International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: https://www.tandfonline.com/loi/tcon20

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To cite this article: G. De Nicolao & S. Strada (1998) Kalman filtering with mixed discrete-continuous observations, International Journal of Control, 70:1, 71-84, DOI: 10.1080/002071798222460

To link to this article: https://doi.org/10.1080/002071798222460

INT. J. CONTROL, 1998, VOL. 70, NO. 1, 71–84

Kalman filtering with mixed discrete–continuous observations

G. DE NICOLAO† and S. STRADA‡

The optimal filtering problem for systems subject to both the discrete and continuous measurements is studied. The observability and detectability properties of such systems are investigated pointing out their connections with the existence of stable discrete–continuous observers. The optimal filter is based on the solution of a suitable matrix differential Riccati equation with jumps. Sufficient and necessary conditions for the existence of periodic stabilizing solutions to such an equation are worked out. The main result states that both detectability and stabilizability are necessary and sufficient for the existence of a unique periodic solution which is also stabilizing. Stabilizability and detectability also guarantee asymptotic convergence of the Kalman filter to the steady-state periodic filter irrespective of the initial state covariance. The results are illustrated by means of a numerical example.

1. Introduction

Kalman filtering equations are usually formulated either for continuous or discrete-time systems. However, in many practical situations the underlying plant is continuous-time while the measurements are taken at discrete time instants. There are also a number of significant filtering problems where both continuous and discrete observations must be optimally combined. For instance, in navigation problems continuous-time velocity measurements may be available together with discrete-time position 'fixes' (Friedland 1980). Another example arises in the control of chemical processes where continuous temperature and pressure measurements may be complemented with discrete concentration analyses from gas chromatography (Lennartson 1988, Lindgärde and Lennartson 1994, 1995). There is also some interest for the dual problem of optimal control with continuous-time cost functionals including additional discrete state penalty terms at specified times (Geering 1976, Mook and Lew 1991). According to Geering (1976), such optimal control problems arise in economics and politics where the discrete times correspond to the end of the fiscal year or the election day.

Some theoretical contributions concerning filtering with mixed discrete–continuous observations can be found in the works of Lipster (1975), Orlov, (1989) and Orlov and Basin (1995). In particular, the Kalman–Bucy filter with discrete–continuous observations calls for the solution of a differential Riccati equation 'with jumps' (Friedland 1980). In between the discrete measurements the state-error covariance is computed according to a differential Riccati equation while 'jumps' of the covariance occur at discrete instants. Correspondingly, the filter is a discrete–continuous dynamical system whose inputs include both discrete and continuous

Received 6 May 1997. Revised 31 October 1997.

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0020-7179/98 $12.00

observations. In the case of equally spaced discrete-time measurements the continuous gain of the steady-state filter is a periodic function of time (Friedland 1980).

The existing literature has not fully clarified theoretical issues such as the existence and uniqueness of steady-state solutions as well as the convergence properties of the filter when it is initialized with a generic initial state covariance. The analysis would be made easier if one could assume that either only continuous-time observations or only discrete-time ones would guarantee system observability. A complete analysis, however, must also encompass the case where the system state can be reconstructed only by jointly using discrete and continuous measurements.

In the present paper a comprehensive treatment of the optimal filtering problem for systems with discrete–continuous observations is provided. In particular, the observability notion for such a class of systems is investigated and its connections with the existence of stable observers clarified. The differential Riccati equation with jumps is studied in order to obtain sufficient and necessary conditions for the existence and uniqueness of a stable periodic steady-state filter which is globally attractive. In view of the periodic structure of the problem, the analysis heavily relies on tools drawn from periodic system theory (Bittanti 1986, Bittanti *et al.* 1990). The fact that periodic system theory only became well established towards the end of the 1980s could possibly explain why some basic questions concerning systems with mixed discrete-continuous measurements have so far remained unanswered.

The paper is organized as follows. Section 2 is devoted to analysis of the structural properties of mixed (discrete–continuous) systems. The filtering problem is introduced in section 3 where the main results on the periodic steady-state filter are derived. A numerical example is given in section 4. Some concluding remarks (section 5) end the paper.

2. Structural properties of 'mixed systems'

2.1. *System definition*

This section is mainly devoted to analysis of the observability properties of continuous-time systems subject to simultaneous discrete and continuous measurements. More precisely, we will be concerned with the system

$$\Sigma_0 \begin{cases} \dot{x}(t) = Ax(t) \\ y_c(t) = C_c x(t) \\ y_d(k) = C_d x(kT) \end{cases}$$

where $A \in R^{n\times n}, C_c \in R^{p_c\times n}, C_d \in R^{p_d\times n}, t \in R, k \in Z$, and $T \in R$ is the sampling period of the discrete measurements. In the following it will also be useful to refer to the dual system with discrete–continuous inputs

$$\Sigma_c \begin{cases} \dot{x}(t) = Ax(t) + B_c u_c(t), \quad t \neq kT \\ x(kT^+) = x(kT^-) + B_d u_d(k) \end{cases}$$

where $B_c \in R^{n\times p_c}$ and $B_d \in R^{n\times p_d}$.

2.2. *Observability and detectability*

Both Σ_0 and Σ_c have an intrinsic periodic structure so that the usual time-invariant observability and reachability notions do not apply. As for observability, the following result establishes the equivalence between three different characterizations of unobservability.

Lemma 1: *The following statements are equivalent.*

(i) *There exists $x(\tau) \neq 0$ such that $y_c(t) = 0, t \geq \tau$, and $y_d(k) = 0, \forall k \geq \tau/T$.*

(ii) *There exist $v \neq 0, \lambda \in C$ such that*

$$\begin{cases} e^{AT} v = \lambda v \\ C_c e^{At} v = 0 \qquad t \in [0, T) \\ C_d v = 0 \end{cases} \tag{1}$$

(iii) *The time-invariant pair $(e^{AT}, W_o(0, T) + C_d^T C_d)$ is not observable, where*

$$W_o(0, T) = \int_0^T e^{A^T t} C_c^T C_c e^{At} \, dt$$

is the observability Grammian of the continuous-time pair (A, C_c) over $(0, T)$.

Proof: See the Appendix. □

In the following, whenever there exist $v \neq 0$ and $\lambda \in C$ such that (1) holds, λ will be said to be an (A, C_c, C_d)-unobservable characteristic multiplier. The term 'characteristic multiplier' is borrowed from periodic system theory where it denotes the eigenvalues of the transition matrix evaluated over one period. By duality, λ will be said to be an (A, B_c, B_d)-unreachable characteristic multiplier if it is an (A^T, B_c^T, B_d^T)-unobservable characteristic multiplier. The following result is a direct consequence of Lemma 1.

Theorem 1—modal and Grammian characterizations of observability: *The following statements are equivalent.*

(i) *The system Σ_0 (or equivalently the triple (A, C_c, C_d)) is observable.*

(ii) *There exists no (A, C_c, C_d)-unobservable characteristic multiplier (modal observability notion).*

(iii) *The time-invariant pair $(e^{AT}, W_0(0, T) + C_d^T C_d)$ is observable (Grammian observability notion).*

The modal notion of observability for the triple (A, C_c, C_d) was first introduced by De Nicolao (1994) where, however, its connection with the standard observability (notion (i)) was not investigated.

In analogy with time-invariant and periodic systems it is also possible to introduce a canonical decomposition for mixed discrete–continuous systems.

Proposition 1—canonical decomposition of discrete–continuous systems: *Given a triple (A, C_c, C_d) there exists a nonsingular. transformation Σ such that*

$$\hat{A} = \Sigma A \Sigma^{-1} = \begin{bmatrix} \bar{A} & \breve{A} \\ 0 & \tilde{A} \end{bmatrix}, \quad \tilde{A} \in R^{\tilde{n} \times \tilde{n}}$$

$$\hat{C}_c = C_c \Sigma^{-1} = \begin{bmatrix} 0 & \tilde{C}_c \end{bmatrix} \quad \tilde{C}_c \in R^{p_c \times \tilde{n}}$$

$$\hat{C}_d = C_d \Sigma^{-1} = \begin{bmatrix} 0 & \tilde{C}_d \end{bmatrix} \quad \tilde{C}_d \in R^{p_d \times \tilde{n}}$$

and the triple $(\tilde{A}, \tilde{C}_c, \tilde{C}_d)$ *is observable.*

Proof: Let $v_1, \dots, v_k$ denote the (A, C_c, C_d)-unobservable eigenvectors (for simplicity distinct eigenvalues are assumed). Then, just let

$$\Sigma = \left[v_1 v_2 \cdots v_k \bar{v}_{k+1} \cdots \bar{v}_n \right]^{1}$$

where the vectors $\bar{v}$ are chosen so as to render T nonsingular. The rest of the proof is standard. □

For future reference it is also useful to define

$$\hat{B} = \Sigma B = \begin{bmatrix} \bar{B} \\ \tilde{B} \end{bmatrix}, \quad \tilde{B} \in R^{\tilde{n} \times m}$$

In view of the notion of an unobservable (unreachable) characteristic multiplier it is straightforward to introduce a modal detectability (stabilizability) notion. Precisely, the triple $(A, C_c, C_d)((A, B_c, B_d))$ is termed H-detectable (H-stabilizable) if there does not exist any (A, C_c, C_d)-unobservable $((A, B_c, B_d)$-unreachable) characteristic multiplier lying outside the open unit disc. In analogy with periodic system theory (Bittanti 1986, Bittanti *et al.* 1990), a further two definitions of detectability can be introduced: K-detectability requires the stability of the unobservable part of the canonical decomposition of (A, C_c, C_d), whereas G-detectability refers to the detectability of the invariant pair $(e^{AT}, W_o(0, T) + C_d^T C_d)$. The equivalence between the three notions is established below (the simple proof is omitted).

Theorem 2: *The following statements are equivalent:*

(i) *The unobservable part* $\bar{A}$ *of the triple* (A, C_c, C_d) *is asymptotically stable.*

(ii) *There does not exist any* (A, C_c, C_d)*-unobservable characteristic multiplier lying outside the open unit disc.*

(iii) *The time-invariant pair* $(e^{AT}, W_o(0, T) + C_d^T C_d)$ *is detectable.*

When any of (i)–(iii) is satisfied, the triple (A, C_c, C_d) is said to be D-detectable.

2.3. *Stabilization*

The notion of D-stabilizability is instrumental in stating the Lyapunov Lemma which will play a fundamental role in all the subsequent stability analyses.

Discrete–continuous Lyapunov Lemma (De Nicolao 1994): *Consider the discrete–continuous Lyapunov equation (DCLE)*

$$\dot{P} = AP + PA^T + B_c B_c^T \tag{2a}$$

$$P(kT^+) = P(kT^-) + B_d B_d^T \tag{2b}$$

(i) *If* A *is stable, then there exists a unique* T*-periodic solution to the DCLE. Moreover such a solution is non-negative definite.*

(ii) *If the triple* (A, B_c, B_d) *satisfies the* D*-stabilizability condition and the DCLE admits a non-negative definite* T*-periodic solution, then* A *is asymptotically stable.*

Hereafter, we will be concerned with the state estimation problem for system Σ_0. In particular we consider discrete–continuous observers of the type.

$$\Omega \begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + L_c(t)\left[y_c(t) - C_c\hat{x}(t)\right] & t \neq kT \\ \hat{x}(kT^+) = \hat{x}(kT^-) + L_d\left[y_d(k) - C_d x(kT^-)\right] \end{cases}$$

where $L_c(t)$ is a T-periodic gain matrix and L_d is a constant gain. Accordingly, letting $e(t) = x(t) - \hat{x}(t)$, the error dynamics are governed by

$$\Omega_e \begin{cases} \dot{e}(t) = (A - L_c(t)C_c)e(t), & t \neq kT \\ e(kT^+) = (I - L_d C_d)e(kT^-) \end{cases}$$

Hereafter $\Psi_{F_c}(t, \tau)$ will denote the transition matrix on $[\tau, t]$ of $F_c(t) = A - L_c(t)C_c$.

The autonomous system Ω_e has a periodic structure and, as such, it is asymptotically stable if and only if the transition matrix over one period, $\Phi_{cl} = (I - L_d C_d)\Psi_{F_c}(T, 0)$, has all its eigenvalues inside the open unit disc (the simple proof is omitted). According to the terminology of periodic system theory, Φ_{cl} is the *monodromy matrix* of Ω_e and its eigenvalues are called the characteristic multipliers of Ω_e. It is apparent that the asymptotic stability of the error dynamics Ω_e is the basic requirement for any observer. In this respect, a sufficient condition for the existence of a stable observer is provided by the following theorem

Theorem 3: *If the system Σ_0 is observable, then there exists a discrete–continuous gain pair $(L_c(t), L_d)$, where $L_c(\cdot)$ is T-periodic, such that Ω_e is asymptotically stable. Moreover, a stabilizing pair is given by*

$$L_c(t) = S(t - nT, t)^{-1} C_c^T, \quad t \in [0, T)$$

$$L_d = S(0, nT)^{-1} C_d^T \left[C_d S(0, nT)^{-1} C_d^T + 1\right]^{-1}$$

where

$$S(t - nT, t) = \int_{t-nT}^{t} e^{A^T(t-\sigma)} C_c^T C_c e^{A(t-\sigma)}\, d\sigma + \sum_{k=\bar{k}}^{\bar{k}+n-1} e^{A^T(t-kT)} C_d^T C_d e^{A(t-kT)}$$

$\bar{k}$ being the minimum integer such that $\bar{k}T > t - nT$.

Proof: See the Appendix. □

Theorem 4: *There exists a discrete–continuous gain pair $(L_c(t), L_d)$ such that Ω_e is asymptotically stable iff the triple (A, C_c, C_d) is D-detectable.*

Proof: The result is a straightforward consequence of Theorem 3 combined with canonical decomposition (Proposition 1). □

3. The discrete–continuous filtering problem

In this section, the optimal filtering problem for systems with discrete–continuous observations is addressed. Consider the stochastic system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ y_c(t) &= C_c x(t) + v_c(t) \\ y_d(k) &= C_d x(kT) + v_d(k) \end{aligned}\right\} \tag{3}$$

where $w(\cdot)$ and $v_c(\cdot)$ are continuous-time white noises with unitary intensity and independent of each other, i.e. $E[w(t)w(t+\tau)^T] = I\delta(\tau)$, $E[v_c(t)v_c(t+\tau)^T] = I\delta(\tau)$, $E[w(t)v_c(\tau)] = 0$, $\forall t, \tau$. The signal $v_d(k)$ represents the measurement error of the discrete observations and is a discrete-time white noise, independent of $w(\cdot)$ and $v_c(\cdot)$, with var $[v_d(k)] = I, \forall k$. Finally, the initial condition $x(\tau) = x_\tau$ is assumed to be a normally distributed random variable, independent of $w(\cdot), v_c(\cdot), v_d(\cdot)$, with $E[x_\tau] = 0$ and var $[x_\tau] = P_\tau$.

A straightforward application of the recursive equations of discrete and continuous Kalman filtering leads to the following expression for the optimal estimate $\hat{x}(\cdot)$ minimizing $E[(x(t) - \hat{x}(t))^2]$ given all the observations (either discrete or continuous) up to time t:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K_c(t)[y_c(t) - C_c\hat{x}(t)] \quad t \neq kT \tag{4 a}$$

$$\hat{x}(t^+) = \hat{x}(t^-) + K_d(k)[y_d(k) - C_d\hat{x}(t)] \quad t = kT \tag{4 b}$$

$$K_c(t) = P(t)C_c^T \tag{5 a}$$

$$K_d(k) = P(t)C_d^T[C_dP(t)C_d^T + I]^{-1}, \qquad t = kT \tag{5 b}$$

where $P(\cdot)$ satisfies the discrete–continuous Riccati equation (DCRE)

$$\dot{P}(t) = AP(t) + P(t)A^T + BB^T - P(t)C_c^TC_cP(t) \tag{6 a}$$

$$P(t^+) = P(t^-) - P(t^-)C_d^T[I + C_dP(t^-)C_d^T]^{-1}C_dP(t^-), \quad t = kT \tag{6 b}$$

Hereafter, we will be concerned with the following basic questions

(i) Does the filter (4)–(6) admit a steady-state configuration?

(ii) Under what conditions is the steady-state filter stabilizing?

(iii) Under what conditions is convergence of the generic filter (4)–(6) towards a stabilizing steady-state configuration ensured?

The practical significance of the above questions is clear as it is a common practice in Kalman filtering to replace the optimal (time varying) filter with a suboptimal steady-state one. Note that in the present discrete–continuous setting, the possible steady-state configurations are obviously T-periodic.

Given a T-periodic solution $\tilde{P}(\cdot)$ of (6), the periodic filter (4)–(5) is asymptotically stable iff its monodromy matrix $\tilde{F}_{dc} = \tilde{F}_d \Psi_{\tilde{F}_c}(T, 0)$ has all its eigenvalues strictly inside the unit circle, where

$$\tilde{F}_d = I - \tilde{K}_dC_d$$

$$\tilde{F}_c(t) = A - \tilde{K}_c(t)C_c$$

with K_d and K_c as in (5) but with $P(\cdot)$ replaced by $\tilde{P}(\cdot)$. When $\tilde{F}_{dc}$ is stable, the T-periodic solution $\tilde{P}(\cdot)$ is said to be *stabilizing*.

The first result shows that D-detectability suffices to ensure the existence of a T-periodic solution of (6) although stability of the filter is not guaranteed.

Theorem 5: *Suppose that (A, C_c, C_d) is D-detectable and consider the sequence of DCLEs*

$$\dot{P}^{(i)}(t) = F_c^{(i)}(t)P(t) + P(t)F_c^{(i)}(t)^T + BB^T + K_c^{(i)}(t)K_c^{(i)}(t)^T, \quad t \neq kT \tag{7 a}$$

$$P^{(i)}(t^+) = F_d^{(i)}P(t^-)F_c^{(i)T} + K_d^{(i)}K_d^{(i)T}, \quad t = kT \tag{7 b}$$

where

$$
\begin{aligned}
F_{\mathrm{c}}^{(i)}(t) &= A - K_{\mathrm{c}}^{(i)}(t)C_{\mathrm{c}}, && i \geq 0\\
F_{\mathrm{d}}^{(i)} &= I - K_{\mathrm{d}}C_{\mathrm{d}}, && i \geq 0\\
K_{\mathrm{c}}^{(i)}(t) &= P^{(i)}(t)C_{\mathrm{c}}^{\mathrm{T}}, && i \geq 1\\
K_{\mathrm{d}}^{(i)} &= P(T)C_{\mathrm{d}}^{\mathrm{T}}\left[I + C_{\mathrm{d}}P(T)C_{\mathrm{d}}^{\mathrm{T}}\right]^{-1}, && i \geq 1
\end{aligned}
$$

and $(K_{\mathrm{c}}^{(0)}(\cdot)K_{\mathrm{d}}^{(0)})$ *are chosen so as to ensure the stability of* $F_{\mathrm{dc}}^{(0)}$. *Then the following hold.*

(i) *For each* $i \geq 0$, (7) *admits a unique real symmetric T-periodic solution. Moreover, this solution is positive semidefinite and such that* $F_{\mathrm{dc}}^{(i+1)}$ *is stable.*

(ii) *For any t,* $0 \leq t \leq T, P^{(i)}(t)$, *as a function of i, is a non-increasing sequence that, for* $i \to \infty$, *converges to* $\bar{P}(t)$, *where* $\bar{P}(t)$ *is a T-periodic solution of the DCRE.*

(iii) $\bar{P}(t)$ *is the maximal T-periodic solution of the DCRE, in the sense that, for any other T-periodic solution* $\tilde{P}(\cdot)$ *of* (6), *it results* $\bar{P}(t) \geq \tilde{P}(t), \forall t$.

Proof: The proof follows the same rationale as that of Theorem 3 in Bittanti *et al.* (1988) where a similar result is proven for the discrete-time periodic Riccati equation. In the proof, Theorem 4 of section 2.3 is used to show that D-detectability ensures the existence of the initial gains $(K_{\mathrm{c}}^{(0)}(\cdot), K_{\mathrm{d}}^{(0)})$. The subsequent steps of the proof rely on extensive use of the discrete–continuous Lyapunov Lemma. □

From point (i) of the above theorem it follows that the closed-loop matrix $\bar{F}_{\mathrm{dc}}$ associated with the maximal solution $\bar{P}(\cdot)$ has all its eigenvalues inside the closed unit disc. According to the standard terminology of Riccati equations (Chan *et al.* 1984, De Souza *et al.* 1986), $\bar{P}(\cdot)$ is the so-called *strong solution* of the DCRE (6). An important connection with the stabilizing solution is clarified by the following lemmas.

Lemma 2: *If the stabilizing solution of* (6) *exists it coincides with the strong and maximal solution* $\bar{P}(\cdot)$ *of Theorem 5.*

Proof: Let $\tilde{P}(\cdot)$ be the periodic stabilizing solution and $(\tilde{K}_{\mathrm{c}}(\cdot), \tilde{K}_{\mathrm{d}})$ the associated continuous and discrete gains. Since these gains are stabilizing, Theorem 4 guarantees that the triple $(A, C_{\mathrm{c}}, C_{\mathrm{d}})$ is D-detectable. Then, the procedure of Theorem 5 can be applied with initial gains $K_{\mathrm{c}}^{(0)}(t) = \tilde{K}_{\mathrm{c}}(t), K_{\mathrm{d}}^{(0)} = \tilde{K}_{\mathrm{d}}$. It is easily seen that the iterative scheme converges in one step giving $\bar{P}(t) = \tilde{P}(t)$ so proving the Lemma. □

Lemma 3: *If* $(A, C_{\mathrm{c}}, C_{\mathrm{d}})$ *is D-detectable and* (A, B) *is stabilizable, then any non-negative definite T-periodic solution* $\tilde{P}(\cdot)$ *of* (6) *is stabilizing.*

Proof: Just rewrite (6) as

$$
\begin{aligned}
\dot{\tilde{P}}(t) &= \tilde{F}_{\mathrm{c}}(t)\tilde{P}(t) + \tilde{P}(t)\tilde{F}_{\mathrm{c}}(t)^{\mathrm{T}} + BB^{\mathrm{T}} + \tilde{K}_{\mathrm{c}}(t)\tilde{K}_{\mathrm{c}}(t)^{\mathrm{T}}, && t \neq kT\\
\tilde{P}(t^{+}) &= \tilde{F}_{\mathrm{d}}\tilde{P}(t^{-})\tilde{F}_{\mathrm{d}}^{\mathrm{T}} + \tilde{K}_{\mathrm{d}}\tilde{K}_{\mathrm{d}}^{\mathrm{T}}, && t = kT
\end{aligned}
$$

Then, letting

$$W = \int_0^T \Psi_{\tilde{F}_c}(T,\tau)\left[BB^T + \tilde{K}_c(t)\tilde{K}_c(t)^T\right]\Psi_{\tilde{F}_c}(T,\tau)^T \, d\tau$$

one obtains

$$\tilde{P}(T^+) = \tilde{F}_{dc}\tilde{P}(0^+)\tilde{F}_{dc}^T + \tilde{K}_d\tilde{K}_d^T + \tilde{F}_d W \tilde{F}_d^T \tag{8}$$

which, recalling the T-periodicity of $P(\cdot)$, yields the algebraic Lyapunov equation

$$\tilde{P}(0^+) = \tilde{F}_{dc}\tilde{P}(0^+)\tilde{F}_{dc}^T + \tilde{K}_d\tilde{K}_d^T + \tilde{F}_d W \tilde{F}_d^T$$

The stability of $\tilde{F}_{dc}$ is then established by applying the Lyapunov Lemma to the above equation. In this respect we just need to show that the pair $(\tilde{F}_{dc}, \tilde{K}_d\tilde{K}_d^T + \tilde{F}_d W \tilde{F}_d^T)$ is stabilizable. By contradiction, assume that there exist λ, with $|\lambda| \geq 1$, and $x \neq 0$ such that

$$\tilde{F}_{dc}^T x = \lambda x \tag{9 a}$$

$$\tilde{K}_d^T x = 0 \tag{9 b}$$

$$W\tilde{F}_d^T x = 0 \tag{9 c}$$

Recalling the definition of $\tilde{F}_{dc}$ and $\tilde{F}_d$, from (9 *a*) and (9 *b*) it follows that

$$\tilde{F}_c^T x = 0$$

$$Wx = 0$$

so contradicting the stabilizability of (A, B). Hence the stabilizability of $(\tilde{F}_{dc}, \tilde{K}_d\tilde{K}_d^T + \tilde{F}_d W \tilde{F}_d^T)$ is proven. □

Before stating the main result on the existence and uniqueness of the periodic filter, a further technical lemma is still needed.

Lemma 4: *Consider the canonical decomposition in observable/unobservable part given in section 2. If the controllable part of $(\underline{A}, \underline{B})$ is stable, the DCRE (6) admits a non-negative definite T-periodic solution $\hat{P}(\cdot)$ of the type*

$$\hat{P}(t) = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P}(t) \end{bmatrix}, \quad \forall t$$

where $\tilde{P}(\cdot)$ is the maximal solution of the reduced-order DCRE associated with the matrices $\tilde{A}, \tilde{B}, \tilde{C}_c, \tilde{C}_d$.

Proof: The proof follows the same rationale of the sufficiency part of Theorem 4 in Bittanti *et al.* (1988). □

Theorem 6: *The following statements are equivalent.*

(i) (A, C_c, C_d) *is D-detectable and* (A, B) *is stabilizable.*

(ii) *The DCRE admits a unique non-negative definite T-periodic solution, which is also stabilizing.*

Proof: (i) $\Longrightarrow$ (ii) In view of Theorem 5, D-detectability implies the existence of a non-negative definite T-periodic solution $\bar{P}(\cdot)$. Moreover, by Lemma 3, $\bar{P}(\cdot)$ is stabilizing. Assume now, by contradiction, that there exists another T-periodic non-negative definite solution $\tilde{P}(\cdot)$. Then, some computations show that

$$\left[\bar{P}(T) - \tilde{P}(T)\right] = \bar{F}_{\mathrm{dc}}\left[\bar{P}(0) - \tilde{P}(0)\right]\tilde{F}_{\mathrm{dc}} \tag{10}$$

where $\bar{F}_{\mathrm{dc}}$ and $\tilde{F}_{\mathrm{dc}}$ are the closed-loop monodromy matrices associated with $\bar{P}(\cdot)$ and $\tilde{P}(\cdot)$ respectively. In view of Lemma 3, also $\tilde{F}_{\mathrm{dc}}$ has all its eigenvalues strictly inside the unit circle. Then, by T-periodicity of $\bar{P}(\cdot)$ and $\tilde{P}(\cdot)$, (10) implies $\bar{P}(0) = \tilde{P}(0)$ so that $\bar{P}(t) = \tilde{P}(t), \forall t$.

(ii) $\Rightarrow$ (i) The proof is similar to the necessity part of Theorem 6 in Bittanti *et al.* (1988). □

Before addressing the issue of convergence, it is useful to recall the statement of the extended Lyapunov Lemma for time-varying systems.

Lemma 5—Time-varying Lyapunov Lemma (Anderson and Moore 1981): *Let $P(\cdot)$ be a bounded solution of the discrete Lyapunov equation*

$$P(k+1) = F(k)P(k)F(k)^{\mathrm{T}} + Q(k), \quad k \geq 0$$

where $F(\cdot)$ and $Q(\cdot)$ are bounded time-varying matrix functions. If the pair $(F(\cdot), Q(\cdot))$ is uniformly stabilizable, then $F(\cdot)$ is exponentially stable.

Theorem 7—Global convergence of the optimal filter: *If $(A, C_{\mathrm{c}}, C_{\mathrm{d}})$ is D-detectable and (A, B) is stabilizable then the non-negative definite T-periodic stabilizing solution is a global attractor for the non-negative definite solutions, i.e. given any non-negative definite T-periodic solution $\tilde{P}(\cdot)$ it results*

$$\lim_{t\to\infty} \tilde{P}(t) - \bar{P}(t) = 0$$

Proof: Using the same rationale leading to (8) one can show that

$$P((k+1)T^{+}) = F_{\mathrm{dc}}(k)P(kT^{+})F_{\mathrm{dc}}(k)T + K_{\mathrm{d}}(k)K_{\mathrm{d}}(k)^{\mathrm{T}} + F_{\mathrm{d}}(k)W(k)F_{\mathrm{d}}(k)^{\mathrm{T}} \tag{11}$$

where

$$F_{\mathrm{d}}(k) = I - K_{\mathrm{d}}(k)C_{\mathrm{d}}, \quad F_{\mathrm{dc}}(k) = F_{\mathrm{d}}(k+1)\Psi_{F_{\mathrm{c}}}((k+1)T^{-}, kT^{+}), \quad F_{\mathrm{c}}(k) = A - K_{\mathrm{c}}(k)C_{\mathrm{c}}$$

and

$$W(k) = \int_{kT}^{(k+1)T} \Psi_{F\mathrm{c}}((k+1)T, \tau)\left[BB^{\mathrm{T}} + K_{\mathrm{c}}(t)K_{\mathrm{c}}(t)^{\mathrm{T}}\right]\Psi_{F\mathrm{c}}((k+1)T, \tau)^{\mathrm{T}}\, \mathrm{d}\tau$$

The first step is proving the exponential stability of $F_{\mathrm{dc}}(\cdot)$ by applying the time-varying Lyapunov Lemma to (11). To this aim we need to show that the pair $F_{\mathrm{dc}}(\cdot), K_{\mathrm{d}}(\cdot)K_{\mathrm{d}}(\cdot)^{\mathrm{T}} + F_{\mathrm{d}}(\cdot)W(\cdot)F_{\mathrm{d}}(\cdot)^{\mathrm{T}})$ is uniformly stabilizable. First, note that stabilizability of (A, B) implies the stabilizability of the discrete-time pair $(\Psi_{F_{\mathrm{c}}}((k+1)T, kT^{+}), W(k))$. Indeed, the stabilizability of such a discrete-time pair is equivalent to that of the continuous-time pair $(F_{\mathrm{c}}(\cdot), \left[B\ K_{\mathrm{c}}(\cdot)\right])$ which easily follows from the stabilizability of (A, B). In turn, the stabilizability of $(\Psi_{F_{\mathrm{c}}}((k+1)T, kT^{+}), W(k))$ implies that of $(F_{\mathrm{dc}}(\cdot), K_{\mathrm{d}}(\cdot)K_{d}(\cdot)^{\mathrm{T}} + F_{\mathrm{d}}(\cdot)W(\cdot)F_{\mathrm{d}}(\cdot)^{\mathrm{T}})$. Indeed, letting $K(\cdot)$ be such that $(\Psi_{F_{\mathrm{c}}}((k+1)T, kT^{+}) - W(k)^{1/2}K(k))$ is stable, it is not difficult to verify that

$$\tilde{K}(k) = \left[-K(k)^{\mathrm{T}} \,\middle|\, (\Psi_{F_{\mathrm{c}}}((k+1)T^{-}, kT^{+}) - W(k)^{1/2}K(k))^{\mathrm{T}}C_{\mathrm{d}}^{\mathrm{T}}\right]^{\mathrm{T}}$$

is such that $(F_{dc}(k) + [F_d(k) W(k)^{1/2} \mid K_d(k)] \tilde{K}(k))$ is stable. Hence, exponential stability of $F_{dc}(\cdot)$ follows.

Now, letting $P(\cdot)$ denote a generic solution of the DCRE, it can be seen that

$$\left[P((k+1)T^+) - \bar{P}((k+1)T^+)\right] = F_{dc}(k)\left[P(kT^-) - \bar{P}(kT^-)\right]\bar{F}_{dc}(k)$$

where $\bar{P}(\cdot)$ is the unique non-negative definite T-periodic solution. Observing that both $F_{dc}(\cdot)$ and $\bar{F}_{dc}(\cdot)$ are exponentially stable it follows that

$$\lim_{t\to\infty} P(kT^-) - \bar{P}(kT^-) = 0$$

and the thesis follows. □

4. Numerical example

In this section the results on discrete–continuous filtering are illustrated by means of a simple example. Consider a tank containing a mixture of two (incompressible) liquids A and B. The flows u_A and u_B are the control variables and the masses contained in the tank are denoted by m_A and m_B respectively:

$$\begin{cases} \dot{m}_A = u_A \\ \dot{m}_B = u_B \end{cases}$$

It is assumed that measurements of the total mass $m_A + m_B$ are available in continuous time (by a level sensor, for instance) while m_A is measured only at the discrete instants $t_j = jT$ (by concentration analysis, for instance). Hence, the continuous and discrete measurements are

$$\begin{cases} y_c(t) = m_A(t) + m_B(t) \\ y_d(j) = m_A(jT) \end{cases}$$

Taking into account the process noise $w(\cdot)$ and the measurement noises $v_c(\cdot)$ and $v_d(\cdot)$, the overall model is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) + Gu(t) \\ y_c(t) = C_c x(t) + v_c(t) \\ y_d(j) = C_d x(jT) + v_d(j) \end{cases}$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad G = I, \quad B = I$$

$$C_c = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In the example, the continuous-time white noise $w(\cdot)$ has unitary intensity, $v_c(\cdot)$ has an intensity equal to 0.1 while the discrete-time white noise $v_d(\cdot)$ has unit variance. The process noise $w(\cdot)$ may account for unmodelled disturbances as well as actuation errors. Note that neither the continuous pair (A, C_c) nor the discrete one (e^{AT}, C_d) is observable. Nevertheless, the triple (A, C_c, C_d) is D-observable so that the results of the paper can be applied.

The discrete–continuous steady-state Kalman filter has been designed assuming that the sampling period of the discrete observations is $T = 0.5$. The time profiles of the two entries of the periodic continuous-time gain $K_c(\cdot)$ are depicted in figure 1(A). In figures 1(B)–(D) the time profiles of the solutions $P(\cdot)$ of the DCRE (6) starting

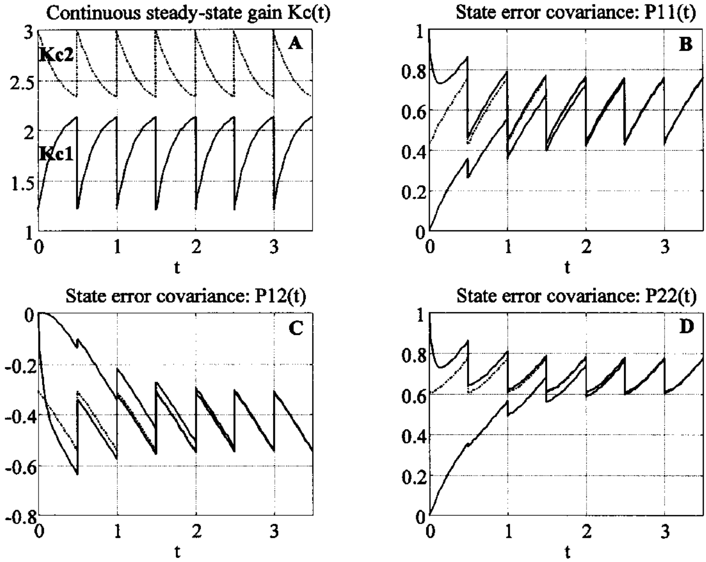

Figure 1. Numerical example (section 4). Panel A time profile of the two entries of the steady-state continuous gain matrix. Panels B–D time profiles of the entries of the solutions $P(\cdot)$ of the DCRE (6) with initial conditions $P(0) = 0$ and $P(0) = I$ (continuous). For comparison, the steady-state periodic solutions $\tilde{P}(\cdot)$ are also reported (chain curves).

with initial conditions $P(0) = 0$ and $P(0) = I$ are plotted together with the (unique) steady-state periodic solution $\tilde{P}(\cdot)$ (panel B, entry $(1,1)$; panel C, entry $(1,2)$; panel D, entry $(2,2)$). As predicted by Theorem 7, the periodic solution asymptotically attracts the other two solutions. Note also the typical sawtooth patterns. Indeed, as soon as they become available, the discrete measurements produce an instantaneous jump of the error covariance matrix $P(\cdot)$.

5. Conclusions

Filtering problems where one must optimally combine discrete and continuous observations have been considered. In this context, a thorough analysis concerning sufficient and necessary conditions for the existence of stable filters and their attractiveness properties was not available. The main results of the paper are as follows. (i) The modal and Grammian characterization of observability and detectability for systems with discrete and continuous outputs (Theorem 1). (ii) The proof that detectability is equivalent to the existence of a stable discrete–continuous observer (Theorem 4). (iii) A necessary and sufficient condition for the existence of a unique periodic filter which is also stable (Theorem 6). (iv) The proof that such a steady-state filter is globally attractive for all other optimal filters irrespective of their initial state-covariance (Theorem 7).

Acknowledgment

This work was partially supported by the MURST Project 'Model Identification, Systems Control, Signal Processing'.

Appendix

Proof of Lemma 1: (ii) $\Rightarrow$ (i). Without loss of generality one can assume $\tau \geq 0$. Letting $x(\tau) = e^{A\tau} v$, it is easy to see that (i) follows.

(i) $\Rightarrow$ (ii). Due to periodicity, without loss of generality one can assume $\tau \leq 0$. For the sake of simplicity, it is assumed that A has n distinct eigenvalues:

$$A v_i = \rho_i v_i, \quad v_i \neq 0, \quad \rho_i \in C^1$$

Then, there always exist n complex numbers $\alpha_i, i = 1, \ldots, n$ such that

$$x(0) = e^{-A\tau} x(\tau) = \sum_{i=1}^{n} \alpha_i v_i$$

Since the coefficients α_i cannot be all equal to zero, there is no loss of generality in assuming $\alpha_1 \neq 0$. Now, $y_c(t) = 0, t \geq \tau$, implies $C_c x(t) = 0, t \geq 0$ or equivalently

$$C_c e^{At} \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} e^{\rho_i t} \alpha_i \beta_i = 0, \quad t \geq 0$$

where $\beta_i \overset{\text{def}}{=} C_c v_i$. Then $\alpha_1 \neq 0$ obviously implies $\beta_1 = C_c v_1 = 0$. Moreover, $C_d x(kT) = 0, k \geq 0$, implies

$$C_d e^{AkT} \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} e^{k v_i T} \alpha_i \gamma_i, \quad k \geq 0$$

where $\gamma_i \overset{\text{def}}{=} C_d v_i$. Then $\alpha_1 \neq 0$ implies $\gamma_1 = C_d v_1 = 0$. Hence,

$$\begin{cases} A v_1 = \rho_1 v_1, \quad v_1 \neq 0 \\ C_c v_1 = 0 \\ C_d v_1 = 0 \end{cases}$$

Recalling that $e^{At} v_1 = e^{\rho_1 t} v_1$, (ii) follows just by letting $v = v_1$ and $\lambda = e^{\rho_1 T}$.

(iii) $\Rightarrow$ (ii). If $(e^{AT} W_o(0, T) + C_d^T C_d)$ is unobservable there exists $w \neq 0$ such that

$$\begin{cases} e^{AT} w = \rho w, \quad \rho \in C^1 \\ (W_o(0, T) + C_d C_d^T) w = 0 \end{cases}$$

Then it follows that

$$\begin{cases} W_o(0, T) w = 0 \\ C_d w = 0 \end{cases}$$

or also

$$\begin{cases} C_c e^{At} w = 0, \quad t \in [0, T) \\ C_d w = 0 \end{cases}$$

Letting $v = w$, (ii) follows.

(ii) $\Rightarrow$ (iii). Just let $w = v$ and reverse the above rationale. □

Proof of Theorem 3: For the sake of convenience, the proof is carried out for the dual system Σ_c. More precisely, assuming Σ_c reachable, we show that the discrete-continuous control law $u_c(t) = K_c(t)x(t), u_d(k) = K_d x(kT^-))$ is stabilizing, where

$$K_c(t) = -B_c^T S(t, t+nT)^{-1}, \quad t \in (0, T]$$

$$K_d = -\left[B_d^T S(0, nT)^{-1} B_d + I\right]^{-1} B_d^T S(0, nT)^{-1}$$

and

$$S(t, t+nT) = \int_t^{t+nT} e^{A(t-\sigma)} B_c B_c^T e^{A^T(t-\sigma)} \, d\sigma + \sum_{k=\bar{k}}^{\bar{k}+n-1} e^{A(t-kT)} B_d B_d^T e^{A^T(t-kT)}$$

$\bar{k}$ being the minimum integer such that $\bar{k}T \geq t$.

It can be seen that the reachability assumption guarantees that the reachability Grammian $S(t, t+nT)$ is positive definite (in general, the reachability Grammian over n periods of any reachable periodic system is nonsingular (Bittanti 1986)), so that $K_c(\cdot)$ and K_d are well defined.

Now, letting $x(t_0) = \bar{x}$, define the function

$$V(\bar{x}, t_0, t_f) = \min_{\{u_c(\cdot), u_d(\cdot)\}} \left\{ \int_{t_0}^{t_f} u_c(\sigma)^T u_c(\sigma) \, d\sigma + \sum_{k=k_1}^{k_2} u_d^T(k) u_d(k) \right\} \tag{A 1}$$

subject to Σ_c and $x(t_f) = 0$, where k_1 is the minimum integer such that $k_1 T \geq t_0$ and k_2 is the maximum integer such that $k_2 T < t_f$.

Denoting by $\{\mathring{u}_c(\cdot), \mathring{u}_d(\cdot)\}$ the optimal solution of (A 1) when $t_f = t_0 + nT$, it turns out that $\mathring{u}_c(t_0) = K_c(t_0)\bar{x}$ and, for $t_0 = kT, \mathring{u}_d(k) = K_d x(kT)$. For given t_0 and t_f, $V(\bar{x}, t_0, t_f)$ is a quadratic function of $\bar{x}$, i.e. there exists $\bar{P} = \bar{P}^T \geq 0$ such that

$$V(\bar{x}, t_0, t_f) = \bar{x}^T \bar{P} \bar{x}, \forall \bar{x}$$

Let the pair $(F_c(t), F_d)$, where $F_c(t) = A + B_c K_c(t)$ and $F_d = I + B_d K_d$, denote the discrete–continuous closed-loop dynamics. Moreover let $\Psi_F(t_0 + T, t_0)$ denote the transition matrix of the closed-loop dynamics over one period, i.e.

$$\Psi_F(t_0 + T, t_0) = \Psi_{Fc}(t_0 + T, t_0)(I + B_d K_d)$$

where $\Psi_{Fc}(t_0 + T, t_0)$ is the transition matrix of $F_c(\cdot)$, if $t_0 = kT$, for some k; otherwise

$$\Psi_F(t_0 + T, t_0) = \Psi_{Fc}(t_0 + T, kT)(I + B_d K_d)\Psi_{Fc}(kT, t_0)$$

if $t_0 < kT < t_0 + T$, for some k.

The stability of $(F_c(\cdot), F_d)$ will be proven by using $\bar{P}$ as a Lyapunov function. More precisely, using the same arguments as in (De Nicolao and Strada 1977), it can be shown that

$$V(\bar{x}, t_0, t_f) \geq V(\Psi_F(t_0 + T, t_0)\bar{x}, t_0 + T, t_f + T)$$

which implies

$$\bar{P} = \Psi_F(t_0 + T, t_0)^T \bar{P} \Psi_F(t_0 + T, t_0) + Q$$

for some non-negative definite Q. Following the same rationale as in De Nicolao and Strada (1997), one can show that the pair $(\Psi_F(t_0 + T, t_0), Q)$ is reconstructible so that, by the Lyapunov Lemma, all the eigenvalues of $\Psi_F(t_0 + T, t_0)$ belong to the open unit disc. □

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