Anisotropic Generalized Kolmogorov Equations (AGKE): A novel tool to describe complex flows

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Our topic

I will introduce the AGKE, explain its advantages and guide you in a tour through example applications

- AGKE is a post-processing tool, you also need (DNS or experimental) data
- AGKE results have their own language

Introducing the AGKE

(A) The Kármán-Howarth equation for homogeneous isotropic turbulence

An exact evolution equation for the longitudinal correlation function $f(r, t)$

Starting point is far, far away...

$$\frac{\partial}{\partial t}\left[\langle u^2\rangle r^4 f(r,t)\right] = \langle u^3\rangle \frac{\partial}{\partial r}\left[r^4 K(r)\right] + 2\nu\langle u^2\rangle \frac{\partial}{\partial r}\left[r^4 \frac{\partial f(r,t)}{\partial r}\right]$$

- First term: evolutive, null for stationary flows
- Second term: inertial processes (energy cascade)
- Third term: viscous processes

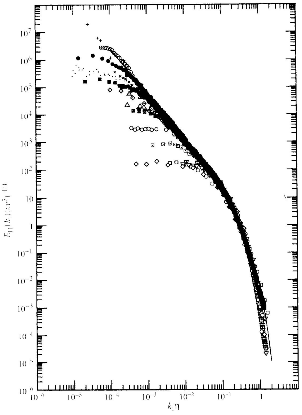
The Kolmogorov equation

An exact evolution equation for the second-order structure function $D_2(r,t)$

The Kármán-Howarth equation is rewritten in terms of structure functions:

- $D_2(r,t) = \langle u^2 \rangle (1-f)$
- $D_3(r,t) = 6\langle u^3 \rangle K(r)$

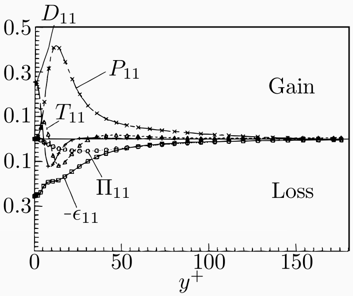
$$6\nu \frac{\partial D_2(r,t)}{\partial r} - D_3(r,t) - \frac{4}{5}\epsilon r = \frac{3}{r^5}\int_0^r s^4 \frac{\partial}{\partial t} D_2(s,t)\, ds$$



Fluxes in scale space

(B) The Reynolds stress equation for a non-homogeneous, non-isotropic flow

$$\left(\frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k}\right)\langle u_i' u_j' \rangle = -\frac{\partial}{\partial x_k}\langle u_i' u_j' u_k' \rangle + P_{ij} + \Pi_{ij} - \epsilon_{ij} + \nu \nabla^2 \langle u_i' u_j' \rangle$$

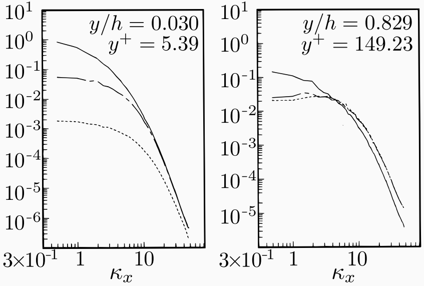


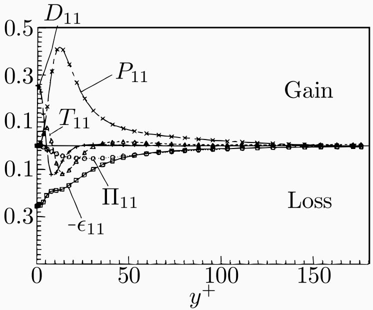
Fluxes in physical space

Are these viewpoints really alternative in a realistic flow?

Typical analysis of a DNS database of turbulent plane channel flow

Fluxes in scale space or physical space



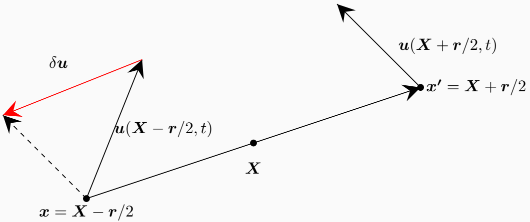


The Generalized Kolmogorov Equation (GKE)

- Equation for the scale energy $\langle \delta u_i \delta u_i \rangle$
- Introduced by Hill (JFM 2000, 2001)
- Several other contributions
- Partially or incorrectly written until Gatti et al. (JoT 2019)
- Large computational cost!!

$\delta u_i(\boldsymbol{X}, \boldsymbol{r}, t) =$
$\left(u_i\left(\boldsymbol{X} + \frac{\boldsymbol{r}}{2}, t\right) - u_i\left(\boldsymbol{X} - \frac{\boldsymbol{r}}{2}, t\right)\right)$

Seven independent variables: $t, \boldsymbol{X} = \frac{\boldsymbol{x}+\boldsymbol{x}'}{2}$ and $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$



Adding anisotropy: the AGKE

The AGKE are exact budget equation for the tensor $\langle \delta u_i \delta u_j \rangle$, with

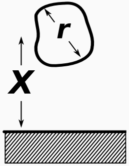
$$\langle \delta u_i \delta u_i \rangle = tr \begin{bmatrix} \langle \delta u \delta u \rangle & \langle \delta u \delta v \rangle & \langle \delta u \delta w \rangle \\ & \langle \delta v \delta v \rangle & \langle \delta v \delta w \rangle \\ sym & & \langle \delta w \delta w \rangle \end{bmatrix}$$

The AGKE have been introduced in:

Gatti, Chiarini, Cimarelli & Quadrio, *Structure function tensor equations in inhomogeneous turbulence*, J. Fluid Mech. **898** A5, 2020

Interpretation

- $\langle \delta u_i \delta u_j \rangle(\boldsymbol{X}, \boldsymbol{r})$ is the amount of turbulent stresses at location $\boldsymbol{X}$ and scale (up to) $\boldsymbol{r}$
- AGKE describe production, transport and dissipation of $\langle \delta u_i \delta u_j \rangle$ in both space of scales & physical space
- Post-processing tool (closure...)



Advantages

- Ultimate tool for second-order statistics
- Unequivocal definition of scale in inhomogeneous directions
- Substantial advantage over spectral analysis

AGKE: budget equation for the structure function tensor

$$\frac{\partial \langle u_i u_j \rangle}{\partial t} + \frac{\partial \phi_{ij,k}}{\partial r_k} + \frac{\partial \psi_{ij,k}}{\partial X_k} = \xi_{ij}$$

- $\phi_{ij,k}$ and $\psi_{ij,k}$: components in the space of scales r_k and in the physical space X_k of a 6-dimensional vector field of fluxes
- ξ_{ij}: source term

The fluxes

$$\phi_{ij,k} = \underbrace{\langle \delta U_k \delta u_i \delta u_j \rangle}_{\text{mean transport}} + \underbrace{\langle \delta u_k \delta u_i \delta u_j \rangle}_{\text{turbulent transport}} \underbrace{-2\nu \frac{\partial}{\partial r_k} \langle \delta u_i \delta u_j \rangle}_{\text{viscous diffusion}} \quad k = 1, 2, 3$$

$$\psi_{ij,k} = \underbrace{\langle U_k^* \delta u_i \delta u_j \rangle}_{\text{mean transport}} + \underbrace{\langle u_k^* \delta u_i \delta u_j \rangle}_{\text{turbulent transport}} \underbrace{+ \frac{1}{\rho} \langle \delta p \delta u_i \rangle \delta_{kj} + \frac{1}{\rho} \langle \delta p \delta u_j \rangle \delta_{ki}}_{\text{pressure transport}} \underbrace{- \frac{\nu}{2} \frac{\partial}{\partial X_k} \langle \delta u_i \delta u_j \rangle}_{\text{viscous diffusion}} \quad k = 1, 2,$$

The source

$$\xi_{ij} = \underbrace{-\langle u_k^* \delta u_j \rangle \delta\left(\frac{\partial U_i}{\partial x_k}\right) - \langle u_k^* \delta u_i \rangle \delta\left(\frac{\partial U_j}{\partial x_k}\right)}_{\text{production } (P_{ij})} +$$

$$\underbrace{-\langle \delta u_k \delta u_j \rangle \left(\frac{\partial U_i}{\partial x_k}\right)^* - \langle \delta u_k \delta u_i \rangle \left(\frac{\partial U_j}{\partial x_k}\right)^*}_{\text{production } (P_{ij})} +$$

$$\underbrace{+\frac{1}{\rho}\left\langle \delta p \frac{\partial \delta u_i}{\partial X_j} \right\rangle + \frac{1}{\rho}\left\langle \delta p \frac{\partial \delta u_j}{\partial X_i} \right\rangle}_{\text{pressure strain } (\Pi_{ij})} \underbrace{-4\epsilon_{ij}^*}_{\text{ps.dissipation } (D_{ij})}$$

Computing the AGKE

A tough computational endeavor

Purely post-processing, but:

- very costly (more than producing the database itself)
- difficult to validate and interpret

Source code is available at: `www.github.com/davecats`

Gatti, Remigi, Chiarini, Cimarelli & Quadrio, *An efficient numerical method for the generalised Kolmogorov equation*, J. Turbulence 2019

Our computational approach

- Products instead of correlations (if possible)
- Symmetries fully exploited, optimizations
- Three-steps sequence
- Three parallel strategies available
- Undersampling, selection of VoI

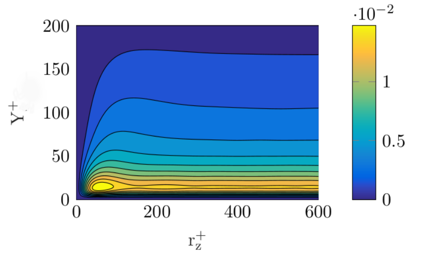
Using the AGKE

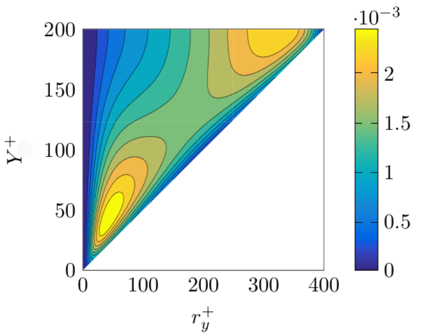
Ex.1) The starting point: low-Re turbulent channel flow

- DNS database of a turbulent channel flow at $Re_\tau = 200$
- Known physics, useful to "learn" the AGKE language
- 4-dimensional analysis, only one inhomogeneous direction

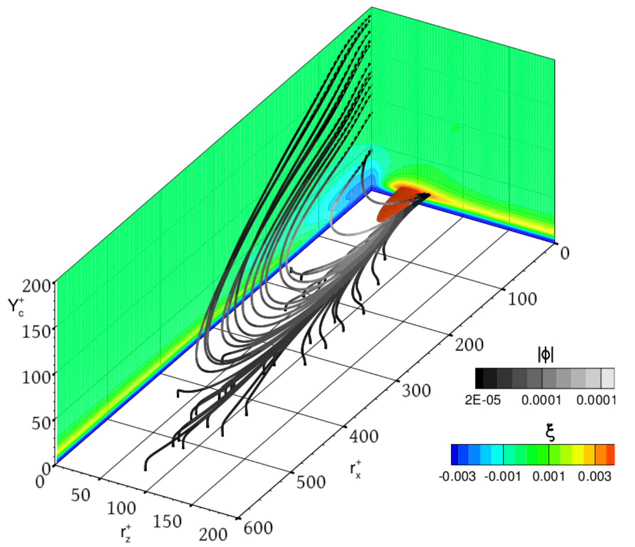
Low-Re Poiseuille: scale energy

Left: $\langle \delta u \delta u \rangle$. Right: $\langle \delta w \delta w \rangle$ (it must be $r_y/2 \leq Y$!!)



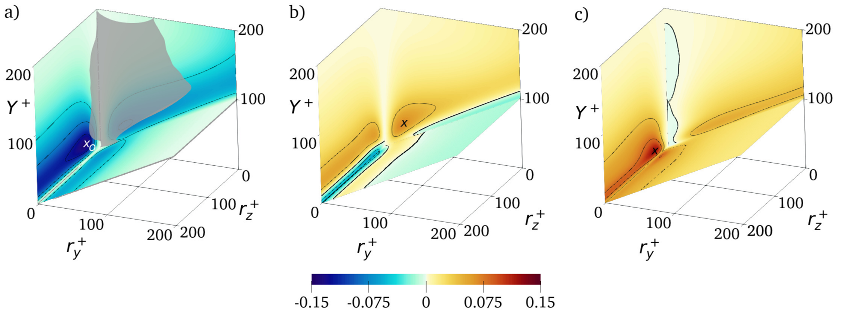


Low-Re Poiseuille: source and fluxes



Low-Re Poiseuille: redistribution by pressure strain

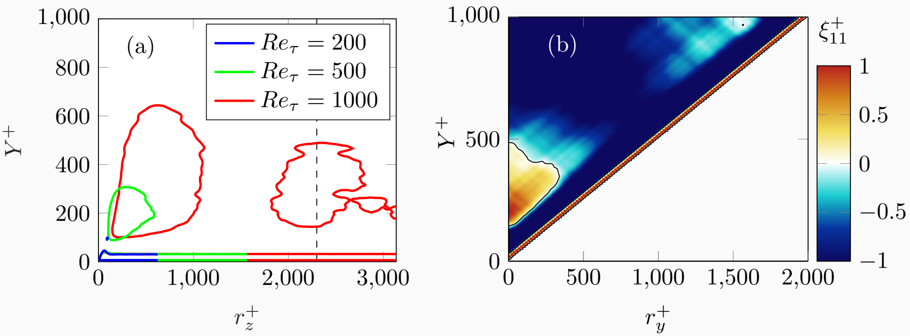
Π_{11}, Π_{22} and Π_{33} (with splatting)



Ex.2) Higher-Re Poiseuille

- DNS database of a turbulent channel flow at $Re_\tau = 1000$
- Useful to investigate the interaction between inner cycle and outer cycle
- Better tool than energy spectrum, and also more sensitive

Contours of $\xi_{11} = 0$: at $Re_\tau = 1000$ both attached structures and superstructures appear

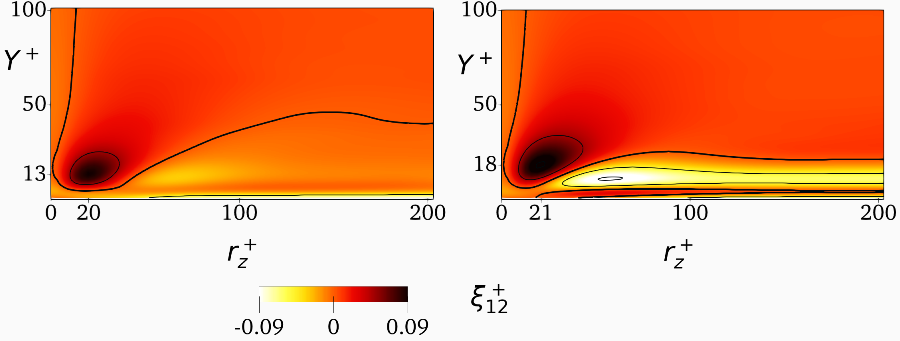


Ex.3) More physics: the spanwise-oscillating wall

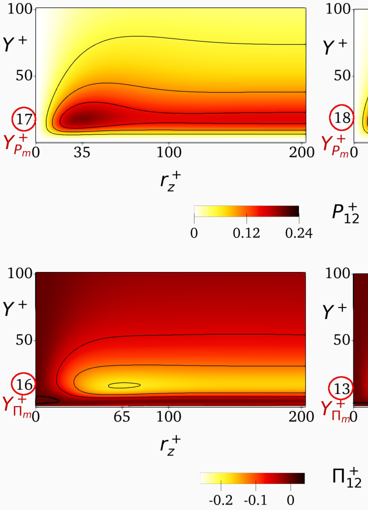
- Wall-bounded flow modified by skin-friction drag reduction (spanwise oscillating wall)
- DNS database at $Re_\tau = 200$
- More complex physics

Channel flow with drag reduction

Source of $-\langle \delta u \delta v \rangle$, REF vs SOW



Identifying the changes

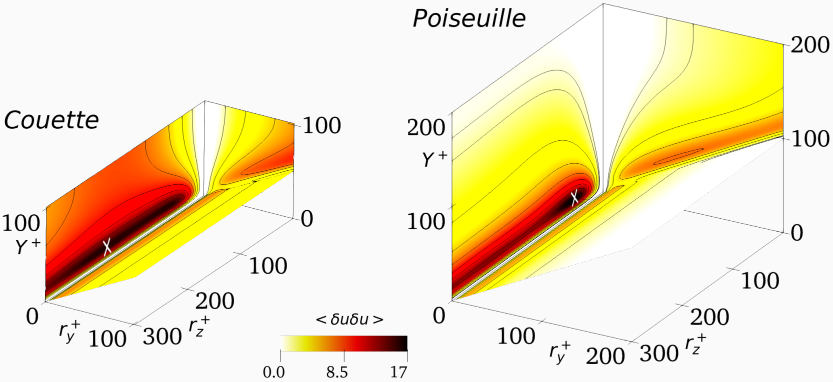


$$\xi_{12} = P_{12} + \Pi_{12} + \epsilon_{12} \approx P_{12} + \Pi_{12}$$

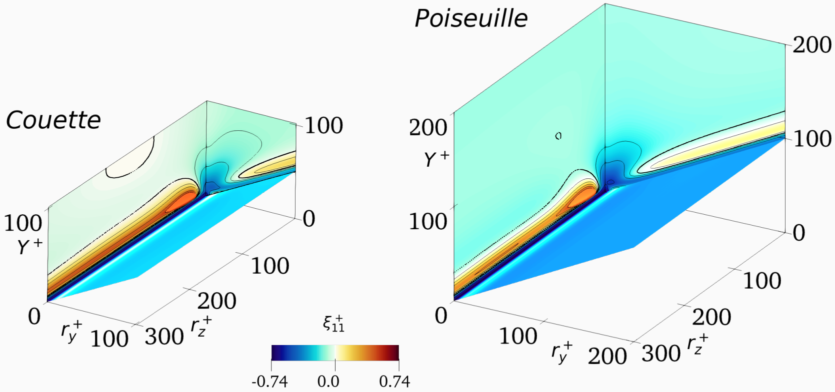
Ex.4) Other flows: turbulent Couette

- Turbulent Couette flow at $Re_\tau = 100$
- Large streamwise rolls
- Understanding the inner/outer interaction
- Ascending-descending and direct-inverse cascades of Reynolds stresses

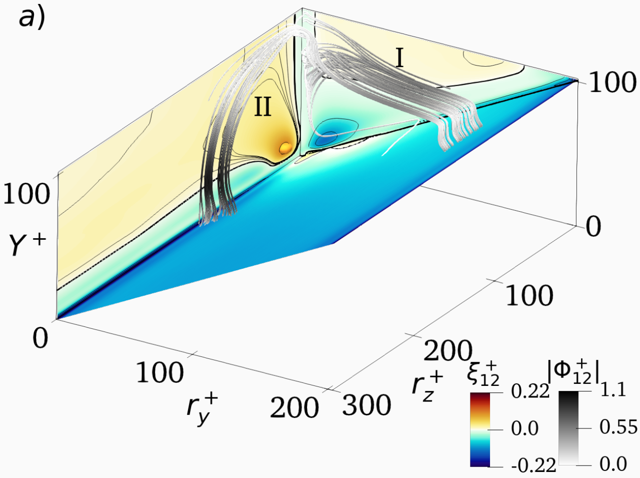
Couette vs Poiseuille



Couette vs Poiseuille



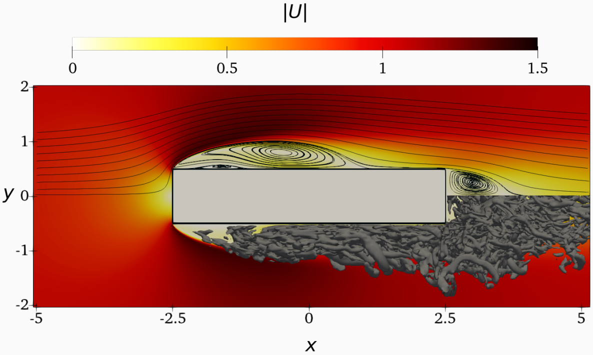
Shear stresses in Couette



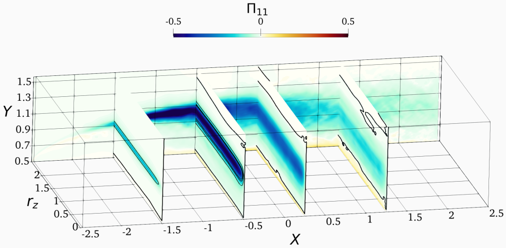
Ex.5) Complex flows: the BARC

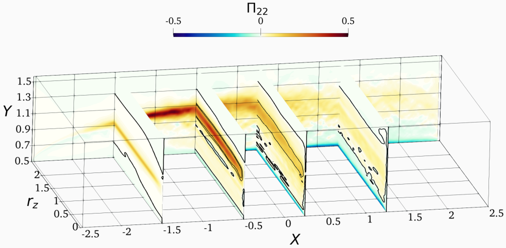
- Benchmark case with several open issues
- Only one homogeneous direction
- Rich physics: laminar separation, reattachment, turbulent boundary layer, 3 separation bubbles, one reverse boundary layer, a turbulent wake...

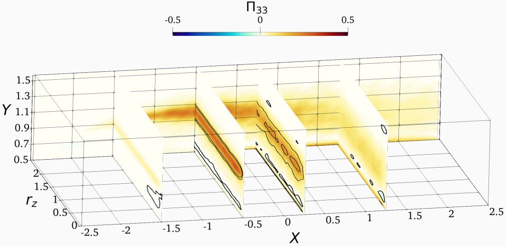
The flow



Redistribution by pressure-strain



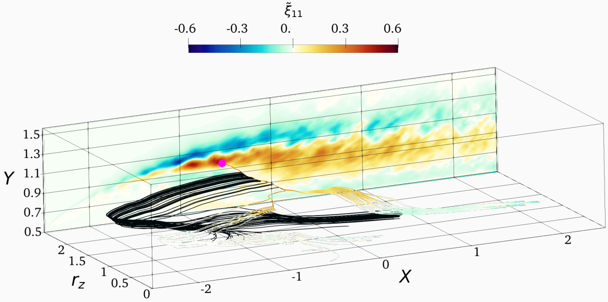




- Difference between outer forward and inner reverse flow
- Splatting in Π_{22}
- In reverse flow Π_{33} larger than Π_{11}

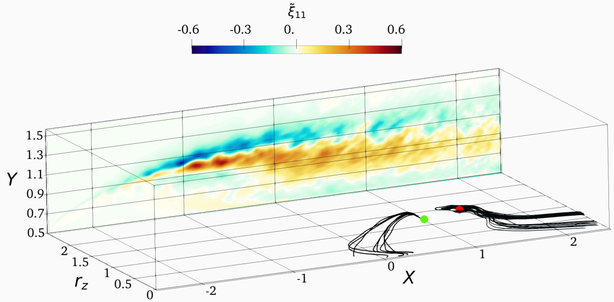
Multiple scales

Extended source!



- Large scale: purple dot at $(X, Y, r_z) = (-0.5, 1, 2.4)$
- Scale of the spanwise modulation of the rolls before breakup

Multiple scales

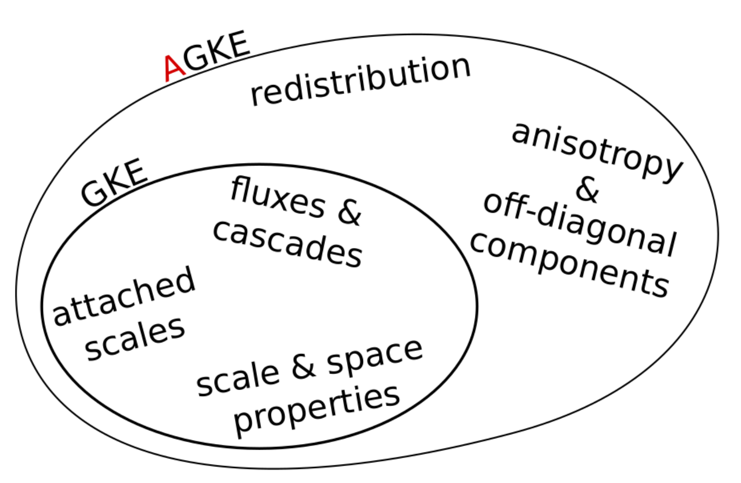


- Small scale: green dot at $(X, Y, r_z) = (0.45, 0.8, 0)$
- Inverse + direct cascade, streamwise vortices
- Small scale: red dot at $(X, Y, r_z) = (1, 0.8, 0.35)$

Wrapping up

- AGKE are a powerful tool
- Extension to increasingly complex flows
- Ongoing work: AGKE with triple-decomposition
- Extract modeling (LES?) information

AGKE in a nutshell



The Facepage



Contributions from several Master students:

- Alberto Remigi
  (code layout, validation)
- Marco Villani
  (higher-*Re* Poiseuille)
- Mariadebora Mauriello
  (Couette)
- Federica Gattere
  (drag reduction, triple decomposition)