g(u,\alpha) := \int_{\mathbb{R}^6} \Big(g(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}(k)\alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dx dk\ . \tag{49}$$

The functional $\mathscr{D}_g$ induces the following Hamiltonian equations of motion:

$$\begin{cases} i\partial_\theta u = A_g u, \\ i\partial_\theta \alpha = gF(|u|^2), \end{cases} \tag{50}$$

where

$$A_g(x) = \int_{\mathbb{R}^3} \Big(g(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}(k)\alpha(k)e^{ik\cdot x}\Big) dk\ , \tag{51}$$

$$F(|u|^2)(k) = \int_{\mathbb{R}^3} e^{-ik\cdot x}|u(x)|^2 dx\ . \tag{52}$$

[5]The two systems are equivalent since $(1+\omega^\varsigma)\mathrm{Re}\alpha \in L^2(\mathbb{R}^3) \Leftrightarrow A \in H^{\varsigma+1/2}(\mathbb{R}^3)$, $(1+\omega^\varsigma)\mathrm{Im}\alpha \in L^2(\mathbb{R}^3) \Leftrightarrow \partial_t A \in H^{\varsigma-1/2}(\mathbb{R}^3)$. In (S-KG) the unknowns are u and α.

[6]We denote by $\partial_{(i)}$ the derivative with respect to the ith component of the variable $x \in \mathbb{R}^3$. Analogously, we denote by $v^{(i)}$ the ith component of a 3-dimensional vector v.

Observe that for any $g \in L^2(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$, $A_g(x) \in \mathbb{R}$. This will lead to an explicit form for the solutions of the Cauchy problem related to (50). The latter can be rewritten in integral form, for any $\theta \in \mathbb{R}$:

$$\begin{cases} u_\theta(x) = u_0(x) \exp\Big\{ -i \int_0^\theta (A_g)_\tau(x)\, d\tau \Big\}, \\ \alpha_\theta(k) = \alpha_0(k) - ig(k) \int_0^\theta F(|u_\tau|^2)(k)\, d\tau, \end{cases} \tag{53}$$

where $(A_g)_\tau$ is defined by (51) with α replaced by α_τ; analogously we define B_g by (51) with α replaced by β.

LEMMA 3.1. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$; $(1+\omega^{\frac{1}{2}})g \in L^2(\mathbb{R}^3)$. Also, let $u, v \in H^s(\mathbb{R}^3)$ and $(1+\omega^\varsigma)\alpha, (1+\omega^\varsigma)\beta \in L^2(\mathbb{R}^3)$. Then there exist constants $C_s, C_\varsigma > 0$ such that*

$$\|(A_g - B_g)u\|_{H^s} \leq C_s \max_{w\in\{u,v\}} \|w\|_{H^s} \|(1+\omega^{\frac{1}{2}})g\|_2 \|(1+\omega^\varsigma)(\alpha-\beta)\|_2 \,, \tag{54}$$

$$\|A_g(u-v)\|_{H^s} \leq C_s \max_{\zeta\in\{\alpha,\beta\}} \|(1+\omega^\varsigma)\zeta\|_2 \|(1+\omega^{\frac{1}{2}})g\|_2 \|u-v\|_{H^s} \,, \tag{55}$$

$$\begin{aligned} \Big\|(1+\omega^\varsigma)g \int_{\mathbb{R}^3} e^{-ik\cdot x}\big((u-v)\bar{v} + (\bar{u}-\bar{v})u\big)dx\Big\|_2 &\leq C_\varsigma \max_{w\in\{u,v\}} \|w\|_{H^s} \\ &\quad \cdot \|(1+\omega^{\frac{1}{2}})g\|_2 \|u-v\|_{H^s} \,. \end{aligned} \tag{56}$$

Proof. If $s \in \mathbb{N}$, the results follow by standard estimates, keeping in mind that $|k| \leq \omega(k) \leq |k| + m_0$. The bounds for noninteger s are then obtained by interpolation. ◻

PROPOSITION 3.2. *Let $\theta \in \mathbb{R}$, $(u_0, \alpha_0) \in L^2 \oplus L^2$. If $(u_\theta, \alpha_\theta) \in \mathcal{C}^0(\mathbb{R}, L^2 \oplus L^2)$ is a solution of* (53)*, then it is unique; i.e., any $(v_\theta, \beta_\theta) \in \mathcal{C}^0(\mathbb{R}, L^2 \oplus L^2)$ that satisfies* (53) *is such that $(v_\theta, \beta_\theta) = (u_\theta, \alpha_\theta)$.*

Proof. We have

$$\begin{aligned} \frac{i}{2}\partial_\theta\Big(\big\|u_\theta - v_\theta\big\|_2^2 + \big\|\alpha_\theta - \beta_\theta\big\|_2^2\Big) = \mathrm{Im}\Big(\Big\langle u_\theta - v_\theta, \big((A_g)_\theta - (B_g)_\theta\big)u_\theta \\ + (B_g)_\theta\big(u_\theta - v_\theta\big)\Big\rangle_2 \\ + \Big\langle \alpha_\theta - \beta_\theta, g\int_{\mathbb{R}^3} e^{-ik\cdot x}\big((u_\theta - v_\theta)\bar{v}_\theta + (\bar{u}_\theta - \bar{v}_\theta)u_\theta\big)dx\Big\rangle_2\Big) \,. \end{aligned}$$

The result hence is an application of the estimates of Lemma 3.1 with $s = 0$ and Gronwall's lemma. ◻

Now that we are assured that the solution of (53) is unique, we can construct it explicitly. Since $A_g(x)$ is real, it follows that for any $\theta \in \mathbb{R}$, $|u_\theta| = |u_0|$. Therefore, $F(|u_\theta|^2) = F(|u_0|^2)$, and

$$\alpha_\theta(k) = \alpha_0(k) - i\theta g(k) F(|u_0|^2)(k) \,.$$

Substituting this explicit form into the expression for u_θ, we obtain the solution for any $(u_0, \alpha_0) \equiv (u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:

$$\begin{cases} u_\theta(x) = u(x)\ \exp\Big\{ -i\theta A_g(x) + i\theta^2 \mathrm{Im} \int_{\mathbb{R}^3} F(|u|^2)(k)|g(k)|^2 e^{ik\cdot x} dk \Big\}, \\ \alpha_\theta(k) = \alpha(k) - i\theta g(k) F(|u|^2)(k). \end{cases} \tag{57}$$

This system of equations defines a nonlinear symplectomorphism: the "classical dressing map" on $L^2 \oplus L^2$.

DEFINITION 3.3. *Let $g \in L^2(\mathbb{R}^3)$. Then $\mathbf{D}_g(\cdot) : \mathbb{R} \times \bigl(L^2 \oplus L^2\bigr) \to L^2 \oplus L^2$ is defined by* (57) *as*

$$\mathbf{D}_g(\theta)(u, \alpha) = (u_\theta, \alpha_\theta)\ .$$

The map $\mathbf{D}_g(\cdot)$ is the Hamiltonian flow generated by $\mathscr{D}_g$.

Using the explicit form (57) and Lemma 3.1, it is straightforward to prove some interesting properties of the classical dressing map. The results are formulated in the following proposition, after the definition of useful classes of subspaces of $L^2 \oplus L^2$.

DEFINITION 3.4. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$. We define the spaces $H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:*

$$H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3) = \Bigl\{(u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\ ,\ u \in H^s(\mathbb{R}^3) \\ \text{and } \mathcal{F}^{-1}(\alpha) \in H^\varsigma(\mathbb{R}^3)\Bigr\}\ .$$

PROPOSITION 3.5. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$; and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then*

$$\mathbf{D}_g : \mathbb{R} \times \bigl(H^s \oplus \mathcal{F}H^\varsigma\bigr) \to H^s \oplus \mathcal{F}H^\varsigma\ ;$$

i.e., the flow preserves the spaces $H^s \oplus \mathcal{F}H^\varsigma$. Furthermore, it is a bijection with inverse $\bigl(\mathbf{D}_g(\theta)\bigr)^{-1} = \mathbf{D}_g(-\theta)$. Hence the classical dressing is a Hamiltonian flow on $H^s \oplus \mathcal{F}H^\varsigma$.

COROLLARY 3.6. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$, $\theta \in \mathbb{R}$, and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then there exist a constant $C(g, \theta) > 0$ and a $\lambda(s) \in \mathbb{N}^*$ such that for any $(u, \alpha) \in H^s \oplus \mathcal{F}H^\varsigma$*

$$\|\mathbf{D}_g(\theta)(u, \alpha)\|_{H^s \oplus \mathcal{F}H^\varsigma} \leq C(g, \theta) \|(u, \alpha)\|^{\lambda(s)}_{H^s \oplus \mathcal{F}H^\varsigma}\ . \tag{58}$$

Using the positivity of both $-\Delta$ and V, and using Corollary 3.6, one also obtains the following result.

COROLLARY 3.7. *Let $V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$; and let $Q(-\Delta + V) \subset L^2(\mathbb{R}^3)$ be the form domain of $-\Delta + V$. Then for any $\frac{1}{2} \leq \varsigma \leq \frac{3}{2}$ and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$*

$$\mathbf{D}_g : \mathbb{R} \times \bigl(Q(-\Delta + V) \oplus \mathcal{F}H^\varsigma\bigr) \to Q(-\Delta + V) \oplus \mathcal{F}H^\varsigma\ .$$

3.2. Classical Hamiltonians. In this section we define the classical Hamiltonian functionals that generate the undressed and dressed dynamics on $L^2 \oplus L^2$. Then we show that they are related by a suitable classical dressing: the quantum procedure described in subsection 2.2 is reproduced, in simplified terms, on the classical level.

DEFINITION 3.8 ($\mathscr{E}$, $\hat{\mathscr{E}}$). *The undressed Hamiltonian (or energy) $\mathscr{E}$ is defined as the following real functional on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:*

$$\mathscr{E}(u, \alpha) := \Bigl\langle u, \Bigl(-\tfrac{\Delta}{2M} + V\Bigr)u\Bigr\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 \\ + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}} \Bigl(\bar{\alpha}(k) e^{-ik\cdot x} + \alpha(k) e^{ik\cdot x}\Bigr) |u(x)|^2 dx dk\ .$$

We denote by $\mathscr{E}_0$ the free part of the classical energy, namely

$$\mathscr{E}_0(u,\alpha) = \Big\langle u, \Big(-\tfrac{\Delta}{2M} + V\Big) u\Big\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 \ .$$

Let $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-1/2}(\mathbb{R}^3)$ such that $\chi_{\sigma_0}(k) = \chi_{\sigma_0}(-k)$ for any $k \in \mathbb{R}^3$. Then (again as a real functional on $L^2 \oplus L^2$) the dressed Hamiltonian $\hat{\mathscr{E}}$ is defined as[7]

$$\begin{aligned}\hat{\mathscr{E}}(u,\alpha) := &\Big\langle u, \Big(-\tfrac{\Delta}{2M} + V\Big) u\Big\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 \\ &+ \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{\chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}} \Big(\bar{\alpha}(k) e^{-ik\cdot x} + \alpha(k) e^{ik\cdot x}\Big) |u(x)|^2 dxdk \\ &+ \tfrac{1}{2M} \int_{\mathbb{R}^9} \Big(r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{r}_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar{\alpha}(l)e^{-il\cdot x} \\ &+ \bar{r}_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2 dxdkdl \\ &- \tfrac{2}{M}\mathrm{Re}\int_{\mathbb{R}^6} r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}\bar{u}(x) D_x u(x) dxdk \\ &+ \tfrac{1}{2}\int_{\mathbb{R}^6} V_\infty(x-y)|u(x)|^2|u(y)|^2 dxdy \ .\end{aligned}$$

Remark 3.9. We denote by $D(\mathscr{E}) \subset L^2 \oplus L^2$ the domain of definition of $\mathscr{E}$ and by $D(\hat{\mathscr{E}}) \subset L^2 \oplus L^2$ the domain of definition of $\hat{\mathscr{E}}$. We have that $D(\mathscr{E}) \supset \mathcal{C}_0^\infty \oplus \mathcal{C}_0^\infty$ and $D(\hat{\mathscr{E}}) \supset \mathcal{C}_0^\infty \oplus \mathcal{C}_0^\infty$. Therefore, both $\mathscr{E}$ and $\hat{\mathscr{E}}$ are densely defined, and $D(\mathscr{E}) \cap D(\hat{\mathscr{E}})$ is dense in $L^2 \oplus L^2$.

We are interested in the action of $\mathscr{E}$ and $\hat{\mathscr{E}}$ on $H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$, since this emerges naturally as the energy space of the system, at least when $V = 0$.

LEMMA 3.10. *Let $\theta \in \mathbb{R}$, $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then for any $u \in Q(V) \cap H^1(\mathbb{R}^3)$ and $\alpha \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$, $\mathbf{D}_g(\theta)(u,\alpha) \in D(\mathscr{E})$.*

Proof. Let $u \in Q(V)$ and $\alpha \in L^2(\mathbb{R}^3)$. Then

$$\langle u_\theta, V u_\theta\rangle_2 = \langle u, V u\rangle_2 \ ,$$

where u_θ is as defined in (57), and it is the first component of $\mathbf{D}_g(\theta)(u,\alpha)$. Also, for any $(u,\alpha) \in H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$ we have that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^6} |u(x)|^2 \frac{1}{\sqrt{\omega(k)}} \alpha(k) e^{ik\cdot x} dxdk\Big| \\ &\quad = C\Big|\int_{\mathbb{R}^3} \frac{1}{|k|\omega(k)} \big(\omega^{1/2}\alpha\big)(k)\Big(\int_{\mathbb{R}^3}\big(D_x|u(x)|^2\big) e^{ik\cdot x} dx\Big) dk\Big| \\ &\quad \leq 2C\Big\|\frac{1}{|k|\omega(k)}\Big\|_2 \|\omega^{1/2}\alpha\|_2 \|u\|_2 \|u\|_{H^1} < +\infty \ .\end{aligned}$$

The result then follows since $\mathbf{D}_g(\theta)$ maps $H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$ into itself by Proposition 3.5. ∎

[7]We recall that $g_\infty(k) = -i\frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}}\frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M}+\omega(k)}$; $V_\infty(x) = 2\mathrm{Re}\int_{\mathbb{R}^3}\omega(k)|g_\infty(k)|^2 e^{-ik\cdot x}dk - 4\mathrm{Im}\int_{\mathbb{R}^3}\frac{\bar{g}_\infty(k)}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(k)}}e^{-ik\cdot x}dk$. Also, $D_x = -i\nabla_x$; $r_\infty(k) = -ikg_\infty(k)$.

The functional $\mathscr{E}$ is independent of g_∞, while $\hat{\mathscr{E}}$ depends on it. In addition, we know that g_∞ has been fixed, at the quantum level, to renormalize the Nelson Hamiltonian, and it is the function that appears in the generator of the dressing transformation U_∞. Hence, since we are establishing a correspondence between the classical and quantum theories, we expect it to be the function that appears in the classical dressing too. Two features of g_∞ are very important in the classical setting: the first is that $g_\infty \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$ for any $\chi_{\sigma_0} \in L^\infty \cap \mathcal{F}H^{-\frac{1}{2}}$; the second is that it is an even function, i.e., $g_\infty(k) = g_\infty(-k)$ for any $k \in \mathbb{R}^3$. Using the first fact, one shows that $\mathbf{D}_{g_\infty}(\cdot)$ maps the energy space into itself (and that will be convenient when discussing global solutions); using the second property we can simplify the explicit form of $\mathbf{D}_{g_\infty}(\cdot)$.

LEMMA 3.11. *Let $\theta \in \mathbb{R}$ and $g \in L^2(\mathbb{R}^3)$. If g is an even or odd function, then the map $\mathbf{D}_g(\theta)$ defined by (57) becomes*

$$\mathbf{D}_g(\theta)(u(x), \alpha(k)) = \Big(u(x) e^{-i\theta A_g(x)} \, , \, \alpha(k) - i\theta g(k) F(|u|^2)(k) \Big) \, . \tag{59}$$

Proof. Consider $I(x) := \int_{\mathbb{R}^3} F(|u|^2)(k) |g(k)|^2 e^{ik\cdot x} dk$. We will show that $\bar{I}(x) = I(x)$. We have that

$$\bar{I}(x) = \int_{\mathbb{R}^6} |u(x')|^2 |g(k)|^2 e^{-ik\cdot(x-x')} dx' dk = \int_{\mathbb{R}^6} |u(x')|^2 |g(-k)|^2 e^{ik\cdot(x-x')} dx' dk \, .$$

Now if g is either even or odd, $|g(-k)| = |g(k)|$. Hence $\bar{I}(x) = I(x)$; therefore, $\mathrm{Im} I(x) = 0$. ∎

We conclude this section proving its main result: $\mathscr{E}$ and $\hat{\mathscr{E}}$ are related by the $\mathbf{D}_{g_\infty}(1)$ classical dressing.[8]

PROPOSITION 3.12. *For any $u \in Q(V) \cap H^1(\mathbb{R}^3)$, $\alpha \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$, and for any $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3)$,*

(1) *$(u, \alpha) \in D(\mathscr{E})$;*

(2) *$(u, \alpha) \in D(\hat{\mathscr{E}})$;*

(3) *$\hat{\mathscr{E}}(u, \alpha) = \mathscr{E} \circ \mathbf{D}_{g_\infty}(1)(u, \alpha)$.*

Remark 3.13. Relation (3) of Proposition 3.12 actually holds for any $(u, \alpha) \in \mathbf{D}_{g_\infty}(-1) D(\mathscr{E})$.

Remark 3.14. The Wick quantization of $\mathscr{E}$ yields the quadratic form $\langle \cdot \, , H_\infty^{(n)} \cdot \rangle_{\mathcal{H}_n}$, which is not closed and not bounded from below for any $n \in \mathbb{N}_*$. On the other hand, if χ_{σ_0} is the ultraviolet cutoff of section 2, then the Wick quantization of $\hat{\mathscr{E}}$ yields directly the *renormalized* quadratic form $\langle \cdot \, , \hat{H}_\infty^{(n)} \cdot \rangle_{\mathcal{H}_n}$ that is closed and bounded from below for any $n \leq \mathfrak{N}(\varepsilon, \sigma_0)$.

Proof of Proposition 3.12. The statement (1) is just an application of Lemma 3.10 when $\theta = 0$. If (3) holds formally, then (2) follows directly, since by Lemma 3.10 the right-hand side of (3) is well defined. It remains to prove that the relation (3) holds formally. This is done by means of a direct calculation, which we will briefly outline

[8] We recall again that $g_\infty = -i \frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}} \frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M} + \omega(k)}$.

here.

$$\mathscr{E}\circ\mathbf{D}_{g_\infty}(1)(u,\alpha)=\Big\langle ue^{-iA_{g_\infty}},\frac{D_x}{2M}D_x\big(ue^{-iA_{g_\infty}}\big)\Big\rangle_2+\langle u,Vu\rangle_2+\langle\alpha,\omega\alpha\rangle_2$$
$$\text{(a)}\qquad +2\mathrm{Im}\langle\alpha,\omega g_\infty F_u\rangle_2+\tfrac{1}{(2\pi)^{3/2}}2\mathrm{Re}\int_{\mathbb{R}^6}\tfrac{1}{\sqrt{2\omega(k)}}\bar\alpha(k)e^{-ik\cdot x}|u(x)|^2dxdk$$
$$\text{(b)}\qquad +\|\omega g_\infty F_u\|_2^2+\tfrac{1}{(2\pi)^{3/2}}2\mathrm{Im}\int_{\mathbb{R}^6}\tfrac{1}{\sqrt{2\omega(k)}}g_\infty(k)F_u(k)e^{ik\cdot x}|u(x)|^2dxdk\;.$$

After some manipulation, taking care of the ordering, the first term on the right-hand side becomes

$$\Big\langle ue^{-iA_{g_\infty}},\frac{D_x}{2M}D_xy\big(ue^{-iA_{g_\infty}}\big)\Big\rangle_2=\Big\langle u,-\frac{\Delta}{2M}u\Big\rangle_2$$
$$\text{(c)}\qquad +\tfrac{1}{2M}\langle A_{r_\infty}u,A_{r_\infty}u\rangle_2$$
$$\text{(d)}\qquad -i\langle u,A_{\frac{k^2}{2M}g_\infty}u\rangle_2$$
$$\text{(e)}\qquad -\tfrac{1}{M}\Big\langle u,\int_{\mathbb{R}^3}dk\Big(D_x\bar r_\infty(k)\alpha(k)e^{ik\cdot x}+r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}D_x\Big)u\Big\rangle_2\;.$$

The proof is concluded by making the following identifications (the other terms sum to the free part):

$$\text{(a)}+\text{(d)}=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^6}\frac{\chi_{\sigma_0}}{\sqrt{2\omega(k)}}\Big(\bar\alpha(k)e^{-ik\cdot x}+\alpha(k)e^{ik\cdot x}\Big)|u(x)|^2dxdk\;;$$
$$\text{(b)}=\frac12\int_{\mathbb{R}^6}V_\infty(x-y)|u(x)|^2|u(y)|^2dxdy\;;$$
$$\text{(c)}=\frac{1}{2M}\int_{\mathbb{R}^9}\Big(r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}+\bar r_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar\alpha(l)e^{-il\cdot x}+\bar r_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2dxdkdl\;;$$
$$\text{(e)}=-\frac{2}{M}\mathrm{Re}\int_{\mathbb{R}^6}r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}\bar u(x)D_xu(x)dxdk\;.$$
∎

3.3. Global existence results. In this section we discuss uniqueness and global existence of the classical dynamical system: using a well-known result on the undressed dynamics, we prove uniqueness and existence also for the dressed system.

The Cauchy problem associated to $\mathscr{E}$ by the Hamilton equations is[9] (S-KG). Theorem 3.15 below is a straightforward extension of [40, 99] that includes a (confining) potential on the nonlinear Schrödinger equation. As proved in [25, 98], the quadratic potential is the maximum we can afford to still have Strichartz estimates and global existence in the energy space. Therefore, we make the following standard assumption on V:

$$\text{(A-V)}\qquad V\in\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}_+)\,,\text{ and }\partial^\alpha V\in L^\infty(\mathbb{R}^3)\text{ for any }\alpha\in\mathbb{N}^3\,,\text{ with }|\alpha|\geq2\,;$$

i.e., it is at most a quadratic positive confining potential.

[9]The Cauchy problem associated to $\hat{\mathscr{E}}$ is equivalent to (S-KG[D]), setting $W=V_\infty$, $\varphi=(2\pi)^{-3/2}\mathcal{F}(\chi_{\sigma_0})$, $\xi=\frac{(2\pi)^{-3/2}}{\sqrt2M}\big(\mathcal{F}(\frac{k^2}{\sqrt\omega}g_\infty)-\mathcal{F}(i\frac{k^2}{\omega}g_\infty)\big)$, $\rho=\frac{\sqrt2}{M}\mathcal{F}(\sqrt\omega kg_\infty)$, and $\zeta=\frac{i}{\sqrt M}\mathcal{F}(\frac{k}{\sqrt\omega}g_\infty)$.

THEOREM 3.15 (undressed global existence). *Assume* (A-V). *Then there is a unique Hamiltonian flow solving* (S-KG)*:*

$$\mathbf{E} : \mathbb{R} \times \big(Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\big) \to Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\ . \tag{60}$$

If $V = 0$, then there is a unique Hamiltonian flow

$$\mathbf{E} : \mathbb{R} \times \big(H^s(\mathbb{R}^3) \oplus \mathcal{F}H^{\varsigma}(\mathbb{R}^3)\big) \to H^s(\mathbb{R}^3) \oplus \mathcal{F}H^{\varsigma}(\mathbb{R}^3) \tag{61}$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$.

THEOREM 3.16 (dressed global existence). *Assume* (A-V). *Then for any $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3)$, there is a unique Hamiltonian flow:*

$$\hat{\mathbf{E}} : \mathbb{R} \times \big(Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\big) \to Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\ . \tag{62}$$

If $V = 0$, then there is a unique Hamiltonian flow

$$\hat{\mathbf{E}} : \mathbb{R} \times \big(H^s(\mathbb{R}^3) \oplus \mathcal{F}H^{\varsigma}(\mathbb{R}^3)\big) \to H^s(\mathbb{R}^3) \oplus \mathcal{F}H^{\varsigma}(\mathbb{R}^3) \tag{63}$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$. For any V that satisfies (A-V)*, the flows $\hat{\mathbf{E}}$ and $\mathbf{E}$ are related by*

$$\hat{\mathbf{E}} = \mathbf{D}_{g_\infty}(-1) \circ \mathbf{E} \circ \mathbf{D}_{g_\infty}(1), \quad \mathbf{E} = \mathbf{D}_{g_\infty}(1) \circ \hat{\mathbf{E}} \circ \mathbf{D}_{g_\infty}(-1). \tag{64}$$

Proof of Theorem 3.16. The theorem is a direct consequence of the global well-posedness result of Theorem 3.15, the relation $\hat{\mathscr{E}} = \mathscr{E} \circ \mathbf{D}_{g_\infty}(1)$ proved in Proposition 3.12, and the regularity properties of the dressing proved in Proposition 3.5. $\square$

3.4. Symplectic character of $\mathbf{D}_{\chi_{\sigma_0}}$. To complete our description of the S-KG system, we explicitly prove that the classical dressing is a (nonlinear) symplectomorphism for the real symplectic structure $\big\{(L^2 \oplus L^2)_{\mathbb{R}}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{L^2\oplus L^2}\big\}$. We denote by $\mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)} \in \mathcal{L}(L^2 \oplus L^2)$ the (Fréchet) derivative of $\mathbf{D}_g(\theta)$ at the point $(u, \alpha) \in L^2 \oplus L^2$.

PROPOSITION 3.17. *Let $g \in L^2(\mathbb{R}^3)$ be an even or odd function. Then for any $\theta \in \mathbb{R}$, $\mathbf{D}_g(\theta)$ is differentiable at any point $(u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. In addition, it satisfies for any $(v_1, \beta_1), (v_2, \beta_2) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$*

$$\mathrm{Im}\langle \mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v_1, \beta_1), \mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v_2, \beta_2)\rangle_{L^2\oplus L^2} = \mathrm{Im}\langle (v_1, \beta_1), (v_2, \beta_2)\rangle_{L^2\oplus L^2}\ .$$

Proof. We recall that with the assumptions on g, $\mathbf{D}_g(\theta)$ has the explicit form

$$\mathbf{D}_g(\theta)(u(x), \alpha(k)) = \Big(\ u(x)e^{-i\theta A_g(x)}\ ,\ \alpha(k) - i\theta g(k) F(|u|^2)(k)\ \Big)\ ,$$

where A_g and F are defined by (51) and (52), respectively. The Fréchet derivative of $\mathbf{D}_g(\theta)$ is easily computed and yields

$$\begin{aligned}\mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v(x), \beta(k)) = \Big(\ \big(v(x) - i\theta B_g(x) u(x)\big)e^{-i\theta A_g(x)}\ ,\ \beta(k)\\ -2i\theta g(k)\mathrm{Re}\big(F(\bar{u}v)(k)\big)\ \Big) = \Big(\ \mathrm{i}(v,\beta)\ ,\ \mathrm{ii}(v,\beta)\ \Big)\ ,\end{aligned}$$

where we recall that $B_g(x)$ is $A_g(x)$ with α substituted by β. Then we have

$$\operatorname{Im}\langle \mathrm{i}(v_1,\beta_1),\mathrm{i}(v_2,\beta_2)\rangle_{L^2} = \operatorname{Im}\langle v_1,v_2\rangle_{L^2} + 2\theta\operatorname{Re}\Big(\langle B_g^{(1)}u, v_2\rangle_{L^2} - \langle v_1, B_g^{(2)}u\rangle_{L^2}\Big)\,,$$

$$\begin{aligned}\operatorname{Im}\langle \mathrm{ii}(v_1,\beta_1),\mathrm{ii}(v_2,\beta_2)\rangle_{L^2} = \operatorname{Im}\langle \beta_1,\beta_2\rangle_{L^2} + 2\theta\operatorname{Re}\Big(&\langle g\operatorname{Re}F(\bar u v_1),\beta_2\rangle_{L^2}\\ &-\langle \beta_1, g\operatorname{Re}F(\bar u v_2)\rangle_{L^2}\Big)\,.\end{aligned}$$

The result then follows, noting that $\langle g\operatorname{Re}F(\bar u v_1),\beta_2\rangle_{L^2} = \langle v_1, B_g^{(2)}u\rangle_{L^2}$ and $\langle \beta_1, g\operatorname{Re}F(\bar u v_2)\rangle_{L^2} = \langle B_g^{(1)}u, v_2\rangle_{L^2}$. ◻

4. The classical limit of the renormalized Nelson model.

In this section we discuss in detail the classical limit of the renormalized Nelson model, both dressed and undressed, and prove the main result Theorem 1.1. A schematic outline of the proof is given in subsection 4.1 to improve readability. Subsections 4.2–4.6 are dedicated to proving the convergence of the dressed dynamics. The obtained results are summarized by Theorem 4.26. In subsection 4.7 we study the classical limit of the dressing transformation. Finally, in subsection 4.8 we put all the pieces together to prove Theorem 1.1.

4.1. Scheme of the proof.

First, let us explain the main ideas behind our proof of Theorem 1.1. Since the explicit form of $H_\varepsilon^{\mathrm{ren}}$ is not known, it seems a very hard task to directly study the limit of a time-evolved family of states $e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}\,\varrho_\varepsilon\, e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}$, at least using established techniques. The introduction of the classical dressing, and the relation $\mathbf{E} = \mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}\circ\mathbf{D}_{g_\infty}(-1)$ (equation (64), proved in Theorem 3.16) play therefore a crucial role. Once we combine them with the convergence of the quantum dressing to the classical dressing "as a dynamical system" (see Proposition 4.25), we can relate the undressed and dressed dynamics throughout the entire limit procedure. The final ingredient is the convergence of a family of states $\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}\hat H_\varepsilon^{\mathrm{ren}}}\,\varrho_\varepsilon\, e^{i\frac{t}{\varepsilon}\hat H_\varepsilon^{\mathrm{ren}}}$ evolved with the quantum *dressed* dynamics to the corresponding Wigner measure $\hat{\mathbf{E}}(t)_{\#}\mu_0$ evolved with the classical dressed dynamics. Despite being technically demanding, the proof of the latter takes advantage of the explicit expression of the quadratic form $\hat h_\varepsilon^{\mathrm{ren}}(\cdot,\cdot) = \langle\,\cdot\,,\hat H_\varepsilon^{\mathrm{ren}}\,\cdot\,\rangle$ associated to the dressed Hamiltonian. We shall from time to time omit the explicit ε-dependence in the quantum operators to avoid heavy notation. The lengthier part of the aforementioned proof is to control each term that arises from the expansion of the quadratic form associated to $[\hat H_I^{\mathrm{ren}}, W(\tilde\xi_s)]$: it is necessary to prove that each associated classical symbol either is compact or can be approximated with a compact one.

In light of the discussion above, the proof of Theorem 1.1 can be schematized through the following steps.

(i) (Subsection 4.2.) Express the average of the Weyl operator $W(\xi)$ with respect to the dressed time-evolved state $\tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_0}\,\varrho_\varepsilon(t)\,e^{-i\frac{t}{\varepsilon}H_0}$ (in the interaction picture) as the integral formula

$$\operatorname{Tr}\Big[\tilde\varrho_\varepsilon(t)W(\xi)\Big] = \operatorname{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \operatorname{Tr}\Big[\varrho_\varepsilon(s)[\hat H_I^{\mathrm{ren}}, W(\tilde\xi_s)]\Big]\,ds.$$

(ii) (Subsection 4.3.) Characterize the quadratic form associated to $[\hat H_I^{\mathrm{ren}}, W(\tilde\xi_s)]$; in particular, prove that the associated classical symbol can be approximated with a compact symbol (Proposition 4.9).

(iii) (Subsections 4.4 and 4.5.) Take the limit $\varepsilon \to 0$ in the integral formula of step (i) (extracting a common subsequence for all times), thus obtaining a time-dependent family $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ of Wigner measures characterized by a transport equation

$$\partial_t \tilde{\mu}_t + \nabla^T \Big(\mathbf{V}(t) \tilde{\mu}_t \Big) = 0 \; .$$

(iv) (Subsection 4.6.) The transport equation of step (iii) is solved by $\mathbf{E}_0(-t)_{\#}\hat{\mathbf{E}}(t)_{\#}\mu_0$. Prove that the family $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ can be uniquely identified with $(\mathbf{E}_0(-t)_{\#}\hat{\mathbf{E}}(t)_{\#}\mu_0)_{t\in\mathbb{R}}$ provided that $\varrho_\varepsilon(0) \to \mu_0$. This is achieved by applying a general uniqueness result for probability measure solutions of transport equations proved in [8].

(v) (Subsection 4.7.) Prove that the dressed state $e^{-i\frac{\theta}{\varepsilon}T_\infty}\,\varrho_\varepsilon\, e^{i\frac{\theta}{\varepsilon}T_\infty}$ converges when $\varepsilon \to 0$ to $\mathbf{D}_{g_\infty}(\theta)_{\#}\mu$ for any $\theta \in \mathbb{R}$, provided that $\varrho_\varepsilon \to \mu$.

(vi) (Subsection 4.8.) Combine the results together, and use the relation $\mathbf{E} = \mathbf{D}_{g_\infty}(1) \circ \hat{\mathbf{E}} \circ \mathbf{D}_{g_\infty}(-1)$ to prove that $\varrho_\varepsilon \to \mu$ yields

$$e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}}\,\varrho_\varepsilon\, e^{i\frac{t}{\varepsilon}H_\varepsilon^{\mathrm{ren}}} \to \mathbf{E}(t)_{\#}\mu \; .$$

4.2. The integral formula for the dressed Hamiltonian. The results of this subsection and the next are similar in spirit to those previously obtained in [5, section 3] for the Nelson model with cutoff and in [12, section 3] for the mean field problem. However, some additional care has to be taken, for in this more singular situation the manipulations below are allowed only in the sense of quadratic forms. We start with a couple of preparatory lemmas. The proof of the first can be essentially obtained following [5, Lemma 6.1]; the second is an equivalent reformulation of assumption (A-n):

$$\exists \mathfrak{C} > 0 \, , \, \forall \varepsilon \in (0,\bar{\varepsilon}) \, , \, \forall k \in \mathbb{N} \, , \, \mathrm{Tr}[\varrho_\varepsilon N_1^k] \leq \mathfrak{C}^k \; .$$

We recall that the Weyl operator $W(\xi)$, $L^2 \oplus L^2 \ni \xi = \xi_1 \oplus \xi_2$, is defined as

$$W(\xi) = e^{\frac{i}{\sqrt{2}}\left(\psi^*(\xi_1)+\psi(\xi_1)\right)} e^{\frac{i}{\sqrt{2}}\left(a^*(\xi_2)+a(\xi_2)\right)} \; . \tag{65}$$

Lemma 4.1. *For any $\xi = \xi_1 \oplus \xi_2$ such that $\xi_1 \in Q(-\Delta + V) \subset H^1$ and $\xi_2 \in D(\omega^{1/2}) \equiv \mathcal{F}H^{1/2}$, there exists $C(\xi) > 0$ that depends only on $\|\xi_1\|_{H^1}$ and $\|\xi_2\|_{\mathcal{F}H^{1/2}}$, such that for any $\varepsilon \in (0,\bar{\varepsilon})$,*

$$\begin{aligned} \|H_0^{1/2} W(\xi)\Psi\| &\leq C(\xi) \|(H_0+\bar{\varepsilon})^{1/2}\Psi\| \;\; \forall \Psi \in Q(H_0) \; ; \\ \|(H_0+1)^{1/2}(N_1+1)^{1/2} W(\xi)\Psi\| &\leq C(\xi)\|(H_0+\bar{\varepsilon})^{1/2}(N_1+\bar{\varepsilon})^{1/2}\Psi\| \\ &\qquad \forall \Psi \in Q(H_0)\cap Q(N_1) \; . \end{aligned}$$

In an analogous fashion, for any $\xi \in L^2 \oplus L^2$, $r > 0$, there exists $C(\xi) > 0$ that depends only on $\|\xi_1\|_2$ and $\|\xi_2\|_2$, such that for any $\varepsilon \in (0,\bar{\varepsilon})$,

$$\|(N_1+N_2)^{r/2} W(\xi)\Psi\| \leq C(\xi)\|(N_1+N_2+\bar{\varepsilon})^{r/2}\Psi\| \;\; \forall \Psi \in Q(N_1^r) \; .$$

Lemma 4.2. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$. Then $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies assumption* (A-n) *if and only if for any $\varepsilon \in (0,\bar{\varepsilon})$ there exists a sequence*

$(\Psi_i(\varepsilon))_{i\in\mathbb{N}}$ of orthonormal vectors in $\mathcal{H}$ with nonzero components only in $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ and a sequence $(\lambda_i(\varepsilon))_{i\in\mathbb{N}}\in l^1$, with each $\lambda_i(\varepsilon)>0$, such that

$$\varrho_\varepsilon = \sum_{i\in\mathbb{N}} \lambda_i(\varepsilon)\lvert\Psi_i(\varepsilon)\rangle\langle\Psi_i(\varepsilon)\rvert \; .$$

The explicit ε-dependence of Ψ_i and λ_i will be often omitted.

Proof. We start assuming (A-n). Let $\varrho_\varepsilon = \sum_{i\in\mathbb{N}}\lambda_i\lvert\Psi_i\rangle\langle\Psi_i\rvert$ be the spectral decomposition of ϱ_ε. Then

$$\operatorname{Tr}\bigl[\varrho_\varepsilon N_1^k\bigr] = \sum_{i\in\mathbb{N}}\lambda_i\langle\Psi_i, N_1^k\Psi_i\rangle \le \mathfrak{C}^k \Rightarrow \sum_{i\in\mathbb{N}}\lambda_i\langle\Psi_i,(N_1/\mathfrak{C})^k\Psi_i\rangle \le 1 \; .$$

Let $\mathbb{1}_{[L,+\infty)}(N_1)$ be the spectral projection of N_1 on the interval $[L,+\infty)$, and choose $L>\mathfrak{C}$. Then it follows that

$$\begin{aligned} 1 \ge \operatorname{Tr}\bigl[\varrho_\varepsilon \mathbb{1}_{[L,+\infty)}(N_1)(N_1/\mathfrak{C})^k\bigr] &= \sum_{i\in\mathbb{N}}\lambda_i\langle\Psi_i,\mathbb{1}_{[L,+\infty)}(N_1)(N_1/\mathfrak{C})^k\Psi_i\rangle \\ &\ge \sum_{i\in\mathbb{N}}\lambda_i(L/\mathfrak{C})^k\langle\Psi_i,\mathbb{1}_{[L,+\infty)}(N_1)\Psi_i\rangle \; . \end{aligned}$$

Therefore, $(L/\mathfrak{C})^k\langle\Psi_i,\mathbb{1}_{[L,+\infty)}(N_1)\Psi_i\rangle \le 1$ for any $k\in\mathbb{N}$ and for any Ψ_i. Now $(L/\mathfrak{C})^k$ diverges when $k\to\infty$, while $\langle\Psi_i,\mathbb{1}_{[L,+\infty)}(N_1)\Psi_i\rangle$ does not depend on k, so their product is uniformly bounded if and only if $\mathbb{1}_{[L,+\infty)}(N_1)\Psi_i=0$ for any $L>\mathfrak{C}$. The result follows immediately, recalling that the eigenvalues of N_1 are of the form εn_1, with $n_1\in\mathbb{N}$.

The converse statement, that assumption (A-n) follows if $\varrho_\varepsilon=\sum_{i\in\mathbb{N}}\lambda_i\lvert\Psi_i\rangle\langle\Psi_i\rvert$, with each Ψ_i with at most $[\mathfrak{C}/\varepsilon]$ particles, is trivial to prove. ∎

In this subsection, we will consider only families of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ that satisfy assumption (A-n) and the following assumption:

(A(h)') $$\exists C>0\,,\;\forall\varepsilon\in(0,\bar\varepsilon)\,,\;\operatorname{Tr}[\varrho_\varepsilon(N_1+H_0)]\le C \; .$$

Definition 4.3 ($\varrho_\varepsilon(t)$, $\tilde\varrho_\varepsilon(t)$). *We define the dressed time evolution of a state ϱ_ε to be*

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}\hat H^{\mathrm{ren}}}\,\varrho_\varepsilon\, e^{i\frac{t}{\varepsilon}\hat H^{\mathrm{ren}}} \; ,$$

where the σ_0 on which $\hat H^{\mathrm{ren}}$ depends is chosen such that the dynamics is nontrivial on the whole subspace with at most $[\mathfrak{C}/\varepsilon]$ nucleons (see Lemma 4.2 *and the discussion in subsection* 1.2*). We also define the dressed evolution in the interaction picture to be*

$$\tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_0}\,\varrho_\varepsilon(t)\,e^{-i\frac{t}{\varepsilon}H_0} \; .$$

To characterize the evolved Wigner measures corresponding to $\tilde\varrho_\varepsilon(t)$, it is sufficient to study its Fourier transform; this is done by studying the evolution of $\operatorname{Tr}[\tilde\varrho_\varepsilon(t)W(\xi)]$ by means of an integral equation.

PROPOSITION 4.4. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *be a family of normal states on* $\mathcal{H}$ *satisfying assumptions* (A-n) *and* (A(h)'). *Then for any* $t\in\mathbb{R}$, $Q(-\Delta+V)\oplus D(\omega^{1/2})\ni\xi=\xi_1\oplus\xi_2$*,*

$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big]=\mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big]+\frac{i}{\varepsilon}\int_0^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]\Big]\,ds\,, \tag{66}$$

where $\tilde{\xi}_s=e^{is(-\Delta+V)}\xi_1\oplus e^{-is\omega}\xi_2$*. The commutator* $[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]$ *has to be intended as a densely defined quadratic form with domain* $Q(H_0)$*, or equivalently as an operator from* $Q(H_0)$ *to* $Q(H_0)^*$*.*

Proof. The family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies assumption (A-n); therefore, by Lemma 4.2,

$$\mathrm{Tr}\big[\tilde{\varrho}_\varepsilon(t)W(\xi)\big]=\sum_{i\in\mathbb{N}}\lambda_i\langle e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,W(\xi)e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle\;.$$

By assumption (A(h)'), it follows that $\Psi_i\in Q(H_0)$ for any $i\in\mathbb{N}$. Hence the right-hand side is differentiable in t by Lemma 4.1, since $Q(H_0)$ is the form domain of both H_0 and $\hat{H}_{\mathrm{ren}}$. Using the Duhamel formula and the fact that $e^{-i\frac{s}{\varepsilon}H_0}W(\xi)e^{i\frac{s}{\varepsilon}H_0}=W(\tilde{\xi}_s)$, we then obtain

$$\begin{aligned}\mathrm{Tr}\big[\tilde{\varrho}_\varepsilon(t)W(\xi)\big]=&\sum_{i\in\mathbb{N}}\lambda_i\bigg(\langle\Psi_i,W(\xi)\Psi_i\rangle\\&+\frac{i}{\varepsilon}\int_0^t\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle ds\bigg)\,,\end{aligned}$$

where $[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]$ makes sense as a quadratic form on $Q(H_0)$. The result is then obtained using Lebesgue's dominated convergence theorem on the right-hand side, by virtue of Assumption (A(h)') and Lemma 4.1. ◻

4.3. The commutator $[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]$. In this subsection we study the explicit form of the commutator $[\hat{H}_I^{\mathrm{ren}},W(\tilde{\xi}_s)]$. The goal is to show that each of its terms converges in the limit $\varepsilon\to0$, either to zero or to a suitable phase space symbol.

For convenience, we recall some terminology related to quantization procedures in infinite dimensional phase spaces (see [9] for additional information). Let $\mathcal{Z}$ be a Hilbert space (the classical phase space). In the language of quantization, we call a densely defined functional $\mathscr{A}:D\subset\mathcal{Z}\to\mathbb{C}$ a (classical) *symbol*. We say that A is a *polynomial symbol* if there are densely defined bilinear forms $b_{p,q}$ on $\mathcal{Z}^{\otimes_s p}\times\mathcal{Z}^{\otimes_s q}$, $0\leq p\leq\bar{p}$, $0\leq q\leq\bar{q}$ (with $p,\bar{p},q,\bar{q}\in\mathbb{N}$), such that

$$\mathscr{A}(z)=\sum_{\substack{0\leq p\leq\bar{p}\\0\leq q\leq\bar{q}}}b_{p,q}(z^{\otimes p},z^{\otimes q})\;. \tag{67}$$

The Wick quantized quadratic form $(\mathscr{A})^{Wick}$ on $\Gamma_s(\mathcal{Z})$ is then obtained, roughly speaking, by replacing each $z(\cdot)$ with the annihilation operator valued distribution $a(\cdot)$ and each $\bar{z}(\cdot)$ with the creation operator valued distribution $a^*(\cdot)$, and putting all the $a^*(\cdot)$ to the left of the $a(\cdot)$. We denote, with a straightforward notation, the class of all polynomial symbols on $\mathcal{Z}$ by $\bigoplus_{(p,q)\in\mathbb{N}^2}^{alg}\mathcal{Q}_{p,q}(\mathcal{Z})$. If $\mathscr{A}:\mathcal{Z}\to\mathbb{C}$ and the bilinear forms $b_{p,q}(z^{\otimes p},z^{\otimes q})$ in (67) can all be written as $\langle z^{\otimes q},\tilde{b}_{p,q}z^{\otimes p}\rangle_{\mathcal{Z}^{\otimes_s q}}$ for some bounded (resp., compact) operator $\tilde{b}_{p,q}:\mathcal{Z}^{\otimes_s p}\to\mathcal{Z}^{\otimes_s q}$, we say that $\mathscr{A}$ is a bounded (resp., compact) polynomial symbol. We denote the class of all bounded (resp., compact) polynomial symbols by $\bigoplus_{(p,q)\in\mathbb{N}^2}^{alg} \mathcal{P}_{p,q}(\mathcal{Z})$ (resp., $\bigoplus_{(p,q)\in\mathbb{N}^2}^{alg} \mathcal{P}_{p,q}^{\infty}(\mathcal{Z})$). We remark that $\mathscr{E}$, $\hat{\mathscr{E}}$, and $\mathscr{D}_g$ defined in section 3 are all polynomial symbols[10] on $L^2 \oplus L^2$.

LEMMA 4.5. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *satisfy the same assumptions as in Proposition* 4.4. *Then there exist maps* $\mathscr{B}_j(\cdot) : Q(-\Delta + V) \oplus D(\omega^{1/2}) \to \bigoplus_{(p,q)\in\mathbb{N}^2}^{alg} \mathcal{Q}_{p,q}\big(L^2 \oplus L^2\big)$, $j = 0, \dots, 3$, *such that for any* $t \in \mathbb{R}$, $\xi \in Q(-\Delta + V) \oplus D(\omega^{1/2})$,

$$
\begin{aligned}
\operatorname{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big] &= \operatorname{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^{3} \varepsilon^j \int_0^t \operatorname{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)\big(\mathscr{B}_j(\tilde{\xi}_s)\big)^{Wick}\Big]ds \\
&= \operatorname{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^{3} \varepsilon^j \int_0^t \operatorname{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds\,,
\end{aligned}
\tag{68}
$$

where the $\big(\mathscr{B}_j(\tilde{\xi}_s)\big)^{Wick}$ *make sense as densely defined quadratic forms. To simplify the notation, we have set* $B_j(\cdot) := \big(\mathscr{B}_j(\cdot)\big)^{Wick}$.

Proof. We only sketch the proof here since it follows the same lines as in [5, section 3.2] for the Nelson model with cutoff; see also [12, 89] for detailed accounts of the general strategy. By (66), the only thing we have to prove is that, in the sense of quadratic forms, $\frac{i}{\varepsilon}[\hat{H}_I^{\mathrm{ren}}, W(\tilde{\xi}_s)] = \sum_{j=0}^3 W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)$. First, we remark that $\hat{H}_I^{\mathrm{ren}} = \hat{H}_I(\infty)$, defined by (16), is the Wick quantization of a polynomial symbol;[11] i.e., $\hat{H}_I^{\mathrm{ren}} = \big(\hat{\mathscr{E}}_I\big)^{Wick}$, with

$$
\begin{aligned}
\hat{\mathscr{E}}_I(u,\alpha) = {} & \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{\chi_{\sigma_0}}{\sqrt{2\omega(k)}} \Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dxdk \\
& + \tfrac{1}{2M} \int_{\mathbb{R}^9} \Big(r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{r}_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar{\alpha}(l)e^{-il\cdot x} \\
& + \bar{r}_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2 dxdkdl \\
& - \tfrac{2}{M}\operatorname{Re} \int_{\mathbb{R}^6} r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}\bar{u}(x)D_x u(x) dxdk \\
& + \tfrac{1}{2} \int_{\mathbb{R}^6} V_\infty(x-y)|u(x)|^2|u(y)|^2 dxdy\,.
\end{aligned}
\tag{69}
$$

We also recall, according to [9, Proposition 2.10 for bounded polynomial symbols] and [89, Proposition 2.1.30 for the general case], that essentially for any $\mathscr{A} \in \bigoplus_{(p,q)\in\mathbb{N}^2}^{alg} \mathcal{Q}_{p,q}(L^2 \oplus L^2)$ the following formula is true, in the sense of forms, for any suitably regular $\xi \in L^2 \oplus L^2$:

$$
W^*(\xi)\big(\mathscr{A}\big)^{Wick}W(\xi) = \Big(\mathscr{A}\big(\cdot + \tfrac{i\varepsilon}{\sqrt{2}}\xi\big)\Big)^{Wick}\,.
\tag{70}
$$

[10] In $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, we adopt the notation $z = (u,\alpha)$; and to each $u(x)$ corresponds the operator valued distribution $\psi(x)$, and to each $\alpha(k)$ the distribution $a(k)$. The Wick quantization is again obtained by substituting each $\big(u^{\#}(x), \alpha^{\#}(k)\big)$ with $\big(\psi^{\#}(x), a^{\#}(k)\big)$, and using the normal ordering of creators to the left of annihilators.

[11] To be precise, we are considering here the quadratic form $\hat{h}_I^{\mathrm{ren}}$, defined and different from zero on the whole space $\mathcal{H}$, since it agrees with $\langle\,\cdot\,, \hat{H}_I^{\mathrm{ren}}\,\cdot\rangle$ when restricted to vectors that belong to $\bigoplus_{n\leq[\mathfrak{C}/\varepsilon]} \mathcal{H}_n$ (this is the case by Lemma 4.2).

Roughly speaking, the Weyl operators $W(\xi)$ translate each creation/annihilation operator by $\mp\frac{i\varepsilon}{\sqrt{2}}\xi$. The result then follows immediately on the states $\varrho_\varepsilon(s)$:

$$[\hat{H}_I^{\mathrm{ren}}, W(\tilde{\xi}_s)] = W(\tilde{\xi}_s)\big(W^*(\tilde{\xi}_s)\hat{H}_I^{\mathrm{ren}}W(\tilde{\xi}_s) - \hat{H}_I^{\mathrm{ren}}\big) = W(\tilde{\xi}_s)\Big(\hat{\mathscr{E}}_I(\cdot + \tfrac{i\varepsilon}{\sqrt{2}}\tilde{\xi}_s) - \hat{\mathscr{E}}_I(\cdot)\Big)^{Wick};$$

finally, we define $\sum_{j=0}^{3}\varepsilon^j \mathscr{B}_j(\xi)(z) = \frac{i}{\varepsilon}(\hat{\mathscr{E}}_I(z + \frac{i\varepsilon}{\sqrt{2}}\xi) - \hat{\mathscr{E}}_I(z))$ to factor out the ε-dependence. ∎

We state the next lemma without giving the tedious proof, which is based on the same type of estimates given in subsection 2.2 for the full operator $\hat{H}_I^{\mathrm{ren}}$.

LEMMA 4.6. *For any $j = 0, 1, 2, 3$, $\xi \in Q(-\Delta + V) \cap D(\omega^{1/2})$, and $\mathfrak{C} > 0$, there exists $C_j(\xi) > 0$ such that for any $\Phi, \Psi \in D(H_0^{1/2}) \cap D(N_1)$, with Φ or Ψ in $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]} \mathcal{H}_n$ and for any $s \in \mathbb{R}$ and $\varepsilon \in (0, \bar{\varepsilon})$,*

$$|\langle \Phi, B_j(\tilde{\xi}_s)\Psi\rangle| \leq C_j(\xi)\|(N_1 + H_0 + \bar{\varepsilon})^{1/2}\Phi\| \cdot \|(N_1 + H_0 + \bar{\varepsilon})^{1/2}\Psi\| . \tag{71}$$

Thanks to this lemma we are now in a position to prove that the higher order terms in ε of (68) (namely those with $j > 0$) vanish in the limit $\varepsilon \to 0$.

PROPOSITION 4.7. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfy assumptions* (A-n) *and* (A(h)')*; let $\xi \in Q(-\Delta + V) \cap D(\omega^{1/2})$. Then the following limit holds for any $t \in \mathbb{R}$:*

$$\lim_{\varepsilon\to 0}\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds = 0 . \tag{72}$$

Proof. By Lemma 4.2 we can write $\varrho_\varepsilon = \sum_i \lambda_i|\Psi_i\rangle\langle\Psi_i|$, where each Ψ_i has nonzero components only in the subspace $\bigoplus_{n\leq[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$, and each $\lambda_i > 0$. Assumption (A(h)') then translates to the fact that each Ψ_i is on the domain $Q(H_0)\cap Q(N_1)$, and in addition $\sum_i \lambda_i\langle\Psi_i, (N_1 + H_0)\Psi_i\rangle \leq C$, uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Therefore, we can write

$$\begin{aligned}&\left|\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds\right| \\ &\qquad \leq \sum_{j=1}^{3}\varepsilon^j\sum_i\lambda_i\int_0^t\Big|\langle W^*(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i, B_j(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle\Big|ds .\end{aligned}$$

Using now Lemma 4.6 and then Lemma 4.1 and the fact that N_1 commutes with $\hat{H}_{\mathrm{ren}}$, we obtain

$$\begin{aligned}
&\left|\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds\right| \\
&\qquad \leq \sum_{j=1}^{3}\varepsilon^j C_j(\xi)\sum_i \lambda_i\int_0^t \|(N_1+H_0+\bar{\varepsilon})^{1/2}W^*(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\| \\
&\qquad \cdot \|(N_1+H_0+\bar{\varepsilon})^{1/2}e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\|ds \\
&\qquad \leq \sum_{j=1}^{3}\varepsilon^j C(\xi)C_j(\xi)\sum_i \lambda_i\int_0^t \langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,(N_1+H_0+\bar{\varepsilon})e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle ds \\
&\qquad \leq \sum_{j=1}^{3}\varepsilon^j C(\xi)C_j(\xi)\sum_i \lambda_i\bigg(t\langle\Psi_i,(N_1+\bar{\varepsilon})\Psi_i\rangle \\
&\qquad + \int_0^t \langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle ds\bigg)\ .
\end{aligned}$$

Now we consider the term $\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle$. First we write it as

$$\begin{aligned}
&\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle = \langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,(\hat{H}^{\mathrm{ren}}-\hat{H}_I^{\mathrm{ren}})e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle \\
&\quad = \sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\left\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\Big(\hat{H}_\infty^{(n)}-\hat{H}_I^{(n)}(\infty)\Big)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\right\rangle \\
&\quad = \sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle - \langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle \\
&\quad \leq \sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\left(\left|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle\right| + \left|\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\right|\right)\ .
\end{aligned} \tag{73}$$

The idea now is to use the bound of (36) on

$$\left|\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\right|\ .$$

The crucial point is that since we have chosen σ_0 such that the dynamics is nontrivial for any $n \leq [\mathfrak{C}/\varepsilon]$, it follows that there exist an $a<1$ and a $b<\infty$ *both independent of* ε *and* n such that the bound (36) holds for any $n\leq[\mathfrak{C}/\varepsilon]$. Therefore, we obtain

$$\begin{aligned}
\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle &\leq a\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle + b\langle\Psi_i,\Psi_i\rangle \\
&\quad + \sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\left|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle\right|\ .
\end{aligned} \tag{74}$$

Now, since $a<1$, we may take it to the left-hand side and use again (36) on $|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle|$:

$$\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,H_0 e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\rangle \leq \frac{1}{1-a}\langle\Psi_i,H_0\Psi_i\rangle + \frac{2b}{1-a}\langle\Psi_i,\Psi_i\rangle\ . \tag{75}$$

Finally, since the state is normalized (i.e., $\sum_i \lambda_i \langle \Psi_i, \Psi_i\rangle = 1$), we conclude that

$$\begin{aligned}
&\left| \sum_{j=1}^{3} \varepsilon^j \int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s) W(\tilde{\xi}_s) B_j(\tilde{\xi}_s)\Big] ds \right| \\
&\quad \leq t \sum_{j=1}^{3} \varepsilon^j C(\xi) C_j(\xi) \sum_i \lambda_i \Big(\langle \Psi_i, N_1 \Psi_i\rangle + \tfrac{1}{1-a} \langle \Psi_i, H_0 \Psi_i\rangle + (\tfrac{2b}{1-a} + \bar{\varepsilon}) \langle \Psi_i, \Psi_i \rangle \Big) \\
&\quad \leq t \sum_{j=1}^{3} \varepsilon^j C(\xi) C_j(\xi) \Big(\Big(1 + \tfrac{1}{1-a}\Big) \sum_i \lambda_i \langle \Psi_i, (N_1 + H_0)\Psi_i \rangle + \tfrac{2b}{1-a} + \bar{\varepsilon} \Big) \\
&\quad \leq t \sum_{j=1}^{3} \varepsilon^j C(\xi) C_j(\xi) \Big(\Big(1 + \tfrac{1}{1-a}\Big) C + \tfrac{2b}{1-a} + \bar{\varepsilon} \Big) .
\end{aligned}$$

The right-hand side has no implicit dependence on ε, so it converges to zero when $\varepsilon \to 0$. ◻

By the same argument used from (73) to (75) above, we can prove the following useful lemma.

Lemma 4.8. *If a family of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies assumptions* (A-n) *and* (A(h)')*, then for any $t \in \mathbb{R}$, $\big(\varrho_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$ and $\big(\tilde{\varrho}_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfy assumptions* (A-n) *and* (A(h)')*. In particular, there exist $a(\mathfrak{C}) < 1$ and $b(\mathfrak{C}) > 0$ such that uniformly on $\varepsilon \in (0, \bar{\varepsilon})$*

$$\mathrm{Tr}[\varrho_\varepsilon(t) N_1^k] \leq \mathfrak{C}^k \quad \forall k \in \mathbb{N} \, , \tag{76}$$

$$\mathrm{Tr}[\varrho_\varepsilon(t)(N_1 + H_0)] \leq \frac{1}{1 - a(\mathfrak{C})} C + \frac{2b(\mathfrak{C})}{1 - a(\mathfrak{C})} \, , \tag{77}$$

and the same bounds hold for $\big(\tilde{\varrho}_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$.

It remains to study the limit of the $B_0(\cdot)$-term in (68). As already pointed out in Lemma 4.5, we know that B_0 is a Wick quantization. More precisely, there exists a densely defined map from the one-particle space to polynomial symbols in $\bigoplus_{(p,q)\in\{(i,j)|0\leq i,j\leq 2; 2\leq i+j\leq 3\}} \mathcal{Q}_{p,q}\big(L^2 \oplus L^2\big)$. In order to apply the convergence results of [9], we need to show that the symbol of B_0 may be approximated by a compact one, with an error that vanishes in the limit $\varepsilon \to 0$.

To improve readability, we will write $B_0(\xi)$ in a schematic fashion. The precise structure of each term will be discussed and analyzed in the proof of the sequent proposition. In addition, as seen in (16), the dressed interaction quadratic form $\hat{H}_I(\infty)$ can be split into three terms: the first is just the interaction term $H_I(\sigma_0)$ of the Nelson model with cutoff (with σ replaced by σ_0), whose classical limit has been analyzed by the authors in [5]; the second is a "mean-field" term for the nucleons, of the same type as the ones analyzed by Ammari and Nier in [12]; the last one has a structure similar to the interaction part of the Pauli–Fierz model (see, e.g., [15, 16, 17, 111]) and thus will be called of "Pauli–Fierz type." We will concentrate on the analysis of the Pauli–Fierz type terms of B_0, while for a precise treatment of the others the reader may refer to [5, 12]. In order to highlight the different parts of $B_0(\xi) = B_0(\xi_1, \xi_2)$, we will use different styles of underlining to distinguish the

Nelson, mean-field, and Pauli–Fierz type terms:

(78)
$$\begin{aligned} B_0(\xi_1,\xi_2) = \big(\mathscr{B}_0(\xi_1,\xi_2)\big)^{Wick} &= \underline{(a^*+a)(\xi_1\psi^* - \bar{\xi}_1\psi) + \mathrm{Im}(\xi_2)\psi^*\psi} \\ &+ \bar{\xi}_1\psi^*\psi\psi - \xi_1\psi^*\psi^*\psi \\ &+ (a^*a^* + aa + a^*a)(\xi_1\psi^* - \bar{\xi}_1\psi) + (\xi_2 a^* - \bar{\xi}_2 a)\psi^*\psi \\ &+ (a^* D_x + D_x a)(\psi^*\xi_1 - \bar{\xi}_1\psi) + \psi^* D_x \xi_2\psi - \psi^*\bar{\xi}_2 D_x\psi\ . \end{aligned}$$

Proposition 4.9. *There exists a family of maps $(\mathscr{B}_0^{(m)})_{m\in\mathbb{N}}$ such that the following hold:*

* *For any $m \in \mathbb{N}$*

$$\mathscr{B}_0^{(m)}(\cdot) : Q(-\Delta+V)\oplus D(\omega^{3/4}) \to \bigoplus_{(p,q)\in\{(i,j)|0\leq i,j\leq 2; 2\leq i+j\leq 3\}} \mathcal{P}^{\infty}_{p,q}(L^2\oplus L^2).$$

* *For any $\xi \in Q(-\Delta+V)\oplus D(\omega^{3/4})$, there exists a sequence $\big(C^{(m)}(\xi)\big)_{m\in\mathbb{N}}$ that depends only on $\|\xi\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})}$ such that $\lim_{m\to\infty} C^{(m)} = 0$, and such that for any two vectors $\Phi,\Psi \in \mathcal{H}\cap D(N_1)$, and for any $\varepsilon \in (0,\bar{\varepsilon})$,*

$$\begin{aligned} &\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \big(\mathscr{B}_0(\xi) - \mathscr{B}_0^{(m)}(\xi)\big)^{Wick}(H_0+1)^{-1/2}\Psi\Big\rangle\Big| \\ &\qquad\leq C^{(m)}(\xi)\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\|\,\big\|(N_1+\bar{\varepsilon})^{1/2}\Psi\big\|\ . \end{aligned} \tag{79}$$

Remark 4.10. Contrary to what it was previously assumed throughout section 4, in this proposition we need additional regularity on ξ_2, namely $\xi_2 \in D(\omega^{3/4}) \subset D(\omega^{1/2})$. This will not be a problem in the following, since we will extend our results to any $\xi \in L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ by a density argument, and $D(\omega^{3/4})$ is still dense in $L^2(\mathbb{R}^3)$.

Proof of Proposition 4.9. To prove the proposition, we need to analyze each term of (78) and prove that either it has a compact symbol or it can be approximated by one, in a way that (79) holds. The analysis for the Nelson terms has been carried out in [5, Proposition 3.11 and Lemma 3.15]. In addition, using Lemma 2.6, we see that V_∞ satisfies the hypotheses of the mean field potentials in [12], and therefore (79) holds also for the mean-field terms; see in particular section 3.2 of [12]. For the sake of completeness, we explicitly write the Nelson and mean field parts of (78):

$$\begin{aligned} &\underline{(a^*+a)(\xi_1\psi^* - \bar{\xi}_1\psi)} \\ &\quad = -\tfrac{1}{\sqrt{2}(2\pi)^{3/2}}\int_{\mathbb{R}^3}\Big(a^*\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_{\sigma_0}\big) + a\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_{\sigma_0}\big)\Big)\Big(\xi_1(x)\psi^*(x) - \bar{\xi}_1(x)\psi(x)\Big)dx\ ; \end{aligned}$$

$$\underline{\mathrm{Im}(\xi_2)\psi^*\psi} = -\tfrac{1}{\sqrt{2}(2\pi)^{3/2}}\int_{\mathbb{R}^6}\bigg(\frac{\chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}}\Big(\xi_2(k)e^{ik\cdot x} - \bar{\xi}_2(k)e^{-ik\cdot x}\Big)\bigg)\psi^*(x)\psi(x)\,dx\,dk\ ;$$

$$\begin{aligned} &\bar{\xi}_1\psi^*\psi\psi - \xi_1\psi^*\psi^*\psi \\ &\qquad = \tfrac{1}{\sqrt{2}}\int_{\mathbb{R}^6} V_\infty(x-y)\Big(\bar{\xi}_1(y)\psi^*(x)\psi(x)\psi(y) - \xi_1(y)\psi^*(x)\psi^*(y)\psi(x)\Big)dx\,dy\ . \end{aligned}$$

It remains to study the terms of Pauli–Fierz type. This is done in six parts; in each part we group terms that are either adjoint of each other or can be treated in a similar fashion.

Part 1 ($\bar{\xi}_1 aa\psi$, $\xi_1 a^* a^* \psi^*$).

$$\bar{\xi}_1 aa\psi = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a\big(r_\infty e^{-ik\cdot x}\big)\Big)^2 \bar{\xi}_1(x)\psi(x)dx\;.$$

We recall that $r_\infty \sim k g_\infty$, where g_σ is defined by (7) for any $\sigma \le \infty$. Let $\bar{\xi}_1\alpha\alpha u$ be the symbol[12] associated to $\bar{\xi}_1 aa\psi$; i.e., $\bar{\xi}_1 aa\psi = (\bar{\xi}_1\alpha\alpha u)^{Wick}$. Now, since $r_\infty \notin L^2(\mathbb{R}^3)$, we cannot expect that $\bar{\xi}_1\alpha\alpha u$ is defined for any $u,\alpha \in L^2(\mathbb{R}^3)$ and therefore that it is a compact symbol. We introduce the approximated symbol $\bar{\xi}_1\alpha\alpha u^{(m)}$ defined by

$$\bar{\xi}_1 aa\psi^{(m)} = (\bar{\xi}_1\alpha\alpha u^{(m)})^{Wick} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a\big(r_{\sigma_m} e^{-ik\cdot x}\big)\Big)^2 \bar{\xi}_1(x)\psi(x)dx\;,$$

with $(\sigma_m)_{m\in\mathbb{N}} \subset \mathbb{R}$ such that $\lim_{m\to\infty}\sigma_m = \infty$. First, we prove that (79) holds for $\bar{\xi}_1 aa\psi - \bar{\xi}_1 aa\psi^{(m)}$:

$$\begin{aligned}\Big|\Big\langle (H_0+1)^{-1/2}\Phi, (\bar{\xi}_1 aa\psi - \bar{\xi}_1 aa\psi^{(m)})(H_0+1)^{-1/2}\Psi\Big\rangle\Big| &\le \tfrac{1}{2\sqrt{2}M}\|\xi_1\|_2\big\|(d\Gamma(\omega)+1)^{1/2}\\ &\qquad (H_0+1)^{-1/2}\Phi\big\|\\ &\cdot \sup_{x\in\mathbb{R}^3}\Big\|(d\Gamma(\omega)+1)^{-1/2}\Big(a\big((r_\infty - r_{\sigma_m})e^{-ik\cdot x}\big)\Big)^2(d\Gamma(\omega)+1)^{-1/2}\\ &\cdot(d\Gamma(\omega)+1)^{1/2}(H_0+1)^{-1/2}(N_1+\varepsilon)^{1/2}\Psi\Big\|\;.\end{aligned}$$

We use (28) of Lemma 2.8 and the fact that $(d\Gamma(\omega)+1)^{1/2}(H_0+1)^{-1/2}$ is bounded with norm smaller than one to obtain

$$\begin{aligned}&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, (\bar{\xi}_1 aa\psi - \bar{\xi}_1 aa\psi^{(m)})(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\ &\qquad\le \tfrac{c}{2\sqrt{2}M}\|\xi_1\|_2\|\omega^{-1/4}(r_\infty - r_{\sigma_m})\|_2^2\|\Phi\|\;\|(N_1+\bar{\varepsilon})^{1/2}\Psi\|\\ &\qquad\le C^{(m)}(\xi_1)\|(N_1+\bar{\varepsilon})^{1/2}\Phi\|\cdot\|(N_1+\bar{\varepsilon})^{1/2}\Psi\|\;,\end{aligned}$$

with $C^{(m)}(\xi_1) = C(\bar{\varepsilon},\xi_1)\|\omega^{-1/4}(r_\infty - r_{\sigma_m})\|_2^2$ for some $C(\bar{\varepsilon},\xi_1) > 0$. The sequence $\big(C^{(m)}(\xi_1)\big)_{m\in\mathbb{N}}$ converges to zero since by our choice of $(\sigma_m)_{m\in\mathbb{N}}$

$$\lim_{m\to\infty}\|\omega^{-1/4}(r_\infty - r_{\sigma_m})\|_2^2 = 0\;.$$

It remains to show that $\bar{\xi}_1\alpha\alpha u^{(m)}$ is a compact symbol. Such a symbol can be written as

$$\bar{\xi}_1\alpha\alpha u^{(m)} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^9}\bar{\xi}_1(x)\bar{r}_{\sigma_m}(k)\bar{r}_{\sigma_m}(k')e^{i(k+k')\cdot x}\alpha(k)\alpha(k')u(x)dxdkdk'\;.$$

Now we can define an operator $\tilde{b}_{\alpha\alpha u} : \big(L^2 \oplus L^2\big)^{\otimes_s 3} \to \mathbb{C}$ in the following way. Let the maps $\pi_1,\pi_2 : L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ be the projections on the first and second

[12]We recall that for the Nelson model $\mathcal{Z} = L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$; thus we denote the variable z by $u\oplus\alpha$.

spaces, respectively. Then we define the operator $\tilde{b}_{\alpha\alpha u}$ as

$$\begin{array}{rcl} \tilde{b}_{\alpha\alpha u} : (u,\alpha)^{\otimes 3} \in \left(L^2 \oplus L^2\right)^{\otimes_s 3} & \xrightarrow{\pi_2\otimes\pi_2\otimes\pi_1} & \alpha(k)\alpha(k')u(x) \in L^2(\mathbb{R}^9) \\ & & \downarrow \\ & & \langle f, \alpha\alpha u\rangle_{L^2(\mathbb{R}^9)} \in \mathbb{C}, \end{array}$$

where $f(k,k',x) = -\frac{1}{2\sqrt{2}M}\bar{\xi}_1(x)\bar{r}_{\sigma_m}(k)\bar{r}_{\sigma_m}(k')e^{i(k+k')\cdot x} \in L^2(\mathbb{R}^9)$. Therefore, $\tilde{b}_{\alpha\alpha u}$ is bounded and of finite rank and therefore compact. The proof for the corresponding adjoint term

$$\underline{\xi_1 a^* a^* \psi^*} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a^*\big(r_\infty e^{-ik\cdot x}\big)\Big)^2\xi_1(x)\psi^*(x)dx$$

can be obtained directly from the above, using the following approximation with compact symbol:

$$\xi_1 a^* a^* \psi^{*(m)} = (\xi_1\bar{\alpha}\bar{\alpha}\bar{u}^{(m)})^{Wick} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a^*\big(r_{\sigma_m} e^{-ik\cdot x}\big)\Big)^2\xi_1(x)\psi^*(x)dx\ .$$

Part 2 $(\xi_1 a a\psi^*,\ \bar{\xi}_1 a^* a^*\psi)$.

$$\underline{\xi_1 a a \psi^*} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a\big(r_\infty e^{-ik\cdot x}\big)\Big)^2\xi_1(x)\psi^*(x)dx\ .$$

Again we approximate this term by

$$\xi_1 a a \psi^{*(m)} = (\xi_1\alpha\alpha\bar{u}^{(m)})^{Wick} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a\big(r_{\sigma_m} e^{-ik\cdot x}\big)\Big)^2\xi_1(x)\psi^*(x)dx$$

as above. The proof that it satisfies (79) is perfectly analogous to the one for the previous term. Therefore, we only prove that $\xi_1\alpha\alpha\bar{u}^{(m)}$ is a compact symbol. We define an operator $b_{\alpha\alpha\bar{u}} : \left(L^2 \oplus L^2\right)^{\otimes_s 2} \to L^2 \oplus L^2$ by

$$\begin{array}{rcl} \tilde{b}_{\alpha\alpha\bar{u}} : (u,\alpha)^{\otimes 2} \in \left(L^2 \oplus L^2\right)^{\otimes_s 2} & \xrightarrow{\qquad\pi_2\otimes\pi_2\qquad} & \alpha(k)\alpha(k') \in L^2(\mathbb{R}^6) \\ & & \Big\downarrow{\scriptstyle \tilde{c}_{\alpha\alpha\bar{u}}} \\ & & \left(\int_{\mathbb{R}^6}\bar{f}(k,k',\cdot)\alpha(k)\alpha(k')dkdk' \ \oplus\ 0\right) \in L^2 \oplus L^2, \end{array}$$

where $f(k,k',x) = -\frac{1}{2\sqrt{2}M}\xi_1(x)r_{\sigma_m}(k)r_{\sigma_m}(k')e^{-i(k+k')\cdot x}$. By definition, we have that

$$\xi_1\alpha\alpha\bar{u}^{(m)} = \langle(u,\alpha), \tilde{b}_{\alpha\alpha\bar{u}}(u,\alpha)^{\otimes 2}\rangle_{L^2\oplus L^2}\ .$$

It is easily seen that the operator $\tilde{c}_{\alpha\alpha\bar{u}}$ is bounded. It is in fact compact: let $\beta_j \rightharpoonup \beta$ in $L^2(\mathbb{R}^6)$ be a weakly convergent (bounded) sequence such that $\max\big\{\big(\sup_j\|\beta_j\|_{L^2(\mathbb{R}^6)}\big), \|\beta\|_{L^2(\mathbb{R}^6)}\big\} = X < \infty$; then

$$\big\|\tilde{c}_{\alpha\alpha\bar{u}}(\beta-\beta_j)\big\|_{L^2\oplus L^2} = \big\|\langle f(k,k',x),(\beta-\beta_j)(k,k')\rangle_{L^2_{k,k'}(\mathbb{R}^6)}\big\|_{L^2_x(\mathbb{R}^3)} \underset{j\to\infty}{\longrightarrow} 0\ ,$$

by Lebesgue's dominated convergence theorem, using the uniform bound

$$\begin{aligned}\Big|\langle f(k,k',x),(\beta-\beta_j)(k,k')\rangle^2_{L^2_{k,k'}(\mathbb{R}^6)}\Big| &\leq \|f(k,k',x)\|^2_{L^2_{k,k'}(\mathbb{R}^6)}\Big(\|\beta\|^2_{L^2(\mathbb{R}^6)}+\|\beta_j\|^2_{L^2(\mathbb{R}^6)}\Big)\\ &\leq \tfrac{2X}{8M^2}\|r_{\sigma_m}\|_2^4|\xi_1(x)|^2\in L^1_x(\mathbb{R}^3)\ .\end{aligned}$$

Therefore, since $\tilde{c}_{\alpha\alpha\bar{u}}$ is compact and $\pi_2\otimes\pi_2$ is bounded, it follows that $\tilde{b}_{\alpha\alpha\bar{u}}$ is compact. Again, that implies the result holds also for the adjoint term

$$\underline{\bar{\xi}_1 a^*a^*\psi}=-\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a^*\big(r_\infty e^{-ik\cdot x}\big)\Big)^2\bar{\xi}_1(x)\psi(x)dx\ .$$

Part 3 ($\bar{\xi}_1 a^*a\psi$, $\xi_1 a^*a\psi^*$).

$$\begin{aligned}\underline{\bar{\xi}_1 a^*a\psi}&=-\tfrac{1}{\sqrt{2}M}\int_{\mathbb{R}^3}a^*\big(r_\infty e^{-ik\cdot x}\big)a\big(r_\infty e^{-ik\cdot x}\big)\bar{\xi}_1(x)\psi(x)dx\ ,\\ \underline{\xi_1 a^*a\psi^*}&=-\tfrac{1}{\sqrt{2}M}\int_{\mathbb{R}^3}a^*\big(r_\infty e^{-ik\cdot x}\big)a\big(r_\infty e^{-ik\cdot x}\big)\xi_1(x)\psi^*(x)dx\ .\end{aligned}$$

The proof for this couple of terms goes on exactly like the previous one, i.e., approximating r_∞ with r_{σ_m} and showing that the corresponding operator $\tilde{c}_{\bar{\alpha}\alpha u}$ is compact, for it maps weakly convergent sequences into strongly convergent ones.

Part 4 ($\bar{\xi}_2 a\psi^*\psi$, $\xi_2 a^*\psi^*\psi$).

$$\underline{\bar{\xi}_2 a\psi^*\psi}=-\tfrac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\mathrm{Im}\big(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\big)a\big(r_\infty e^{-ik\cdot x}\big)\psi^*(x)\psi(x)dxdk'\ .$$

We approximate it by the symbol $\bar{\xi}_2\alpha\bar{u}u^{(m)}$ defined by

$$\begin{aligned}\bar{\xi}_2 a\psi^*\psi^{(m)}&=(\bar{\xi}_2\alpha\bar{u}u^{(m)})^{Wick}\\ &=-\tfrac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\psi^*(x)\chi_m(D_x)\mathrm{Im}\big(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\big)a\big(r_{\sigma_m}e^{-ik\cdot x}\big)\psi(x)dxdk'\ ,\end{aligned}$$

where χ_m is the smooth cutoff function defined at the beginning of section 2, while r_{σ_m} is the usual regularization of r_∞ defined above. First we check that the approximation satisfies (79). By the chain rule, two parts have to be checked:

$$\begin{aligned}&\Big|\Big\langle(H_0+1)^{-1/2}\Phi,(\bar{\xi}_2 a\psi^*\psi-\bar{\xi}_2 a\psi^*\psi^{(m)})(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\ &\qquad\leq\tfrac{\sqrt{2}(2\pi)^{3/2}}{M}\bigg(\Big|\Big\langle(H_0+1)^{-1/2}\Phi,\int_{\mathbb{R}^3}dx\ \psi^*(x)\big(1-\chi_m(D_x)\big)\\ &\qquad\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big(r_\infty e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\ &\qquad+\Big|\Big\langle(H_0+1)^{-1/2}\Phi,\int_{\mathbb{R}^3}dx\ \psi^*(x)\chi_m(D_x)\\ &\qquad\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big((r_\infty-r_{\sigma_m})e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\bigg)\ ;\end{aligned}$$

and we will consider them separately. For the first part we have

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\big(1-\chi_m(D_x)\big)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big(r_\infty e^{-ik\cdot x}\big) \\
&\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big| \\
&\leq \sum_{n=0}^{\infty} n\varepsilon \Big|\Big\langle (H_0^{(n)}+1)^{-1/2}\Phi_n, (1-\chi_m(D_{x_1}))\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x_1)a\big(r_\infty e^{-ik\cdot x_1}\big) \\
&(H_0^{(n)}+1)^{-1/2}\Psi_n\Big\rangle_{\mathcal{H}_n}\Big| \\
&\leq \sum_{n=0}^{\infty} n\varepsilon \big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\big\|\mathcal{F}^{-1}(\xi_2\bar{r}_\infty)\big\|_\infty \\
&\big\|\omega^{-1/2}r_\infty\big\|_2\cdot\big\|(1-D_{x_1}^2)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}\cdot\big\|d\Gamma(\omega)^{1/2}(H_0^{(n)}+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n} \\
&\leq (1+\bar{\varepsilon})\|\xi_2\|_{\mathcal{F}H^{1/2}}\cdot\big\|\omega^{-1/2}r_\infty\big\|_2^2\cdot\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))} \\
&\cdot\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\|\cdot\big\|N_1^{1/2}\Psi\big\|\ ,
\end{aligned}$$

where in the last inequality we have utilized the following bound:

$$\begin{aligned}
n\varepsilon&\big\|(1-D_{x_1}^2)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}^2 \\
&= \big\langle\Phi_n, (H_0^{(n)}+1)^{-1/2}d\Gamma(1-\Delta)(H_0^{(n)}+1)^{-1/2}\Phi_n\big\rangle_{\mathcal{H}_n} \\
&\leq \big\|N_1^{1/2}\Phi_n\big\|_{\mathcal{H}_n}+\big\|d\Gamma(-\Delta)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n} \\
&\leq (1+\bar{\varepsilon})\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi_n\big\|_{\mathcal{H}_n}\ .
\end{aligned}$$

So the first part satisfies (79), since

$$\lim_{m\to\infty}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}=0\ .$$

A similar procedure for the second part yields

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\chi_m(D_x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big((r_\infty-r_{\sigma_m})e^{-ik\cdot x}\big) \\
&\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big| \\
&\leq \|\xi_2\|_{\mathcal{F}H^{1/2}}\cdot\big\|\omega^{-1/2}r_\infty\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\big\|_2\big\|N_1^{1/2}\Phi\big\|\cdot\big\|N_1^{1/2}\Psi\big\|\ ;
\end{aligned}$$

i.e., it satisfies (79) for $\lim_{m\to\infty}\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\|_2=0$. Now it remains to show that $\bar{\xi}_2\alpha\bar{u}u^{(m)}$ is a compact symbol:

$$\bar{\xi}_2\alpha\bar{u}u^{(m)}=-\tfrac{(2\pi)^{3/2}\sqrt{2}i}{M}\int_{\mathbb{R}^6}\bar{u}(x)\chi_m(D_x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)\bar{r}_{\sigma_m}(k)e^{ik\cdot x}\alpha(k)u(x)dxdk\ .$$

As for the previous terms, we define an operator $b_{\alpha\bar{u}u}:\big(L^2\oplus L^2\big)^{\otimes_s 2}\to L^2\oplus L^2$ by

$$\begin{aligned}
\tilde{b}_{\alpha\bar{u}u}:(u,\alpha)^{\otimes 2}\in\big(L^2\oplus L^2\big)^{\otimes_s 2}\xrightarrow{\pi_2\otimes\pi_1}\ &\alpha(k)u(x)\in L^2(\mathbb{R}^6) \\
&\Big\downarrow{\scriptstyle \tilde{c}_{\alpha\bar{u}u}} \\
&\Big(f'(x,D_x)u(x)\oplus 0\Big)\in L^2\oplus L^2,
\end{aligned}$$

where $f'(x,D_x)=-\frac{(2\pi)^3\sqrt{2}i}{M}\chi_m(D_x)\mathcal{F}^{-1}\big(\bar{r}_{\sigma_m}\alpha\big)(x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)$. We can easily prove that $f':L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is a compact operator. The cutoff function χ_m belongs to $L_0^\infty(\mathbb{R}^3)$ by hypothesis.[13] Now both $\bar{r}_{\sigma_m}\alpha$ and $\xi_2\bar{r}_\infty$ belong to $L^1(\mathbb{R}^3)$, since $r_{\sigma_m},\alpha,\omega^{1/2}\xi_2,\omega^{-1/2}r_\infty\in L^2(\mathbb{R}^3)$. Therefore, $\mathcal{F}^{-1}\big(\bar{r}_{\sigma_m}\alpha\big)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)\in L_0^\infty(\mathbb{R}^3)$; hence $f'(x,D_x)\in\mathcal{K}(L^2(\mathbb{R}^3))$. It immediately follows that $\tilde{b}_{\alpha\bar{u}u}$ is compact, and the proof is complete. As usual, this result implies the one for the adjoint term

$$\underline{\xi_2a^*\psi^*\psi}=-\frac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\mathrm{Im}\big(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\big)a^*\big(r_\infty e^{-ik\cdot x}\big)\psi^*(x)\psi(x)dxdk'\ .$$

Part 5 ($D_xa\bar{\xi}_1\psi$, $a^*D_x\psi^*\xi_1$, $D_xa\psi^*\xi_1$, $a^*D_x\bar{\xi}_1\psi$).

$$\underline{D_xa\bar{\xi}_1\psi}=\frac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)D_xa\big(r_\infty e^{-ik\cdot x}\big)\psi(x)dx\ .$$

The approximated symbol $D_xa\bar{\xi}_1\psi^{(m)}$ is given by

$$D_xa\bar{\xi}_1\psi^{(m)}=\frac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)D_xa\big(r_{\sigma_m}e^{-ik\cdot x}\big)\psi(x)dx\ .$$

First we prove that (79) is satisfied. Given $\Phi\in\mathcal{H}$, we denote by $\Phi_{n,p}$ its restriction to the subspace $\mathcal{H}_{n,p}=(L^2(\mathbb{R}^3))^{\otimes_s n}\otimes(L^2(\mathbb{R}^3))^{\otimes_s p}$ with n nucleons and p mesons. We also denote by $X_n=\{x_1,\dots,x_n\}$ a set of variables, and $dX_n=dx_1,\dots,dx_n$ the corresponding Lebesgue measure (and analogously for K_p, dK_p). The proof is obtained by a direct calculation on the Fock space as follows:

$$\begin{aligned}
&\Big|\Big\langle(H_0+1)^{-1/2}\Phi,\int_{\mathbb{R}^3}\bar{\xi}_1(x)D_xa\big((r_\infty-r_{\sigma_m})e^{-ik\cdot x}\big)\psi(x)dx(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\
=&\Big|\sum_{n,p=0}^\infty\varepsilon\sqrt{(n+1)(p+1)}\int_{\mathbb{R}^{(n+p+2)d}}\overline{\Big((H_0+1)^{-1/2}\Phi\Big)}_{n,p}(X_n;K_p)\\
&\bar{\xi}_1(x)D_x(\bar{r}_\infty-\bar{r}_{\sigma_m})(k)e^{ik\cdot x}\overline{\Big((H_0+1)^{-1/2}\Psi\Big)}_{n+1,p+1}(x,X_n;k,K_p)dxdX_ndkdK_p\Big|\\
\leq&\sum_{n,p=0}^\infty\sqrt{\varepsilon(n+1)}\Big|\int_{\mathbb{R}^{(n+p+2)d}}\overline{\Big((H_0+1)^{-1/2}\Phi\Big)}_{n,p}(X_n;K_p)\overline{D_x\xi_1}(x)\overline{\tfrac{r_\infty-r_{\sigma_m}}{\sqrt{\omega}}}(k)\\
&e^{ik\cdot x}\sqrt{\varepsilon(p+1)\omega(k)}\overline{\Big((H_0+1)^{-1/2}\Psi\Big)}_{n+1,p+1}(x,X_n;k,K_p)dxdX_ndkdK_p\Big|\\
\leq&\sum_{n,p=0}^\infty\sqrt{\varepsilon(n+1)}\big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\big\|_2\cdot\big\|(H_0+1)^{-1/2}\Phi_{n,p}\big\|_{\mathcal{H}_{n,p}}\\
&\cdot\big\|e^{ik\cdot x}\sqrt{\varepsilon(p+1)\omega(k_1)}(H_0+1)^{-1/2}\Psi_{n+1,p+1}(X_{n+1};K_{p+1})\big\|_{\mathcal{H}_{n+1,p+1}}\\
\leq&\big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\big\|_2\cdot\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\|\cdot\big\|\Psi\big\|\ ,
\end{aligned}$$

where in the last bound we have used Schwarz's inequality and the fact that $p\omega(k_1)\equiv\sum_{j=1}^p\omega(k_j)$ when acting on vectors of $\mathcal{H}_{n,p}$. Now, since $\lim_{m\to\infty}\big\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\big\|_2=0$, (79) holds with $C^{(m)}(\xi_1)=\frac{1}{\sqrt{2M}}\big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty-r_{\sigma_m})\big\|_2$.

[13]We denote by $L_0^\infty(\mathbb{R}^3)$ the set of bounded functions on $\mathbb{R}^3$ that vanish at infinity.

It remains to show that the classical symbol

$$D_x\alpha\bar{\xi}_1 u^{(m)} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^6}\bar{\xi}_1(x)D_x\alpha(k)\bar{r}_{\sigma_m}(k)e^{ik\cdot x}u(x)dxdk$$

is compact. Here we have written $D_x\alpha\bar{\xi}_1 u^{(m)} = \langle\xi_1, D_x v\rangle_2$, with $v(x) = \frac{(2\pi)^{3/2}}{\sqrt{2M}}\mathcal{F}^{-1}\big(\alpha\bar{r}_{\sigma_m}\big)(x)u(x)$; and that is defined for any $v\in\dot{H}^1(\mathbb{R}^3)$. However, since $\xi_1\in Q(-\Delta+V)\subset H^1(\mathbb{R}^3)$ and D_x is self-adjoint, we can write $D_x\alpha\bar{\xi}_1 u^{(m)} = \langle D_x\xi_1, v\rangle_2$ for any $v\in L^2(\mathbb{R}^3)$. It follows that $D_x\alpha\bar{\xi}_1 u^{(m)}$ is defined for any $u,\alpha\in L^2(\mathbb{R}^3)$, since $\alpha, r_{\sigma_m}\in L^2$ implies $\alpha\bar{r}_{\sigma_m}\in L^1$, and therefore $\mathcal{F}^{-1}\big(\alpha\bar{r}_{\sigma_m}\big)\in L^\infty$. It follows that the operator $\tilde{b}_{D_x\alpha u} : \big(L^2\oplus L^2\big)^{\otimes_s 2}\to\mathbb{C}$ defined as

$$\tilde{b}_{D_x\alpha u} : (u,\alpha)^{\otimes 2}\in\big(L^2\oplus L^2\big)^{\otimes_s 2}\underset{\pi_2\otimes\pi_1}{\longrightarrow}\alpha(k)u(x)\in L^2(\mathbb{R}^6)\longrightarrow\langle f'',\alpha u\rangle_{L^2(\mathbb{R}^6)}\in\mathbb{C}\,,$$

with $f''(x,k) = \frac{1}{\sqrt{2M}}(D_x\xi_1)(x)r_{\sigma_m}(k)e^{-ik\cdot x}$, is bounded and of finite rank and therefore compact.

$$\underline{a^*D_x\bar{\xi}_1\psi} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)a^*\big(r_\infty e^{-ik\cdot x}\big)D_x\psi(x)dx\,.$$

Again, the approximated symbol $a^*D_x\bar{\xi}_1\psi$ is given by

$$a^*D_x\bar{\xi}_1\psi^{(m)} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)a^*\big(r_{\sigma_m}e^{-ik\cdot x}\big)D_x\psi(x)dx\,.$$

Inequality (79) is satisfied, and the proof follows the same guidelines as the one for the previous term $D_x a\bar{\xi}_1\psi$. We give the compactness proof for the symbol

$$\bar{\alpha}D_x\bar{\xi}_1 u^{(m)} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^6}\bar{\xi}_1(x)\bar{\alpha}(k)r_{\sigma_m}(k)e^{-ik\cdot x}D_x u(x)dxdk\,.$$

We rewrite it as $\bar{\alpha}D_x\bar{\xi}_1 u^{(m)} = \langle(u,\alpha),\tilde{b}_{\bar{\alpha}D_x u}(u,\alpha)\rangle_{L^2\oplus L^2}$, with $\tilde{b}_{\bar{\alpha}D_x u} : L^2\oplus L^2\to L^2\oplus L^2$ defined as

$$\tilde{b}_{\bar{\alpha}D_x u} : (u,\alpha)\in L^2\oplus L^2\underset{\pi_1}{\longrightarrow}u(x)\in L^2(\mathbb{R}^3)\underset{\tilde{c}_{\bar{\alpha}D_x u}}{\longrightarrow}\Big(0\,\oplus\,f'''(k)\Big)\in L^2\oplus L^2\,,$$

where $f'''(k) = \frac{1}{\sqrt{2M}}r_{\sigma_m}(k)(k\langle e^{ik\cdot x}\xi_1, u\rangle_{L^2_x} + \langle e^{ik\cdot x}D_x\xi_1, u\rangle_{L^2_x})$. Now suppose that $u_j\rightharpoonup u$ is a weakly convergent (bounded) sequence with bound X. It follows that, uniformly in j,

$$\begin{aligned}|f_j'''(k)|^2 &= \Big|\tfrac{1}{\sqrt{2M}}r_{\sigma_m}(k)\Big(k\langle e^{ik\cdot x}\xi_1,u_j\rangle_{L^2_x} + \langle e^{ik\cdot x}D_x\xi_1,u_j\rangle_{L^2_x}\Big)\Big|^2\\ &\leq \tfrac{1}{2M^2}X^2|r_{\sigma_m}(k)|^2(k^2+1)\|\xi_1\|_{H^1}^2\in L^1_k(\mathbb{R}^3)\,.\end{aligned}$$

In addition, $\lim_{j\to\infty}|f'''(k) - f_j'''(k)|^2 = 0$; therefore, $\tilde{c}_{\bar{\alpha}D_x u}$ is a compact operator by Lebesgue's dominated convergence theorem. So $\tilde{b}_{\bar{\alpha}D_x u}$ is compact. The proofs above extend immediately to the adjoint terms

$$\underline{a^*D_x\psi^*\xi_1} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\psi^*(x)a^*\big(r_\infty e^{-ik\cdot x}\big)D_x\xi_1(x)dx\,,$$

$$\underline{D_x a\psi^*\xi_1} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\psi^*(x)D_x a\big(r_\infty e^{-ik\cdot x}\big)\xi_1(x)dx\,.$$

Part 6 ($\psi^* D_x\xi_2\psi$, $\psi^*\bar{\xi}_2 D_x\psi$).

$$\underline{\psi^* D_x\xi_2\psi} = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\psi^*(x)D_x\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)\psi(x)dx\,.$$

The approximated symbol, as for the terms of part 4, contains $\chi_m(D_x)$:

$$\psi^* D_x\xi_2\psi^{(m)} = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\psi^*(x)\chi_m(D_x)D_x\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)\psi(x)dx\,.$$

As usual, we start proving that (79) holds. We remark that this is the only term where we need $\xi_2\in D(\omega^{3/4})$ instead of $D(\omega^{1/2})$.

$$\begin{aligned}
&\Big|\Big\langle(H_0+1)^{-1/2}\Phi,\int_{\mathbb{R}^3}\psi^*(x)\big(1-\chi_m(D_x)\big)D_x\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)\psi(x)dx(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\
&\leq\sum_{n=0}^\infty n\varepsilon\Big|\Big\langle(H_0+1)^{-1/2}\Phi_n,\big(1-\chi_m(D_{x_1})\big)D_{x_1}\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x_1)(H_0+1)^{-1/2}\Psi_n\Big\rangle\Big|\\
&\leq\sum_{n=0}^\infty n\varepsilon\big\|(1-\Delta)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\Big(\big\|\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)\big\|_\infty+\big\|\mathcal{F}^{-1}\Big(k\xi_2\bar{r}_\infty\Big)\big\|_\infty\Big)\\
&\quad\cdot\big\|(1-\Delta_{x_1})^{1/2}(H_0+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}\cdot\Big(\big\|D_{x_1}(H_0+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n}\\
&\quad+\big\|(H_0+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n}\Big)\\
&\leq 2\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\big\|\omega^{3/4}\xi_2\big\|_2\cdot\big\|\omega^{-1/4}r_\infty\big\|_2\cdot\|\Phi\|\\
&\quad\cdot\Big(\big\|(N_1+\bar{\varepsilon})^{1/2}\Psi\big\|+\|\Psi\|\Big)\,;
\end{aligned}$$

hence the result follows with

$$C^{(m)}(\xi_2)=\tfrac{2\sqrt{2}(2\pi)^{3/2}}{M}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\big\|\omega^{3/4}\xi_2\big\|_2\big\|\omega^{-1/4}r_\infty\big\|_2\,,$$

since $\lim_{m\to\infty}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}=0$. It remains to show that the symbol

$$\bar{u}D_x\xi_2 u^{(m)} = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\bar{u}(x)\chi_m(D_x)D_x\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)u(x)dx$$

is compact. We introduce the operator $\tilde{b}_{\bar{u}D_x u}: L^2\oplus L^2\to L^2\oplus L^2$ such that $\bar{u}D_x\xi_2 u^{(m)}=\langle(u,\alpha),\tilde{b}_{\bar{u}D_x u}(u,\alpha)\rangle_{L^2\oplus L^2}$:

$$\tilde{b}_{\bar{u}D_x u}:(u,\alpha)\in L^2\oplus L^2\xrightarrow[\pi_1]{}u(x)\in L^2(\mathbb{R}^3)\xrightarrow[\tilde{c}_{\bar{u}D_x u}]{}\Big(f''''(x,D_x)u(x)\,\oplus\,0\Big)\in L^2\oplus L^2\,,$$

where $f''''(x,D_x)=\frac{(2\pi)^{3/2}}{\sqrt{2}M}D_x\,\chi_m(D_x)\mathcal{F}^{-1}(\xi_2\bar{r}_\infty)(x)$. Now $f''''(x,D_x)$ is a compact operator: both $x\,\chi_m(x)$ and $\mathcal{F}^{-1}(\xi_2\bar{r}_\infty)(x)$ are in $L_0^\infty(\mathbb{R}^3)$. Therefore, $\tilde{b}_{\bar{u}D_x u}$ is compact. The proof extends immediately to the adjoint term

$$\underline{\psi^*\bar{\xi}_2 D_x\psi} = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\psi^*(x)\mathcal{F}\Big(\bar{\xi}_2 r_\infty\Big)(x)D_x\psi(x)dx\,.$$

4.4. Defining the time-dependent family of Wigner measures. The last tool we need in order to take the limit $\varepsilon \to 0$ of the integral formula (68) is Wigner measures. Throughout this section, we will leave some statements unproven; the reader may refer to [9, section 6] for the proofs and a detailed discussion of Wigner measures' properties. We recall the definition of a Wigner measure associated with a family of states on $\mathcal{H} = \Gamma_s\big(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\big)$.

DEFINITION 4.11. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \subset \mathcal{L}^1\big(\mathcal{H}\big)$ be a family of normal states; $\mu \in \mathfrak{P}\big(L^2 \oplus L^2\big)$ is a Borel probability measure. We say that μ is a Wigner (or semiclassical) measure associated to $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, or in symbols $\mu \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)\big)$, if there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar\varepsilon)$ such that $\lim_{k\to\infty} \varepsilon_k = 0$ and*

$$\lim_{k\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_k} W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle_{L^2\oplus L^2}} d\mu(z) \ \ \forall \xi \in L^2 \oplus L^2 \,. \tag{80}$$

We remark that the right-hand side is essentially the Fourier transform of the measure μ, so considering the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ there is at most one probability measure that could satisfy (80)*. If* (80) *is satisfied, we say that to the sequence $(\varrho_{\varepsilon_k})_{k\in\mathbb{N}}$ corresponds a single Wigner (or semiclassical) measure μ, or simply $\varrho_{\varepsilon_k} \to \mu$.*

First, it is necessary to ensure that such a definition of Wigner measures is meaningful, i.e., that under suitable conditions the set of Wigner measures $\mathcal{M}$ associated to a family of states is not empty. Since $m_0 > 0$, it turns out that assumption (A(h)') is sufficient. Assumption (A-h) would be sufficient as well, even if we will not use it for the moment.

LEMMA 4.12. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ that satisfies assumptions* (A(h)') *and* (A-n)*. Then for any $t \in \mathbb{R}$, the following hold:*

(i) $\mathcal{M}\big(\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big) \neq \emptyset$; $\mathcal{M}\big(\tilde\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big) \neq \emptyset$.

(ii) *Any $\mu \in \mathcal{M}\big(\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big)$ or in $\mathcal{M}\big(\tilde\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big)$*[14] *satisfies*

$$\mu\Big(B_u(0,\sqrt{\mathfrak{C}}) \cap Q(-\Delta + V) \oplus D(\omega^{1/2})\Big) = 1.$$

(iii) *Moreover,*

$$\int_{z=(u,\alpha)\in L^2\oplus L^2} \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}\, d\mu(z) < +\infty\,.$$

We recall that $B_u(0,\sqrt{\mathfrak{C}}) = \{(u,\alpha) \in L^2 \oplus L^2, \|u\|_2 \le \sqrt{\mathfrak{C}}\}$.

Proof. By (77) of Lemma 4.8, we see that $\varrho_\varepsilon(t)$ and $\tilde\varrho_\varepsilon(t)$ satisfy (A-n) and (A(h)') at any time. Now (i) follows by [9, Theorem 6.2] and (ii) by (iii) and [11, Lemma 2.14]. The third point is essentially a consequence of [12, Lemma 3.12]. However, the latter result requires more regularity on the states ϱ_ε. So we indicate here how to adapt the argument to our case. It is enough to assume $t = 0$ and $\{\mu\} = \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)\big)$. The operators $-\frac{\Delta}{2M} + V$ and ω are positive (self-adjoint). So one can find nondecreasing sequences of finite rank operators A_k and B_k that converge weakly to $-\frac{\Delta}{2M} + V$ and ω, respectively. In particular,

$$b_k^{Wick} = d\Gamma(A_k) \otimes 1 + 1 \otimes d\Gamma(B_k) \le d\Gamma\bigg(-\frac{\Delta}{2M} + V\bigg) \otimes 1 + 1 \otimes d\Gamma(\omega) = H_0\,,$$

[14] In this section, we have used mostly the notation $D(\omega^{1/2})$; however, $D(\omega^{1/2}) = \mathcal{F}H^{1/2}$, where the latter is defined in Definition 3.4.

where $b_k(u,\alpha) = \langle u, A_k u\rangle + \langle \alpha, B_k\alpha\rangle \in \mathcal{P}^\infty_{1,1}(L^2 \oplus L^2)$. Let P_k and Q_k be the orthogonal projections on $\mathrm{Ran}(A_k)$ and $\mathrm{Ran}(B_k)$, respectively. Using the Fock space decomposition $\Gamma_s(L^2 \oplus L^2) \equiv \Gamma_s(P_k L^2 \oplus Q_k L^2) \otimes \Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)$ where $P_k^\perp = 1 - P_k$ and $Q_k^\perp = 1 - Q_k$, one can write $b_k^{Wick} \equiv (b_k)^{Wick}_{|\Gamma_s(P_k L^2 \oplus Q_k L^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)}$ and $\varrho_\varepsilon \equiv \hat{\varrho}_\varepsilon$. Hence

$$\mathrm{Tr}\Big[\varrho_\varepsilon b_k^{Wick}\Big] = \mathrm{Tr}\Big[\hat{\varrho}_\varepsilon b^{Wick}_{k|\Gamma_s(P_k L^2 \oplus Q_k L^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)}\Big] = \mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k b_k^{Wick}\Big],$$

where $\varrho^k_{\varepsilon_j}$ is a given reduced density matrix which is trace-class in $\Gamma_s(P_k L^2 \oplus Q_k L^2)$. So the problem is in some sense reduced to finite dimension. Now using Wick calculus (in finite dimension) b_k^{Wick} can be written as an anti-Wick operator by moving all the a^* to the right of a. So, one obtains that $b_k^{Wick} = b_k^{A-Wick} + \varepsilon T$ with $T(d\Gamma(P_k \oplus Q_k) + 1)^{-1}$ is bounded uniformly with respect to $\varepsilon \in (0,\bar{\varepsilon})$. Hence

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}}\, \mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big] &= \overline{\lim_{\varepsilon\to 0}}\, \mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k b_k^{Wick}\Big] \\ &\leq \overline{\lim_{\varepsilon\to 0}}\, \mathrm{Tr}\Big[\varrho_\varepsilon H_0\Big] \leq C\,.\end{aligned}$$

For details on the anti-Wick quantization we refer the reader to [9]; in particular, it is a positive quantization (see, e.g., [9, Proposition 3.6]). Hence, we see that

$$\mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k (b_{k,\chi})^{A-Wick}\Big] \leq \mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big],$$

where $b_{k,\chi}(u,\alpha) = \chi(u)\langle u, A_k u\rangle + \chi(\alpha)\langle \alpha, B_k \alpha\rangle$ for any cutoff function $\chi \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \chi \leq 1$. Finally, [9, Theorem 6.2] gives

$$\begin{aligned}\int_{z=(u,\alpha)\in L^2\oplus L^2} b_{k,\chi}(u,\alpha)\, d\mu(z) &= \lim_{\varepsilon\to 0} \mathrm{Tr}\Big[\varrho_\varepsilon (b_{k,\chi})^{A-Wick}\Big] \\ &= \lim_{\varepsilon\to 0} \mathrm{Tr}_{\Gamma_s(P_k L^2 \oplus Q_k L^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big] \leq C\,,\end{aligned}$$

and the monotone convergence theorem proves (iii). ◻

As we said above, our aim is to take the limit $\varepsilon_k \to 0$ on the integral equation (68) for a suitable sequence contained in $(0,\bar{\varepsilon})$. We may suppose that the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ is chosen in such a way that there exists $\mu_0 \in \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon})\big)$ such that (80) holds, i.e., $\mathcal{M}\big(\varrho_{\varepsilon_k}, k \in \mathbb{N}\big) = \{\mu_0\}$. However, nothing a priori ensures that the sequence, or one of its subsequences $(\varepsilon_{k_i})_{i\in\mathbb{N}} \subset (\varepsilon_k)_{k\in\mathbb{N}}$, is such that *for any* $t \in \mathbb{R}$

$$\lim_{i\to\infty} \mathrm{Tr}\Big[\tilde{\varrho}_{\varepsilon_{k_i}}(t) W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle \xi, z\rangle} d\tilde{\mu}_t(z) \;\; \forall \xi \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)\,,$$

where $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ is a map such that $\tilde{\mu}_0 = \mu_0$. The possibility of extracting such a common subsequence is crucial, since the integral equation involves all measures from zero to an arbitrary time t. To prove it is possible, we exploit the uniform continuity properties of $\mathrm{Tr}[\tilde{\varrho}_\varepsilon(t) W(\xi)]$ in both t and ξ, proved in the following lemma.

LEMMA 4.13. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *be a family of quantum states on* $\mathcal{H}$ *that satisfies assumptions* (A-n) *and* (A(h)'). *Then the family of functions* $(t,\xi) \mapsto \tilde{G}_\varepsilon(t,\xi) := \mathrm{Tr}[\tilde{\varrho}_\varepsilon(t) W(\xi)]$ *is uniformly equicontinuous on bounded subsets of* $\mathbb{R} \times \big(Q(-\Delta + V) \oplus D(\omega^{1/2})\big)$.

Proof. Let $(t,\xi),(s,\eta)\in\mathbb{R}\times\big(Q(-\Delta+V)\oplus D(\omega^{1/2})\big)$. Without loss of generality, we may suppose that $s\leq t$. We write

$$\Big|\tilde{G}_\varepsilon(t,\xi)-\tilde{G}_\varepsilon(s,\eta)\Big|\leq\Big|\tilde{G}_\varepsilon(t,\eta)-\tilde{G}_\varepsilon(s,\eta)\Big|+\Big|\tilde{G}_\varepsilon(t,\xi)-\tilde{G}_\varepsilon(t,\eta)\Big|$$

and define $X_1:=|\tilde{G}_\varepsilon(t,\eta)-\tilde{G}_\varepsilon(s,\eta)|$, $X_2:=|\tilde{G}_\varepsilon(t,\xi)-\tilde{G}_\varepsilon(t,\eta)|$. Consider X_1; we get by standard manipulations and Lemma 4.2

$$X_1\leq\sum_{j=0}^{3}\varepsilon^j\sum_{i\in\mathbb{N}}\lambda_i\int_s^t\Big|\big\langle e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i,W\big(\tilde{(\eta)}_s\big)B_j\big(\tilde{(\eta)}_s\big)e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\big\rangle\Big|ds\,.$$

Now using Lemma 4.6 we obtain

$$\begin{aligned}X_1\leq\sum_{j=0}^{3}\varepsilon^jC_j(\eta)\sum_{i\in\mathbb{N}}\lambda_i\int_s^t&\Big\|(N_1+H_0+\bar{\varepsilon})^{1/2}W^*\big(\tilde{(\eta)}_s\big)e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\Big\|\\&\cdot\Big\|(N_1+H_0+\bar{\varepsilon})^{1/2}e^{-i\frac{s}{\varepsilon}\hat{H}^{\mathrm{ren}}}\Psi_i\Big\|ds\,;\end{aligned}$$

then using Lemma 4.1, and the fact that $\|\tilde{(\eta_1)}_s\|_{H^1}=\|\eta_1\|_{H^1}$, $\|\tilde{(\eta_2)}_s\|_{\mathcal{F}H^{1/2}}=\|\eta_2\|_{\mathcal{F}H^{1/2}}$ we get

$$\begin{aligned}X_1&\leq C(\eta)\sum_{j=0}^{3}\varepsilon^jC_j(\eta)\int_s^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)(N_1+H_0+\bar{\varepsilon})\Big]ds\\&\leq|t-s|C(\eta)\sum_{j=0}^{3}\bar{\varepsilon}^jC_j(\eta)\Big(\tfrac{C}{1-a(\mathfrak{C})}+\tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}+\bar{\varepsilon}\Big)\,,\end{aligned}$$

where in the last inequality we used (77) of Lemma 4.8. Now let us consider X_2; a standard manipulation using Weyl's relation yields

$$\begin{aligned}X_2\leq\Big\|\big(e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\xi,\eta\rangle_{L^2\oplus L^2}}W(\xi-\eta)-1\big)(N_1+N_2+1)^{-1}\Big\|_{\mathcal{L}\left(\Gamma_s(L^2\oplus L^2)\right)}&\\\cdot\,\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)(N_1+N_2+1)\Big]\,.&\end{aligned}$$

Now we use the estimate in [9, Lemma 3.1] and obtain

$$\begin{aligned}X_2&\leq\|\xi-\eta\|_{L^2\oplus L^2}\Big(\bar{\varepsilon}\|\eta\|_{L^2\oplus L^2}+1\Big)\,\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)(N_1+N_2+1)\Big]\\&\leq\|\xi-\eta\|_{L^2\oplus L^2}\Big(\bar{\varepsilon}\|\eta\|_{L^2\oplus L^2}+1\Big)\Big(\tfrac{C}{1-a(\mathfrak{C})}+\tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}+1\Big)\,,\end{aligned}$$

where in the last inequality we used again (77) of Lemma 4.8, keeping in mind that $N_2\leq d\Gamma(\omega)\leq H_0$. ∎

Now using Lemma 4.13 with the estimates on X_1,X_2 above and a diagonal extraction argument, we prove the following proposition. We omit the proof since it is similar to [12, Proposition 3.9].

PROPOSITION 4.14. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of quantum states on $\mathcal{H}$ that satisfies assumptions* (A-n) *and* (A(h)'). *Then for any sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar{\varepsilon})$ with*

$\lim_{k\to\infty} \varepsilon_k = 0$, *there exists a subsequence* $(\varepsilon_{k_i})_{i\in\mathbb{N}}$ *such that there exists a map* $\mu_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ *verifying the following statements:*

$$\varrho_{\varepsilon_{k_i}}(t) \to \mu_t \ \ \forall t \in \mathbb{R}\ ; \tag{81}$$

$$\tilde{\varrho}_{\varepsilon_{k_i}}(t) \to \tilde{\mu}_t \ \ \forall t \in \mathbb{R}\ ,\ \text{with } \tilde{\mu}_t = \mathbf{E}_0(-t)_{\#}\mu_t\ ; \tag{82}$$

$$\varrho_{\varepsilon_{k_i}}(t)W(\tilde{\xi}_t) \to \mu_{\xi,t}\ \forall t \in \mathbb{R}\,, \forall \xi \in L^2 \oplus L^2,\ \text{with } d\mu_{\xi,t}(z) = e^{i\sqrt{2}\mathrm{Re}\langle \tilde{\xi}_t, z\rangle} d\mu_t(z)\ , \tag{83}$$

where $\mathbf{E}_0(t)z = e^{-it(-\Delta+V)}u \oplus e^{-it\omega}\alpha$ *is the Hamiltonian flow associated with the free classical energy* $\mathscr{E}_0$*, and* $\tilde{\xi}_t = \mathbf{E}_0(-t)\xi$*. Moreover,* μ_t *and* $\tilde{\mu}_t$ *are both Borel probability measures on* $Q(-\Delta + V) \oplus D(\omega^{1/2})$*.*

4.5. The classical limit of the integral formula. We are finally ready to discuss the limit $\varepsilon \to 0$ of the integral formula (68). As a final preparation, we state a couple of preliminary lemmas. The first is a slight improvement of [9, Theorem 6.13]. The second can be easily proved by standard estimates on the symbol $\mathscr{B}_0^{(m)}(\xi)$ which we recall for convenience:

$$\begin{aligned}
\mathscr{B}_0^{(m)}(\xi)(u,\alpha) = {} & 2i\sqrt{2}\Big\langle \mathrm{Re}\mathcal{F}\big(\tfrac{\chi_{\sigma_0}\bar{\alpha}}{\sqrt{2\omega}}\big)(x), \mathrm{Im}\big(\bar{\xi}_1 u\big)(x)\Big\rangle_2 \\
& + i\sqrt{2}\Big\langle u(x), \chi_m(D_x)\mathrm{Im}\Big(\mathcal{F}\big(\tfrac{\chi_{\sigma_0}\bar{\xi}_2}{\sqrt{2\omega}}\big)\Big)(x)u(x)\Big\rangle_2 \\
& + i\sqrt{2}\mathrm{Im}\Big\langle u(x), \big(\chi_m(D_{(\cdot)})V_\infty * \bar{\xi}_1 u\big)(x)u(x)\Big\rangle_2 \\
& + \tfrac{i(2\pi)^{3/2}}{2M}\mathrm{Im}\Big\langle \xi_1(x), \Big(\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)^2 + \mathcal{F}(r_{\sigma_m}\bar{\alpha})^2 \\
& + \mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)\mathcal{F}(r_{\sigma_m}\bar{\alpha})\Big)(x)u(x)\Big\rangle_2 \\
& - \tfrac{2\sqrt{2}(2\pi)^3}{M}\mathrm{Im}\Big\langle u(x), \chi_m(D_x)\mathrm{Im}\Big(\mathcal{F}^{-1}(\bar{r}_\infty\xi_2)\Big)(x)\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)(x)u(x)\Big\rangle_2 \\
& - \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle \xi_1(x), D_x\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)(x)u(x)\Big\rangle_2 \\
& - \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle \xi_1(x), \mathcal{F}(r_{\sigma_m}\bar{\alpha})(x)D_x u(x)\Big\rangle_2 \\
& + \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle u(x), \chi_m(D_x)D_x\mathcal{F}^{-1}(\bar{r}_\infty\xi_2)(x)u(x)\Big\rangle_2\ .
\end{aligned} \tag{84}$$

LEMMA 4.15. *Let* $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, \bar{\varepsilon})$, $\lim_{j\to\infty}\varepsilon_j = 0$, *and* $\delta > 0$*. Furthermore, let* $(\varrho_{\varepsilon_j})_{j\in\mathbb{N}}$ *be a sequence of normal states in* $\mathcal{H}$ *such that for some* $C(\delta) > 0$,

$$\Big\|(N_1 + N_2)^{\delta/2}\varrho_{\varepsilon_j}(N_1 + N_2)^{\delta/2}\Big\|_{\mathcal{L}^1(L^2\oplus L^2)} \leq C(\delta)\,, \tag{85}$$

uniformly in $\varepsilon \in (0, \bar{\varepsilon})$*. Suppose that* $\varrho_{\varepsilon_j} \to \mu \in \mathfrak{P}(L^2 \oplus L^2)$*; then the following statement is true:*

$$\Bigg(\forall \mathscr{A} \in \bigoplus_{\substack{(p,q)\in\mathbb{N}^2\\ p+q<2\delta}} \mathcal{P}_{p,q}^{\infty}\big(L^2 \oplus L^2\big)\ ,\ \lim_{j\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\Big] = \int_{L^2\oplus L^2}\mathscr{A}(z)d\mu(z)\Bigg)\ .$$

Proof. By linearity it is enough to assume $\mathscr{A} \in \mathcal{P}_{p,q}^{\infty}\big(L^2 \oplus L^2\big)$ for $(p,q) \in \mathbb{N}^2$ with $p + q < 2\delta$. Let $(P_R)_{R>0}$ be an increasing family of finite rank orthogonal projections on L^2 such that the strong limit $s-\lim_{R\to+\infty} P_R = 1$ holds. Let $\mathscr{A}_R(z) := \mathscr{A}(P_R \oplus P_R z)$ for any $z \in L^2 \oplus L^2$. One writes

$$\left|\mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\right] - \int_{L^2\oplus L^2} \mathscr{A}(z) d\mu(z)\right|$$

$$\leq \left|\mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\right] - \mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right]\right| \tag{86}$$

$$+ \left|\mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right] - \int_{L^2\oplus L^2} \mathscr{A}_R(z) d\mu(z)\right| \tag{87}$$

$$+ \left|\int_{L^2\oplus L^2} \mathscr{A}_R(z) d\mu(z) - \int_{L^2\oplus L^2} \mathscr{A}(z) d\mu(z)\right|. \tag{88}$$

Using standard number estimates and the regularity of the states $(\varrho_{\varepsilon_j})_j$, one shows

$$\left|\mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A} - \mathscr{A}_R)^{Wick}\right]\right| \leq ||(N_1 + N_2)^{\delta/2} \varrho_{\varepsilon_j} (N_1 + N_2)^{\delta/2}||_{\mathcal{L}^1(L^2\oplus L^2)} \;\; ||\tilde{\mathscr{A}} - \tilde{\mathscr{A}_R}||\,,$$

where $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}_R}$ denote the compact operators satisfying $\mathscr{A}(z) = \langle z^{\otimes q}, \tilde{\mathscr{A}} z^{\otimes p}\rangle$ and $\mathscr{A}_R(z) = \langle z^{\otimes q}, \tilde{\mathscr{A}_R} z^{\otimes p}\rangle$, respectively. Since $\tilde{\mathscr{A}_R} = (P_R \oplus P_R)^{\otimes q} \tilde{\mathscr{A}} (P_R \oplus P_R)^{\otimes p}$ and $\tilde{\mathscr{A}}$ is compact, one shows that $\lim_{R\to+\infty} ||\tilde{\mathscr{A}} - \tilde{\mathscr{A}_R}|| = 0$. So the right-hand side of (86) can be made arbitrarily small by choosing R large enough.

According to [9, Theorem 6.2], the regularity of $(\varrho_{\varepsilon_j})_j$ ensures the bound

$$\int_{L^2\oplus L^2} ||z||^{2\delta}_{L^2\oplus L^2}\, d\mu(z) \leq C(\delta)\,.$$

Hence by dominated convergence the right-hand side of (88) can also be made arbitrarily small when R is large enough since $\mathscr{A}(z)$ and $\mathscr{A}_R(z)$ are both bounded by $c||z||^{p+q}_{L^2\oplus L^2}$ and $\mathscr{A}_R(z)$ converges pointwise to $\mathscr{A}(z)$.

To handle the right-hand side of (87), we use a further regularization. Let $\chi \in C_0^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ in a neighborhood of 0 and $\chi_m(x) = \chi(\frac{x}{m})$ for $m > 0$. Recall that the Fock space has the decomposition $\Gamma_s(L^2 \oplus L^2) \equiv \Gamma_s(P_R L^2 \oplus P_R L^2) \otimes \Gamma_s(P_R^\perp L^2 \oplus P_R^\perp L^2)$, where $P_R^\perp = 1 - P_R$. In this representation $\mathscr{A}_R^{Wick} \equiv (\mathscr{A}_R)^{Wick}_{|\Gamma_s(P_R L^2\oplus P_R L^2)} \otimes 1_{\Gamma_s(P_R^\perp L^2 \oplus P_R^\perp L^2)}$ and $\varrho_{\varepsilon_j} \equiv \hat{\varrho}_{\varepsilon_j}$. Hence using reduced density matrices $\varrho^R_{\varepsilon_j}$ that are normalized positive trace-class operators in $\Gamma_s(P_R L^2 \oplus P_R L^2)$, one writes

$$\begin{aligned}\mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right] &= \mathrm{Tr}\left[\hat{\varrho}_{\varepsilon_j}(\mathscr{A}_R)^{Wick}_{|\Gamma_s(P_R L^2\oplus P_R L^2)} \otimes 1_{\Gamma_s(P_R^\perp L^2\oplus P_R^\perp L^2)}\right] \\ &= \mathrm{Tr}_{\Gamma_s(P_R L^2\oplus P_R L^2)}\left[\varrho^R_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right].\end{aligned}$$

As in the proof of Lemma 4.12, the Wick calculus gives that $(\mathscr{A}_R)^{Wick}$ can be written as an anti-Wick operator by moving all the a^* to the right of a. So, one obtains that $(\mathscr{A}_R)^{Wick} = (\mathscr{A}_R)^{A-Wick} + \varepsilon T$ with $T(d\Gamma(P_R \oplus P_R) + 1)^{-\frac{p+q}{2}}$ is bounded uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$. We refer the reader to [9], where Weyl and anti-Wick quantization are explained for "cylindrical" symbols. Hence

$$\begin{aligned}\lim_{j\to\infty} \mathrm{Tr}\left[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right] &= \lim_{j\to\infty} \mathrm{Tr}_{\Gamma_s(P_R L^2\oplus P_R L^2)}\left[\varrho^R_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\right] \\ &= \lim_{j\to\infty} \mathrm{Tr}_{\Gamma_s(P_R L^2\oplus P_R L^2)}\left[\varrho^R_{\varepsilon_j}(\mathscr{A}_R)^{A-Wick}\right].\end{aligned}$$

Now we define $\chi_{m,R}(z) := \chi_m(|P_R \oplus P_R z|^2)$ and

$$\varrho_{\varepsilon_j}^{R,m} := \chi_{m,R}(z)^{Weyl}\, \varrho_{\varepsilon_j}^{R}\, \chi_{m,R}(z)^{Weyl}\,.$$

So one writes

$$\left|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R}(\mathscr{A})^{A-Wick}\Big] - \int_{L^2\oplus L^2} \mathscr{A}(z)d\mu(z)\right|$$

$$\leq \left|\mathrm{Tr}\Big[(\varrho_{\varepsilon_j}^{R} - \varrho_{\varepsilon_j}^{R,m})(\mathscr{A})^{A-Wick}\Big]\right| \tag{89}$$

$$+ \left|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] - \int \chi_{m,R}^2(z)\mathscr{A}_R(z)d\mu(z)\right| \tag{90}$$

$$+ \left|\int \chi_{m,R}^2(z)\mathscr{A}_R(z)d\mu(z) - \int \mathscr{A}_R(z)d\mu(z)\right|\,, \tag{91}$$

where the traces are on the Fock space $\Gamma_s(P_R L^2 \oplus P_R L^2)$ and the integrals are over $L^2 \oplus L^2$. By dominated convergence the right-hand side of (91) tends to 0 when $m \to \infty$ at fixed R. The right-hand side of (89) can be made arbitrarily small when $m \to \infty$ using the following decomposition:

$$(\varrho_{\varepsilon_j}^{R,m} - \varrho_{\varepsilon_j}^{R}) = \underbrace{(\chi_{m,R}^{Weyl} - 1)\, \varrho_{\varepsilon_j}^{R}\, \chi_{m,R}^{Weyl}}_{(A)} + \underbrace{\varrho_{\varepsilon_j}^{R}\, (\chi_{m,R}^{Weyl} - 1)}_{(B)}\,,$$

which gives $\mathrm{Tr}[(A)\,(\mathscr{A}_R)^{A-Wick}] = \mathrm{Tr}[T_1T_2T_3T_4]$ and a similar expression for (B) with

$T_1 = (N_R+1)^{\frac{p+q}{4}}(\chi_{m,R}^{Weyl} - 1)(N_R+1)^{-\frac{\delta}{2}}$, $T_2 = (N_R+1)^{\frac{\delta}{2}}\varrho_{\varepsilon_j}^{R}(N_R+1)^{\frac{\delta}{2}}$,
$T_3 = (N_R+1)^{-\frac{\delta}{2}}\chi_{m,R}^{Weyl}(N_R+1)^{\frac{p+q}{4}}$, $T_4 = (N_R+1)^{-\frac{p+q}{4}}(\mathscr{A}_R)^{A-Wick}(N_R+1)^{-\frac{p+q}{4}}$,

where $N_R = d\Gamma(P_R \oplus P_R)$. The Weyl–Hörmander pseudodifferential calculus gives that $T_1 \to_{m\to\infty} 0$ in norm (since $\delta > p+q$) and that T_i, $i = 2,3,4$, are uniformly bounded with respect $j \in \mathbb{N}$ and $m > 0$ at fixed R (see, e.g., [9, Proposition 3.2 and 3.3]).

To complete the proof, we remark that

$$\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] = \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R}\ \chi_{m,R}^{Weyl}\ (\mathscr{A}_R)^{A-Wick}\ \chi_{m,R}^{Weyl}\Big]\,.$$

So again by pseudodifferential calculus we know $(\mathscr{A}_R)^{A-Wick} = (\mathscr{A}_R)^{Weyl} + \varepsilon\, b(\varepsilon)^{Weyl}$ with $b(\varepsilon)$ belonging to the Weyl–Hörmander class symbol $S_{P_R\oplus P_R}(\langle z\rangle^{p+q-2}, \frac{dz^2}{\langle z\rangle^2})$ uniformly in ε (see [9, sections 3.2 and 3.4]). Therefore,

$$\lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] = \lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R}\ \chi_{m,R}^{Weyl}(\mathscr{A}_R)^{Weyl}\ \chi_{m,R}^{Weyl}\Big]\,,$$

since $(d\Gamma(P_R \oplus P_R) + 1)^{-(q+p)/2}\ b(\varepsilon)^{Weyl}\ (d\Gamma(P_R \oplus P_R) + 1)^{-(p+q)/2}$ is uniformly bounded with respect to ε. The Weyl–Hörmander pseudodifferential calculus gives $\chi_{m,R}^{Weyl}(\mathscr{A}_R)^{Weyl}\ \chi_{m,R}^{Weyl} = (\chi_{m,R}^2\mathscr{A}_R)^{Weyl} + \varepsilon\, c(\varepsilon)^{Weyl}$ with $c(\varepsilon) \in S_{P_R\oplus P_R}(1, dz^2)$ uniformly in ε (see, e.g., [9, Proposition 3.2]). Hence, according to [9, Theorem 6.2] one obtains

$$\begin{aligned}\lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] &= \lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}\ (\chi_{m,R}^2\,\mathscr{A}_R)^{Weyl}\Big]\\ &= \int_{L^2\oplus L^2} \chi_{m,R}^2(z)\mathscr{A}_R(z)d\mu(z)\,.\end{aligned}$$

This yields the intended bound on (87) and completes the proof. ∎

LEMMA 4.16. *There exists $C(\sigma_0) > 0$ depending only on $\sigma_0 \in \mathbb{R}_+$ such that the following bound holds for $\mathscr{B}_0^{(m)}$ uniformly in $m \in \mathbb{N}$:*

$$\begin{aligned}\Big|\mathscr{B}_0^{(m)}(\xi)(u,\alpha)\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(&\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}\\ &+\|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2\cdot\|\alpha\|^2_{\mathcal{F}H^{1/2}}\\ &+\|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2\cdot\|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.\end{aligned}\tag{92}$$

It follows that

- *for any $\xi \in L^2\oplus L^2$, for any $(u,\alpha) \in Q(-\Delta+V)\oplus D(\omega^{1/2})$, $\lim_{m\to\infty}\mathscr{B}_0^{(m)}(\xi)(u,\alpha) = \mathscr{B}_0(\xi)(u,\alpha)$, and therefore the bound (92) holds also for $\mathscr{B}_0$;*
- *for any $m \in \mathbb{N}$, $\mathscr{B}_0^{(m)}(\cdot), \mathscr{B}_0(\cdot)$ are are jointly continuous with respect to $\xi \in L^2\oplus L^2$ and $(u,\alpha) \in Q(-\Delta+V)\oplus D(\omega^{1/2})$.*

Recall that for any $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar\varepsilon)$ there exists $b>0$ such that the operator $\hat H_\varepsilon^{\mathrm{ren}} + b$ is nonnegative uniformly for $\varepsilon \in (0,\bar\varepsilon)$. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3))$; we consider the additional assumption

$$\exists C>0\,,\ \forall\varepsilon\in(0,\bar\varepsilon)\,,\ \mathrm{Tr}[\varrho_\varepsilon\,(\hat H_\varepsilon^{\mathrm{ren}}+b)^2] \leq C\,. \tag{A(h)"}$$

PROPOSITION 4.17. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfy assumptions* (A-n), (A(h)'), *and* (A(h)") *such that*[15] $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar\varepsilon)$. *Then the following hold:*

(i) *For any sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar\varepsilon)$ converging to zero, there exist a subsequence $(\varepsilon_{k_\iota})_{\iota\in\mathbb{N}}$ and a map $\mu_t : \mathbb{R} \to \mathfrak{P}\big(L^2\oplus L^2\big)$ such that $\varrho_{\varepsilon_{k_\iota}}(t) \to \mu_t$ and $\tilde\varrho_{\varepsilon_{k_\iota}}(t) \to \tilde\mu_t = \mathbf{E}_0(-t)_{\#}\mu_t$ for any $t\in\mathbb{R}$.*

(ii) *The action of $e^{-i\frac{t}{\varepsilon}\hat H_\varepsilon^{\mathrm{ren}}}$ is nontrivial on the states ϱ_ε.*

(iii) *The Fourier transform of $\tilde\mu_{(\cdot)}$ satisfies the following transport equation for all $\xi \in L^2\oplus L^2$:*

$$\begin{aligned}\int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\,d\tilde\mu_t(z) = &\int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\,d\mu_0(z)\\ &+\int_0^t\Big(\int_{L^2\oplus L^2}\mathscr{B}_0(\tilde\xi_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde\xi_s,z\rangle}\,d\mu_s(z)\Big)ds\,,\end{aligned}\tag{93}$$

where the right-hand side makes sense since $\mathscr{B}_0(\tilde\xi_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde\xi_t,z\rangle} \in L_t^\infty(\mathbb{R}, L_z^1\big[L^2\oplus L^2, d\mu_t(z)\big])$ for any $\xi\in L^2\oplus L^2$.

Proof. The first part of the proposition, points (i) and (ii), is just a partial restatement of Proposition 4.14. We discuss the last assertion in (iii) about $\mathscr{B}_0(\tilde\xi_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde\xi_t,z\rangle}$ before proving (93). Recall the fact that for any $\xi\in L^2\oplus L^2$

[15] We recall that $\mathfrak{C}$ appears in assumption (A-n) and σ_0 in Definition 2.13 of $\hat H^{\mathrm{ren}}_{\mathrm{ren}}$. The condition $\sigma_0 \geq K(\mathfrak{C}+1)$ ensures that the dressed dynamics is nontrivial on $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ and hence nontrivial on the state ϱ_ε according to Lemma 4.2.

and for any $t \in \mathbb{R}$, $\|\tilde{\xi}_t\|_{L^2\oplus L^2} = \|\xi\|_{L^2\oplus L^2}$. Using bound (92) of Lemma 4.16, we obtain, setting $Q(-\Delta+V)\oplus D(\omega^{1/2}) \ni z = (u,\alpha)$,

$$\Big|\mathscr{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_t,z\rangle}\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}$$
$$+\|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2\cdot\|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2\cdot\|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.$$

Now $\mu_t \in \mathcal{M}\big(\varrho_\varepsilon(t), \varepsilon\in(0,\bar{\varepsilon})\big)$; therefore by Lemma 4.12, $\mu_t(B_u(0,\sqrt{\mathfrak{C}})\cap Q(-\Delta+V)\oplus D(\omega^{1/2})) = 1$ for any $t\in\mathbb{R}$. Then it follows that there exists $C(\mathfrak{C})>0$ such that

$$\begin{aligned}&\Big|\int_{L^2\oplus L^2}\mathscr{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_t,z\rangle}d\mu_t(z)\Big|\\ &\qquad\leq C(\mathfrak{C})\|\xi\|_{L^2\oplus L^2}\int_{L^2\oplus L^2}\Big(\|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}\Big)d\mu_t(z)\\ &\qquad\leq C(\mathfrak{C})\|\xi\|_{L^2\oplus L^2}J(t)\,,\end{aligned}$$

where $J(t)<\infty$ by Lemma 4.12. Actually, using the fact that the bound (77) is independent of t, it is easily proved that $J(t)$ does not depend on t as well, i.e., $J(t)\in L^\infty(\mathbb{R})$.

We prove (93) by successive approximations. Consider $\mathrm{Tr}[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)W(\xi)]$, $\xi\in L^2\oplus L^2$. We can approximate ξ with $(\xi^{(l)})_{l\in\mathbb{N}}\subset Q(-\Delta+V)\oplus D(\omega^{3/4})$, since the latter is dense in $L^2\oplus L^2$, and $\lim_{l\to\infty}\mathrm{Tr}[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)\big(W(\xi)-W(\xi^{(l)})\big)]=0$ uniformly in ε_{k_ι} by Lemma 4.13. Now, for $\mathrm{Tr}[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)W(\xi^{(l)})]$ the integral equation (68) holds. Proposition 4.14 implies that $\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)\to\tilde{\mu}_t = \mathbf{E}_0(t)_{\#}\mu_t$ for any $t\in\mathbb{R}$. Therefore, the left-hand side of (68) converges when $\iota\to\infty$ to $\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi^{(l)},z\rangle}d\tilde{\mu}_t(z)$; and that in turn converges when $l\to\infty$ to $\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\tilde{\mu}_t(z)$ by dominated convergence theorem. In addition,

$$\lim_{\iota\to\infty}\sum_{j=1}^{3}\varepsilon^j\int_0^t\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)B_j(\tilde{\xi^{(l)}}_s)\Big]ds = 0$$

by Proposition 4.7. It remains to show the convergence of the B_0 term in (68). We approximate $\mathscr{B}_0$ by the compact $\mathscr{B}_0^{(m)}$, because using Lemma 4.2 and (79) of Proposition 4.9 we obtain

$$\begin{aligned}&\Big|\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Big(B_0(\tilde{\xi^{(l)}}_s)-B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)\Big]\Big|\\ &\qquad\leq\sum_{i\in\mathbb{N}}\lambda_i\Big|\Big\langle W^*(\tilde{\xi^{(l)}}_s)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}^{\mathrm{ren}}}\Psi_i,\Big(B_0(\tilde{\xi^{(l)}}_s)-B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}^{\mathrm{ren}}}\Psi_i\Big\rangle\Big|\\ &\qquad\leq\sum_{i\in\mathbb{N}}\lambda_iC^{(m)}(\tilde{\xi^{(l)}}_s)\Big\|(H_0+1)^{1/2}(N_1+\bar{\varepsilon})^{1/2}W^*(\tilde{\xi^{(l)}}_s)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}^{\mathrm{ren}}}\Psi_i\Big\|\\ &\qquad\cdot\Big\|(H_0+1)^{1/2}(N_1+\bar{\varepsilon})^{1/2}e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}^{\mathrm{ren}}}\Psi_i\Big\|\,.\end{aligned}$$

Now, using the fact that $C^{(m)}(\tilde{\xi^{(l)}}_s)$ depends only on $\|\tilde{\xi^{(l)}}_s\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})} =$

$\|\xi^{(l)}\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})}$ and Lemma 4.1, we obtain

$$\left|\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Big(B_0(\tilde{\xi^{(l)}}_s)-B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)\Big]\right| \leq \sum_{i\in\mathbb{N}}\lambda_i C^{(m)}(\xi^{(l)})C(\xi^{(l)})\Big\|(H_0+1)^{1/2}e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}^{\mathrm{ren}}}(N_1+\bar{\varepsilon})^{1/2}\Psi_i\Big\|^2 .$$

We then use (77) of Lemma 4.8:

$$\left|\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Big(B_0(\tilde{\xi^{(l)}}_s)-B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)\Big]\right| \leq \sum_{i\in\mathbb{N}}\lambda_i C^{(m)}(\xi^{(l)})C(\xi^{(l)})(\mathfrak{C}+\bar{\varepsilon})\tfrac{1}{1-a(\mathfrak{C})}C+\tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})} .$$

The right-hand side goes to zero when $m\to\infty$ uniformly with respect to ε_{k_ι} and s by Proposition 4.9, and therefore

$$\lim_{m\to\infty}\int_0^t \mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Big(B_0(\tilde{\xi^{(l)}}_s)-B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)\Big]ds=0 .$$

So the next step is to prove

$$\lim_{\iota\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Big(\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)^{Wick}\Big]=\int_{L^2\oplus L^2}\mathscr{B}_0^{(m)}(\tilde{\xi}^{(l)}_s)(z)\,e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_s^{(l)},z\rangle}d\mu_s(z).$$

This statement follows by applying Lemma 4.15 with $\delta=2$ and by checking the assumption

$$||(N_1+N_2)\,\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\,(N_1+N_2)||_{\mathcal{L}^1(L^2\oplus L^2)}\leq C\,, \tag{94}$$

uniformly in k_ι for some $C>0$. In fact, (94) holds true by assumptions (A-n)–(A(h)"), the higher order estimate of Proposition A.4 and Lemma 4.1. Note that while $\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)$ is not a nonnegative trace-class operator, one can still apply Lemma 4.15. In fact, one can write

$$\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\;B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big]=\mathrm{Tr}\Big[W(\eta)\varrho_{\varepsilon_{k_\iota}}(s)W(\eta)\;\mathscr{A}^{Wick}\Big]$$

for some $\mathscr{A}\in\bigoplus_{p+q<4}\mathcal{P}^\infty_{p,q}\big(L^2\oplus L^2\big)$ and with $\eta=\frac{1}{2}\tilde{\xi^{(l)}}_s$. We note now that $W(\eta)\varrho_{\varepsilon_{k_\iota}}(s)W(\eta)$ decomposes explicitly into a linear combination of nonnegative trace-class operators satisfying the assumption (85) of Lemma 4.15. Note that the Wigner measures of $\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)$ are identified through (83). Hence the dominated convergence theorem yields

$$\lim_{\iota\to\infty}\int_0^t\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big]ds =\int_0^t\bigg(\int_{L^2\oplus L^2}\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s,z\rangle}d\mu_s(z)\bigg)ds .$$

By Lemma 4.16, $\lim_{m\to\infty}\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z)=\mathscr{B}_0(\tilde{\xi^{(l)}}_s)(z)$, so by the dominated convergence theorem

$$\lim_{m\to\infty}\int_0^t\bigg(\int_{L^2\oplus L^2}\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s,z\rangle}d\mu_s(z)\bigg)ds =\int_0^t\bigg(\int_{L^2\oplus L^2}\mathscr{B}_0(\tilde{\xi^{(l)}}_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s,z\rangle}d\mu_s(z)\bigg)ds .$$

Above it is possible to apply the dominated convergence theorem due to a reasoning analogous to that done at the beginning of this proof: roughly speaking, we have that $\mathscr{B}_0^{(m)}(\xi^{\tilde{(l)}}{}_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle \xi^{\tilde{(l)}}{}_t, z\rangle} \in L_t^\infty(\mathbb{R}, L_z^1[L^2\oplus L^2, d\mu_t(z)])$ uniformly with respect to $m\in\mathbb{N}$. In an analogous fashion we finally obtain

$$\lim_{l\to\infty}\int_0^t\Big(\int_{L^2\oplus L^2}\mathscr{B}_0(\xi^{\tilde{(l)}}{}_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle \xi^{\tilde{(l)}}{}_s, z\rangle}d\mu_s(z)\Big)ds = \int_0^t\Big(\int_{L^2\oplus L^2}\mathscr{B}_0(\tilde{\xi}_s)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_s,z\rangle}d\mu_s(z)\Big)ds\ . \qquad \square$$

COROLLARY 4.18. *The transport equation* (93) *may be rewritten as*

$$\begin{aligned}\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\tilde{\mu}_t(z) &= \int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\mu_0(z)\\ &\quad + i\sqrt{2}\int_0^t\Big(\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\mathrm{Re}\langle\xi,\mathbf{V}(s)(z)\rangle_{L^2\oplus L^2}\ d\tilde{\mu}_s(z)\Big)ds\ ,\end{aligned}\tag{95}$$

with the vector field $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t)\circ\partial_{\bar{z}}\big(\hat{\mathscr{E}}-\mathscr{E}_0\big)\circ\mathbf{E}_0(t)(z)$*. In addition,* $\tilde{\mu}_t = \mathbf{E}_0(-t)_\#\hat{\mathbf{E}}(t)_\#\mu_0$ *is a solution of* (95).

Proof. The proof is by direct calculation, since $\mu_t\big(Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}\big)=1$ for any $t\in\mathbb{R}$ by Lemma 4.12, and $\hat{\mathbf{E}}(t),\mathbf{E}_0(t)$ are globally well defined on this space (for $\hat{\mathbf{E}}(t)$, it is proved in Theorem 3.16; for $\mathbf{E}_0(t)$ it is trivial). The second point is proved by differentiating with respect to time and using Lemmas 4.16 and 4.12(iii). $\square$

4.6. Uniqueness of solutions for the transport equation. As discussed in Corollary 4.18, the dressed flow yields in the classical limit a solution of the transport equation (95). The second part of the same corollary suggests that it is important to study uniqueness properties of (95): it is by means of uniqueness that we can close the argument and reach a satisfactory characterization of the dynamics of classical states (Wigner measures). This subsection is devoted to proving that the family of Wigner measures $\tilde{\mu}_t$ of Proposition 4.17 satisfies sufficient conditions, induced by the properties of $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, to be uniquely identified with $\mathbf{E}_0(-t)_\#\hat{\mathbf{E}}(t)_\#\mu_0$. We use an optimal transport technique introduced by [3], then extended by [12] to propagation of Wigner measures, and improved recently by [8] (see also [89, 92]).

In order to do that, we need to introduce a suitable topology on $\mathfrak{P}\big(L^2\oplus L^2\big)$. Let $(e_j)_{j\in\mathbb{N}}\subset L^2\oplus L^2$ be an orthonormal basis. Then

$$d_w(z_1,z_2) = \Big(\sum_{j\in\mathbb{N}}\frac{|\langle z_1-z_2,e_j\rangle_{L^2\oplus L^2}|^2}{(1+j)^2}\Big)^{1/2}\ ,\tag{96}$$

where $z_1,z_2\in L^2\oplus L^2$, defines a distance on $L^2\oplus L^2$. The topology induced by $\big(L^2\oplus L^2,d_w\big)$ is homeomorphic to the weak topology on bounded sets.

DEFINITION 4.19 (weak narrow convergence of probability measures). *Let* $(\mu_i)_{i\in\mathbb{N}}\subset\mathfrak{P}\big(L^2\oplus L^2\big)$*. Then* $(\mu_i)_{i\in\mathbb{N}}$ *weakly narrowly converges to* $\mu\in\mathfrak{P}\big(L^2\oplus L^2\big)$*, in symbols* $\mu_i\overset{n}{\rightharpoonup}\mu$*, if*

$$\forall f\in\mathcal{C}_b\Big(\big(L^2\oplus L^2,d_w\big),\mathbb{R}\Big)\ ,\ \lim_{i\to\infty}\int_{L^2\oplus L^2}f(z)d\mu_i(z) = \int_{L^2\oplus L^2}f(z)d\mu(z)\ ,$$

where $\mathcal{C}_b((L^2 \oplus L^2, d_w), \mathbb{R})$ *is the space of bounded continuous real-valued functions on* $(L^2 \oplus L^2, d_w)$.

It is actually more convenient to use cylindrical functions to prove narrow continuity properties. We define below two useful spaces of smooth cylindrical functions on $L^2 \oplus L^2$.

DEFINITION 4.20 (spaces of cylindrical functions). *Let* $f : L^2 \oplus L^2 \to \mathbb{R}$. *Then* $f \in \mathcal{S}_{cyl}(L^2 \oplus L^2)$ *if there exists an orthogonal projection* $\mathbf{p} : L^2 \oplus L^2 \to L^2 \oplus L^2$, $\dim(\operatorname{Ran}\mathbf{p}) = d < \infty$, *and a rapidly decreasing function* g *in the Schwartz space* $\mathcal{S}(\operatorname{Ran}\mathbf{p})$, *such that*

$$\forall z \in L^2 \oplus L^2 \, , \; f(z) = g(\mathbf{p}z) \, .$$

Analogously, if $g \in \mathcal{C}_0^\infty(\operatorname{Ran}\mathbf{p})$, *then* $f \in \mathcal{C}_{0,cyl}^\infty(L^2 \oplus L^2)$, *the cylindrical smooth functions with compact support.*

We remark that neither $\mathcal{S}_{cyl}(L^2 \oplus L^2)$ nor $\mathcal{C}_{0,cyl}^\infty(L^2 \oplus L^2)$ possesses a vector space structure. Finally, for cylindrical Schwartz functions we define the Fourier transform:

$$\mathcal{F}[f](\eta) = \int_{\operatorname{Ran}\mathbf{p}} e^{-2\pi i \operatorname{Re}\langle \eta, z\rangle_{L^2\oplus L^2}} f(z) dL_{\mathbf{p}}(z) \, ,$$

where $dL_{\mathbf{p}}$ denotes integration with respect to the Lebesgue measure on $\operatorname{Ran}\mathbf{p}$. The inversion formula is then

$$f(z) = \int_{\operatorname{Ran}\mathbf{p}} e^{2\pi i \operatorname{Re}\langle \eta, z\rangle_{L^2\oplus L^2}} \mathcal{F}[f](\eta) dL_{\mathbf{p}}(\eta) \, .$$

With these definitions in mind, we can prove the following lemma.

LEMMA 4.21. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})$ *be a family of normal states that satisfies assumptions* (A-n), (A(h)'), *and* (A(h)"); $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}(L^2 \oplus L^2)$ *such that for any* $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon}))$. *If, in addition,* $\tilde{\mu}_t$ *satisfies the integral equation* (95), *then the following statements are true:*

* *For any* $t \in \mathbb{R}$, *and for any* $(t_i)_{i\in\mathbb{R}} \subset \mathbb{R}$ *such that* $\lim_{i\to\infty} t_i = t$,

$$\tilde{\mu}_{t_i} \xrightarrow{n} \tilde{\mu}_t \; ;$$

i.e., $\tilde{\mu}_t$ *is a weakly narrowly continuous map in* $\mathfrak{P}(L^2 \oplus L^2)$.

* *The map* $\tilde{\mu}_t$ *solves the transport equation*[16]

$$\partial_t \tilde{\mu}_t + \nabla^T\Big(\mathbf{V}(t)\tilde{\mu}_t\Big) = 0$$

in the weak sense, i.e., for any $f \in \mathcal{C}_{0,cyl}^\infty(\mathbb{R} \times (L^2 \oplus L^2))$,

$$\int_{\mathbb{R}}\int_{L^2\oplus L^2} \Big(\partial_t f + \operatorname{Re}\langle \nabla f, \mathbf{V}(t)\rangle_{L^2\oplus L^2}\Big) d\tilde{\mu}_t dt = 0 \, . \tag{97}$$

Proof. Let $f \in \mathcal{S}_{cyl}(L^2 \oplus L^2)$. Fubini's theorem gives

$$\int_{L^2\oplus L^2} f(z) d\tilde{\mu}_t(z) = \int_{\operatorname{Ran}\mathbf{p}} \mathcal{F}[f](\xi) \left(\int_{L^2\oplus L^2} e^{2\pi i \operatorname{Re}\langle \xi, z\rangle} d\tilde{\mu}_t(z)\right) dL_{\operatorname{Ran}\mathbf{p}}(\xi) \, ,$$

[16]Recall that $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t) \circ \partial_{\bar{z}}(\hat{\mathscr{E}} - \mathscr{E}_0) \circ \mathbf{E}_0(t)(z)$.

where $dL_{\mathrm{Ran}\,\mathbf{p}}$ is the Lebesgue measure on $\mathrm{Ran}\,\mathbf{p}$ and

$$\mathcal{F}(f)(\xi) = \int_{\mathrm{Ran}\,\mathbf{p}} f(z) e^{-2\pi i \mathrm{Re}\langle \xi, z\rangle} dL_{\mathrm{Ran}\,\mathbf{p}}(z) \,.$$

Now we define $\tilde{G}_0(t,\xi) := \int_{L^2\oplus L^2} e^{2\pi i \mathrm{Re}\langle \xi, z\rangle} d\tilde{\mu}_t(z)$. Hence (93) of Proposition 4.17 gives

$$\tilde{G}_0(t,\xi) - \tilde{G}_0(s,\xi) = \int_s^t \Big(\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde{\xi}_\tau)(z) e^{i\sqrt{2}\mathrm{Re}\langle \tilde{\xi}_\tau, z\rangle} d\mu_\tau(z) \Big) d\tau \,, \tag{98}$$

and this proves that $t \mapsto \tilde{G}_0(t,\xi)$ is continuous for any $\xi \in L^2 \oplus L^2$ since the integrand in the right-hand side of (98) is bounded with respect to τ by Proposition 4.17. Note that $\tilde{G}_0(t,\xi)$ is bounded by one for any $(t,\xi) \in \mathbb{R}\times(L^2\oplus L^2)$. Therefore, the map $t \mapsto \int_{L^2\oplus L^2} f(z) d\tilde{\mu}_t(z)$ is continuous for any $f \in \mathcal{S}_{cyl}(L^2\oplus L^2)$. Finally, by an argument analogous to the one used at the beginning of the proof of Proposition 4.17, it is easy to prove that $\int_{L^2\oplus L^2} \|z\|^2_{L^2\oplus L^2} d\tilde{\mu}_t(z) \in L^\infty_t(\mathbb{R})$. In fact, we know that $\tilde{\mu}_t(B_u(0,\sqrt{\mathfrak{C}}) \cap Q(-\Delta+V) \oplus D(\omega^{1/2})) = 1$ by Lemma 4.12; and if $z=(u,\alpha)$, then the functions $\alpha \mapsto \|\alpha\|_2^2 \leq \|\alpha\|^2_{\mathcal{F}H^{1/2}}$ belong to $L^1_z[L^2\oplus L^2, d\tilde{\mu}_t(z)]$ uniformly in t by Lemmas 4.8 and 4.12. Then it follows that $\tilde{\mu}_t$ is weakly narrowly continuous by [3, Lemma 5.1.12 - f], thus proving the first point.

Now we prove the second point by a similar argument as in [12] which we reproduce here for completeness. Let $g \in \mathcal{C}^\infty_{0,cyl}(L^2\oplus L^2)$; we integrate (95) with respect to the measure $\mathcal{F}[g](\eta) dL_{\mathbf{p}}$ obtaining

$$\begin{aligned}\int_{L^2\oplus L^2} g(z) d\tilde{\mu}_t(z) &= \int_{L^2\oplus L^2} g(z) d\tilde{\mu}_0(z) \\ &\quad + 2\pi i \int_0^t \int_{\mathrm{Ran}\,\mathbf{p}} \Big(\int_{L^2\oplus L^2} \mathrm{Re}\langle \eta, \mathbf{V}(s)(z)\rangle_{L^2\oplus L^2} d\tilde{\mu}_s(z) \Big) \mathcal{F}[g](\eta) dL_{\mathbf{p}}(\eta) ds \,.\end{aligned}$$

Let ∇g be the differential of $g : L^2\oplus L^2 \to \mathbb{R}$, where here $L^2\oplus L^2$ is considered as a real Hilbert space with scalar product $\mathrm{Re}\langle\cdot,\cdot\rangle_{L^2\oplus L^2}$. Then, by Fubini's theorem and the properties of the Fourier transform, we get

$$\begin{aligned}\int_{L^2\oplus L^2} g(z) d\tilde{\mu}_t(z) &= \int_{L^2\oplus L^2} g(z) d\tilde{\mu}_0(z) \\ &\quad + \int_0^t \int_{L^2\oplus L^2} \mathrm{Re}\langle \nabla g(z), \mathbf{V}(s)(z)\rangle_{L^2\oplus L^2} d\tilde{\mu}_s(z) ds \,.\end{aligned}$$

By Lebesgue's differentiation theorem (with respect to t), we obtain

$$\partial_t \int_{L^2\oplus L^2} g(z) d\tilde{\mu}_t(z) - \int_{L^2\oplus L^2} \mathrm{Re}\langle \nabla g(z), \mathbf{V}(t)(z)\rangle_{L^2\oplus L^2} d\tilde{\mu}_t(z) = 0 \,.$$

Equation (97) is then obtained for $f(t,z) = \varphi(t) g(z)$, multiplying by $\varphi(t) \in \mathcal{C}^\infty_0(\mathbb{R},\mathbb{R})$, integrating with respect to t, and finally using integration by parts. The result for a generic $f \in \mathcal{C}^\infty_{0,cyl}(\mathbb{R}\times(L^2\oplus L^2))$ follows immediately: $f(t,z) = g(t,\mathbf{p}z)$ for some $g \in \mathcal{C}^\infty_0(\mathbb{R}\times\mathrm{Ran}\,\mathbf{p})$, and the latter can be approximated by a sequence $(g_j(t,\mathbf{p}z))_{j\in\mathbb{N}} \subset \mathcal{C}^\infty_0(\mathbb{R}) \overset{alg}{\otimes} \mathcal{C}^\infty_0(\mathrm{Ran}\,\mathbf{p})$. ∎

We need to check a hypothesis on the vector field $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t) \circ \partial_{\bar{z}}\big(\hat{\mathscr{E}} - \mathscr{E}_0\big) \circ \mathbf{E}_0(t)(z)$ to prove the sought uniqueness result. This is done in the following lemma.

LEMMA 4.22. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfies assumptions* (A-n) *and* (A(h)')*; $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}\big(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})\big)$. Then $\|\mathbf{V}(t)(z)\|_{L^2\oplus L^2} \in L_t^\infty(\mathbb{R}, L_z^1\big[L^2 \oplus L^2, d\mu_t(z)\big])$, i.e., the norm of the vector field is integrable with respect to $\tilde{\mu}_t$, uniformly in $t \in \mathbb{R}$.*

Proof. By (92) of Lemma 4.16 and the definition of $\mathbf{V}(t)$, we have that for any $\xi \in L^2 \oplus L^2$,

$$\begin{aligned}\Big|\mathrm{Re}\langle \xi, \mathbf{V}(t)(z)\rangle\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(&\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 \\ &+ \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} \\ &+ \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.\end{aligned}$$

It is easy to prove an equivalent bound for the imaginary part and hence obtain for any $\xi \in L^2 \oplus L^2$

$$\begin{aligned}\Big|\langle \xi, \mathbf{V}(t)(z)\rangle\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(&\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} \\ &+ \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} \\ &+ \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.\end{aligned}$$

Therefore, it follows immediately that

$$\begin{aligned}\|\mathbf{V}(t)(z)\|_{L^2\oplus L^2} \leq C(\sigma_0)\Big(&\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} \\ &+ \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} \\ &+ \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.\end{aligned}$$

The right-hand side of the above equation is in $L_t^\infty(\mathbb{R}, L_z^1\big[L^2 \oplus L^2, d\mu_t(z)\big])$, as shown at the beginning of the proof of Proposition 4.17. ∎

At this stage, we appeal to a result proved in [8, Proposition 4.1] concerning the uniqueness of measure-valued solutions of the Liouville equation (97), which we briefly recall in Appendix B.

PROPOSITION 4.23. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfies assumptions* (A-n), (A(h)')*, and* (A(h)")*. In addition, let $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}\big(\tilde{\varrho}_{\varepsilon_k}(t), k \in \mathbb{N}\big)$ for some sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\varepsilon_k \to 0$ and $\tilde{\mu}_t$ satisfies the integral equation* (95)*. Then $\tilde{\mu}_t = (\mathbf{E}_0(-t) \circ \hat{\mathbf{E}}(t))_{\#}\mu_0$.*

Proof. Observe that Lemmas 4.21 and 4.22 and Lemma 4.12 (ii)–(iii) are sufficient to apply Proposition B.1 with $v(t,z) = \mathbf{V}(t)(z)$ and $(\tilde{\mu}_t)_{t\in\mathbb{R}}$. Hence, we obtain the existence of a probability measure η verifying the properties (i)–(ii) in Proposition B.1. The next step is to show that η is concentrated on solutions of the dressed equation (S-KG[D]) written in the interaction representation. For simplicity one can take the interval I such that $[0,T] \subset I$ for some $T > 0$.

By Hölder's inequality, Lemma 4.12(iii), and Proposition B.1(ii),

$$\int_{\mathfrak{X}} \left(\int_I ||\gamma(t)||^2_{H^1\oplus\mathcal{F}H^{1/2}} dt\right)^{1/2} d\eta(x,\gamma) \leq \left(\int_I \int_{H^1\oplus\mathcal{F}H^{1/2}} ||z||^2_{H^1\oplus\mathcal{F}H^{1/2}} d\tilde{\mu}_t\right)^{1/2} < \infty\,.$$

This means that $\gamma \in L^2(I, H^1 \oplus \mathcal{F}H^{1/2})$ for η-a.e. So we conclude that there exists an η-negligible set $\mathscr{N}$ such that for any $(x,\gamma) \in \mathfrak{X} \smallsetminus \mathscr{N}$, $\gamma \in W^{1,1}(I, L^2 \oplus L^2)$ satisfy the equation

$$\gamma(t) = x + \int_0^t \mathbf{V}(s)(\gamma(s))ds \quad \forall t \in I\,;$$

and furthermore $\gamma \in L^2(I, H^1 \oplus \mathcal{F}H^{1/2}) \cap L^\infty(I, L^2 \oplus L^2)$ and $\mathbf{V}(\cdot)(\gamma(\cdot)) \in L^1(I, L^2 \oplus L^2)$. Remember that $\mathbf{D}_{g_\infty}(-1)$ and $\mathbf{E}_0(t)$ preserve the spaces $H^1 \oplus \mathcal{F}H^{1/2}$ and $L^2 \oplus L^2$ (see Proposition 3.5). So by a simple computation one checks that for any γ as before, the curve

$$t \to \tilde{\gamma}(t) := \mathbf{D}_{g_\infty}(1) \circ \mathbf{E}_0(t)(\gamma(t)) \in L^2(I, H^1 \oplus \mathcal{F}H^{1/2}) \cap L^\infty(I, L^2 \oplus L^2)$$

and satisfies the Duhamel formula,

$$\tilde{\gamma}(t) = \mathbf{E}_0(t) \circ \mathbf{D}_{g_\infty}(1)x - i\int_0^t \mathbf{E}_0(t-s)\partial_{\bar{z}}(\mathscr{E} - \mathscr{E}_0)(\tilde{\gamma}(s))ds \quad \forall t \in I\,,$$

which is the original Cauchy problem (S-KG[Y]) with the energy $\mathscr{E}$ given by Definition 3.8. Remember that we have already checked that $\mathbf{D}_{g_\infty}(\theta)$ are nonlinear symplectomorphisms on the phase-space $L^2 \oplus L^2$ (see Proposition 3.17). Now appealing to the result [40, Theorem 1.3], we need to show that $\tilde{\gamma}_1 \in L^{10/3}([0,T], L^{10/3}(\mathbb{R}^3)) \cap L^8([0,T], L^{12/5}(\mathbb{R}^3))$ where $\tilde{\gamma} = (\tilde{\gamma}_1, \tilde{\gamma}_2)$ in order to conclude that $\tilde{\gamma}$ is actually the unique strong and global solution of the S-KG equation with initial condition $\gamma(0) = x \in H^1 \oplus \mathcal{F}H^{1/2}$ and belonging to $C(\mathbb{R}, H^1 \oplus \mathcal{F}H^{1/2})$. The last statement follows by Strichartz estimates since $\tilde{\gamma}_1 \in L^2([0,T], L^6(\mathbb{R}^3) \cap L^\infty([0,T], L^2(\mathbb{R}^3)$. So going back to γ, we conclude that $\gamma(t) = \mathbf{E}_0(-t)_\# \hat{\mathbf{E}}(t)(x)$. Hence, for any Borel bounded function φ on $L^2 \oplus L^2$ and $t \in \mathbb{R}$,

$$\begin{aligned}\int_{L^2\oplus L^2} \varphi(x) d\tilde{\mu}_t = \int_{\mathfrak{X}} \varphi(\gamma(t))d\eta &= \int_{\mathfrak{X}} \varphi \circ \mathbf{E}_0(-t) \circ \hat{\mathbf{E}}(t)(x)d\eta \\ &= \int_{L^2\oplus L^2} \varphi(\mathbf{E}_0(-t) \circ \hat{\mathbf{E}}(t)(x))\, d\mu_0(x)\,.\end{aligned}$$

∎

4.7. The classical limit of the dressing transformation. Let us consider now the dressing transformation $U_\infty(\theta) = e^{-i\frac{\theta}{\varepsilon}T_\infty}$ on $\mathcal{H}$, with self-adjoint generator:

$$T_\infty = \big(\mathscr{D}_{g_\infty}\big)^{Wick} = \int_{\mathbb{R}^3} \psi^*(x)\Big(a^*(g_\infty e^{-ik\cdot x}) + a(g_\infty e^{-ik\cdot x})\Big)\psi(x)dx\;,$$

$$g_\infty(k) = -\frac{i}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(k)}}\frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M}+\omega(k)} \in L^2(\mathbb{R}^3)\;.$$

The family $\big(e^{-i\frac{\theta}{\varepsilon}T_\infty}\big)_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ is a strongly continuous unitary group and therefore can be seen as a dynamical system acting on quantum states. Therefore, given a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ of normal quantum states on $\mathcal{H}$, we could determine the Wigner measures of

$$\hat{\varrho}_\varepsilon(\theta) = e^{-i\frac{\theta}{\varepsilon}T_\infty}\, \varrho_\varepsilon\, e^{i\frac{\theta}{\varepsilon}T_\infty}\;. \tag{99}$$

Since $T_\infty = \big(\mathscr{D}_{g_\infty}\big)^{Wick}$, where $\mathscr{D}_{g_\infty}$ is the classical dressing generator defined in subsection 3.1, we expect that under suitable assumptions, $(\varrho_{\varepsilon_k} \to \mu \Rightarrow \hat{\varrho}_{\varepsilon_k}(\theta) \to$

$\mathbf{D}_{g_\infty}(\theta)_{\#}\mu)$, where $\mathbf{D}_{g_\infty}(\theta)$ is the classical dressing transformation. The last assertion is indeed true, as explained in the following. Observe that the dressing generator T_∞ is equal to the interaction part $H_I(\sigma)$ of the Nelson model with cutoff, where $\frac{\chi_\sigma}{\sqrt{2\omega}}$ is replaced by g_∞, i.e., $T_\infty = H_I(\sigma)\big|_{\frac{\chi_\sigma}{\sqrt{2\omega}}=g_\infty}$. The classical limit of the Nelson model with cutoff has been treated by the authors in [5]; thus the results below can be immediately deduced by the results in [5, $d=3$, $H_0=0$, and $\frac{\chi}{\sqrt{\omega}} = g_\infty$]. We recall also that g_∞ and therefore also T_∞ and $\mathscr{D}_{g_\infty}$ depend on $\sigma_0 \in \mathbb{R}_+$.

LEMMA 4.24. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal (quantum) states on $\mathcal{H}$ that satisfies assumptions* (A-n) *and* (A-h). *Then for any $\sigma_0 \in \mathbb{R}_+$, $(\hat\varrho_\varepsilon(-1))_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies assumptions* (A-n) *and* (A(h)').

PROPOSITION 4.25. *Let $\mathbf{D}_{g_\infty} : \mathbb{R}\times Q(-\Delta+V)\oplus\mathcal{F}H^{1/2} \to Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}$ be the classical dressing transformation. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal quantum states on $\mathcal{H}$ that satisfies assumption* (A-n) *and assumption* (A-h) *or* (A(h)'). *Then $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon\in(0,\bar\varepsilon)\big) \neq \emptyset$; and for any $\sigma_0\in\mathbb{R}_+$ and $\theta\in\mathbb{R}$,*

$$\mathcal{M}\Big(\hat\varrho_\varepsilon(\theta), \varepsilon\in(0,\bar\varepsilon)\Big) = \Big\{\mathbf{D}_{g_\infty}(\theta)_{\#}\mu\,,\, \mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)\Big\}\,. \tag{100}$$

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar\varepsilon)$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k = 0$. Then the following statement is true:

$$\varrho_{\varepsilon_k}\to\mu \Leftrightarrow \Big(\forall\theta\in\mathbb{R}\,,\ \forall\sigma_0\in\mathbb{R}_+\,,\ \hat\varrho_{\varepsilon_k}(\theta)\to\mathbf{D}_{g_\infty}(\theta)_{\#}\mu\Big)\,. \tag{101}$$

4.8. Overview of the results: Linking the dressed and undressed systems. Since as discussed in the previous subsection we can treat the dressing as a dynamical transformation with its own "time" parameter θ, we are able to link the classical limit of the dressed and undressed quantum dynamics via the classical dressing. In this way we are able to recover the expected classical S-KG dynamics for the undressed dynamics and finally prove Theorem 1.1.

First, we put together the results proved from subsection 4.2 to subsection 4.6 on the renormalized dressed dynamics and remove assumption (A(h)") with the help of an approximation argument worked out in [11]. This is done in the following theorem.

THEOREM 4.26. *Let $\hat{\mathbf{E}} : \mathbb{R}\times Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}\to Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}$ be the dressed S-KG flow associated to $\hat{\mathscr{E}}$. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states in $\mathcal{H}$ that satisfies assumptions* (A-n) *and* (A(h)'). *Then for any $\sigma_0\geq 2K(\mathfrak{C}+1+\bar\varepsilon)$ the dynamics $e^{-i\frac{t}{\varepsilon}\hat H^{\mathrm{ren}}_\varepsilon}$ is nontrivial on every relevant sector with fixed nucleons of the state ϱ_ε; $\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)\neq\emptyset$; and for any $t\in\mathbb{R}$*

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}\hat H^{\mathrm{ren}}_\varepsilon}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}\hat H^{\mathrm{ren}}_\varepsilon},\varepsilon\in(0,\bar\varepsilon)\Big) = \Big\{\hat{\mathbf{E}}(t)_{\#}\mu\,,\,\mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)\Big\}\,. \tag{102}$$

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar\varepsilon)$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k=0$. Then the following statement is true:

$$\varrho_{\varepsilon_k}\to\mu\Leftrightarrow\Big(\forall t\in\mathbb{R}\,,\ e^{-i\frac{t}{\varepsilon_k}\hat H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}\hat H^{\mathrm{ren}}_{\varepsilon_k}}\to\hat{\mathbf{E}}(t)_{\#}\mu\Big)\,. \tag{103}$$

Proof. Thanks to the argument briefly sketched below, we no longer need assumption (A(h)"). Let $\chi\in C_0^\infty(\mathbb{R})$ such that $0\leq\chi\leq1$, $\chi\equiv1$, in a neighborhood of 0 and $\chi_R(x) = \chi(\frac{x}{R})$. The approximation

$$\varrho_{\varepsilon,R} = \frac{\chi_R(\hat{H}_\varepsilon^{\rm ren})\ \varrho_\varepsilon\ \chi_R(\hat{H}_\varepsilon^{\rm ren})}{\mathrm{Tr}\Big[\chi_R(\hat{H}_\varepsilon^{\rm ren})\ \varrho_\varepsilon\ \chi_R(\hat{H}_\varepsilon^{\rm ren})\Big]}$$

satisfies assumptions (A-n), (A(h)'), and (A(h)") and the property

$$||e^{-i\frac{t}{\varepsilon}\hat{H}_\varepsilon^{\rm ren}}\,(\varrho_\varepsilon - \varrho_{\varepsilon,R})\,e^{i\frac{t}{\varepsilon}\hat{H}_\varepsilon^{\rm ren}}||_{\mathcal{L}^1(\mathscr{H})} = ||\varrho_\varepsilon - \varrho_{\varepsilon,R}||_{\mathcal{L}^1(\mathcal{H})} \le \nu(R)\,,$$

where $\nu(R)$ is independent of ε and $\lim_{R\to\infty}\nu(R) = 0$. The last claim follows by assumption (A(h)'), Theorem 2.11, and Definition 2.13. Up to extracting a sequence which a priori depends on R and t, we can suppose that $\mathscr{M}(\varrho_{\varepsilon_n,R}, n\in\mathbb{N}) = \{\mu_{0,R}\}$, $\mathscr{M}(\varrho_{\varepsilon_n}, n\in\mathbb{N}) = \{\mu_0\}$, and $\mathcal{M}(\varrho_{\varepsilon_n}(t), n\in\mathbb{N}) = \{\mu_t\}$. In particular, applying Proposition 4.23, we obtain

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}\ \varrho_{\varepsilon_n,R}\ e^{i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}, n\in\mathbb{N}\Big) = \Big\{\hat{\mathbf{E}}(t)_{\#}\mu_{0,R}\Big\}\ .$$

A general estimate proved in [11, Proposition 2.10] compares the total variation distance of Wigner (probability) measures with the trace distance of their associated quantum states. In our case, it implies

$$\int_{L^2\oplus L^2}|\mu_t - \hat{\mathbf{E}}(t)_{\#}\mu_{0,R}| \le \liminf_{n\to\infty}||e^{-i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}\,(\varrho_{\varepsilon_n} - \varrho_{\varepsilon_n,R})\,e^{i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}||_{\mathcal{L}^1(\mathcal{H})} \le \nu(R)\,,$$
$$\int_{L^2\oplus L^2}|\mu_0 - \mu_{0,R}| \le \liminf_{n\to\infty}||\varrho_{\varepsilon_n} - \varrho_{\varepsilon_n,R}||_{\mathcal{L}^1(\mathcal{H})} \le \nu(R)\,,$$

where the left-hand side denotes the total variation of the signed measures $\mu_t - \hat{\mathbf{E}}(t)_{\#}\mu_{0,R}$ and $\mu_0 - \mu_{0,R}$, respectively. Hence, by the triangle inequality, we obtain

$$\int_{L^2\oplus L^2}|\mu_t - \hat{\mathbf{E}}(t)_{\#}\mu_0| \le \int_{L^2\oplus L^2}|\mu_t - \hat{\mathbf{E}}(t)_{\#}\mu_{0,R}| + \int_{L^2\oplus L^2}|\mu_{0,R} - \mu_0| \le 2\nu(R)\,.$$

This proves that

$$\Big\{\hat{\mathbf{E}}(t)_{\#}\mu_0\Big\} \subset \mathcal{M}(e^{-i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}\varrho_{\varepsilon_n}e^{i\frac{t}{\varepsilon_n}\hat{H}_{\varepsilon_n}^{\rm ren}}, n\in\mathbb{N})\,.$$

By reversing time and utilizing the analogous inclusion above, we prove (103). □

Proof of Theorem 1.1. Observe that using the definition of the renormalized dressed evolution $\varrho_\varepsilon(t)$ (Definition 4.3) and the definition of the "dressing dynamics" $\hat{\varrho}_\varepsilon(\theta)$ (equation (99)), we obtain

$$\begin{aligned} e^{-i\frac{t}{\varepsilon}H_\varepsilon^{\rm ren}}\ \varrho_\varepsilon\ e^{i\frac{t}{\varepsilon}H_\varepsilon^{\rm ren}} &= e^{-\frac{i}{\varepsilon}T_\infty}e^{-i\frac{t}{\varepsilon}\hat{H}_\varepsilon^{\rm ren}}e^{\frac{i}{\varepsilon}T_\infty}\ \varrho_\varepsilon\ e^{-\frac{i}{\varepsilon}T_\infty}e^{i\frac{t}{\varepsilon}\hat{H}_\varepsilon^{\rm ren}}e^{\frac{i}{\varepsilon}T_\infty} \\ &= \Big(\big(\hat{\varrho}_\varepsilon(-1)\big)(t)\Big)^{\hat{}}(1)\ . \end{aligned}$$

Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states in $\mathcal{H}$ that satisfies assumptions (A-n) and (A-h). In addition, as usual, let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar{\varepsilon})$ be a sequence such that

$\lim_{k\to\infty}\varepsilon_k = 0$. Then we can use Lemma 4.24, Proposition 4.25, and Theorem 4.26 to prove the following statement:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \Big(\forall t\in\mathbb{R}\,,\; e^{-i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}} \to \mathbf{D}_{g_\infty}(1)_\#\hat{\mathbf{E}}(t)_\#\mathbf{D}_{g_\infty}(-1)_\#\mu = [\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}(t)\circ\mathbf{D}_{g_\infty}(-1)]_\#\mu\Big)\,.$$

Therefore, Theorem 1.1 is proved, since by (64) of Theorem 3.16 $\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}(t)\circ\mathbf{D}_{g_\infty}(-1) = \mathbf{E}(t)$. To be more precise, we use the following chain of inferences:

$$\begin{aligned}
\Big(\varrho_{\varepsilon_k}\to\mu\Big) &\overset{\substack{\text{Lem. 4.24}\\ \text{Prop. 4.25}}}{\Longrightarrow} \Big(\forall\sigma_0\in\mathbb{R}_+\,,\; \hat{\varrho}_{\varepsilon_k}(-1)\to\mathbf{D}_{g_\infty}(-1)_\#\mu \text{ and } \big(\hat{\varrho}_{\varepsilon_k}(-1)\big)_{k\in\mathbb{N}} \\
&\qquad\qquad \text{satisfies ass. (A-n), (A(h)')}\Big) \\
&\overset{\substack{\text{Thm. 4.26}\\ \text{Lem. 4.8}}}{\Longrightarrow} \Big(\exists\sigma_0\in\mathbb{R}_+\,,\;\forall t\in\mathbb{R}\,,\; \big(\hat{\varrho}_{\varepsilon_k}(-1)\big)(t)\to\hat{\mathbf{E}}(t)_\#\mathbf{D}_{g_\infty}(-1)_\#\mu \\
&\qquad\qquad \text{and } \Big(\big(\hat{\varrho}_{\varepsilon_k}(-1)\big)(t)\Big)_{k\in\mathbb{N}} \text{ satisfies ass. (A-n), (A(h)')}\Big) \\
&\overset{\text{Prop. 4.25}}{\Longrightarrow} \Big(\forall t\in\mathbb{R}\,,\; \Big(\big(\hat{\varrho}_{\varepsilon_k}(-1)\big)(t)\Big)^{\hat{}}(1)\to\mathbf{D}_{g_\infty}(1)_\#\hat{\mathbf{E}}(t)_\#\mathbf{D}_{g_\infty}(-1)_\#\mu\Big) \\
&\overset{\text{Thm. 3.16}}{\Longrightarrow} \Big(\forall t\in\mathbb{R}\,,\; e^{-i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}H^{\mathrm{ren}}_{\varepsilon_k}}\to\mathbf{E}(t)_\#\mu\Big)\,.
\end{aligned}$$

The inference in the opposite sense is trivial.

As has become evident with the above discussion, we do not prove Theorem 1.1 directly; and it would be very difficult to do so, due to the fact that we do not know the explicit form of the generator $H^{\mathrm{ren}}_\varepsilon$ of the undressed dynamics. We know instead how the dressed generator $\hat{H}^{\mathrm{ren}}_\varepsilon$ acts as a quadratic form, and that is sufficient to characterize its dynamics in the classical limit and obtain the results of Theorem 4.26. The properties of the dressing transformation and of its classical counterpart are then crucial to translate the results on the dressed dynamics to the corresponding results on the undressed one.

Appendix A. Uniform higher-order estimate. We prove in this appendix a higher-order estimate that bounds the meson number operator N_2 by the dressed Hamiltonian $\hat{H}^{(n)}_\sigma$ uniformly with respect to the effective (semiclassical) parameter ε and the cutoff parameter σ. Such types of estimates rely on the pull-through formula, and they are known for the $P(\varphi)_2$ model [105] and for the Nelson model [4]. However, since the dependence of the dressed Hamiltonian $\hat{H}^{(n)}_\sigma$ on ε is somewhat nontrivial, we briefly indicate in this appendix how to obtain a uniform estimate.

LEMMA A.1. *For any $\varepsilon\in(0,\bar{\varepsilon})$ and any $\psi\in D(N_2)\subset\mathcal{H}$,*

$$\big\|N_2\psi\big\|^2 = \int_{\mathbb{R}^3}\big\|(N_2+\varepsilon)^{\frac12}a(k)\psi\big\|^2\,dk\,.$$

Proof. Recall that N_2 and $a(k)$ depend on the parameter ε according to the notation of subsection 1.1. Taking care of domain issues as in [4, Lemma 2.1], one proves

$$\|N_2\psi\|^2 = \left\langle N_2^{\frac{1}{2}}\psi, \int_{\mathbb{R}^3} a^*(k)a(k)\,dk\, N_2^{\frac{1}{2}}\psi \right\rangle = \int_{\mathbb{R}^3} \|a(k)N_2^{\frac{1}{2}}\psi\|^2\,dk$$
$$= \int_{\mathbb{R}^3} \|(N_2+\varepsilon)^{\frac{1}{2}}a(k)\psi\|^2\,dk\,. \qquad \square$$

Recall that the interaction term $\hat{H}_I(\sigma)^{(n)}$ is given by (13). A simple computation yields

$$[a(k), \hat{H}_I(\sigma)^{(n)}] = \varepsilon^2 \Bigg[\sum_{j=1}^n \frac{1}{2\sqrt{(2\pi)^3}} \frac{\chi_\sigma(k)}{\sqrt{\omega(k)}} e^{-ik.x_j} + \frac{1}{M} \sum_{j=1}^n r_\sigma(k) e^{-ik\cdot x_j} a^*(r_\sigma e^{-ik\cdot x_j})$$
$$+ r_\sigma(k) e^{-ik\cdot x_j} a(r_\sigma e^{-ik\cdot x_j}) - r_\sigma(k) e^{-ik\cdot x_j} D_{x_j} \Bigg]\,.$$

Lemma A.2. *For any $\mathfrak{C} > 0$ and $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ there exist $c, b > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$, and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$, we have*

$$\left\|(b+\varepsilon\omega(k)+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}} [a(k), \hat{H}_I(\sigma)^{(n)}] (b+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}}\right\| \leq c\left(\left|\tfrac{\chi_\sigma(k)}{\sqrt{\omega(k)}}\right| + |r_\sigma(k)|\omega(k)^{-1/4} \right).$$

Proof. According to Proposition 2.10 and Theorem 2.11, $\hat{H}_I(\sigma)^{(n)}$ is $H_0^{(n)}$-form bounded with small bound that is uniform with respect to $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$, and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$. Hence $(H_0^{(n)})^{\frac{1}{2}}(b + \hat{H}_\sigma^{(n)})^{-\frac{1}{2}}$ is uniformly bounded for some $b > 0$. So it is enough to prove the claimed bound with $H_0^{(n)}$ instead of $\hat{H}_\sigma^{(n)}$. Now using similar estimates as in Lemma 2.7 and the fact that $\sqrt{\varepsilon\omega(k)}(b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-\frac{1}{2}}$ is uniformly bounded, one correctly bounds all the terms of the commutator except the one with a^*. Note that the commutator contains the power ε^2 that controls the sum over $1 \leq j \leq n$ and the factor $1/\sqrt{\varepsilon\omega(k)}$. In order to bound the term with a^*, one uses the type of estimate in [4, Lemma 3.3(ii)] with $s = 1/2$. Note that one gets an ε-dependent estimate from [4, Lemma 3.3(ii)] by noticing that $\varepsilon^{1/4}(H_0^{(n)}+1)^{-1/4}(d\Gamma_1(\omega)+1)^{1/4}$ and $\varepsilon^{1/4}(N_2+1)^{-1/4}(d\Gamma_1(1)+1)^{1/4}$ are uniformly bounded[17] and that a^* contains $\sqrt{\varepsilon}$ which cancels the latter $\varepsilon^{-1/4}\cdot\varepsilon^{-1/4}$. $\square$

Let $\mathfrak{C} > 0$ and $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ as in the above lemma. In particular, $\hat{H}_\sigma^{(n)}$ is a self-adjoint operator for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma_0 < \sigma \leq +\infty$, and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$.

Lemma A.3 (the pull-through formula). *The following identity holds true for some $b < 0$, any $\phi \in D(N_2^{\frac{1}{2}}) \cap \mathcal{H}_n$, and k almost everywhere in $\mathbb{R}^3$:*

$$a(k)(b - \hat{H}_\sigma^{(n)})^{-1}\phi = (b - \varepsilon\omega(k) - \hat{H}_\sigma^{(n)})^{-1} a(k)\phi$$
$$+ (b - \varepsilon\omega(k) - \hat{H}_\sigma^{(n)})^{-1}\, [a(k), \hat{H}_I(\sigma)^{(n)}]\, (b - \hat{H}_\sigma^{(n)})^{-1}\phi\,.$$

Proof. According to [4, Lemma 4.4] there exists $\psi \in (H_0^{(n)}+1)^{-1}D(N^{\frac{1}{2}})$ such that $\phi = (b - \hat{H}_\sigma^{(n)})\psi$ for some $b < 0$. So the claimed formula is equivalent to

$$(b - \varepsilon\omega(k) - \hat{H}_\sigma^{(n)})a(k)\psi = a(k)(b - \hat{H}_\sigma^{(n)})\psi + [a(k), \hat{H}_I(\sigma)^{(n)}]\psi\,.$$

[17] $d\Gamma_1(\cdot)$ is the ε-independent second quantization operator in [4].

The latter identity follows by a simple computation. ∎

PROPOSITION A.4. *For any $\mathfrak{C} > 0$ and $\sigma_0 \geq 2K(\mathfrak{C} + 1 + \bar{\varepsilon})$ there exist $c, b > 0$ such that the operator $\hat{H}_\sigma^{(n)}$ is self-adjoint and the following bound holds true:*

$$\big\|N_2\psi\big\| \leq c\big\|(\hat{H}_\sigma^{(n)} + b)\psi\big\| \quad \forall\psi \in D(\hat{H}_\sigma^{(n)}),$$

for any $\varepsilon \in (0, \bar{\varepsilon})$, $\sigma \in (\sigma_0, +\infty]$, $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$.

Proof. The operator $\hat{H}_\sigma^{(n)}$ is uniformly bounded from below. So by choosing $b > 0$ large enough one can take $\psi = (-b - \hat{H}_\sigma^{(n)})^{-1}\phi$. Now it is enough to prove the estimate for $\phi \in (H_0^{(n)} + 1)^{-1/2} D(N_2^{\frac{1}{2}})$. Using Lemmas A.1 and A.3,

$$\big\|N_2\psi\big\|^2 = \int_{\mathbb{R}^3} \big\|(N_2 + \varepsilon)^{\frac{1}{2}} a(k)(b + \hat{H}_\sigma^{(n)})^{-1}\phi\big\|^2\, dk$$

$$\leq 2\int_{\mathbb{R}^3} \big\|(N_2 + \varepsilon)^{\frac{1}{2}} (b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1} a(k)\phi\big\|^2\, dk \tag{104}$$

$$+ 2\int_{\mathbb{R}^3} \big\|(N_2 + \varepsilon)^{\frac{1}{2}} (b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1} [a(k), \hat{H}_I(\sigma)^{(n)}]\,(b + \hat{H}_\sigma^{(n)})^{-1}\phi\big\|^2\, dk\,. \tag{105}$$

Since $(N_2+\varepsilon)^{\frac{1}{2}}(b+\varepsilon\omega(k)+\hat{H}_\sigma^{(n)})^{-1/2}$ is uniformly bounded, by Lemma A.2 one shows

$$(105) \leq c\int_{\mathbb{R}^3} |\tfrac{\chi_\sigma(k)}{\sqrt{\omega(k)}}| + |r_\sigma(k)|\omega(k)^{-1/4}\, dk \;\cdot\; \big\|(b + \hat{H}_\sigma^{(n)})^{-1/2}\phi\big\|^2.$$

For simplicity we denote by c any constant. In the same way, one also shows

$$\begin{aligned}(104) &\leq c\int_{\mathbb{R}^3} \big\|(b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1/2} a(k)\phi\big\|^2\, dk \\ &\leq c\int_{\mathbb{R}^3} \big\|(b + \varepsilon\omega(k) + H_0^{(n)})^{-1/2} a(k)\phi\big\|^2\, dk = c\big\|N_2^{1/2}(b + H_0^{(n)})^{-1/2}\phi\big\|^2\,.\end{aligned}$$

The last equality follows by an argument similar to that in the proof of Lemma A.1. Hence, one obtains

$$\begin{aligned}\big\|N_2\psi\big\|^2 &\leq c\left(\big\|\phi\big\|^2 + \big\|(b + H_0^{(n)})^{-1/2}\phi\big\|^2\right) = c\left(\big\|(b + \hat{H}_\sigma^{(n)})\psi\big\|^2 + \big\|(b + H_\sigma^{(n)})^{1/2}\psi\big\|^2\right) \\ &\leq c\big\|(b + \hat{H}_\sigma^{(n)})\psi\big\|^2\,.\end{aligned}$$

The last inequality is a consequence of the uniform boundedness of the operator $(b + H_0^{(n)})^{-1/2}(b + \hat{H}_\sigma^{(n)})^{-1/2}$ with respect to ε, σ, and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$. ∎

Appendix B. Probabilistic representation. For any open bounded interval I, we denote by Γ_I the space of all continuous curves from $\bar{I}$ into $(L^2 \oplus L^2, ||\cdot||_{L^2\oplus L^2})$ and define the following metric space:

$$\mathfrak{X} = \big(L^2 \oplus L^2 \times \Gamma_I, ||\cdot||_{(L^2\oplus L^2, d_w)} + \sup_{t\in\bar{I}} ||\cdot||_{(L^2\oplus L^2, d_w)}\big), \tag{106}$$

where the norm $||\cdot||_{(L^2\oplus L^2, d_w)}$ is associated to the distance introduced in (96). For each $t \in I$, we define the continuous evaluation map,

$$e_t : (x, \gamma) \in E \times \Gamma_I(E) \mapsto \gamma(t) \in E\,.$$

Consider the transport or Liouville equation,

$$\partial_t \mu_t + \nabla^T (v.\mu_t) = 0 \,,$$

understood in a weak sense as the integral equation: for any $\varphi \in \mathcal{C}^\infty_{0,cyl}(I \times L^2 \oplus L^2)$,

$$\int_I \int_{L^2\oplus L^2} \partial_t \varphi(t,x) + \mathrm{Re}\langle v(t,x), \nabla\varphi(t,x)\rangle_{L^2\oplus L^2} \, d\mu_t(x)\, dt = 0. \tag{107}$$

The following result is an adaptation of [8, Propositon 4.1].

Proposition B.1. *Let $v : \mathbb{R} \times H^1 \oplus \mathcal{F}H^{1/2} \to L^2 \oplus L^2$ be a Borel vector field such that v is bounded on bounded sets. Let $t \in I \to \mu_t \in \mathfrak{P}(H^1 \oplus \mathcal{F}H^{1/2})$ be a weakly narrowly continuous solution in $\mathfrak{P}(L^2 \oplus L^2)$ of the Liouville equation* (107) *defined on an open bounded interval I with the following estimate satisfied:*

$$\int_I \int_{H^1\oplus\mathcal{F}H^{1/2}} \|v(t,x)\|_{L^2\oplus L^2} \, d\mu_t(x)\, dt < \infty \,.$$

Then there exists a Borel probability measure η on the space $\mathfrak{X}$ given in (106) *satisfying the following:*

(i) *η is concentrated on the set of $(x,\gamma) \in H^1 \oplus \mathcal{F}H^{1/2} \times \Gamma_I$ such that $\gamma \in W^{1,1}(I, L^2 \oplus L^2)$ and γ are solutions of the initial value problem $\dot\gamma(t) = v(t,\gamma(t))$ for a.e. $t \in I$ and $\gamma(t) \in H^1 \oplus \mathcal{F}H^{1/2}$ for a.e. $t \in I$ with $\gamma(s) = x$ for some fixed $s \in I$.*

(ii) *$\mu_t = (e_t)_\sharp \eta$ for any $t \in I$.*

Here $W^{1,1}(I, L^2 \oplus L^2)$ is the Sobolev space of functions in $L^1(I, L^2 \oplus L^2)$ with distributional first derivatives in $L^1(I, L^2 \oplus L^2)$. In particular, functions in $W^{1,1}(I, L^2 \oplus L^2)$ are absolutely continuous curves in $L^2 \oplus L^2$.

REFERENCES

[1] A. Abdesselam and D. Hasler, *Analyticity of the ground state energy for massless Nelson models*, Comm. Math. Phys., 310 (2012), pp. 511–536, https://doi.org/10.1007/s00220-011-1407-6.

[2] L. Ambrosio, A. Figalli, G. Friesecke, J. Giannoulis, and T. Paul, *Semiclassical limit of quantum dynamics with rough potentials and well-posedness of transport equations with measure initial data*, Comm. Pure Appl. Math., 64 (2011), pp. 1199–1242, https://doi.org/10.1002/cpa.20371.

[3] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2005.

[4] Z. Ammari, *Asymptotic completeness for a renormalized nonrelativistic Hamiltonian in quantum field theory: The Nelson model*, Math. Phys. Anal. Geom., 3 (2000), pp. 217–285, https://doi.org/10.1023/A:1011408618527.

[5] Z. Ammari and M. Falconi, *Wigner measures approach to the classical limit of the Nelson model: Convergence of dynamics and ground state energy*, J. Stat. Phys., 157 (2014), pp. 330–362, https://doi.org/10.1007/s10955-014-1079-7.

[6] Z. Ammari and M. Falconi, *Bohr's Correspondence Principle in Quantum Field Theory and Classical Renormalization Scheme: The Nelson Model*, preprint, https://arxiv.org/abs/1602.03212, 2016.

[7] Z. Ammari, M. Falconi, and B. Pawilowski, *On the rate of convergence for the mean field approximation of Bosonic many-body quantum dynamics*, Commun. Math. Sci., 14 (2016), pp. 1417–1442.

[8] Z. Ammari and Q. Liard, *On the Uniqueness of Probability Measure Solutions to Liouville's Equation of Hamiltonian PDEs*, preprint, https://arxiv.org/abs/1602.06716, 2016.

[9] Z. Ammari and F. Nier, *Mean field limit for bosons and infinite dimensional phase-space analysis*, Ann. Henri Poincaré, 9 (2008), pp. 1503–1574, https://doi.org/10.1007/s00023-008-0393-5.

[10] Z. Ammari and F. Nier, *Mean field limit for bosons and propagation of Wigner measures*, J. Math. Phys., 50 (2009), 042107, https://doi.org/10.1063/1.3115046.

[11] Z. Ammari and F. Nier, *Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states*, J. Math. Pures Appl. (9), 95 (2011), pp. 585–626, https://doi.org/10.1016/j.matpur.2010.12.004.

[12] Z. Ammari and F. Nier, *Mean field propagation of infinite dimensional Wigner measures with a singular two-body interaction potential*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 14 (2015), pp. 155–220, https://doi.org/10.2422/2036-2145.201112_004.

[13] Z. Ammari and M. Zerzeri, *On the classical limit of self-interacting quantum field Hamiltonians with cutoffs*, Hokkaido Math. J., 43 (2014), pp. 385–425, https://doi.org/10.14492/hokmj/1416837571.

[14] A. Arai, *Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation*, Rev. Math. Phys., 13 (2001), pp. 1075–1094, https://doi.org/10.1142/S0129055X01000934.

[15] V. Bach, J. Fröhlich, and I. M. Sigal, *Quantum electrodynamics of confined nonrelativistic particles*, Adv. Math., 137 (1998), pp. 299–395, https://doi.org/10.1006/aima.1998.1734.

[16] V. Bach, J. Fröhlich, and I. M. Sigal, *Renormalization group analysis of spectral problems in quantum field theory*, Adv. Math., 137 (1998), pp. 205–298, https://doi.org/10.1006/aima.1998.1733.

[17] V. Bach, J. Fröhlich, and I. M. Sigal, *Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field*, Comm. Math. Phys., 207 (1999), pp. 249–290, https://doi.org/10.1007/s002200050726.

[18] A. Bachelot, *Problème de Cauchy pour des systèmes hyperboliques semi-linéaires*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 453–478, http://www.numdam.org/item?id=AIHPC_1984__1_6_453_0.

[19] J.-B. Baillon and J. M. Chadam, *The Cauchy problem for the coupled Schroedinger-Klein-Gordon equations*, in Contemporary Developments in Continuum Mechanics and Partial Differential Equations, North–Holland Math. Stud. 30, North–Holland, Amsterdam, New York, 1978, pp. 37–44.

[20] N. Benedikter, G. de Oliveira, and B. Schlein, *Quantitative derivation of the Gross-Pitaevskii equation*, Comm. Pure Appl. Math., 68 (2015), pp. 1399–1482, https://doi.org/10.1002/cpa.21542.

[21] G. Benfatto and G. Gallavotti, *Renormalization Group*, Phys. Notes 1, Princeton University Press, Princeton, NJ, 1995.

[22] F. A. Berezin, *The Method of Second Quantization*, translated from the Russian by Nobumichi Mugibayashi and Alan Jeffrey, Pure Appl. Phys. 24, Academic Press, New York, 1966.

[23] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos, and H. Spohn, *Ground state properties of the Nelson Hamiltonian: A Gibbs measure-based approach*, Rev. Math. Phys., 14 (2002), pp. 173–198, https://doi.org/10.1142/S0129055X02001119.

[24] J. T. Cannon, *Quantum field theoretic properties of a model of Nelson: Domain and eigenvector stability for perturbed linear operators*, J. Funct. Anal., 8 (1971), pp. 101–152.

[25] R. Carles, *Sharp Weights in the Cauchy Problem for Nonlinear Schrodinger Equations with Potential*, preprint, https://arxiv.org/abs/1409.5759, 2014.

[26] T. Chen, C. Hainzl, N. Pavlović, and R. Seiringer, *Unconditional uniqueness for the cubic Gross-Pitaevskii hierarchy via quantum de Finetti*, Comm. Pure Appl. Math., 68 (2015), pp. 1845–1884, https://doi.org/10.1002/cpa.21552.

[27] T. Chen and N. Pavlović, *The quintic NLS as the mean field limit of a boson gas with three-body interactions*, J. Funct. Anal., 260 (2011), pp. 959–997, https://doi.org/10.1016/j.jfa.2010.11.003.

[28] T. Chen and N. Pavlović, *Derivation of the cubic NLS and Gross–Pitaevskii hierarchy from manybody dynamics in $d = 3$ based on spacetime norms*, Ann. Henri Poincaré, 15 (2014), pp. 543–588, https://doi.org/10.1007/s00023-013-0248-6.

[29] T. Chen and N. Pavlović, *On the Cauchy problem for focusing and defocusing Gross-Pitaevskii hierarchies*, Discrete Contin. Dyn. Syst., 27 (2010), pp. 715–739, https://doi.org/10.3934/dcds.2010.27.715.

[30] X. Chen, *Collapsing estimates and the rigorous derivation of the 2d cubic nonlinear Schrödinger equation with anisotropic switchable quadratic traps*, J. Math. Pures Appl. (9), 98 (2012), pp. 450–478, https://doi.org/10.1016/j.matpur.2012.02.003.

[31] X. Chen, *Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions*, Arch. Ration. Mech. Anal., 203 (2012), pp. 455–497, https://doi.org/10.1007/s00205-011-0453-8.

[32] X. Chen, *On the rigorous derivation of the* 3*D cubic nonlinear Schrödinger equation with a quadratic trap*, Arch. Ration. Mech. Anal., 210 (2013), pp. 365–408, https://doi.org/10.1007/s00205-013-0645-5.

[33] X. Chen and J. Holmer, *On the rigorous derivation of the* 2*D cubic nonlinear Schrödinger equation from* 3*D quantum many-body dynamics*, Arch. Ration. Mech. Anal., 210 (2013), pp. 909–954, https://doi.org/10.1007/s00205-013-0667-z.

[34] X. Chen and J. Holmer, *Correlation structures, many-body scattering processes, and the derivation of the Gross-Pitaevskii hierarchy*, Int. Math. Res. Not. IMRN, 10 (2016), pp. 3051–3110, https://doi.org/10.1093/imrn/rnv228.

[35] X. Chen and J. Holmer, *Focusing quantum many-body dynamics: The rigorous derivation of the* 1*D focusing cubic nonlinear Schrödinger equation*, Arch. Ration. Mech. Anal., 221 (2016), pp. 631–676, https://doi.org/10.1007/s00205-016-0970-6.

[36] X. Chen and J. Holmer, *On the Klainerman-Machedon conjecture for the quantum BBGKY hierarchy with self-interaction*, J. Eur. Math. Soc. (JEMS), 18 (2016), pp. 1161–1200, https://doi.org/10.4171/JEMS/610.

[37] X. Chen and J. Holmer, *Focusing quantum many-body dynamics,* II*: The rigorous derivation of the* 1*D focusing cubic nonlinear Schrödinger equation from* 3*D*, Anal. PDE, 10 (2017), pp. 589–633, https://doi.org/10.2140/apde.2017.10.589.

[38] X. Chen and P. Smith, *On the unconditional uniqueness of solutions to the infinite radial Chern-Simons-Schrödinger hierarchy*, Anal. PDE, 7 (2014), pp. 1683–1712, https://doi.org/10.2140/apde.2014.7.1683.

[39] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys., 102 (1985), pp. 497–502, http://projecteuclid.org/euclid.cmp/1104114465.

[40] J. Colliander, J. Holmer, and N. Tzirakis, *Low regularity global well-posedness for the Zakharov and Klein-Gordon-Schrödinger systems*, Trans. Amer. Math. Soc., 360 (2008), pp. 4619–4638, https://doi.org/10.1090/S0002-9947-08-04295-5.

[41] M. Combescure, J. Ralston, and D. Robert, *A proof of the Gutzwiller semiclassical trace formula using coherent states decomposition*, Comm. Math. Phys., 202 (1999), pp. 463–480, https://doi.org/10.1007/s002200050591.

[42] J. M. Cook, *The mathematics of second quantization*, Proc. Natl. Acad. Sci. USA, 37 (1951), pp. 417–420.

[43] J. Dereziński and C. Gérard, *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians*, Rev. Math. Phys., 11 (1999), pp. 383–450, https://doi.org/10.1142/S0129055X99000155.

[44] J. Dereziński and C. Gérard, *Mathematics of Quantization and Quantum Fields*, Cambridge Monogr. Math. Phys., Cambridge University Press, Cambridge, UK, 2013, https://doi.org/10.1017/CBO9780511894541.

[45] A. Elgart, L. Erdös, B. Schlein, and H.-T. Yau, *Gross-Pitaevskii equation as the mean field limit of weakly coupled bosons*, Arch. Ration. Mech. Anal., 179 (2006), pp. 265–283, https://doi.org/10.1007/s00205-005-0388-z.

[46] L. Erdős, B. Schlein, and H.-T. Yau, *Derivation of the Gross-Pitaevskii hierarchy for the dynamics of Bose-Einstein condensate*, Comm. Pure Appl. Math., 59 (2006), pp. 1659–1741, https://doi.org/10.1002/cpa.20123.

[47] L. Erdős, B. Schlein, and H.-T. Yau, *Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential*, J. Amer. Math. Soc., 22 (2009), pp. 1099–1156, https://doi.org/10.1090/S0894-0347-09-00635-3.

[48] L. Erdős, B. Schlein, and H.-T. Yau, *Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems*, Invent. Math., 167 (2007), pp. 515–614, https://doi.org/10.1007/s00222-006-0022-1.

[49] L. Erdős, B. Schlein, and H.-T. Yau, *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate*, Ann. of Math. (2), 172 (2010), pp. 291–370, https://doi.org/10.4007/annals.2010.172.291.

[50] L. Erdős and H.-T. Yau, *Derivation of the nonlinear Schrödinger equation from a many body Coulomb system*, Adv. Theor. Math. Phys., 5 (2001), pp. 1169–1205, https://doi.org/10.4310/ATMP.2001.v5.n6.a6.

[51] M. Falconi, *Classical limit of the Nelson model with cutoff*, J. Math. Phys., 54 (2013), 012303, 012303, https://doi.org/10.1063/1.4775716.

[52] M. Falconi, *Self-adjointness criterion for operators in Fock spaces*, Math. Phys. Anal. Geom., 18 (2015), 2, https://doi.org/10.1007/s11040-015-9173-x.

[53] J. Faupin and I. M. Sigal, *On Rayleigh scattering in non-relativistic quantum electrodynamics*, Comm. Math. Phys., 328 (2014), pp. 1199–1254, https://doi.org/10.1007/s00220-014-1883-6.

[54] V. Fock, *Konfigurationsraum und zweite quantelung*, Zeitschrift für Physik, 75 (1932), pp. 622–647.

[55] G. B. Folland, *Quantum Field Theory: A Tourist Guide for Mathematicians*, Math. Surveys Monogr. 149, AMS, Providence, RI, 2008, https://doi.org/10.1090/surv/149.

[56] R. Frank and B. Schlein, *Dynamics of a strongly coupled polaron*, Lett. Math. Phys., 104 (2014), pp. 911–929, https://doi.org/10.1007/s11005-014-0700-7.

[57] R. L. Frank and Z. Gang, *Derivation of an effective evolution equation for a strongly coupled polaron*, Anal. PDE, 10 (2017), pp. 379–422.

[58] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic completeness for Compton scattering*, Comm. Math. Phys., 252 (2004), pp. 415–476, https://doi.org/10.1007/s00220-004-1180-x.

[59] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations.* I, Bull. Sci. Engrg. Res. Lab. Waseda Univ., 69 (1975), pp. 51–62.

[60] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations.* II, J. Math. Anal. Appl., 66 (1978), pp. 358–378, https://doi.org/10.1016/0022-247X(78)90239-1.

[61] I. Fukuda and M. Tsutsumi, *On coupled Klein-Gordon-Schrödinger equations.* III. *Higher order interaction, decay and blow-up*, Math. Japon., 24 (1979/80), pp. 307–321.

[62] V. Georgescu, C. Gérard, and J. S. Møller, *Spectral theory of massless Pauli-Fierz models*, Comm. Math. Phys., 249 (2004), pp. 29–78, https://doi.org/10.1007/s00220-004-1111-x.

[63] C. Gérard, *On the scattering theory of massless Nelson models*, Rev. Math. Phys., 14 (2002), pp. 1165–1280, https://doi.org/10.1142/S0129055X02001508.

[64] C. Gérard, F. Hiroshima, A. Panati, and A. Suzuki, *Infrared problem for the Nelson model on static space-times*, Comm. Math. Phys., 308 (2011), pp. 543–566, https://doi.org/10.1007/s00220-011-1289-7.

[65] P. Gérard, *Mesures semi-classiques et ondes de Bloch*, in Séminaire sur les Équations aux Dérivées Partielles, 1990–1991, École Polytech., Palaiseau, 1991, Exp. XVI, 19.

[66] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math., 50 (1997), pp. 323–379, https://doi.org/10.1002/(SICI)1097-0312(199704)50:4<323::AID-CPA4>3.0.CO;2-C.

[67] J. Ginibre, F. Nironi, and G. Velo, *Partially classical limit of the Nelson model*, Ann. Henri Poincaré, 7 (2006), pp. 21–43, https://doi.org/10.1007/s00023-005-0240-x.

[68] J. Ginibre and G. Velo, *Renormalization of a quadratic interaction in the Hamiltonian formalism*, Comm. Math. Phys., 18 (1970), pp. 65–81, http://projecteuclid.org/euclid.cmp/1103842445.

[69] J. Ginibre and G. Velo, *The classical field limit of scattering theory for nonrelativistic many-boson systems.* I, Comm. Math. Phys., 66 (1979), pp. 37–76, http://projecteuclid.org/getRecord?id=euclid.cmp/1103904940.

[70] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for the wave-Schrödinger system*, Ann. Henri Poincaré, 3 (2002), pp. 537–612, https://doi.org/10.1007/s00023-002-8627-4.

[71] J. Glimm, *The foundations of quantum field theory*, Adv. Math., 3 (1969), pp. 101–125, https://doi.org/10.1016/0001-8708(69)90004-8.

[72] J. Glimm and A. Jaffe, *Collected papers. Vol.* 2. *Constructive Quantum Field Theory. Selected Papers*, Reprint of articles published in 1968–1980, Birkhäuser Boston, Boston, 1985.

[73] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer-Verlag, New York, 1987, https://doi.org/10.1007/978-1-4612-4728-9.

[74] O. Greenberg and S. Schweber, *Clothed particle operators in simple models of quantum field theory*, Nuovo Cimento (10), 8 (1958), pp. 378–406, https://doi.org/10.1007/BF02828746.

[75] P. Gressman, V. Sohinger, and G. Staffilani, *On the uniqueness of solutions to the periodic 3D Gross-Pitaevskii hierarchy*, J. Funct. Anal., 266 (2014), pp. 4705–4764, https://doi.org/10.1016/j.jfa.2014.02.006.

[76] M. Grillakis and M. Machedon, *Pair excitations and the mean field approximation of interacting bosons,* I, Comm. Math. Phys., 324 (2013), pp. 601–636, https://doi.org/10.1007/s00220-013-1818-7.

[77] M. Grillakis, M. Machedon, and D. Margetis, *Second-order corrections to mean field evolution of weakly interacting bosons.* I, Comm. Math. Phys., 294 (2010), pp. 273–301, https://doi.org/10.1007/s00220-009-0933-y.

[78] M. Grillakis, M. Machedon, and D. Margetis, *Second-order corrections to mean field evolution of weakly interacting bosons.* II, Adv. Math., 228 (2011), pp. 1788–1815, https://doi.org/10.1016/j.aim.2011.06.028.

[79] E. P. Gross, *Particle-like solutions in field theory*, Ann. Phys., 19 (1962), pp. 219–233, https://doi.org/10.1016/0003-4916(62)90216-6.

[80] B. Helffer, A. Martinez, and D. Robert, *Ergodicité et limite semi-classique*, Comm. Math. Phys., 109 (1987), pp. 313–326, http://projecteuclid.org/getRecord?id=euclid.cmp/1104116844.

[81] K. Hepp, *Théorie de la renormalisation*, Lecture Notes in Phys. 2, Springer-Verlag, Berlin, 1969.

[82] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Comm. Math. Phys., 35 (1974), pp. 265–277, http://projecteuclid.org/euclid.cmp/1103859623.

[83] Y. Hong, K. Taliaferro, and Z. Xie, *Unconditional uniqueness of the cubic Gross–Pitaevskii hierarchy with low regularity*, SIAM J. Math. Anal., 47 (2015), pp. 3314–3341, https://doi.org/10.1137/140964898.

[84] L. Hörmander, *The Analysis of Linear Partial Differential Operators* III*: Pseudo-Differential Operators*, Reprint of 1994 ed., Classics in Mathematics, Springer, Berlin, 2007.

[85] A. Jaffe, *Constructive quantum field theory*, in Mathematical Physics 2000, Imperial College Press, London, 2000, pp. 111–127, https://doi.org/10.1142/9781848160224_0007.

[86] K. Kirkpatrick, B. Schlein, and G. Staffilani, *Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics*, Amer. J. Math., 133 (2011), pp. 91–130, https://doi.org/10.1353/ajm.2011.0004.

[87] S. Klainerman and M. Machedon, *On the uniqueness of solutions to the Gross-Pitaevskii hierarchy*, Comm. Math. Phys., 279 (2008), pp. 169–185, https://doi.org/10.1007/s00220-008-0426-4.

[88] E. Kuz, *Rate of convergence to mean field for interacting bosons*, Comm. Partial Differential Equations, 40 (2015), pp. 1831–1854, https://doi.org/10.1080/03605302.2015.1053568.

[89] Q. Liard, *Dérivation des équations de Schrödinger non-linéaires par une méthode des caractéristiques en dimension infinie.*, Ph.D. thesis, Université de Rennes I, Rennes, France, 2015.

[90] E. H. Lieb, R. Seiringer, J. P. Solovej, and J. Yngvason, *The Mathematics of the Bose Gas and Its Condensation*, Oberwolfach Seminars 34, Birkhäuser Verlag, Basel, 2005.

[91] P.-L. Lions and T. Paul, *Sur les mesures de Wigner*, Rev. Mat. Iberoam., 9 (1993), pp. 553–618, https://doi.org/10.4171/RMI/143.

[92] S. Maniglia, *Probabilistic representation and uniqueness results for measure-valued solutions of transport equations*, J. Math. Pures Appl. (9), 87 (2007), pp. 601–626, https://doi.org/10.1016/j.matpur.2007.04.001.

[93] A. Martinez, *An Introduction to Semiclassical and Microlocal Analysis*, Universitext, Springer-Verlag, New York, 2002.

[94] V. Mastropietro, *Non-Perturbative Renormalization*, World Scientific, Hackensack, NJ, 2008, https://doi.org/10.1142/9789812792402.

[95] D. Mendelson, A. Nahmod, N. Pavlović, and G. Staffilani, *An Infinite Sequence of Conserved Quantities for the Cubic Gross-Pitaevskii Hierarchy on* $\mathbb{R}$, preprint, https://arxiv.org/abs/1612.09236, 2016.

[96] J. S. Møller, *The translation invariant massive Nelson model.* I*. The bottom of the spectrum*, Ann. Henri Poincaré, 6 (2005), pp. 1091–1135, https://doi.org/10.1007/s00023-005-0234-8.

[97] E. Nelson, *Interaction of nonrelativistic particles with a quantized scalar field*, J. Math. Phys., 5 (1964), pp. 1190–1197, https://doi.org/10.1063/1.1704225.

[98] Y.-G. Oh, *Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials*, J. Differential Equations, 81 (1989), pp. 255–274, https://doi.org/10.1016/0022-0396(89)90123-X.

[99] H. Pecher, *Some new well-posedness results for the Klein-Gordon-Schrödinger system*, Differential Integral Equations, 25 (2012), pp. 117–142.

[100] A. Pizzo, *One-particle (improper) states in Nelson's massless model*, Ann. Henri Poincaré, 4 (2003), pp. 439–486, https://doi.org/10.1007/s00023-003-0136-6.

[101] M. Porta, S. Rademacher, C. Saffirio, and B. Schlein, *Mean field evolution of Fermions with Coulomb interaction*, J. Stat. Phys., 166 (2017), pp. 1345–1364, https://doi.org/10.1007/s10955-017-1725-y.

[102] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* II*. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[103] V. Rivasseau, *Constructive field theory and applications: Perspectives and open problems*, J. Math. Phys., 41 (2000), pp. 3764–3775, https://doi.org/10.1063/1.533326.

[104] D. Robert, *Autour de l'approximation semi-classique*, Progr. Math. 68, Birkhäuser Boston, Boston, 1987.

[105] L. Rosen, *The* $(\phi^{2n})_2$ *quantum field theory: Higher order estimates*, Comm. Pure Appl. Math., 24 (1971), pp. 417–457.

[106] I. E. Segal, *Foundations of the theory of dynamical systems of infinitely many degrees of freedom.* I, Mat.-Fys. Medd. Danske Vid. Selsk., 31 (1959), p. 39.

[107] V. Sohinger, *A rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on* $\mathbb{T}^3$ *from the dynamics of many-body quantum systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32 (2015), pp. 1337–1365, https://doi.org/10.1016/j.anihpc.2014.09.005.

[108] V. Sohinger, *Local existence of solutions to randomized Gross-Pitaevskii hierarchies*, Trans. Amer. Math. Soc., 368 (2016), pp. 1759–1835, https://doi.org/10.1090/tran/6479.

[109] V. Sohinger and G. Staffilani, *Randomization and the Gross-Pitaevskii hierarchy*, Arch. Ration. Mech. Anal., 218 (2015), pp. 417–485, https://doi.org/10.1007/s00205-015-0863-0.

[110] H. Spohn, *Ground state of a quantum particle coupled to a scalar Bose field*, Lett. Math. Phys., 44 (1998), pp. 9–16, https://doi.org/10.1023/A:1007473300274.

[111] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field*, Cambridge University Press, Cambridge, UK, 2004, https://doi.org/10.1017/CBO9780511535178.

[112] L. van Hove, *Energy corrections and persistent perturbation effects in continuous spectra*, Physica, 21 (1955), pp. 901–923.

[113] L. van Hove, *Energy corrections and persistent perturbation effects in continuous spectra.* II. *The perturbed stationary states*, Physica, 22 (1956), pp. 343–354.