Journal of Stochastic Analysis

Volume 1 | Number 4 Article 3

December 2020

# Invariant Projections for Covariant Quantum Markov Semigroups

Franco Fagnola
*Politecnico di Milano*, franco.fagnola@polimi.it

Emanuela Sasso
*University of Genova, 16100 Genova, Italy*, sasso@dima.unige.it

Veronica Umanità
*University of Genova, 16100 Genova, Italy*, umanita@dima.unige.it

Journal of Stochastic Analysis
Vol. 1, No. 4 (2020) Article 3 (14 pages)
DOI: 10.31390/josa.1.4.03



# INVARIANT PROJECTIONS FOR COVARIANT QUANTUM MARKOV SEMIGROUPS

FRANCO FAGNOLA, EMANUELA SASSO*, AND VERONICA UMANITÀ

*Dedicated to Professor Leonard Gross on the occasion of his 88th birthday*

Abstract. In this paper we investigate consequences of covariance of a uniformly Quantum Markov Semigroup, under a group action, on the structure of its minimal invariant projections. We obtain that, under suitable hypotheses, minimal invariant projections correspond to irreducible sub-representations in which the initial covariant representation is decomposed. We apply this results in the study circulant Quantum Markov Semigroups.

## 1. Introduction

Semigroups of completely positive unital maps describe irreversible dynamics of a open quantum system. They often arise from scaling limits of a quantum system interacting with an external environment and covariance properties reflect the symmetries of the total Hamiltonian such as spatial isotropy or translation invariance. A semigroup $(\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{B}(\mathsf{h})$ is called *covariant* with respect to the action $g \to \pi(g)$ of a group on $\mathcal{B}(\mathsf{h})$ if

$$\mathcal{T}_t(\pi(g)^* x\pi(g)) = \pi(g)^*\mathcal{T}_t(x)\pi(g),$$

for all $x \in \mathcal{B}(\mathsf{h})$, $g \in G$ and $t \geq 0$. It is clear that the generator $\mathcal{L}$ satisfies the corresponding covariance property and it is natural to ask if this property influences the structure of $\mathcal{L}$. A.S. Holevo investigated the structure of the infinitesimal generator $\mathcal{L}$ ([14, 15, 16, 17]). In particular, he proved that the covariance property imposes strong restrictions on $\mathcal{L}$. Indeed, in the uniformly continuous case, if we consider a (covariant) (Gorini-Kossakowski-Sudharshan-Lindblad) [18] GKSL representation of $\mathcal{L}$, $(H, (L_k)_k)$, we have that $H$ commutes with the family of operators $\pi(g)$ $(g \in G)$ of the representation $\pi$.

A key step in the study of a QMS is determining whether it is irreducible and, if not, its irreducible sub-semigroups. These are in one to one correspondence with common invariant subspaces of operators $\sum_k L_k^* L_k + 2\mathrm{i}H$ and $L_k$ (see [9, Theorem III.1]). Finding all common invariant subspaces of a set of operators, however, is typically a difficult problem (see [21]) therefore it is worth investigating the

Received 2019-12-10; Accepted 2020-6-24; Communicated by A. Stan et al.
2010 *Mathematics Subject Classification.* 82C10; 47D06; 46L55.
*Key words and phrases.* Quantum Markov semi-groups, covariance, atomic algebras, irreducibility.
* Corresponding author.

implications of covariance in order to gain insight on the structure of QMSs by the interplay of representation theory and invariant subspace problems.

Other objects are strongly connected to the structure of the infinitesimal generator. The most important for the study of the asymptotic behavior of the semigroup are the Decoherence-free sub algebra $\mathcal{N}(\mathcal{T})$ and the set of the fixed points $\mathcal{F}(\mathcal{T})$.

In [13], the authors analyzed the structure of the decoherence free-subalgebra $\mathcal{N}(\mathcal{T})$ of a uniformly continuous *covariant* semigroup with respect to a representation $\pi$ of a compact group $G$ on $\mathsf{h}$. In particular, they obtained that, when $\pi$ is irreducible, $\mathcal{N}(\mathcal{T})$ is isomorphic to $(\mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}})^d$ for suitable Hilbert spaces $\mathsf{k}$ and $\mathsf{m}$, and a integer $d$ related to the number of connected components of $G$. This is the easy case, because if $\pi$ is irreducible, the Hilbert space $\mathsf{h}$ has to be finite dimensional and so $\mathcal{N}(\mathcal{T})$ is an atomic algebra. In order to extend this result when $\pi$ is reducible, they had to add the additional hypothesis that $\mathcal{N}(\mathcal{T})$ is atomic. As we showed in [7], the reason is that the atomicity of the decoherence-free subalgebra of a uniformly-continuous QMS forces the Lindblad operators to have a block-diagonal structure, inherited by the same atomic decomposition of $\mathcal{N}(\mathcal{T})$ in type I factors.

In this paper, we want to compare the covariance property with the set of the fixed points. The latter, in general, is not an algebra, but when there exists a faithful normal invariant state, it is a subalgebra of $\mathcal{N}(\mathcal{T})$. Moreover, we have that $\mathcal{F}(\mathcal{T})$ is the image of a conditional expectation and, as a consequence, it is always atomic. So the aim of this work is to show what consequences the covariance property has for the invariant projections of a uniformly continuous QMS.

In Section 2 we introduce preliminary definitions and results about atomic decoherence-free subalgebras, the set of the fixed points and covariant QMSs. In Section 3, we investigate the GKSL representation of a covariant QMS, showing that, if $\mathcal{T}$ is covariant, there exists a *covariant* GKSL representation, that is "compatible" with respect to the representation of $G$. But it can happen that for a covariant QMS there exists a GKLS representations that it is not covariant. Then in Section 4, we focus on the relationship between the representation $\pi$ and the minimal projections $p_i$ in the center of $\mathcal{F}(\mathcal{T})$, that appear in its atomic decomposition. When the representation is reducible, we obtain that the projections, $(q_j)_j$, on the invariant spaces for the representation $\pi$ are fixed points and, when we have that every invariant projections commutes with $\pi(g)$ for every $g \in G$, we have some important consequences. For example, that the $q_j$'s are the unique minimal invariant projections for $\mathcal{T}$ and every minimal projection in the center of $\mathcal{F}(\mathcal{T})$ can be written as sum of $q_j$. In Section 5, we show that the circulant QMSs satisfies this condition and in this context we have a description of $\mathcal{F}(\mathcal{T})(= \mathcal{N}(\mathcal{T}))$ as direct sum of the family $q_j$.

## 2. Preliminary Results

Let $\mathsf{h}$ be a complex separable Hilbert space. A Quantum Markov Semigroup on the algebra $\mathcal{B}(\mathsf{h})$ of all bounded operators on $\mathsf{h}$ is a weakly*-continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of completely positive, identity preserving normal maps. Such semigroup is uniformly continuous if it satisfies

$$\lim_{t\to 0^+} \|\mathcal{T}_t - \mathbb{1}\| = 0.$$

In this case its generator $\mathcal{L}$ is a linear and bounded operator such that $\mathcal{T}_t = e^{t\mathcal{L}}$ in the uniform topology. Moreover, we can express $\mathcal{L}$ in a GKSL representation [18]

$$\mathcal{L}(x) = \mathrm{i}[H,x] - \frac{1}{2}\sum_{k\geq 1}(L_k^*L_kx - 2L_k^*xL_k + xL_k^*L_k)$$

for some $H = H^*$ and $(L_k)_k$ operators in $\mathcal{B}(\mathsf{h})$ for which the series $\sum_k L_k^*L_k$ is strongly convergent. In particular, we consider here *special representations* of $\mathcal{L}$, by means of operators $(L_k)_k$ such that $\mathbb{1}, L_1, L_2, \ldots$ are linearly independent. More precisely, we have the following characterization (see [20], Proposition 30.14 and the discussion below the proof of Theorem 30.16):

**Theorem 2.1.** *Let $\mathcal{L}$ be the generator of a uniformly continuous QMS on $\mathcal{B}(\mathsf{h})$. Then there exist a bounded selfadjoint operator $H$ and a sequence $(L_k)_{k\geq 1}$ of elements in $\mathcal{B}(\mathsf{h})$ such that:*

1. *$\sum_{k\geq 1} L_k^*L_k$ is strongly convergent,*
2. *if $\sum_{k\geq 0}|c_k|^2 < \infty$ and $c_0\mathbb{1} + \sum_{k\geq 1} c_kL_k = 0$ for scalars $(c_k)_{k\geq 0}$ then $c_k = 0$ for every $k \geq 0$,*
3. *$\mathcal{L}(x) = \mathrm{i}[H,x] - \frac{1}{2}\sum_{k\geq 1}(L_k^*L_kx - 2L_k^*xL_k + xL_k^*L_k)$ for all $x \in \mathcal{B}(\mathsf{h})$.*

*Moreover, if $H', (L'_k)_{k\geq 1}$ is another family of bounded operators in $\mathcal{B}(\mathsf{h})$ with $H'$ selfadjoint, then it satisfies conditions (1)-(3) if and only if the lengths of sequences $(L_k)_{k\geq 1}, (L'_k)_{k\geq 1}$ are equal and there exists a sequence $(\alpha_k)_{k\geq 1} \subseteq \mathbb{C}$ with $\sum_k |\alpha_k|^2 < \infty$ and $\beta \in \mathbb{R}$ such that*

$$H' = H + \beta\mathbb{1} + \frac{1}{2\mathrm{i}}(S - S^*), \qquad L'_k = \sum_j u_{kj}L_j + \alpha_k\mathbb{1} \tag{2.1}$$

*for some unitary matrix $U = (u_{kj})_{kj}$, where $S := \sum_{kj}\overline{\alpha}_k u_{kj}L_j$.*

We conclude this section recalling some basic definitions about two important subspaces of $\mathcal{B}(\mathsf{h})$ associated with $\mathcal{T}$. They are strongly connected to the phenomenon of decoherence (see e.g. [4, 6, 1, 19]) and the asymptotic properties of the semigroup ([5, 8, 11, 12]).

The *decoherence-free* (DF) subalgebra of $\mathcal{T}$ is denoted by $\mathcal{N}(\mathcal{T})$ and is defined by

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \,:\, \mathcal{T}_t(x^*x) = \mathcal{T}_t(x)^*\mathcal{T}_t(x), \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x)^* \ \forall\, t \geq 0\}\,. \tag{2.2}$$

Since $\mathcal{T}$ is uniformly continuous, $\mathcal{N}(\mathcal{T})$ is the biggest subalgebra of $\mathcal{B}(\mathsf{h})$ on which every operator $\mathcal{T}_t$ is a $*$-automorphism, and we have $\mathcal{T}_t(x) = e^{\mathrm{i}tH}xe^{-\mathrm{i}tH} \quad \forall\, x \in \mathcal{N}(\mathcal{T})$ ([10]).

Moreover we denote by $\mathcal{F}(\mathcal{T})$ the set of fixed points of $\mathcal{T}$, i.e.

$$\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \,:\, T_t(x) = x \ \forall\, t \geq 0\} = \{x \in \mathcal{B}(\mathsf{h}) \,:\, \mathcal{L}(x) = 0\}\,. \tag{2.3}$$

When $\mathcal{N}(\mathcal{T})$ is atomic (i.e. for every non-zero projection $p \in \mathcal{N}(\mathcal{T})$ there exists a non-zero minimal projection $q \in \mathcal{N}(\mathcal{T})$ such that $q \leq p$), we obtain some additional information on the structure of the semigroup (see [7]). In general, $\mathcal{N}(\mathcal{T})$ could be no atomic, while if there exists a faithful invariant state, $\mathcal{F}(\mathcal{T})$ is always atomic.

## 3. Structure of the Generator of a Covariant QMS

The following definition establishes the main property of $\mathcal{T}$ that we study in this paper.

**Definition 3.1.** Let $G$ be a compact group and $\pi\colon g \mapsto \pi(g)$ a continuous unitary representation of $G$ on $\mathsf{h}$. The uniformly continuous QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ is said to be *covariant* with respect to the representation $\pi$ if

$$\mathcal{T}_t(\pi(g)^* x\, \pi(g)) = \pi(g)^* \mathcal{T}_t(x)\pi(g) \tag{3.1}$$

for all $x \in \mathcal{B}(\mathsf{h})$, $g \in G$ and $t \geq 0$.

This property can be expressed equivalently in terms of the generator as

$$\mathcal{L}(\pi(g)^* x \pi(g)) = \pi(g)^* \mathcal{L}(x)\pi(g) \tag{3.2}$$

for all $x \in \mathcal{B}(\mathsf{h})$ and $g \in G$.

The structure of the generator of a covariant uniformly continuous QMS was fully characterized by Holevo in [14], Section 2, in the case of amenable locally compact groups. When $G$ is compact, the result can be restated as follows.

**Theorem 3.2.** *Let $G$ be a compact group, $\pi\colon g \mapsto \pi(g)$ a continuous unitary representation of $G$ on $\mathsf{h}$. If $\mathcal{T}$ is a uniformly continuous QMS, then it is covariant if and only if there exists a GKSL representation of $\mathcal{L}$ (called* covariant*) given by operators $\{H, L_k : k \geq 1\}$ satisfying:*

1. $\sum_k L_k^* \pi(g)^* x \pi(g) L_k = \sum_k \pi(g)^* L_k^* x L_k \pi(g)$ *for all* $x \in \mathcal{B}(\mathsf{h})$ *and* $g \in G$,
2. $H \in \{\pi(g) : g \in G\}'$.

*Moreover, the condition in Item 1 is equivalent to*

$$\pi(g)^* L_j \pi(g) = \sum_k v(g)_{jk} L_k \tag{3.3}$$

*for all $g \in G$, where $V(g) = (v(g)_{jk})_{jk}$ is a unitary matrix. In particular, for all $g \in G$, the operators $\{H, \pi(g)^* L_k \pi(g) : k \geq 1\}$ give another GKSL representation of $\mathcal{L}$.*

Theorem 3.2 directly implies the following corollary that we will widely use in the reminder of the paper.

**Corollary 3.3.** *Let $H$, $(L_k)_k$ be in a covariant GKSL representation of $\mathcal{L}$. Then $H$ and $\sum_k L_k^* L_k$ intertwine the representation $\pi$.*

*Proof.* By Theorem 3.2 we already know that $\pi(g)H = H\pi(g)$ for all $g \in G$. To conclude the proof is enough to note that by item 1 of the same theorem and the unitarity of $\pi$ we have

$$\pi(g)^* \left( \sum_k L_k^* L_k \right) \pi(g) = \sum_k L_k^* (\pi(g)^* \pi(g)) L_k = \sum_k L_k^* L_k.$$

Therefore $\pi(g) \left(\sum_k L_k^* L_k\right) = \left(\sum_k L_k^* L_k\right) \pi(g)$ for all $g \in G$. □

Theorem 3.2 gives the existence of a covariant GKSL representation: but what happens if we consider another GKSL representation given by operators $H', (L'_k)_k$? The following result shows that in general it is not covariant.

**Proposition 3.4.** *Let $\mathcal{T}$ be a uniformly continuous covariant QMS w.r.t. a unitary representation $\pi$ of a compact group $G$, and let $\{H, L_k \,:\, k \geq 1\}$ be in a covariant GKSL representation of the generator $\mathcal{L}$.*
*Then another GKSL representation $\{H', L'_k \,:\, k \geq 1\}$ is covariant if and only if it is connected to the former by equation (2.1) with*

$$\alpha_k = (UV(g)U^*\alpha)_k, \tag{3.4}$$

*where $V(g) = (v(g)_{ij})_{ij}$ is the unitary matrix of equation (3.3) for the operators $(L_k)_{k\geq 0}$.*

*Proof.* We begin by proving that equation (3.4) ensures the existence of a unitary matrix $V'(g)$ such that equation (3.3) holds also for the operators $(L'_k)$.
Since $\{H, L_k \,:\, k \geq 1\}$ and $\{H', L'_k \,:\, k \geq 1\}$ give two GKSL representations of $\mathcal{L}$, by equation (2.1) we have

$$\begin{aligned}
\pi(g)^* L'_k \pi(g) &= \sum_h u_{kh} \pi(g)^* L_h \pi(g) + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g))_{kh} L_h + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g)U^*)_{kh} (L'_h - \alpha_h \mathbb{1}) + \alpha_k \mathbb{1} \\
&= \sum_h (UV(g)U^*)_{kh} L'_h + \left( \alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right) \mathbb{1}.
\end{aligned}$$

If condition (3.4) holds, then $(L'_k)_k$ clearly satisfies the covariance condition (3.3) with respect to the unitary matrix $V'(g) = UV(g)U^*$.
On the other hand, relation $\pi(g)^* L'_k \pi(g) = \sum_h w(g)_{kh} L'_h$ for a unitary matrix $W(g) = (w(g)_{kh})_{kh}$ implies

$$\sum_h (UV(g)U^* - W(g))_{kh} L'_h + \left( \alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right) \mathbb{1} = 0.$$

Since $\{\mathbb{1}, L'_k \,:\, k \geq 1\}$ are linearly independent and

$$\sum_k |(UV(g)U^* - W(g))_{kh}|^2 \leq \sum_k |(UV(g)U^*)_{kh}|^2 + \sum_k |(W(g))_{kh}|^2 = 2 < \infty,$$

$$\sum_k \left| \alpha_k - \sum_h (UV(g)U^*)_{kh} \alpha_h \right|^2 \leq \sum_k |\alpha_k|^2 + \sum_k \left| \sum_h (UV(g)U^*)_{kh} \alpha_h \right|^2 < \infty.$$

We immediately have $UV(g)U^* = W(g)$ and $\alpha_k = \sum_h (UV(g)U^*)_{kh} \alpha_h$, i.e. equation (3.4) is fulfilled. This condition is also sufficient for $[H', \pi(g)] = 0$ to hold for all $g \in G$. First of all we note that

$$[H', \pi(g)] = [H + \beta \mathbb{1} + \frac{1}{2\mathrm{i}}(S - S^*), \pi(g)] = \frac{1}{2\mathrm{i}}[S - S^*, \pi(g)]$$

for all $g \in G$ and $\beta \in \mathbb{R}$. Moreover, since $[S^*, \pi(g)] = -[S, \pi(g^{-1})]^*$, it is enough to prove that $[S, \pi(g)] = 0$ for all $g \in G$ to conclude the proof. Indeed we have

$$\begin{aligned} S\pi(g) &= \sum_{i,j} \overline{\alpha}_i u_{ij} L_j \pi(g) \\ &= \pi(g) \sum_{i,j} \overline{\alpha}_i (UV(g))_{ij} L_j \\ &= \pi(g) \sum_{h,j} \overline{\alpha}_h (UV(g)^* U^* UV(g))_{hj} L_j \\ &= \pi(g) S \end{aligned}$$

for all $g \in G$ and therefore $[H', \pi(g)] = 0$ for all $g \in G$. □

We conclude the section proving that $\mathcal{N}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T})$ behave nicely with respect to the covariance property.

**Proposition 3.5.** *Let $\mathcal{T}$ be a covariant QMS w.r.t. a unitary representation $\pi$ of a compact group $G$. Then*

$$\pi(g)^* \mathcal{N}(\mathcal{T}) \pi(g) = \mathcal{N}(\mathcal{T}) \quad \textit{and} \quad \pi(g)^* \mathcal{F}(\mathcal{T}) \pi(g) = \mathcal{F}(\mathcal{T})$$

*for all $g \in G$.*

*Proof.* Let $g \in G$ and $x$ be in $\mathcal{N}(\mathcal{T})$. We have to prove that $y := \pi(g)^* x\, \pi(g)$ belongs to $\mathcal{N}(\mathcal{T})$. The covariance of $\mathcal{T}$ gives

$$\begin{aligned} \mathcal{T}(y^* y) &= \mathcal{T}_t(\pi(g)^* x^* x\, \pi(g)) = \pi(g)^* \mathcal{T}_t(x^* x) \pi(g) \\ &= (\pi(g)^* \mathcal{T}_t(x^*)\, \pi(g))\, (\pi(g)^* \mathcal{T}_t(x) \pi(g)) = \mathcal{T}_t(\pi(g)^* x^* \pi(g)) \mathcal{T}_t(\pi(g)^* x\, \pi(g)) \\ &= \mathcal{T}_t(y^*) \mathcal{T}_t(y). \end{aligned}$$

In the same way we can show the equality $\mathcal{T}_t(yy^*) = \mathcal{T}_t(y)\mathcal{T}_t(y^*)$, i.e. $y \in \mathcal{N}(\mathcal{T})$.

For the set of fixed points $\mathcal{F}(\mathcal{T})$ the proof is very similar. □

Whenever the representation $\pi$ is irreducible it is possible to further specify the structure of the generator $\mathcal{L}$.

Indeed, in this case, Corollary 3.3 and Schur's Lemma give

$$\sum_k L_k^* L_k \in \mathbb{C}\mathbb{1} \quad \text{and} \quad H \in \mathbb{C}\mathbb{1},$$

so that the following result immediately follows.

**Proposition 3.6.** *Let $G$ be a compact group, $\pi \colon g \mapsto \pi(g)$ an irreducible unitary representation of $G$ on a finite dimensional Hilbert space $\mathsf{h}$. Let also $H, (L_k)_k$ be operators in a covariant GKSL representation of $\mathcal{L}$. Then $\mathcal{L}$ can be written as*

$$\mathcal{L}(x) = \sum_k L_k^* x L_k - \epsilon x \tag{3.5}$$

*where $\epsilon$ is a real positive constant such that $\sum_k L_k^* L_k = \epsilon \mathbb{1}$.*

## 4. Structure of Invariant Projections for a Covariant QMS

In this section, assuming the existence of a faithful normal invariant state for the covariant QMS $\mathcal{T}$, we clarify the relationships between the structure of $\mathcal{F}(\mathcal{T})$ and operators $\pi(g)$.

First of all we recall the following facts about the set of fixed points of an arbitrary QMS (not necessarily covariant).

*a*) A projection $p$ belongs to $\mathcal{F}(\mathcal{T})$ if and only if it commutes with operators $H$ and $(L_k)_k$ in any GKSL representation of $\mathcal{L}$,

*b*) If $p$ in as invariant projection, then we have $\mathcal{T}_t(pxp) = p\mathcal{T}_t(x)p$ for every $x \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$. This means that the algebra $p\mathcal{B}(\mathsf{h})\,p = \mathcal{B}(p(\mathsf{h}))$ is preserved by the semigroup, and so we obtain by restriction a QMS $\mathcal{T}^p$ on $\mathcal{B}(p(\mathsf{h}))$, i.e.

$$\mathcal{T}_t^p(x) = \mathcal{T}_t(x) = \mathcal{T}_t(pxp) = p\mathcal{T}_t(x)p \qquad \forall\, x \in \mathcal{B}(p(\mathsf{h})).$$

Moreover we have $\mathcal{F}(\mathcal{T}^p) = p\mathcal{F}(\mathcal{T})p$.

*c*) If $\mathcal{T}$ has a faithful normal invariant state, then $\mathcal{F}(\mathcal{T})$ is the image of a normal conditional expectation and so it is an atomic algebra. Therefore there exist two countable sequences $(\mathsf{k}_i)_{i\in I}$ and $(\mathsf{m}_i)_{i\in I}$ of separable Hilbert spaces such that

$$\mathcal{F}(\mathcal{T}) = \oplus_{i\in I}\left(\mathcal{B}(\mathsf{k}_i) \otimes \mathbb{1}_{\mathsf{m}_i}\right)$$

in accordance to the decomposition of $\mathsf{h}$ given by $\mathsf{h} = \oplus_{i\in I}\left(\mathsf{k}_i \otimes \mathsf{m}_i\right)$. Moreover, denoting by $p_i$ the orthogonal projection onto $\mathsf{k}_i \otimes \mathsf{m}_i$ for $i \in I$, the collection $(p_i)_{i\in I}$ is a family of mutually orthogonal minimal projections in $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$ such that $\sum_{i\in I} p_i = \mathbb{1}$.

On the other hand, if $\mathcal{T}$ is covariant (but non necessarily with a faithful invariant state), since $\pi$ can be decomposed into the direct sum of finite-dimensional irreducible sub-representations acting on orthogonal subspaces, there exists a collection $(V_i)_{j\in J}$ of pairwise orthogonal, finite-dimensional and $\pi$-invariant subspaces of $\mathsf{h}$ such that:

(1) $\mathsf{h} = \oplus_{j\in J} V_j$,
(2) the restriction $\pi_j$ of $\pi$ to every $V_j$ is an irreducible unitary representation,
(3) each orthogonal projection $q_j$ onto $V_j$ belongs to $\mathcal{F}(\mathcal{T})$ and

$$\pi_j(g) = q_j\pi(g)q_j = \pi(g)q_j = q_j\pi(g) \quad \forall\, g \in G, \tag{4.1}$$

(4) the restriction $\mathcal{T}^j$ of $\mathcal{T}$ to $\mathcal{B}(V_j)$ is covariant with respect to $\pi_j$.

For more details we refer the reader to [13, Theorem 8].

We will prove that the family $(p_i)_{i\in I}$ determines the structure of non zero central invariant projections.

**Proposition 4.1.** *Assume $\mathcal{T}$ with a faithful normal invariant state. Every non zero invariant projection $q$ in $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$ can be written as*

$$q = \sum_{i\in I_0} p_i$$

*for some non empty subset $I_0$ of $I$.*

*Proof.* Since both $p_i$ and $q$ belong to $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$ for all $i \in I$, each $p_i q$ is a projection in $\mathcal{Z}(\mathcal{F}(\mathcal{T}))$. Therefore, set $I_0 = \{i \in I \,:\, p_i q \neq 0\}$ and taken $i \in I_0$, the inequality $p_i q = p_i q p_i \leq p_i$ implies $p_i q = p_i$ by the minimality of $p_i$ in the center of $\mathcal{F}(\mathcal{T})$. So

$$q = \sum_i p_i q = \sum_{i \in I_0} p_i q = \sum_{i \in I_0} p_i,$$

concluding the proof. □

When the semigroup satisfies the additional condition

$$\mathcal{F}(\mathcal{T}) \subseteq \{\pi(g) \,:\, g \in G\}' \tag{4.2}$$

we can provide a more accurate description of invariant central projections and the general structure of $\mathcal{F}(\mathcal{T})$.

*Remark* 4.2. In general condition (4.2) does not imply that the semigroup $\mathcal{T}$ is covariant with respect to the representation $\pi$, but it forces an invariant projection $p$ for the semigroup to be invariant with respect the conjugation with $\pi$, i.e. $\pi(g)^* p \pi(g) = p$ for every $g \in G$.

**Proposition 4.3.** *Let $\pi$ be a unitary representation of a compact group $G$ on $\mathsf{h}$, and let $\mathcal{T}$ be a uniformly continuous (not necessarily covariant) QMS on $\mathcal{B}(\mathsf{h})$ possessing a faithful normal invariant state and satisfying $\mathcal{F}(\mathcal{T}) \subseteq \{\pi(g) \,:\, g \in G\}'$. If $p$ is an invariant projection, its range $p(\mathsf{h})$ is a $\pi$-invariant subspace and the restriction $\pi_p : G \to \mathcal{B}(p(\mathsf{h}))$ of $\pi$ to $p(\mathsf{h})$ is the sub-representation given by*

$$\pi_p(g) = p\,\pi(g) = \pi(g)p = p\pi(g)p \qquad \forall\, g \in G. \tag{4.3}$$

*Moreover, if $\mathcal{T}$ is covariant with respect to $\pi$, $\mathcal{T}^p$ is covariant with respect to $\pi_p$.*

*Proof.* Since $p$ is an invariant projection, by assumption it commutes with every $\pi(g)$ and so $p(\mathsf{h})$ is $\pi$-invariant and equation (4.3) immediately follows.
If $\mathcal{T}$ is $\pi$-covariant, the covariance of $\mathcal{T}^p$ with respect to $\pi_p$ is clear since $\mathcal{T}$ preserves $\mathcal{B}(p(\mathsf{h}))$. □

In addition, given an invariant projection $p$, equation (4.2) allows us to prove the equivalence between the irreducibility of the representation $\pi_p$ and that one of the semigroup $\mathcal{T}^p$.

Recall that a QMS $\mathcal{T}$ is irreducible if there does not exist non-trivial projections $q$ such that $\mathcal{T}_t(q) \geq q$ for all $t \geq 0$ (see [9] Definition II.2) In particular, if $\mathcal{T}$ possesses a faithful invariant state, since a projection $q$ satisfying $\mathcal{T}_t(q) \geq q$, by $\mathrm{tr}(\rho(\mathcal{T}_t(q) - q)) = 0$, turns out to be invariant, irreducibility is equivalent to the non-existence of non-trivial invariant projections.

**Proposition 4.4.** *Assume $\mathcal{T}$ covariant with respect to the representation $\pi$ of $G$ with a faithful normal invariant state $\rho$ and $\mathcal{F}(\mathcal{T}) \subseteq \{\pi(g) \,:\, g \in G\}'$. Given an invariant projection $p$, then the following facts are equivalent*

(1) *$p$ is minimal in $\mathcal{F}(\mathcal{T})$;*
(2) *$\mathcal{T}^p$ is irreducible;*
(3) *$\mathcal{F}(\mathcal{T}^p) = \mathbb{C}p$;*
(4) *$\pi_p$ is irreducible.*

*Proof.* 1. $\Rightarrow$ 2. Assume $p$ minimal in $\mathcal{F}(\mathcal{T})$ and take $q \in \mathcal{B}(p(\mathsf{h}))$ an invariant projection for $\mathcal{T}^p$. Then $q \leq p$ and $q = \mathcal{T}^p_t(q) = \mathcal{T}_t(q)$ by definition of $\mathcal{T}^p$. This clearly means that $q$ belongs to $\mathcal{F}(\mathcal{T})$, and so either $q = 0$ or $q = p$ by the minimality of $p$ in this algebra. Therefore $\mathcal{T}^p$ is irreducible.

2. $\Rightarrow$ 3. Denote by $\rho$ the faithful normal invariant state of $\mathcal{T}$. Since $\mathrm{tr}\,(\rho p) \neq 0$ (otherwise $p = 0$ by the fidelity), then $p\rho p(\mathrm{tr}\,(p\rho))^{-1}$ is a faithful normal invariant state for $\mathcal{T}^p$. Therefore, the irreducibility of $\mathcal{T}^p$ and the atomicity of $\mathcal{F}(\mathcal{T}^p)$ force to have $\mathcal{F}(\mathcal{T}^p) = \mathbb{C}p$.

3. $\Rightarrow$ 4. Assume $\pi_p$ reducible. Then by Peter-Weyl Theorem there exists a non trivial projection $q \in \mathcal{F}(\mathcal{T}^p) \subseteq \mathcal{B}(p(\mathsf{h}))$ (see (4.1) here) such that $\pi_q$ is irreducible and $p \neq q$. This contradicts the assumption $\mathcal{F}(\mathcal{T}^p) = \mathbb{C}p$.

4. $\Rightarrow$ 1. Assume $\pi_p$ irreducible and let $q$ be a non zero projection in $\mathcal{F}(\mathcal{T})$ such that $q \leq p$. Hence, by Proposition 4.3, $q(\mathsf{h})$ is a $\pi$-invariant subspace of $p(\mathsf{h})$, and so it coincides with $p(\mathsf{h})$, i.e. $q = p$. This proves the minimality of $p$ in $\mathcal{F}(\mathcal{T})$. □

Finally we can characterize the structure of $\mathcal{F}(\mathcal{T})$ showing that every invariant projection $p$ coincides with some $q_j$.

**Theorem 4.5.** *Let $\mathcal{T}$ be a covariant uniformly continuous QMS with a faithful normal invariant state and satisfying*

$$\mathcal{F}(\mathcal{T}) \subseteq \{\pi(g) \,:\, g \in G\}'.$$

*If $p$ is a non zero minimal projection in $\mathcal{F}(\mathcal{T})$, then there exists a unique index $j \in J$ such that $p = q_j$. In particular, the $q_j$'s are the unique minimal invariant projections for $\mathcal{T}$.*

*Proof.* First of all we claim that each $q_j$ is a minimal projection in $\mathcal{F}(\mathcal{T})$. Indeed, if $q$ is a non zero projection in $\mathcal{F}(\mathcal{T})$ such that $q \leq q_j$ (so that $q \in \mathcal{B}(V_j)$), since the restriction $\mathcal{T}^j$ of $\mathcal{T}$ to $\mathcal{B}(V_j)$ is $\pi_j$-covariant and $\pi_j$ is irreducible, statement 2 in Proposition 4.4 implies $p = \mathbb{1}_{V_j} = q_j$, so that $q_j$ is minimal.

Now we consider $\widetilde{p}_j := q_j \wedge p$, the orthogonal projection onto $q_j(\mathsf{h}) \cap p(\mathsf{h})$. Since we clearly have $\widetilde{p}_j \leq q_j$ and $\widetilde{p}_j \leq p$ and both $q_j$ and $p$ are in $\mathcal{F}(\mathcal{T})$, we get $\mathcal{T}_t(\widetilde{p}_j) \leq q_j$ and $\mathcal{T}_t(\widetilde{p}_j) \leq p$ for all $t \geq 0$. Consequently $\mathcal{T}_t(\widetilde{p}_j) \leq q_j \wedge p = \widetilde{p}_j$ for all $t \geq 0$, so that $\widetilde{p}_j$ belongs to $\mathcal{F}(\mathcal{T})$ thanks to existence of a faithful invariant state. Since $q_j$ is minimal in $\mathcal{F}(\mathcal{T})$, this implies either $\widetilde{p}_j = 0$ or $\widetilde{p}_j = q_j$. If the first case happens for every $j \in J$, the relation

$$p(\mathsf{h}) = \big(\cup_{j\in J}\, q_j(\mathsf{h})\big) \cap p(\mathsf{h}) = \cup_{j\in J}\big(q_j(\mathsf{h}) \cap p(\mathsf{h})\big) = \cup_{j\in J}\, \widetilde{p}_j(\mathsf{h})$$

gives the contradiction $p = 0$. Therefore, there exists at least one $j \in J$ such that $\widetilde{p}_j = q_j$. Finally, since the projections $q_j$ are mutually orthogonal, we can have $\widetilde{p}_j = q_j$ for a unique $j \in J$. □

**Theorem 4.6.** *Assume $\mathcal{T}$ covariant with a faithful normal invariant state and $\mathcal{F}(\mathcal{T}) \subseteq \{\pi(g) \,:\, g \in G\}'$. Then, for all $i \in I$ there exists a subset $J_i \subseteq J$ such that*

$$p_i = \sum_{j \in J_i} q_j.$$

*Moreover, for different $i, k \in I$, we have $J_i \cap J_k = \emptyset$, i.e. every $q_j$ belongs to a unique block $\mathcal{B}(\mathsf{k}_{i_j} \otimes \mathsf{m}_{i_j})$.*

*Proof.* Given $i \in I$ define $J_i = \{j \in J : p_i q_j \neq 0\}$. Then, since each $q_j$ belongs to $\mathcal{F}(\mathcal{T})$ and $p_i$ is in the center of this algebra, $p_i q_j = q_j p_i q_j$ is an invariant non zero projection in $\mathcal{B}(V_j)$ for all $j \in J_i$. Now, by the commutation between $\mathcal{F}(\mathcal{T})$ and every $\pi(g)$ implies

$$\pi_j(g)^* p_i q_j \pi_j(g) = \pi(g)^* p_i q_j \pi(g) = p_i q_j \quad \forall\, g \in G,$$

i.e. $p_i q_j$ intertwines the representation $\pi_j$. Therefore, by Schur's Lemma the equality $p_i q_j = q_j$ follows for all $j \in J_i$. So

$$p_i = \sum_{j \in J} p_i q_j = \sum_{j \in J_i} p_i q_j = \sum_{j \in J_i} q_j$$

Finally, if there exists $l \in J_i \cap J_k$ for some $i, k \in I$ with $i \neq k$, then $q_l \leq p_i$ and $q_l \leq p_k$, giving the contradiction $q_l = 0$. □

As the last result of this section, we analyze the action of the conjugation with the representation $\pi$ on the minimal projections $p_i$ in the center of $\mathcal{F}(\mathcal{T})$, that appear in the atomic decomposition of the algebra.

**Proposition 4.7.** *For all $g \in G$ there exists a unique permutation $\sigma_g$ of $I$ such that*

$$\pi(g) p_i \pi(g)^* = p_{\sigma_g(i)} \qquad \forall\, i \in I. \tag{4.4}$$

*Moreover, if $\pi$ is irreducible and $\mathcal{F}(\mathcal{T})$ is not a factor, there is at least one $g \in G$ such that $\sigma_g(i) \neq i$ for all $i \in I$.*

*Proof.* Assume $\mathcal{F}(\mathcal{T})$ is not a factor (otherwise equation (4.4) is trivially satisfied), so that $I$ has cardinality greater than one. Let $i \in I$ and $g \in G$. Since $\pi(g)$ is unitary and $\pi(g)^* \mathcal{F}(\mathcal{T}) \pi(g) = \mathcal{F}(\mathcal{T})$ by Proposition 3.5, there exists a unique projection $q_i \in \mathcal{F}(\mathcal{T})$ depending on $g$, such that

$$p_i = \pi(g)^* q_i \pi(g). \tag{4.5}$$

We claim that the minimality of $p_i$ implies that one of $q_i$. Indeed, taken a non zero projection $q_i' \in \mathcal{F}(\mathcal{T})$ satisfying $q_i' \leq q_i$, since $\pi(g)^* q_i' \pi(g)$ belongs to $\mathcal{F}(\mathcal{T})$ we have

$$\pi(g)^* q_i' \pi(g) \leq \pi(g)^* q_i \pi(g) = p_i,$$

so that the minimality of $p_i$ gives either $\pi(g)^* q_i' \pi(g) = 0$ or $\pi(g)^* q_i' \pi(g) = p_i$. The first equality contradicts the assumption $q_i' \neq 0$, so that the equation

$$\pi(g)^* q_i' \pi(g) = p_i = \pi(g)^* q_i \pi(g)$$

holds. This means $q_i' = q_i$, i.e. $q_i$ is minimal in $\mathcal{F}(\mathcal{T})$, proving the claim.

Now, since the atomicity of $\mathcal{F}(\mathcal{T})$ gives

$$q_i = \sum_{j \in I} q_{j(i)} \otimes \mathbb{1}_{\mathsf{m}_{j(i)}}$$

for some projection $q_{j(i)} \in \mathcal{B}(\mathsf{k}_{j(i)})$, we immediately obtain $q_i \geq q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}}$ for all $l(i) \in I$. But $q_i$ is minimal in $\mathcal{F}(\mathcal{T})$, to which also $q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}}$ belongs, and so we have either $q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}} = 0$ or $q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}} = q_i$. Now, if $q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}} = 0$ for all $l(i) \in I$, we have $q_i = 0$ contradicting the assumption $p_i \neq 0$. Hence, there exists a unique $l(i) \in I$ satisfying $q_{l(i)} \otimes \mathbb{1}_{\mathsf{m}_{l(i)}} = q_i$, and we can put $\sigma(i) = l(i)$, with

$\sigma\colon I \to I$. The uniqueness of $l(i)$ is a consequence of the the fact that $\{q_i\}_{i\in I}$ is a set of orthogonal projections. It follows that

$$p_i = \pi(g)^* \left(q_{\sigma(i)} \otimes \mathbb{1}_{\mathsf{m}_{\sigma(i)}}\right) \pi(g) \leq \pi(g)^* \left(p_{\sigma(i)}\right) \pi(g),$$

being $q_{\sigma(i)} \otimes \mathbb{1}_{\mathsf{m}_{\sigma(i)}}$ a projection in $\mathcal{B}(\mathsf{k}_{\sigma(i)} \otimes \mathsf{m}_{\sigma(i)})$ and $p_{\sigma(i)}$ the unit of this space. Since $p_{\sigma(i)}$ is minimal in $\mathcal{F}(\mathcal{T})$ we get $p_i = \pi(g)^* p_{\sigma(i)} \pi(g)$, i.e. equality (4.4) holds. Moreover $\sigma$ is a permutation. Indeed $\sigma(i) = \sigma(j)$ with $i \neq j$ implies

$$p_i = \pi(g)^* p_{\sigma(i)} \pi(g) = \pi(g)^* p_{\sigma(j)} \pi(g) = p_j,$$

and this is not possible.

Finally assume $\pi$ irreducible. Since $\mathcal{F}(\mathcal{T})$ is not a factor, we have that $p_i \neq \mathbb{1}$ for all $i \in I$. If for all $g \in G$ there exists a permutation $\sigma_g$ of $I$ such that $\sigma_g(i) = i$ for at least one $i \in I$, then

$$p_i = \pi(g)^* p_i \, \pi(g) \quad \forall\, g \in G,$$

i.e. each $p_i$ intertwines the representation. Therefore, Schur's Lemma implies $p_i \in \mathbb{C}\mathbb{1}$, contradicting the assumption. □

The meaning of this result is the following: for all $g \in G$ the conjugation with $\pi(g)^*$ changes the block $\mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i)$ in the block $\mathcal{B}(\mathsf{k}_j \otimes \mathsf{m}_j)$, where $j = \sigma_g(i)$. We can eventually have $j = i$, i.e. $\mathcal{B}(\mathsf{k}_i \otimes \mathsf{m}_i)$ is invariant with respect to the transformation. In particular, if $\pi$ is irreducible and $\mathcal{F}(\mathcal{T})$ is not a factor, there is no invariant blocks with respect to the conjugation with $\pi(g)^*$.

## 5. Application to Circulant Quantum Markov Semigroups

J. R. Bolaños-Servín and R. Quezada [3] introduced the class of circulant QMSs whose properties have been further studied in [2]. The name of this family of uniformly continuous QMS depends the GKSL form of their generator involving circulant matrices. We fix a dimension $p \geq 2$ and we choose as our von Neumann algebra the algebra of the $p \times p$ matrices with complex entries, i.e. $\mathcal{B}(\mathsf{h}) = \mathcal{M}_p(\mathbb{C})$. We indicate with $(e_k)_{k\in\mathbb{Z}_p}$ the canonical basis of $\mathbb{C}^p$. We underline that, since the index set is $\mathbb{Z}_p$, all the operations among indices should be meant modulus $p$. Let $J_c$ be a circulant matrix associated to a cycle $c$ of order $p$ ( i.e. $c^p = id$ and $c^k \neq id$ for $k = 1, \ldots, p-1$). We can express $J_c$ in terms of the canonical basis as

$$J_c = \sum_{k\in\mathbb{Z}_p} |e_{c(k)}\rangle\langle e_{c(k+1)}|.$$

Without loss of generality, we can consider as $c$, the cycle $c = (0, 1, 2, 3, \ldots .p-1)$. In this case $J = J_c$ (the primary permutation matrix) has an easier formula, since $J = \sum_{k\in\mathbb{Z}_p} |e_k\rangle\langle e_{k+1}|$. Now we can define a circulant generator $\mathcal{L}$ on $\mathcal{M}_p(\mathbb{C})$ as

$$\mathcal{L}(x) = \sum_{k=1}^{p-1} \gamma(p-k) J^{*k} x J^k - x$$

with $\gamma$ a vector in $\mathbb{C}^{p-1}$ such that $\gamma(k) > 0$ for $k = 1, \ldots, p-1$ and $\sum_{k=1}^{p-1} \gamma(k) = 1$. Clearly $\mathcal{L}$ is written in the GKSL representation by means of operators $L_k =$

$\sqrt{\gamma(p-k)}\,J^k$ for $k=1,\dots,p-1$, and $H=\mathbb{1}$. The (uniformly continuous) QMS $\mathcal{T}$ generated by $\mathcal{L}$ is the circulant semigroup associated with the cycle $c$.

*Remark* 5.1. Since $J^*J=\mathbb{1}$, the state $p^{-1}\mathbb{1}$ is a faithful invariant state for the circulant QMS $\mathcal{T}$. So $\mathcal{F}(\mathcal{T})$ is an algebra and

$$\mathcal{N}(\mathcal{T})=\mathcal{F}(\mathcal{T})=\{J^k,J^{*k}\,:\,k\in\mathbb{Z}_p\}'.$$

In particular every invariant projection $p$ satisfies

$$J^kpJ^{*k}=p \qquad \forall\, k\in\mathbb{Z}_p. \tag{5.1}$$

From (5.1) we can deduce a necessary condition for a projection to be invariant.

**Proposition 5.2.** *Every invariant projection $p$, with $p=\sum p_{h,j}|e_h\rangle\langle e_j|$, for a circulant QMS satisfies $p_{h+1,j+1}=p_{h,j}$ for every $h,j\in\mathbb{Z}_p$.*

*Proof.* Every invariant projections $p$ has to commute with $J^k$ and $J^{*k}$ for every $k\in\mathbb{Z}_p$, but it is equivalent to require that $p$ commutes with $J$, i.e. $pJ=Jp$. Now

$$\begin{aligned}
pJ &= \sum_{h,j} p_{h,j}|e_h\rangle\langle e_j| \sum_i |e_{i+1}\rangle\langle e_i| \\
&= \sum_{h,j} p_{h,j}|e_h\rangle\langle e_{j-1}| \\
&= \sum_{h,j} p_{h+1,j+1}|e_{h+1}\rangle\langle e_j|,
\end{aligned}$$

and, analogously,

$$\begin{aligned}
Jp &= \sum_i |e_{i+1}\rangle\langle e_i| \sum_{h,j} p_{h,j}|e_h\rangle\langle e_j| \\
&= \sum_{h,j} p_{h,j}|e_{h+1}\rangle\langle e_j|.
\end{aligned}$$

This conclude the proof. □

Define now a representation $\pi:\mathbb{Z}_p\to M_d(\mathbb{C})$ by setting $\pi(k)f=J^kf$ for all $k\in\mathbb{Z}_p$ and $f\in\mathsf{h}=\mathbb{C}^p$. Clearly $\mathcal{T}$ is covariant with respect to $\pi$ and $\mathcal{F}(\mathcal{T})=\{\pi(k)\,:\,k\in\mathbb{Z}_p\}'$, i.e. condition (4.2) is trivially satisfied.

Moreover, the representation $\pi$ is not irreducible, since, for example, the space $\operatorname{span}\{(1,\dots,1)\}$ is invariant for every $\pi(g)$, with $g\in\mathbb{Z}_p$. Therefore, we can decompose $\pi$ into the direct sum of irreducible sub-representations $(\pi_j)_{j\in J}$ on the mutually orthogonal subspaces $(V_j)_{j\in j}$ such that the orthogonal projection $q_j$ onto $V_j$ is minimal in $\mathcal{F}(\mathcal{T})$ and $\mathcal{F}(\mathcal{T}^j)=q_j\mathcal{F}(\mathcal{T})q_j$, being $\mathcal{T}^j$ the restriction of $\mathcal{T}$ to $V_j$ (see Proposition 4.4).

In order to determine projections $q_j$, we recall the following result:

**Proposition 5.3.** *The permutation matrix $J$ is diagonalized by $F$, the discrete Fourier transform, i.e.*

$$FJF^*=\sum_{i=0}^{p-1}\overline{\omega}^i|e_i\rangle\langle e_i|,$$

*where $\omega$ is a primitive $p$-root of the unit and $F=\frac{1}{\sqrt{p}}\sum_{k,l=0}^{p-1}\omega^{kl}|e_k\rangle\langle e_j|$.*

This clearly means that the family of vectors $\{Fe_i\}_{i=0}^{p-1}$ is an orthonormal basis of $\mathsf{h}$ given by eigenvectors of $J$ related to the eigenvalue $\overline{\omega}^i$. Therefore each projection

$$q_i := |Fe_i\rangle\langle Fe_i| = \frac{1}{p}\sum_{k,l=0}^{p-1}\omega^{ki}\overline{\omega}^{li}|e_k\rangle\langle e_l| = \frac{1}{p}\sum_{k,l=0}^{p-1}\omega^{(k-l)i}|e_k\rangle\langle e_l| \tag{5.2}$$

is a minimal invariant projection for $\mathcal{T}$, and the family $\{q_i\}_{i=0}^{p-1}$ is a collection of mutually orthogonal minimal invariant projections such that $\sum_i q_i = \mathbb{1}$.

Since each $q_i$ commutes with $J^k = \pi(k)$ for all $k = 0, \ldots, p-1$, its range $V_i := q_i(\mathsf{h})$ is $\pi$-invariant, and so the restriction of $\pi$ to $V_i$ gives a sub-representation $\pi_i$ of $\pi$ with $\pi_i(k) = q_i\pi(k)q_i$ for all $i = 0, \ldots, p-1$.

Finally, note that $\pi_i$ is irreducible, being $V_i$ an one-dimensional space.

Summarizing:

**Proposition 5.4.** *The unitary representation $\pi$ splits in the direct sum $\oplus_{i=0}^{p-1}\pi_i$, where each $\pi_i : G \to \mathcal{B}(q_i(\mathsf{h}))$ is an irreducible representation of $G$ and the semigroup $\mathcal{T}^{q_i}$, given by restricting $\mathcal{T}$ to $\mathcal{B}(q_i(\mathsf{h}))$, is $\pi_i$-covariant.*

We now show that the family $\{q_i\}_{i=0}^{p-1}$ gives an atomic decomposition of $\mathcal{N}(\mathcal{T})$.

**Lemma 5.5.** *Every $q_i$ belongs to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$, the center of $\mathcal{N}(\mathcal{T})$.*

*Proof.* Let $p$ be a projection in $\mathcal{N}(\mathcal{T})$. Since it commutes with $J$, $p$ has to commute with its spectral projections and so, in particular, with each $q_i = |Fe_i\rangle\langle Fe_i|$. Therefore, since $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ is generated by its projections, we immediately obtain $q_i \in \mathcal{Z}(\mathcal{N}(\mathcal{T}))$ for all $i = 0, \ldots, p-1$. □

**Theorem 5.6.** *We have $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) = \mathbb{C}q_0 \oplus \ldots \oplus \mathbb{C}q_{p-1}$.*

*Proof.* By Lemma 5.5 projections $q_i$'s belong to $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$, so that

$$\mathcal{N}(\mathcal{T}) = \oplus_{i=0}^{p-1}\, q_i\mathcal{N}(\mathcal{T})q_i,$$

being $\sum_i q_i = \mathbb{1}$. Finally, since each $q_i$ has a one-dimensional range, we obtain $q_i\mathcal{N}(\mathcal{T})q_i = \mathbb{C}q_i$, concluding the proof. □

**Acknowledgment.** The authors would like to thank the organizers for the invitation to the *40th International Conference on Quantum Probability and Infinite Dimensional Analysis* held at Ohio State University, Columbus on August 2019. Special thanks go to Professor Aurel Stan for support, invaluable assistance and a lively atmosphere for discussion and exchange of knowledge.

## References

1. Blanchard, Ph. and Olkiewicz, R.: Decoherence induced transition from quantum to classical dynamics, *Rev. Math.Phys.* **15** (2003), no. 3, 217–243.
2. Bolãnos-Servin, J. R. and Carbone, R.: Spectral properties of circulant quantum Markov semigroups, *Open Syst. Inf. Dyn.* **21** (2014), no. 4, 1450007–1450015.
3. Bolãnos-Servin, J. R. and Quezada, R.: A cycle decomposition and entropy production for circulant quantum Markov semigroups, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **16** (2013), no. 2, 1350016.
4. Carbone, R., Sasso, E., and Umanità, V.: Decoherence for Quantum Markov Semi-Groups on Matrix Algebras, *Ann. Henri Poincaré* **14** (2013), no. 4, 681–697.

5. Carbone, R., Sasso, E., and Umanità, V.: Ergodic Quantum Markov SemiGroups and decoherence, *J. Oper. Theory* **72** (2014), no. 2, 293–312.
6. Carbone, R., Sasso, E., and Umanità, V.: Environment induced decoherence for Markovian evolutions, *J. Math. Phys.* **56** (2015), no. 9, 092704.
7. Deschamps, J., Fagnola, F., Sasso, E., and Umanità, V.: Structure of uniformly continuous quantum Markov semigroups, *Rev. Math. Phys.* **28** (2016), no. 1, 1650003.
8. Evans, D. E.: Irreducible quantum dynamical semigroups, *Comm. Math. Phys.* **54** (1977), no. 3, 293–297.
9. Fagnola, F. and Rebolledo, R.: Subharmonic projections for a quantum Markov semigroup, *J. Math. Phys.* **43** (2002), no. 2, 1074–1082.
10. Fagnola, F., Sasso, E., and Umanità, V.: Structure of Uniformly Continuous Quantum Markov Semigroups with Atomic Decoherence-free Subalgebra, *Open Syst. Inf. Dyn.* **24** (2017), no. 3, 1740005.
11. Frigerio, A.: Stationary states of quantum dynamical semigroups, *Comm. Math. Phys.* **63** (1978), no. 3, 269–276.
12. Frigerio, A. and Verri, M.: Long-time asymptotic properties of dynamical semigroups on W*-algebras, *Math. Z.* **180** (1982), no. 3, 275–286.
13. Ginatta, N., Sasso, E., and Umanità, V.: Covariant Uniformly Continuous Quantum Markov Semigroups, *Rep. Math. Phys.* **84** (2019), no. 2, 131–150.
14. Holevo, A. S.: A note on covariant dynamical semigroups, *Rep. Math. Phys.* **32** (1993), no. 2, 211–216.
15. Holevo, A. S.: On the Structure of Covariant Dynamical Semigroups, *J. Funct. Anal.* **131** (1995), no. 2, 255–278.
16. Holevo, A. S.: Covariant quantum Markovian evolutions, *J. Math. Phys.* **37** (1996), no. 4, 1812–1832.
17. Holevo, A. S.: On the constrained classical capacity of infinite-dimensional covariant quantum channels, *J. Math. Phys.* **57** (2016), no. 1, 015203.
18. Lindblad, G.: On the generators of quantum dynamical semigroups, *Comm. Math. Phys.* **48** (1976), no. 2, 119–130.
19. Olkiewicz, R.: Structure of the Algebra of Effective Observables in Quantum Mechanics, *Ann. Physics* **286** (2000), no. 1, 10–22.
20. Parthasarathy, K. R.: *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Boston, 1992.
21. Radjavi, H. and Rosenthal, P.: *Invariant Subspaces*, Springer-Verlag, Berlin, 1973.

Franco Fagnola: Department of Mathematics, Politecnico di Milano, 20133 Milano, Italy
*E-mail address*: franco.fagnola@polimi.it

Emanuela Sasso: Department of Mathematics, University of Genova, 16100 Genova, Italy
*E-mail address*: sasso@dima.unige.it

Veronica Umanità: Department of Mathematics, University of Genova, 16100 Genova, Italy
*E-mail address*: umanita@dima.unige.it