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C^*-Non-Linear Second Quantization

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Abstract. We construct an inductive system of C^*-algebras each of which is isomorphic to a finite tensor product of copies of the one-mode n-th degree polynomial extension of the usual Weyl algebra constructed in our previous paper (Accardi and Dhahri in Open Syst Inf Dyn 22(3):1550001, 2015). We prove that the inductive limit C^*-algebra is factorizable and has a natural localization given by a family of C^*-sub-algebras each of which is localized on a bounded Borel subset of $\mathbb{R}$. Finally, we prove that the corresponding family of Fock states, defined on the inductive family of C^*-algebras, is projective if and only if $n = 1$. This is a weak form of the no-go theorems which emerge in the study of representations of current algebras over Lie algebras.

1. Introduction: The C^*-Non-Linear Quantization Program

The present paper is a contribution to the program of constructing a theory of renormalized higher powers of quantum white noise (RPWN) or equivalently of non-relativistic free Boson fields.

This program has an old history, but the approach discussed here started in 1999 with the construction of the Fock representation for the renormalized square of Boson white noise [2]. This result motivated a large number of papers extending it in different directions and exhibiting connections with almost all fields of mathematics, see for example [22] for the case of free white noise, [1] for the connection with infinite divisibility and for the identification of the vacuum distributions of the generalized fields with the three nonstandard Meixner classes, [3] and [21] for finite temperature representations, [9] for the construction of the Fock functor, the survey [5] and the paper [6] for the connections with conformal field theory and with the Virasoro–Zamolodchikov hierarchy, [7] for the connections between renormalization and central extensions.

The problem is the following. One starts with the Schrödinger representation of the Heisenberg real $*$-Lie algebra $heis_{\mathbb{R}}(1,1)$ with skew-adjoint generators iq (imaginary unit times position), $-ip$ ($-i$ times momentum), $E := i1$

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(i times central element) and relations $[iq, -ip] = i1$ (we use these notations to make a bridge between the abstract notations used in Lie algebra theory and and those commonly used in physics). By a $*$-*Lie algebra*, we mean a Lie algebra with an involution, denoted $*$, compatible with the Lie brackets in the sense that for any pair of elements a, b, of the Lie algebra, one has

$$[a, b]^* = [b^*, a^*]$$

The universal enveloping algebra of $heis_{\mathbb{R}}(1, 1)$ called for brevity the *full oscillator algebra* (FOA) has a natural structure of $*$-Lie algebra and can be identified with the algebra of differential operators in one real variable with complex polynomial coefficients.

The continuous analog of the Heisenberg Lie algebra is the non-relativistic free boson field algebra, also called the current algebra over $\mathbb{R}$ of the Heisenberg algebra, whose only non-zero commutation relations are, in the sense of operator-valued distributions:

$$[q_s, p_t] = \delta(s - t)1; \quad s, t \in \mathbb{R}$$

The notion of current algebra has been generalized from the Heisenberg algebra to more general $*$-Lie algebras (see Araki's paper [13] for a mathematical treatment and additional references): in this case, the self-adjoint generators of the Cartan sub-algebras are called generalized fields.

Notice that the definition of current algebra of a given Lie algebra is independent of any representation of this algebra, i.e., it does not require to fix a priori a class of states on this algebra.

One can speak of $*$-*Lie algebra second quantization* to denote the transition from the construction of unitary representations of a $*$-Lie algebra to the construction of unitary representations of its current algebra over a measurable space (typically $\mathbb{R}$ with its Borel structure).

Contrarily to the discrete case, the universal enveloping algebra of the current algebra over $\mathbb{R}$ of the Heisenberg algebra is ill defined because of the emergence of higher powers of the δ-function. This is the mathematical counterpart of the old problem of defining powers of local quantum fields.

Any rule that, giving a meaning to these powers, defines a $*$-Lie algebra structure is called a renormalization procedure. The survey [5] describes two inequivalent renormalization procedures and the more recent paper [7] shows the connection between them.

The second step of the program, after renormalization, is the construction of unitary representations of the resulting $*$-Lie algebra. This step, which is the most difficult one because of the no-go theorems (see discussion below), is usually done by fixing a state and considering the associated cyclic representation. At the moment, even in the first-order case, i.e., for usual fields, the explicitly constructed representations are not many, they are essentially reduced to gaussian (quasi-free) representations. Moreover, any gaussian representation can be obtained, by means of a standard construction, from the Fock representation which is characterized by the property that the cyclic vector, called vacuum, is in the kernel of the annihilation operators.

This property has been taken as an heuristic principle to define the notion of *Fock state* also in the higher order situations (see [5] for a precise definition).

It can be proved that, for all renormalization procedures considered up to now, the Fock representation and the Fock state are factorizable, in the sense of Araki and Woods [12]. This property poses an obstruction to the existence of such representation, namely that the restriction of the Fock state on any factorizable Cartan sub-algebra must give rise to a classical infinitely divisible process. If this is not the case then *no Fock representation, and more generally no cyclic representation associated to a factorizable state, can exist.*

When this is the case we say that a no-go theorem holds. Nowadays, several instances of no-go theorems are available. The simplest, and probably most illuminating one, concerns the Schrödinger algebra, which is the Lie algebra generated by the powers ≤ 2 of p and q (see [22], also [1] and [8] for stronger results). This result implies that there is *no natural analog of the Fock representation for the current algebra over* $\mathbb{R}$ *(for any* $d \in \mathbb{N}$*) of the FOA.*

On the other side, we know (see [1,2] and the above discussion) that *for some sub-algebras of current algebras of the FOA such a representation exists.* This naturally rises the problem to characterize these sub-algebras.

Since a full characterization at the moment is not available, a natural intermediate step towards such a characterization is to produce non-trivial examples.

To this goal a family of natural candidates is provided by the $*$-Lie sub-algebras of the FOA consisting of the real linear combinations of the derivation operator and the polynomials of degree less or equal than a fixed natural integer n. Thus, the generic element of such an algebra has the form

$$up + P(q); \quad u \in \mathbb{R}$$

where P is a polynomial of degree less or equal than n and with real coefficients. For $n = 1$, one finds the Heisenberg algebra; for $n = 2$ the Galilei algebra and for $n > 2$, some nilpotent Lie algebras discussed in mathematics [14–16,19] but up to now, with the notable exception of the Galilei algebra ($n = 2$), not considered in physics.

These $*$-Lie algebras enjoy two very special properties:

(i) no renormalization is required in the definition of the associated current algebra over $\mathbb{R}$;

(ii) in the Schrödinger representation of the FOA the skew-adjoint elements of these sub-algebras can be explicitly exponentiated giving rise to a non-linear generalization of the Weyl relations and of the corresponding Heisenberg group. This was done in the paper [10].

Property (i) supports the hope of the existence of the Fock representation for the above-mentioned current algebra. A direct proof of this fact could be obtained by proving the infinite divisibility of all the vacuum characteristic functions of the generalized fields. Unfortunately, even in the case $n = 2$, in which this function can be explicitly calculated, a direct proof of infinite divisibility can be obtained only for a subset of the parameters which define the generalized fields, but not for all, and this problem is challenging the experts of infinite divisibility since several years. More precisely, the situation with the vacuum distributions of the generalized field operators is presently the following. It is proved, in [11], and extended in [10] with a different technique of proof, that the characteristic function of the generalized fields

$$A(\sqrt{2}q)^2 + B(\sqrt{2}q) + C(\sqrt{2}p), \quad A, B, C \in \mathbb{R} \tag{1}$$

with respect to the vacuum vector is given by

$$(1-2itA)^{-\frac{1}{2}} e^{\frac{4C^2(A^2t^4+2iAt^3)-3|M|^2t^2}{6(1-2iAt)}}, \quad M = B + iC \tag{2}$$

Then, we have the following situation:

- the case $n = 1$, corresponding to $A = 0$ in (1), is reduced to standard, i.e., linear, quantization. Hence, all non-trivial distributions in (2) ($|M| \neq 0$) are Gaussian.
- For $n = 2$, the 3-real-parameter family (2) interpolates between the Gaussian distribution, corresponding to $A = 0$, $|M| \neq 0$ (i.e., the previous case) and the Gamma distribution, corresponding to $A \neq 0$ and $B = C = 0$. On the structure of the interpolating distributions little is known at the moment. For example, Accardi et al. [11] have proved that if $B = 0$ the characteristic functions (2) are infinitely divisible, but the case $B \neq 0$ is an open challenge since several years.
- For $n \geq 3$, the generalized fields are an $(n+1)$-real-parameter family of real-valued random variables the form

$$A_n(\sqrt{2}q)^n + A_{n-1}(\sqrt{2}q)^{n-1} + \cdots + A_1(\sqrt{2}q) + A_0(\sqrt{2}p), \quad A_n, \ldots, A_0 \in \mathbb{R} \tag{3}$$

  and practically nothing is known on their vacuum distributions except for very special values of the parameters (corresponding to powers of the standard Gaussian).

In the present paper, we exploit property (ii) and the following heuristic considerations are aimed at making a bridge between the mathematical construction below and its potential physical interpretation.

Our goal is to construct a C^*-algebra whose generators can be *naturally* identified with the following formal expressions that we call the *non-linear Weyl operators*:

$$e^{\sum_{j=0}^{n+1} L_j(f_j)} \tag{4}$$

The formal generators of the non-linear Weyl operators (called *non-linear fields*) are heuristically expressed as *powers of the standard quantum white noise (or free Boson field)*, i.e., the pair of operator-valued distributions q_t, p_t with commutation relations

$$[q_s, p_t] = i\delta(t-s)$$

in the following way:

$$L_{n+1}(f_{n+1}) := ip(f_{n+1}) = \int_{\mathbb{R}} f_{n+1}(t) i p_t \mathrm{d}t;$$

$$L_0(f) := \int_{\mathbb{R}} f_0(t) i q_t^0 \mathrm{d}t := i1 \cdot \int f_0(t) \mathrm{d}t$$

$$L_k(f_k) := ikq^k(f_k) = \int_{\mathbb{R}} f_k(t) i q_t^k \mathrm{d}t; \quad k \in \{1, \ldots, n\} \tag{5}$$

For $n > 1$, the expressions in (5) are ill defined because q_t^k is the k-th power of the standard white noise, that is a distribution-valued process. Our goal is to give a meaning these expressions. When $n = 1$, the expressions (5) are well understood and various forms of second quantization are known. For example, one can prove the unitarity of the Fock representation, i.e., the exponentiability, inside it, of the generators (5) and deduce the commutation relations satisfied by these exponentials. A different example, for $n = 1$, is provided by Weyl second quantization: in it, by heuristic calculations, one guesses the commutation relations that should be satisfied by any representation of the exponentials (4) and then one proves the existence of a C^*-algebra which realizes these commutation relations.

In the present paper, we apply this approach to give a meaning to the exponentials (4) when the test functions f_j ($j \in \{0, 1, \ldots, n+1\}$) are step functions with a finite range. To this goal we exploit the fact that, if π is a finite Borel partition of a bounded Borel subset I of $\mathbb{R}$, then there is a natural way to give a meaning to the generalized Weyl algebra with test functions constant on the sets of π. This is based on the identification of this algebra with the tensor product of $|\pi|$ (cardinality of π) copies of the one-mode generalized Weyl algebra (see Sect. 8). Intuitively, if

$$f_j = \sum_{J \in \pi} a_{j,J} \chi_J$$

where for any bounded Borel set $J \subset \mathbb{R}$

$$\chi_J(x) := \begin{cases} 1 & \text{if } x \in J \\ 0 & \text{if } x \notin J \end{cases} \tag{6}$$

then

$$\begin{aligned} e^{\int_{\mathbb{R}} \sum_{j=0}^{n+1} f_j(t) q_t^j dt} &= e^{\sum_{j=0}^{n+1} L_j(f_j)} = e^{\sum_{j=0}^{n+1} \sum_{J \in \pi} a_{j,J} L_j(\chi_J)} \\ &\equiv \bigotimes_{J \in \pi} e^{\sum_{j=0}^{n+1} a_{j,J} L_j(\chi_J)} \end{aligned} \tag{7}$$

where the $L_j(\chi_J)$ ($J \in \pi$) are identified to *rescaled copies* of the generators of the one-mode generalized Weyl algebra. This identification strongly depends on the specific structure of the Lie algebra considered (see Sect. 6 below).

Using this, we construct an inductive system of C^*-algebras each of which is isomorphic to a finite tensor product of copies of the one-mode generalized Weyl algebra *but the embeddings defining the inductive system are not the usual tensor product embeddings because they depend on the above-mentioned identifications*. The identities (7) provide an intuitive bridge between our inductive construction and our goal to give a meaning to the formal expressions (5). Once a representation with *good* continuity properties is obtained, one can extend the test function space to more general functions thus obtaining *non-linear white noise integrals*.

The C^*-algebra, obtained as inductive limit from the above construction, is naturally interpreted as a C^**-quantization*, of the current algebra over $\mathbb{R}$, of the non-linear Weyl $*$-Lie algebra with test functions given by the finitely valued step functions with compact support. This C^*-algebra has a *localization* given by a family of C^*-sub-algebras, each of which has a natural localization on bounded Borel subset of $\mathbb{R}$. Moreover, this system of local algebras is factorizable in the sense of Definition 8 below.

With this construction, the problem of constructing unitary representations, which guarantee the existence of the generalized fields, of the current algebra over $\mathbb{R}$ of the initial $*$-Lie algebra is reduced to the problem of finding representations of this C^*-algebra with the above-mentioned continuity property. The advantages of the transition from the unbounded formulation of the commutation relations to the bounded one are well known in the case of standard, linear, quantization and our hope is that these advantages could be put to use also in the non-linear case.

The simplest candidate for such a representation would be the (appropriately defined) non-linear generalization of the Fock representation. However, in the last section of the paper, it is shown that although the Fock state is defined (in the usual way) on each of the C^*-algebras of the inductive family, due to the above-mentioned identifications, the corresponding family of states is projective if and only if $n = 1$ (i.e., for the usual Weyl algebra).

This result can be considered as a no-go theorem pointing out in the same direction as the no-go theorems proved in [1,8,22] although quite different, not only because the algebras involved are different, but also in its formulation and in the tools used for its proof. It implies that, in the limit C^*-algebra, there is no factorizable state whose restrictions to the finite dimensional approximating sub-algebras coincide with the corresponding Fock states. On the other hand, experience on all the known examples suggests the factorizability condition as a natural one for the notion of Fock state.

In any case, it should be emphasized that the problem to construct a representation of the non-linear Weyl C^*-algebra, with the above-mentioned continuity properties, is at the moment open.

The basic construction of the present paper can be extended to more general classes of $*$-Lie algebras [for example the C^*-algebras associated to the renormalized square of white noise (RSWN)] and more general spaces (i.e., $\mathbb{R}^d$ instead of $\mathbb{R}$).

2. The 1-Mode n-th Degree Heisenberg $*$-Lie Algebra $heis_{\mathbb{R}}(1, n)$

Definition 1. For $n \in \mathbb{N}^*$, the 1-*mode* n-*th degree Heisenberg algebra*, denoted $heis_{\mathbb{R}}(1, n)$, is the pair

$$\left\{V_{n+2}, (L_j)_{j=0}^{n+1}\right\}$$

where:

– V_{n+2} is a $(n+2)$-dimensional real $*$-Lie algebra;
– $\{L_j\}_{j=0}^{n+1}$ is a skew-adjoint basis of V_{n+2};
– the Lie brackets among the generators are given by

$$[L_i, L_j] = 0; \quad \forall i, j \in \{0, 1, \ldots, n\}$$
$$[L_{n+1}, L_k] = kL_{k-1}; \quad \forall k \in \{1, \ldots, n\}, \quad L_{-1} := 0$$

Remark 1. $(L'_j)_{j=0}^{n+1}$

Denoting $\mathbb{R}_n[X]$ the vector space of polynomials in one indeterminate with real coefficients and degree less than or equal n. Identifying L_n with the power of the indeterminate X

$$X^n \to L_n$$

leads to a (linear) isomorphism $\mathbb{R} \times \mathbb{R}_n[X] \cong \mathbb{R}^{n+2} \to heis_{\mathbb{R}}(1, n)$ defined by the map

$$\mathbb{R}^{n+2} \ni (u, (a_k)_{0 \le k \le n}) \mapsto \ell_0(u, P) \in heis_{\mathbb{R}}(1, n) \tag{8}$$

where

$$\ell_0(u, P) := uL_{n+1} + \sum_{k=0}^{n} a_k L_k := uL_{n+1} + P(L) \equiv (u, (a_k)_{0 \le k \le n})$$

between the algebra $(u, P) \in u \in \mathbb{R} \times P \in \mathbb{R}_n[X]$ and $heis_{\mathbb{R}}(1, n)$.

3. The Schrödinger Representation and the Polynomial Heisenberg Group Heis$(1, n)$

In the 1-dimensional case (more generally, in the finite dimensional case), there is no problem in giving a meaning to the higher powers of the position operator: they are well-defined self-adjoint operators in the Fock representation. In this sense, as explained in the introduction, the 1-dimensional Fock representation will be the building block of the construction that follows.

Let $p, q, 1$ be the usual momentum, position and identity operators acting on the one-mode boson Fock space $\Gamma(\mathbb{C})$

$$\Gamma(\mathbb{C}) \cong L^2(\mathbb{R}) \tag{9}$$

The maximal algebraic domain $\mathcal{D}_{max}$ (see [4]), consisting of the linear combinations of vectors of the form

$$q^n p^k \psi_z; \quad k, n \in \mathbb{N}, \quad z \in \mathbb{C}$$

where ψ_z is the exponential vector associated to $z \in \mathbb{C}$, is a dense subspace of $\Gamma(\mathbb{C})$ invariant under the action of p and q hence of all the polynomials in the two non-commuting variables p and q. In particular, for each $n \in \mathbb{N}$, the real linear span of the set $\{i1, ip, iq, \ldots, iq^n\}$, denoted $heis_{\mathbb{R}}(F, 1, n)$, leaves $\mathcal{D}_{\max}$ invariant. Hence, the commutators of elements of this space are well defined on this domain and one easily verifies that they define a structure of $*$-Lie algebra on $heis_{\mathbb{R}}(F, 1, n)$.

Lemma 1. *In the above notations, the map*

$$L_{n+1} \mapsto ip, \quad L_0 \mapsto i1, \quad L_k \mapsto iq^k; \quad k \in \{1, \dots, n\} \tag{10}$$

admits a unique linear extension from $heis_{\mathbb{R}}(1,n)$ onto $heis_{\mathbb{R}}(F,1,n)$ which is a $*$-Lie algebra isomorphism called the Schrödinger representation of the n-th degree Heisenberg algebra $heis_{\mathbb{R}}(1,n)$.

Proof. The linear space isomorphism property follows from the linear independence of the set $\{1, p, q, \dots, q^n\}$. The $*$-Lie algebra isomorphism property follows from direct computation. $\square$

In [10] (Theorem 1), it is proved that the unitary operators

$$W(u,P) := e^{i(up+P(q))} \in \mathrm{Un}(L^2(\mathbb{R})); \quad (u,P) \in \mathbb{R} \times \mathbb{R}_n[X] \tag{11}$$

satisfy the following polynomial extension of the Weyl relations:

$$W(u,P)W(v,Q) = W((u,P) \circ (v,Q)); \quad \forall (u,P), (v,Q) \in \mathbb{R} \times \mathbb{R}_n[X] \tag{12}$$

where

$$(u,P) \circ (v,Q) := (u+v, T_{u+v}^{-1}(T_u P + T_v S_u Q)) \tag{13}$$

and for any $u, w \in \mathbb{R}$, the linear operators $T_w, S_u : \mathbb{R}_n[X] \to \mathbb{R}_n[X]$ are defined by the following prescriptions:

$$\begin{aligned} T_w 1 &= 1 \\ T_w(X^k) &= \sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} w^{k-h} X^h + X^k; \qquad \forall k \in \{1, \dots, n\} \\ (S_u P)(X) &:= P(X+u) \quad \text{translation operator on } \mathbb{R}_n[X] \end{aligned} \tag{14}$$

Denote

$\mathcal{W}_{F,1,n} :=$ norm closure in $\mathcal{B}(\Gamma(\mathbb{C}))$ of the linear span of the operators (11).

The identity (12) implies that $\mathcal{W}_{F,1,n}$ is a C^*-algebra.

In [10], it is proved that the composition law (13) is a Lie group law on $\mathbb{R} \times \mathbb{R}_n[X]$ whose Lie algebra is $heis_{\mathbb{R}}(1,n)$. Since the elements of $heis_{\mathbb{R}}(1,n)$ are parameterized by the pairs $(u,P) \in \mathbb{R} \times \mathbb{R}_n[X]$, it is natural to introduce the following notation.

Definition 2 (see [10]). The 1-*mode n-th degree Heisenberg group* is the set

$$\mathrm{Heis}(1,n) := \left\{ e^{\ell_0(u,P)} \ : \ (u,P) \in \mathbb{R} \times \mathbb{R}_n[X] \right\} \tag{15}$$

with composition law

$$e^{\ell_0(u,P)} \circ e^{\ell_0(v,Q)} := e^{\ell_0((u+v, T_{u+v}^{-1}(T_u P + T_v S_u Q)))}$$

The name $\mathrm{Heis}(1,n)$ is motivated by the fact that, for $n=1$, $\mathrm{Heis}(1,n)$ reduces to the usual the 1-mode Heisenberg group.

4. The Free Group-C^*-Algebra of Heis$(1, n)$

There are many well-known ways to associate a $*$-algebra (C^*-, Banach, ...,) to a group. We want to construct a C^*-algebra, canonically generated by the elements of the group Heis$(1, n)$ and in which these elements are linearly independent (this is used in Corollary 1 below). In the case $n = 1$, this property holds in the Fock representation. For $n > 1$, we conjecture that this property is still valid, but at the moment no proof is available. Therefore, we introduce a more abstract C^*-algebra, obtained by extending to the non-linear Weyl operators a construction used by Petz ([20], p. 14) for the usual Weyl algebra (see [17] for the origins of this construction). In particular, we too refer to [18] Ch. IV, Sect. 18.3 for the notion of *minimal regular norm*.

Definition 3. Let G be a group. The free complex vector space generated by the set

$$\{W_g \ : \ g \in G\}$$

has a unique structure of unital $*$-algebra, denoted $\mathcal{W}^0(G)$ and defined by the prescription that the map $g \mapsto W_g$ defines a unitary representation of G onto $\mathcal{W}^0(G)$, equivalently:

$$\begin{aligned} W_g W_h &:= W_{gh}; \qquad g, h \in G \\ (W_g)^* &:= W_{g^{-1}}; \qquad g \in G \\ 1 &:= W_e \end{aligned} \tag{16}$$

The completion of $\mathcal{W}^0(G)$ under the (minimal regular) C^*-norm

$$\|x\| := \sup\{\|\pi(x)\| \ : \ \pi \in \{*\text{-representations of } G\}\}; \quad x \in \mathcal{W}^0(G)$$

(where $*$-*representation* means weakly continuous $*$-homomorphism of $\mathcal{W}^0(G)$ into the bounded operators of some separable Hilbert space) will be called the *free group—C^*-algebra of G* and denoted $\mathcal{W}(G)$. Experience shows that it is worth to emphasize that the term *free* is referred to *algebra* and not to *group*.

Remark 2. Because of (16), a $*$-representation of $\mathcal{W}(G)$ maps the generators W_g ($g \in G$), into unitary operators.

Remark 3. If G, G' are groups, then any group homomorphism (resp. isomorphism) $\alpha \ : \ G \to G'$, extends uniquely to a C^*-algebra homomorphism (resp. isomorphism) $\tilde{\alpha} \ : \ \mathcal{W}(G) \to \mathcal{W}(G')$ characterized by the condition

$$\tilde{\alpha}(W_g) := W_{\alpha g}; \quad g \in G$$

Definition 4. If $G = \text{Heis}(1, n)$, its free group—C^*-algebra is called the 1-*mode n-th degree Weyl algebra* and denoted

$$\mathcal{W}^0_{1,n} := \mathcal{W}^0(\text{Heis}(1, n)) \tag{17}$$

For its generators, called the 1-*mode n-th degree Weyl operators*, we will use the notation

$$W^0(u, P) := W_{e^{\ell_0(u,P)}}; \quad (u, P) \in \mathbb{R} \times \mathbb{R}_n[X] \tag{18}$$

By construction, the map

$$u_F : W^0(u,P) \in \mathcal{W}^0_{1,n} \mapsto W(u,P) \in \mathcal{W}_{F,1,n} \tag{19}$$

where the operators $W(u,P)$ are those defined in (11), is a group isomorphism. Hence the definition of free group-C^*-algebra implies that it can be extended to a surjective $*$-representation called *the Fock representation of* $\mathcal{W}^0_{1,n}$. We will use the same symbol u_F for this extension.

We conjecture that, in analogy with the case $n = 1$, the $*$-homomorphism of $\mathcal{W}^0_{1,n}$ onto $\mathcal{W}_{F,1,n}$ is in fact an isomorphism and that there is a unique C^*-norm on $\mathcal{W}^0_{1,n}$.

5. The Current Algebra of $heis_{\mathbb{R}}(1, n)$ Over $\mathbb{R}$

Denote

$$\mathcal{H}_0(\mathbb{R}) := L^1_{\mathbb{R}}(\mathbb{R}) \cap L^\infty_{\mathbb{R}}(\mathbb{R}) = \bigcap_{1\le p\le\infty} L^p_{\mathbb{R}}(\mathbb{R})$$

$\mathcal{H}_0(\mathbb{R})$ has a natural structure of real pre-Hilbert algebra with the pointwise operations and the L^2-scalar product.

Lemma 2. *For any $*$-sub-algebra $\mathcal{T}$ of $\mathcal{H}_0(\mathbb{R})$ and $n \in \mathbb{N}$, there exists a unique real $*$-Lie algebra with skew-adjoint generators*

$$\{L_0\ ,\ L_k(f)\ :\ k \in \{1, \dots, n+1\};\quad f \in \mathcal{T}\}$$

where, with the notation

$$L_0(f) := L_0 \int_{\mathbb{R}} f(t)dt; \quad L_{-1}(f) = 0; \quad \forall f \in \mathcal{T} \tag{20}$$

the maps $f \mapsto L_k(f)$ $(k \in \{0, 1, \dots, n\})$ are real linear on $\mathcal{T}$ and the Lie brackets are given, for all $f, g \in \mathcal{T}$, by

$$[L_i(f), L_j(g)] = 0; \quad i, j \in \{0, 1, \dots, n\} \tag{21}$$

$$[L_{n+1}(f), L_k(g)] = kL_{k-1}(fg); \quad k \in \{0, 1, 2, \dots, n\}, \quad L_{-1}(f) = 0 \tag{22}$$

Proof. By definition, the Lie brackets of two generators defined by (21), (22) are a multiple of the generators. To verify that the Jacobi identity is satisfied notice that, for any $i, j, k \in \{0, 1, \dots, n\}$

$$[L_i(f_1), [L_j(f_2), L_k(f_3)]] = 0$$

unless exactly 2 among the indices i, j, k are equal to $n + 1$. Moreover, up to change of sign one can assume that $i = j = n + 1$. In this case, one verifies that

$$\begin{aligned}
&[L_{n+1}(f_1), [L_{n+1}(f_2), L_k(f_3)]] = k(k-1)L_{k-2}(f_1f_2f_3)\\
&[L_{n+1}(f_2), [L_k(f_3), L_{n+1}(f_1)]] = -k(k-1)L_{k-2}(f_1f_2f_3)\\
&[L_k(f_3), [L_{n+1}(f_1), L_{n+1}(f_2)]] = 0
\end{aligned}$$

and adding these identities side by side the Jacobi identity follows. $\square$

Definition 5. The real $*$-Lie algebra defined in Lemma 2 will be denoted $heis_{\mathbb{R}}(1,n,\mathcal{T})$. If $I \subset \mathbb{R}$ is a bounded Borel subset, we denote

$$\mathcal{T}_I := \text{the sub-algebra of } \mathcal{T} \text{ of functions with support in } I \tag{23}$$

In analogy with the notation (8), we write the generic element of $heis_{\mathbb{R}}(1,n,\mathcal{T})$ in the form

$$\ell(\tilde{f}) := L_{n+1}(f_{n+1}) + \sum_{k=0}^{n} L_k(f_k); \quad f_0,\dots,f_{n+1} \in \mathcal{T} \tag{24}$$

where, here and in the following, if $(f_0,\dots,f_{n+1})$ is an ordered $(n+2)$-tuple of elements of $\mathcal{T}$, we will use the notation

$$\tilde{f} := (f_0,\dots,f_{n+1}) \tag{25}$$

6. Isomorphisms Between the Current Algebras $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$ and $heis_{\mathbb{R}}(1,n)$

In the notations of the previous section, of Definition 5 and (6), for a bounded Borel subset I of $\mathbb{R}$, we define

$$\mathbb{R}\chi_I := \{\text{the real algebra of multiples of } \chi_I\}$$

Thus,

$$heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I) \subset heis_{\mathbb{R}}(1,n,\mathcal{H}_0(\mathbb{R}))$$

is the $*$-Lie sub-algebra of $heis_{\mathbb{R}}(1,n,\mathcal{H}_0(\mathbb{R}))$ with linear skew-adjoint generators

$$\{L_k(\chi_I): \quad k \in \{0,1,\dots,n\}\}$$

and brackets

$$[L_{n+1}(\chi_I), L_k(\chi_I)] = kL_{k-1}(\chi_I); \quad k \in \{0\} \cup \{2,\dots,n\} \tag{26}$$

for $k \in \{2,\dots,n\}$ and the other commutators vanish. Recalling the notation (20), one must have

$$L_0(\chi_I) = |I| L_0$$

Lemma 3. *In the notations of Sect. 3, a real linear map $\hat{s}_I : heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I) \to heis_{\mathbb{R}}(F,1,n)$ satisfying for some constants $a_I, b_I, c_{k,I} \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$ and for each $k \in \{1,\dots,n\}$*

$$\hat{s}_I(L_0) = a_I i1 \tag{27}$$

$$\hat{s}_I(L_{n+1}(\chi_I)) = b_I ip \tag{28}$$

$$\hat{s}_I(L_k(\chi_I)) = c_{k,I} iq^k; \quad \forall k \in \{1,\dots,n\} \tag{29}$$

is a real $*$-Lie algebra isomorphism if and only if

$$c_{k,I} = b_I^{-k} |I| a_I; \quad \forall k \in \{1,\dots,n\} \tag{30}$$

The additional condition

$$c_{1,I} = b_I \tag{31}$$

implies that a_I *must be* >0 *and*

$$c_{k,I} = |I|^{1-\frac{k}{2}} a_I^{1-\frac{k}{2}}; \quad \forall k \in \{1, \dots, n\} \tag{32}$$

Remark 4. In the above statement, $heis_{\mathbb{R}}(F,1,n)$ can be replaced by $heis_{\mathbb{R}}(1,n)$ because of the real $*$-Lie algebra isomorphism between the two.

Proof. By definition, $\hat{s}_I$ maps a basis of $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$ into a basis of $heis_{\mathbb{R}}(F,1,n)$ because the constants $b_I, c_{k,I}$ are non-zero hence it defines a unique vector space isomorphism which is a $*$-map because the constants are real. Moreover, (27), (28), and (29) imply that

$$[\hat{s}_I(L_{n+1}(\chi_I)), \hat{s}_I(L_1(\chi_I))] = [b_I ip, c_{1,I} iq] = b_I c_{1,I}[ip, iq] = b_I c_{1,I} i1$$

while (26) and (29) imply that

$$\hat{s}_I([L_{n+1}(\chi_I), L_1(\chi_I)]) = \hat{s}_I(|I|L_0) = |I|\hat{s}_I(L_0) = |I|a_I i1$$

The isomorphism condition then implies that

$$b_I c_{1,I} = |I|a_I \tag{33}$$

The same argument, using (26), shows that for all $k \in \{2, \dots, n\}$

$$[\hat{s}_I(L_{n+1}(\chi_I)), \hat{s}_I(L_k(\chi_I))] = [b_I ip, c_{k,I} iq^k] = b_I c_{k,I}[ip, iq^k] = b_I c_{k,I} kiq^{k-1}$$
$$\hat{s}_I([L_{n+1}(\chi_I), L_k(\chi_I)]) = \hat{s}_I(kL_{k-1}(\chi_I)) = k\hat{s}_I(L_{k-1}(\chi_I)) = kc_{k-1,I} iq^{k-1}$$

and the isomorphism condition implies that

$$b_I c_{k,I} = c_{k-1,I}$$
$$\Leftrightarrow c_{k,I} = b_I^{-1} c_{k-1,I} = b_I^{-2} c_{k-2,I} = \cdots = b_I^{-(k-1)} c_{1,I} = b_I^{-k}|I|a_I$$

which is (30). Finally, if (31) holds, then (33) becomes

$$b_I^2 = |I|a_I$$

Thus, a_I must be >0 and $b_I = |I|^{1/2} a_I^{1/2}$ which implies (32). □

Remark 5. In the following, we fix condition (31) and put

$$a_I = 1 \tag{34}$$

for all I so that the real $*$-Lie algebras isomorphism $\hat{s}_I$ is given by (27) and (32). Therefore, its inverse $\hat{s}_I^{-1}$ is given, on the generators, by:

$$\hat{s}_I^{-1}(i1) = L_0$$
$$\hat{s}_I^{-1}(ip) = |I|^{-\frac{1}{2}} L_{n+1}(\chi_I)$$
$$\hat{s}_I^{-1}(iq^k) = |I|^{k/2-1} L_k(\chi_I); \quad \forall k \in \{1, \dots, n\}$$

The reason why we introduce the additional conditions (31) and (34) will be explained in Remark 7 at the end of Sect. 9.

Remark 6. Lemma 3 and condition (34) mean that, for any bounded Borel set $I \subset \mathbb{R}$, $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$ can be identified to a copy of $heis_{\mathbb{R}}(1,n)$ with the rescaled basis

$$\{i|I|L_0, \quad i|I|^{\frac{1}{2}} L_{n+1}, \quad i|I|^{1-\frac{k}{2}} L_k, \quad k = 1, \dots, n\} \tag{35}$$

In analogy with (8), we parameterize the elements of $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$, with elements of $\mathbb{R}\times\mathbb{R}_n[X]$, and we write

$$\ell_I(u,P) := uL_{n+1}(\chi_I) + P(L(\chi_I)); \qquad u\in\mathbb{R} \tag{36}$$

where $P := \sum_{j=0}^n a_j X^j$ is a polynomial in one indeterminate and we use the convention

$$P(L(\chi_I)) := \sum_{j=0}^n a_j L_j(\chi_I) := a_0|I|L_0 + \sum_{j=1}^n a_j L_j(\chi_I) \tag{37}$$

The image of such an element under the isomorphism $\hat{s}_I$ is

$$\hat{s}_I(\ell_I(u,P)) = i(u|I|^{\frac{1}{2}}p + P_I(q)) \tag{38}$$

where by definition:

$$P_I(X) := \sum_{j=0}^n a_j|I|^{1-\frac{j}{2}}X^j = a_0|I|1 + \sum_{j=1}^n a_j|I|^{1-\frac{j}{2}}X^j \tag{39}$$

Introducing the linear change of coordinates in $\mathbb{R}\times\mathbb{R}_n[X]$ defined by

$$\hat{k}_I(u,P) := (u|I|^{\frac{1}{2}}, P_I) \equiv \left(u|I|^{\frac{1}{2}}, (a_j|I|^{1-\frac{j}{2}})\right) \tag{40}$$

where P_I is defined by (39) we see that, in the notations (8) and (36), one has

$$\hat{s}_I\circ\ell_I = \ell_0\circ\hat{k}_I \tag{41}$$

7. The Group $\mathrm{Heis}(1,n,\mathbb{R}\chi_I)$ and Its C^*-Algebra

In the notations and assumptions of Sect. 6, we have seen that $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$ is isomorphic to $heis_{\mathbb{R}}(1,n)$. Since $\mathbb{R}^{n+2}$ is connected and simply connected, the Lie group of $heis_{\mathbb{R}}(1,n,\mathbb{R}\chi_I)$, denoted $\mathrm{Heis}(1,n,\mathbb{R}\chi_I)$, is isomorphic to $\mathrm{Heis}(1,n)$. In analogy with the notation (15), the generic element of $\mathrm{Heis}(1,n,\mathbb{R}\chi_I)$ will be denoted

$$e^{\ell_I(u,P)}; \quad (u,P)\in\mathbb{R}\times\mathbb{R}_n[X] \tag{42}$$

Definition 6. For any bounded Borel set $I\subset\mathbb{R}$, we denote

$$\mathcal{W}^0_{1,n;I} := \mathcal{W}(\mathrm{Heis}(1,n,\mathbb{R}\chi_I))$$

the free group-C^*-algebra of the group $\mathrm{Heis}(1,n,\mathbb{R}\chi_I))$. In analogy with (18), its generators will be called the *one-mode n-th degree Weyl operators localized on I* and denoted

$$W^0_I(u,P) := W_{e^{\ell_I(u,P)}} \in \mathcal{W}^0_{1,n;I} \tag{43}$$

Remark 7. Since the groups $\mathrm{Heis}(1,n,\mathbb{R}\chi_I))$ and $\mathrm{Heis}(1,n)$ are isomorphic, the same is true for the corresponding free group-C^*-algebras.

In the following section, we show that, in these C^*-algebra isomorphisms, the group generators of $\mathcal{W}^0_{1,n;I}$ are mapped into a set of group generators of $\mathcal{W}^0_{1,n}$ which depends on I and we introduce a construction that allows to get rid of this dependence.

7.1. C^*-Algebras Isomorphism

In the notations (8) and (42), the map

$$e^{\ell_I(u,P)} \in \mathrm{Heis}(1,n,\mathbb{R}\chi_I) \;\mapsto\; e^{\hat{s}_I(\ell_I(u,P))} \in \mathrm{Heis}(1,n)$$

where $\hat{s}_I$ the isomorphism defined in Lemma 3, is a Lie group isomorphism, hence it can be extended to a C^*-isomorphism of the corresponding free group-C^*-algebras.

This extension will be denoted with the symbol:

$$s_I^0 : \mathcal{W}^0_{1,n;I} \to \mathcal{W}^0_{1,n}$$

In view of the identity (41), and in the notations (18) and (43), the explicit form of s_I^0 on the generators is given by

$$s_I^0(W_I^0(u,P)) = W^0(\hat{k}_I(u,P)) \tag{44}$$

where $\hat{k}_I$ is the linear map defined by (40) and $(u,P) \in \mathbb{R} \times \mathbb{R}_n[X]$.

It is clear from (40) and (44) that, as a vector space, $s_I^0(\mathcal{W}^0_{1,n;I})$ coincides with $\mathcal{W}^0_{1,n}$. In this section, we will prove that the map

$$W^0(u,P) \in \mathcal{W}^0_{1,n} \mapsto W^0(\hat{k}_I(u,P)) \in \mathcal{W}^0_{1,n} \tag{45}$$

induces a C^*-algebra automorphism denoted k_I. To this goal, we use

$$W^0(\hat{k}_I(u,P))W^0(\hat{k}_I(v,Q)) = W^0(\hat{k}_I(u,P) \circ \hat{k}_I(v,Q))$$

and the following result.

Lemma 4. *For all $u \in \mathbb{R}$ and $P \in \mathbb{R}_n[X]$, let $\hat{k}_I$ be the linear map defined by (40). Then, denoting with the same symbol $\hat{k}_I$ its restriction on $\mathbb{R}_n[X]$, one has*:

$$\hat{k}_I \circ T_u(P) = T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I(P)$$
$$\hat{k}_I^{-1} \circ T_u^{-1}(P) = T^{-1}_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1}(P)$$
$$\hat{k}_I \circ T_u^{-1}(P) = T^{-1}_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I(P)$$
$$\hat{k}_I^{-1} \circ T_u(P) = T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1}(P)$$

Proof. Since both T_u and $\hat{k}_I$ are linear maps, it is sufficient to prove the lemma for $P(X) = X^k$ $(k \in \{0,\ldots,n\})$. For $k = 0$, all the identities in the lemma are obviously true. Let $k \in \{1,\ldots,n\}$. Then from the identity (14) one has

$$\begin{aligned}
T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I(X^k) &= T_{u|I|^{\frac{1}{2}}}(|I|^{1-\frac{k}{2}} X^k) \\
&= |I|^{1-\frac{k}{2}} T_{u|I|^{\frac{1}{2}}}(X^k) \\
&= |I|^{1-\frac{k}{2}} \left[\sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} u^{k-h} |I|^{\frac{k-h}{2}} X^h + X^k\right]
\end{aligned}$$

$$= \sum_{h=0}^{k-1} \frac{k!}{(k+1-h)!h!} u^{k-h} |I|^{1-\frac{h}{2}} X^h + |I|^{1-\frac{k}{2}} X^k$$
$$= \hat{k}_I \circ T_u(X^k) \tag{46}$$

(46) is equivalent to

$$T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I = \hat{k}_I \circ T_u \Leftrightarrow \hat{k}_I^{-1} \circ T^{-1}_{u|I|^{\frac{1}{2}}} = T_u^{-1} \circ \hat{k}_I^{-1}$$

Replacing u by $u|I|^{-\frac{1}{2}}$, this yields

$$\hat{k}_I^{-1} \circ T_u^{-1} = T^{-1}_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1} \tag{47}$$

From identities (46) and (47), one gets

$$T_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I \circ T_u^{-1} = \hat{k}_I$$
$$T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1} \circ T_u^{-1} = \hat{k}_I^{-1}$$

or equivalently

$$\hat{k}_I \circ T_u^{-1} = T^{-1}_{u|I|^{\frac{1}{2}}} \circ \hat{k}_I$$
$$\hat{k}_I^{-1} \circ T_u = T_{u|I|^{-\frac{1}{2}}} \circ \hat{k}_I^{-1}$$

□

Proposition 1. *$\hat{k}_I$ is a group automorphism for the composition law (13).*

Proof. We have to prove that for all (u, P), $(v, Q) \in \mathbb{R} \times \mathbb{R}_n[X]$, one has

$$(\hat{k}_I(u, P) \circ \hat{k}_I(v, Q)) = \hat{k}_I\Big((u+v); T^{-1}_{(u+v)}\big(T_u P + T_v S_u Q\big)\Big)$$

We know that

$$\hat{k}_I(u, P) \circ \hat{k}_I(v, Q) = (u|I|^{\frac{1}{2}}, P_I) \circ (v|I|^{\frac{1}{2}}, Q_I)$$

where $P_I(X) = P(|I|^{-\frac{1}{2}} X)$ and $Q_I(X) = Q(|I|^{-\frac{1}{2}} X)$. But from (2), we know that

$$(u|I|^{\frac{1}{2}}, P_I) \circ (v|I|^{\frac{1}{2}}, Q_I)$$
$$= \Big((u+v)|I|^{\frac{1}{2}}\ ,\ T^{-1}_{(u+v)|I|^{\frac{1}{2}}}\big(T_{u|I|^{\frac{1}{2}}} P_I + T_{v|I|^{\frac{1}{2}}} S_u Q_I\big)\Big)$$
$$= \Big((u+v)|I|^{\frac{1}{2}}, T^{-1}_{(u+v)|I|^{\frac{1}{2}}}\big(T_{u|I|^{\frac{1}{2}}} \hat{k}_I(P) + T_{v|I|^{\frac{1}{2}}} S_u \hat{k}_I(Q)\big)\Big)$$

Furthermore, from Lemma 4, we know that

$$T^{-1}_{(u+v)|I|^{\frac{1}{2}}} T_{u|I|^{\frac{1}{2}}} \hat{k}_I(P) = \hat{k}_I T^{-1}_{(u+v)} T_u(P)$$

Moreover, using

$$S_u \hat{k}_I(Q) = \hat{k}_I S_u(Q)$$

We also have

$$\begin{aligned}T^{-1}_{(u+v)|I|^{\frac{1}{2}}}T_{v|I|^{\frac{1}{2}}}S_u\hat{k}_I(Q) &= T^{-1}_{(u+v)|I|^{\frac{1}{2}}}T_{v|I|^{\frac{1}{2}}}\hat{k}_I S_u(Q)\\ &= T^{-1}_{(u+v)|I|^{\frac{1}{2}}}\hat{k}_I T_v S_u(Q)\\ &= \hat{k}_I T^{-1}_{(u+v)}T_v S_u(Q)\end{aligned}$$

Hence, one gets

$$\begin{aligned}(u|I|^{\frac{1}{2}}, P_I)\circ(v|I|^{\frac{1}{2}}, Q_I) &= \Big((u+v)|I|^{\frac{1}{2}}\ ,\ \hat{k}_I T^{-1}_{(u+v)}\big(T_u(P)+T_v S_u(Q)\big)\Big)\\ &= \hat{k}_I\Big((u+v),\quad T^{-1}_{(u+v)}\big(T_u(P)+T_v S_u(Q)\big)\Big)\\ &= \hat{k}_I\big((u,P)\circ(v,Q)\big)\end{aligned}$$

and this proves the statement. □

Corollary 1. *The map*:

$$s_I := k_I^{-1}\circ s_I^0\ :\ \mathcal{W}^0_{1,n;I}\to\mathcal{W}^0_{1,n}$$

is a C^*-algebra isomorphism characterized by the condition

$$s_I(W_I^0(u,P)) = W^0(u,P);\quad \forall(u,P)\in\mathbb{R}^{n+2} \tag{48}$$

Proof. (48) is clear from (44) and the definition (45) of k_I.
We know that s_I^0 is a C^*-algebra isomorphism. From Proposition 1, we know that $\hat{k}_I$ is a group automorphism for the composition law defined by (13). Because of the linear independence of the free group algebra generators k_I extends to a C^*-algebra automorphism. Thus, s_I is composed of an isomorphism with an automorphism and the thesis follows. □

8. The Inductive Limit

In the following, when speaking of tensor products of C^*-algebras, it will be understood that a choice of a cross norm has been fixed and that all tensor products are referred to the same choice.

For a bounded Borel subset I of $\mathbb{R}$, let $\mathcal{W}^0_{1,n;I}$ be the C^*-algebra in Definition 6 and let the isomorphisms $s_I : \mathcal{W}^0_{1,n;I}\to\mathcal{W}^0_{1,n}$ defined by (48). For $\pi=(I_j)_{j\in F}\in Part_{fin}(I)$ define the C^*-algebra

$$\mathcal{W}^0_{1,n;I;\pi} := \bigotimes_{j\in F}\mathcal{W}^0_{1,n;I_j} \tag{49}$$

the injective C^*-homomorphism (C^*-embedding)

$$z_{I,\pi} := \left(\bigotimes_{j\in F}^{(\mathrm{diag})} s^{-1}_{I_j}\right)\circ s_I\ :\ \mathcal{W}^0_{1,n;I}\to\bigotimes_{j\in F}\mathcal{W}^0_{1,n;I_j} = \mathcal{W}^0_{1,n;I;\pi}$$

Then, for any $\pi \prec \pi' \in Part_{fin}(I)$, the map

$$
\begin{aligned}
z_{I;\pi,\pi'} := \bigotimes_{j\in F} \left(\bigotimes_{I_j \supseteq I' \in \pi'}^{(\mathrm{diag})} s_{I'}^{-1} \right) \circ s_{I_j} \ : \mathcal{W}^0_{1,n;I;\pi} \to \bigotimes_{j\in F} \bigotimes_{I_j \supseteq I' \in \pi'} \mathcal{W}^0_{1,n;I'} \\
= \mathcal{W}^0_{1,n;I;\pi'} \qquad (50)
\end{aligned}
$$

is a C^*-embedding. Moreover, by construction and in the notations of Definition 6, for all $u \in \mathbb{R}$ and $P = \sum_{j=0}^{n} a_j X^n \in \mathbb{R}_n[X]$, one has

$$
z_{I;\pi,\pi'} z_{I,\pi}(W^0_I(u,P)) := \bigotimes_{I'\in\pi'} W^0_{I'}(u,P) = z_{I,\pi'}(W^0_I(u,P)) \in \mathcal{W}^0_{1,n;I;\pi'}
$$

Lemma 5. *The family*

$$
\left\{ (\mathcal{W}^0_{1,n;I;\pi})_{\pi\in Part_{fin}(I)},\ (z_{I;\pi,\pi'})_{\pi\prec\pi'\in Part_{fin}(I)} \right\} \qquad (51)
$$

is an inductive system of C^*-algebras, i.e., for all $\pi \prec \pi'$, $z_{I;\pi,\pi'}$ is a morphism and if $\pi \prec \pi' \prec \pi'' \in Part_{fin}(I)$ one has

$$
z_{I;\pi',\pi''} z_{I;\pi,\pi'} = z_{I;\pi,\pi''} \qquad (52)
$$

Proof. We have already proved that the $z_{I,\pi,\pi'}$ are C^*-embeddings. Therefore, it remains to prove (52). To this goal, for π, π', π'' as in the statement, using the identity

$$
\bigotimes_{I'\in\pi'} = \bigotimes_{I\in\pi} \bigotimes_{\pi'\ni I' \subseteq I}
$$

one finds

$$
\begin{aligned}
z_{I;\pi'',\pi'} z_{I;\pi,\pi'} &= \left(\bigotimes_{I'\in\pi'} \left(\bigotimes_{\pi''\ni I''\subseteq I'}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\bigotimes_{I\in\pi} \left(\bigotimes_{\pi'\ni I'\subseteq I}^{(\mathrm{diag})} s_{I'}^{-1} \right) \circ s_I \right) \\
&= \left(\bigotimes_{I\in\pi} \bigotimes_{\pi'\ni I'\subseteq I} \left(\bigotimes_{\pi''\ni I''\subseteq I'}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\bigotimes_{I\in\pi} \left(\bigotimes_{\pi'\ni I'\subseteq I}^{(\mathrm{diag})} s_{I'}^{-1} \right) \circ s_I \right) \\
&= \bigotimes_{I\in\pi} \left(\bigotimes_{\pi'\ni I'\subseteq I} \left(\bigotimes_{\pi''\ni I''\subseteq I'}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_{I'} \right) \left(\left(\bigotimes_{\pi'\ni I'\subseteq I}^{(\mathrm{diag})} s_{I'}^{-1} \right) \circ s_I \right) \\
&= \bigotimes_{I\in\pi} \bigotimes_{\pi'\ni I'\subseteq I} \left(\left(\bigotimes_{\pi''\ni I''\subseteq I'}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_{I'} \left(s_{I'}^{-1} \right) \circ s_I \right) \\
&= \bigotimes_{I\in\pi} \left(\left(\bigotimes_{\pi'\ni I'\subseteq I} \bigotimes_{\pi''\ni I''\subseteq I'}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_I \right) \\
&= \bigotimes_{I\in\pi} \left(\left(\bigotimes_{\pi''\ni I''\subseteq I}^{(\mathrm{diag})} s_{I''}^{-1} \right) \circ s_I \right) = z_{I;\pi'',\pi}
\end{aligned}
$$

$\square$

Definition 7. For any bounded Borel subset I of $\mathbb{R}$, we denote

$$\{\mathcal{W}_{1,n;I}, (\tilde{z}_{I;\pi})_{\pi\in \mathrm{Part}_{\mathrm{fin}}(I)}\}$$

the inductive limit of the family (51) i.e., $\mathcal{W}_{1,n;I}$ is a C^*-algebra and for any $\pi \in \mathrm{Part}_{\mathrm{fin}}(I)$ and in the notation (49),

$$\tilde{z}_{I;\pi} : \mathcal{W}^0_{1,n;I;\pi} \to \mathcal{W}_{1,n;I}$$

is an embedding satisfying

$$\tilde{z}_{I;\pi'} z_{I;\pi,\pi'} = \tilde{z}_{I;\pi}; \quad \forall \pi \prec \pi' \in \mathrm{Part}_{\mathrm{fin}}(I)$$

Remark. Intuitively one can think of the elements of $\mathcal{W}_{1,n;I}$ as a realization of the *non-linear Weyl operators*: (4) with finitely valued, compact support, test functions.

8.1. Factorizable Families of C^*-Algebras

Definition 8. A family of C^*-algebras $\{\mathcal{W}_I\}$, indexed by the bounded Borel subsets of $\mathbb{R}$, is called *factorizable* if, for every bounded Borel $I \subset \mathbb{R}$ and every Borel partition π of I, there is an isomorphism

$$u_{I,\pi} : \bigotimes_{I_j\in\pi} \mathcal{W}_{I_j} \to \mathcal{W}_I$$

If this is the case, an operator $w_I \in \mathcal{W}_I$ is called factorizable if there exist operators $w_{I_j} \in \mathcal{W}_{I_j}$ $(I_j \in \pi)$ such that

$$u^{-1}_{I,\pi}(w_I) = \bigotimes_{I_j\in\pi} w_{I_j} \tag{53}$$

Remark 8. In the following, for a given bounded Borel set I, when $\pi \equiv \{I\}$ is the partition of I, consisting of the only set I, we will use the notation

$$\tilde{z}_I := \tilde{z}_{I;\{I\}} \ : \ \mathcal{W}^0_{1,n;I} \to \mathcal{W}_{1,n;I}$$

We want to prove that:

(i) the family of C^*-algebras

$$\{\mathcal{W}_{1,n;I} \ : \ I\text{-bounded Borel subset of } \mathbb{R}\} \tag{54}$$

where the algebras $\mathcal{W}_{1,n;I}$ are those introduced in Definition 7, is factorizable in the sense of Definition 8;

(ii) for any bounded Borel set I, the operators

$$W_I(u,P) := \tilde{z}_I(W^0_I(u,P)) \in \mathcal{W}_{1,n;I}; \quad W^0_I(u,P) \in \mathcal{W}^0_{1,n;I} \tag{55}$$

are factorizable in the sense of (53).

To this goal let us remark that, if I, J are disjoint bounded Borel sets in $\mathbb{R}$, then the map

$$(\pi_I,\pi_J) \in \mathrm{Part}_{\mathrm{fin}}(I) \times \mathrm{Part}_{\mathrm{fin}}(J) \mapsto \ \pi_{I\cup J} := \{\pi_I \cup \pi_J\} \in \mathrm{Part}_{\mathrm{fin}}(I\cup J) \tag{56}$$

defines a canonical bijection between $\mathrm{Part}_{\mathrm{fin}}(I)\times\mathrm{Part}_{\mathrm{fin}}(J)$ and $\mathrm{Part}_{\mathrm{fin}}(I\cup J)$ such that, if $\pi_I \prec \pi'_I \in \mathrm{Part}_{\mathrm{fin}}(I)$ and $\pi_J \prec \pi'_J \in \mathrm{Part}_{\mathrm{fin}}(J)$, then $\pi_{I\cup J} \prec \pi'_{I\cup J} \in \mathrm{Part}_{\mathrm{fin}}(I\cup J)$.

Lemma 6. *Let I, J be disjoint bounded Borel sets in $\mathbb{R}$. Then the inductive system of C^*-algebras*

$$\left\{(\mathcal{W}^0_{1,n;I\cup J;\pi_{I\cup J}})_{\pi_{I\cup J}\in Part_{fin}(I\cup J)},\ (z_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}})_{\pi_{I\cup J}\prec\pi'_{I\cup J}\in Part_{fin}(I\cup J)}\right\} \tag{57}$$

is isomorphic to the inductive system of C^*-algebras

$$\begin{aligned}&\left\{(\mathcal{W}^0_{1,n;I;\pi_I}\otimes\mathcal{W}^0_{1,n;J;\pi_J})_{(\pi_I,\pi_J)\in Part_{fin}(I)\times Part_{fin}(J)}\ ,\right.\\ &\left.(z_{I;\pi_I,\pi'_I}\otimes z_{J;\pi_J,\pi'_J})_{\pi_I\prec\pi'_I\in Part_{fin}(I),\pi_J\prec\pi'_J\in Part_{fin}(J)}\right\}\end{aligned} \tag{58}$$

in the sense that, for each $\pi_I \in Part_{fin}(I)$ and $\pi_J \in Part_{fin}(J)$, then there exists a C^*-algebra isomorphism

$$u_{I,J,\pi_I,\pi_J}\ :\ \mathcal{W}^0_{1,n;I;\pi_I}\otimes\mathcal{W}^0_{1,n;J;\pi_J}\to\mathcal{W}^0_{1,n;I\cup J;\pi_{I\cup J}}$$

such that, for each $\pi_I \prec \pi'_I \in Part_{fin}(I)$ and $\pi_J \prec \pi'_J \in Part_{fin}(J)$, one has in the notation (56)

$$u_{I,J,\pi_I,\pi_J}\circ(z_{I;\pi_I,\pi'_I}\otimes z_{J;\pi_J,\pi'_J}) = z_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}} \tag{59}$$

Proof. With the notations above, from (49) one deduces that

$$\begin{aligned}\mathcal{W}^0_{1,n;I;\pi_I}\otimes\mathcal{W}^0_{1,n;J;\pi_J} &:= \left(\bigotimes_{I'\in\pi_I}\mathcal{W}^0_{1,n;I'}\right)\otimes\left(\bigotimes_{J'\in\pi_J}\mathcal{W}^0_{1,n;J'}\right)\\ &\equiv\bigotimes_{K\in\pi_{I\cup J}}\mathcal{W}^0_{1,n;K}=\mathcal{W}^0_{1,n;I\cup J;\pi_{I\cup J}}\end{aligned} \tag{60}$$

Denote

$$u_{I\otimes J,I\cup J}:\mathcal{W}^0_{1,n;I;\pi_I}\otimes\mathcal{W}^0_{1,n;J;\pi_J}\to\mathcal{W}^0_{1,n;I\cup J;\pi_{I\cup J}}$$

the isomorphism defined by (60). If $\pi_I \prec \pi'_I \in \mathrm{Part}_{\mathrm{fin}}(I)$ and $\pi_J \prec \pi'_J \in \mathrm{Part}_{\mathrm{fin}}(J)$, then clearly $\pi_{I\cup J} \prec \pi'_{I\cup J} \in \mathrm{Part}_{\mathrm{fin}}(I\cup J)$ and from (50) we see that

$$\begin{aligned}&u_{I,J,\pi_I,\pi_J}\circ(z_{I;\pi_I,\pi'_I}\otimes z_{J;\pi_J,\pi'_J})\\ &=u_{I,J,\pi_I,\pi_J}\circ\left(\left(\bigotimes_{I_j\in\pi_I}\left(\bigotimes^{(\mathrm{diag})}_{I_j\supseteq I'\in\pi'_I}s^{-1}_{I'}\right)\circ s_{I_j}\right)\right.\\ &\qquad\left.\otimes\left(\bigotimes_{J_h\in\pi_J}\left(\bigotimes^{(\mathrm{diag})}_{J_h\supseteq J'\in\pi'_J}s^{-1}_{J'}\right)\circ s_{J_h}\right)\right)\\ &=\bigotimes_{H_l\in\pi_{I\cup J}}\left(\bigotimes^{(\mathrm{diag})}_{H_l\supseteq K\in\pi'_{I\cup J}}s^{-1}_{K}\right)\circ s_{H_l}=z_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}}\end{aligned}$$

which proves (59). □

Theorem 1. (i) *The family of C^*-algebras defined by (54) is factorizable.*
(ii) *The operators defined by (55) are factorizable.*

Proof. We apply Definition 8 to the case in which the family $\mathcal{F}$ is the family of bounded Borel sets in $\mathbb{R}$. By induction it will be sufficient to prove that, if I, J are disjoint bounded Borel sets in $\mathbb{R}$, then there exists a C^*-algebra isomorphism

$$u_{I,J} \ : \ \mathcal{W}_{1,n;I} \otimes \mathcal{W}_{1,n;J} \to \mathcal{W}_{1,n;I\cup J}$$

Since $\mathcal{W}_{1,n;I} \otimes \mathcal{W}_{1,n;J}$ is the inductive limit of the system (58) and $\mathcal{W}_{1,n;I\cup J}$ is the inductive limit of the system (57), the statement follows from Lemma 6 because isomorphic inductive systems have isomorphic inductive limits.

The factorizability of the operators (55) follows from the identity (54). $\square$

From Theorem 1, it follows that, if $I \subset J$ are bounded Borel sets in $\mathbb{R}$, then the map

$$j_{I;J} \ : \ w_I \in \mathcal{W}_{1,n;I} \to w_I \otimes 1_{J\setminus I} \in \mathcal{W}_{1,n;J} \tag{61}$$

is a C^*-algebra isomorphism. Since clearly, for $I \subset J \subset K$ bounded Borel sets in $\mathbb{R}$, $1_{J\setminus I} \otimes 1_{K\setminus J} \equiv 1_{K\setminus I}$, it follows that

$$\{(\mathcal{W}_{1,n;I}), \quad (j_{I;J}), \quad I \subset J \in \text{bounded Borel sets in } \mathbb{R}\} \tag{62}$$

is an inductive system of C^*-algebras.

Notation: The inductive limit of the system (62) will be denoted

$$\{\mathcal{W}_{1,n;\mathbb{R}}, \quad (j_I), \quad I \in \text{bounded Borel sets in } \mathbb{R}\}$$

Since the $j_I : \mathcal{W}_{1,n;I} \to \mathcal{W}_{1,n;\mathbb{R}}$ are injective embeddings, the family $(j_I(\mathcal{W}_{1,n;I}))$ is factorizable and one can introduce the more intuitive notation:

$$j_I(\mathcal{W}_{1,n;I}) \equiv \mathcal{W}_{1,n;I} \otimes 1_{I^c}$$

9. Existence of Factorizable States on $\mathcal{W}_{1,n;\mathbb{R}}$

In the notation (43) and with the operators $W_I(u, P)$ defined by (55), using factorizability of the family $(\mathcal{W}_{1,n;I})$ and of the corresponding generators, for any $I \subset \mathbb{R}$ bounded Borel and any finite partition π of I, we will use the identifications

$$\begin{aligned} \mathcal{W}_{1,n;I} &\equiv j_I(\mathcal{W}_{1,n;I}) \equiv \mathcal{W}_{1,n;I} \otimes 1_{I^c} \subset \mathcal{W}_{1,n;\mathbb{R}} \\ \mathcal{W}_I(u,P) &\equiv \bigotimes_{I_0\in\pi} \mathcal{W}_{I_0}(u,P); \quad \forall (u,P) \in \mathbb{R} \times \mathbb{R}_n[X] \\ W_I(u,P) &\equiv \bigotimes_{I_0\in\pi} W_{I_0}(u,P); \quad \forall (u,P) \in \mathbb{R} \times \mathbb{R}_n[X] \end{aligned} \tag{63}$$

omitting from the notations the isomorphisms implementing these identifications.

Definition 9. A state φ on $\mathcal{W}_{1,n;\mathbb{R}}$ is called factorizable if for every $I \subset \mathbb{R}$ bounded Borel, for every finite partition $\pi = (I_j)_{j\in F}$ of I and for every $W_I(u,P)$ as in (63), one has:

$$\varphi(W_I(u,P)) = \prod_{j\in F} \varphi(W_{I_j}(u,P)); \quad \forall (u,P) \in \mathbb{R}\times\mathbb{R}_n[X] \tag{64}$$

The map (19) can be used to lift the Fock state φ_F on $\mathcal{W}_{F,1,n}$ to a state, denoted φ_0, on $\mathcal{W}^0_{1,n}$ through the prescription

$$\varphi_0(W^0(u,P)) := \varphi_F(W(u,P)) \tag{65}$$

$(W^0(u,P) \in \mathcal{W}^0_{1,n}\ ,\ W(u,P) \in \mathrm{Un}(\Gamma(\mathbb{C})))$. Then, using the maps $\tilde{z}_I$ defined by (55), for each bounded Borel set $I \subset \mathbb{R}$, one can define the state φ_I on $\tilde{z}_I(\mathcal{W}^0_{1,n;I}) \subset \mathcal{W}_{1,n;I}$ through the prescription that, for each $W^0_I(u,P) \in \mathcal{W}^0_{1,n;I}$, one has

$$\varphi_I(W_I(u,P)) = \varphi_I(\tilde{z}_I(W^0_I(u,P))) := \varphi_F(W(\hat{k}_I(u,P))) \tag{66}$$

Theorem 2. *Under the assumption* (34), *if* $n = 1$ *then there exists a factorizable state* φ *on* $\mathcal{W}_{1,n;\mathbb{R}}$ *such that, for each bounded Borel set* $I \subset \mathbb{R}$, *one has*

$$\varphi(W_I(u,P)) = \varphi_F(W(\hat{k}_I(u,P))); \quad \forall (u,P) \in \mathbb{R}\times\mathbb{R}_n[X] \tag{67}$$

If $n \geq 2$, *no such state exists.*

Proof. Let I be a fixed bounded Borel set in $\mathbb{R}$ and let π be a finite partition of I. From Definition 9, we know that φ is factorizable if and only if for every $I \subset \mathbb{R}$ bounded Borel set, for every finite partition π of I and for every $W_I(u,P)$ as in (63), (64) holds. If condition (67) is satisfied, the identity (64) becomes equivalent to:

$$\varphi_F(W(\hat{k}_I(u,P))) = \prod_{I_j\in\pi} \varphi_F(W(\hat{k}_{I_j}(u,P))); \quad \forall (u,P) \in \mathbb{R}\times\mathbb{R}_n[X] \tag{68}$$

Thus, the statement of the theorem is equivalent to say that, for $n = 1$ the identity (68) is satisfied and, for $n \geq 2$, not.

- **Case** $n = 1$. For $P = a_0 + a_1X$ and $u \in \mathbb{R}$, recalling the definition (40) of $\hat{k}_I$, one knows that

  $$W(\hat{k}_I(u,P)) = e^{i(u|I|^{\frac{1}{2}}p + a_0|I|1 + a_1|I|^{\frac{1}{2}}q)}$$

  whose Fock expectation is known to be

  $$\varphi_F(W(\hat{k}_I(u,P))) = e^{-|I|(u^2+a_1^2)/4}e^{ia_0|I|} = (\varphi_F(W(u,P)))^{|I|} \tag{69}$$

  It follows that

  $$\prod_{I_j\in\pi} \varphi_F(W(\hat{k}_{I_j}(u,P))) = \prod_{I_j\in\pi} (\varphi_F(W(u,P)))^{|I_j|}$$

  Therefore, if $a_I = 1$, then

  $$\varphi_F(W(\hat{k}_I(u,P))) = (\varphi_F(W(u,P)))^{|I|} = \varphi_F(W(u,P))$$

- **Case** $n \geq 2$. Since, for $n \geq 2$, the 1-mode n-th degree Heisenberg $*$-Lie algebra $heis_{\mathbb{R}}(1,n)$ contains a copy of $heis(1,2)$ (see Definition 1), the algebra $\mathcal{W}_{1,n;\mathbb{R}}$ contains a copy of $\mathcal{W}_{1,2;\mathbb{R}}$. Therefore, the non-existence of a factorizable state on $\mathcal{W}_{1,2;\mathbb{R}}$, satisfying (67), will imply the same conclusion for $\mathcal{W}_{1,n;\mathbb{R}}$. In the case $n = 2$, let $P = a_0 + a_1 X + a_2 X^2$ and $u \in \mathbb{R}$. Then, using again $a_I = 1$, (11) and (35) one has

$$W(\hat{k}_I(u,P)) = e^{i(|I|^{\frac{1}{2}}up + a_0|I|1 + a_1|I|^{\frac{1}{2}}q + a_2 q^2)}$$

and from [10] (Theorem 2), one knows that

$$\begin{aligned}\varphi_F(W(\hat{k}_I(u,P))) &= (1-2iA)^{-\frac{1}{2}} e^{ia_0|I|} e^{\frac{4C^2(A^2+2iA)-3|M|^2}{6(1-2iA)}|I|} \\ &= (1-2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2+2iA)-3|M|^2}{6(1-2iA)}}\right)^{|I|}\end{aligned}$$

where $A = \frac{a_2}{\sqrt{2}}$, $B = \frac{a_1}{\sqrt{2}}$, $C = \frac{u}{\sqrt{2}}$ and $M = B + iC$. On the other hand, if $\pi \in \mathrm{Part}_{\mathrm{fin}}(I)$ with $|\pi| > 1$, then

$$\begin{aligned}\prod_{I_j \in \pi} \varphi_F(W(\hat{k}_{I_j}(u,P)) &= \prod_{I_j \in \pi} \left((1-2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2+2iA)-3|M|^2}{6(1-2iA)}}\right)^{|I_j|}\right) \\ &= (1-2iA)^{-\frac{|\pi|}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2+2iA)-3|M|^2}{6(1-2iA)}}\right)^{|I|} \\ &\neq (1-2iA)^{-\frac{1}{2}} \left(e^{ia_0} e^{\frac{4C^2(A^2+2iA)-3|M|^2}{6(1-2iA)}}\right)^{|I|} \\ &= \varphi_F(W(\hat{k}_I(u,P)))\end{aligned} \tag{70}$$

□

Remark. Notice that the inequality in (70) holds *for all values of the parameters* A, B, C, including those for which infinite divisibility has been proved. This suggests that the infinite divisibility of the vacuum distributions of the generalized field operators is a necessary, but not sufficient, condition for the possibility to induce a state on the C^*-algebra from states on its finite dimensional approximations.

Lemma 7. *In the case* $n = 1$, *the choice of the isomorphism* $\hat{s}_I$ (*see Lemma* 3) *given by*

$$\begin{aligned}\hat{s}_I(L_0((\chi_I))) &= a_I i|I|1 \\ \hat{s}_I(L_2(\chi_I)) &= a_I i|I|^{\frac{1}{2}} p \\ \hat{s}_I(L_1(\chi_I)) &= a_I i|I|^{\frac{1}{2}} q\end{aligned}$$

gives rise to a factorizable state satisfying (67) *if and only if the map* $I \subset \mathbb{R} \mapsto a_I$ *has the form*

$$a_I := \frac{1}{|I|} \int_I p(s) ds$$

for all Borel subsets $I \subseteq \mathbb{R}$ where $p(\,\cdot\,)$ is a locally integrable almost everywhere strictly positive function on $\mathbb{R}$. In this case, the factorizable state will be translation invariant if and only if $p(\,\cdot\,)$ is a strictly positive constant.

Proof. In the case $n = 1$, if $a_I \neq 1$, then the expression for $W(\hat{k}_I(u,P))$ becomes

$$W(\hat{k}_I(u,P)) = e^{i(u|I|^{\frac{1}{2}} a_I^{1/2} p + a_0 |I| a_I 1 + a_1 |I|^{\frac{1}{2}} a_I^{1/2} q)}$$

consequently its Fock expectation is

$$\varphi_F(W(\hat{k}_I(u,P))) = e^{-|I| a_I (u^2 + a_1^2)/4} e^{i a_0 a_I |I|} = (\varphi_F(W(u,P)))^{a_I |I|} \tag{71}$$

Therefore, the factorizability condition (68) can hold if and only if the map $I \subset \mathbb{R} \mapsto a_I|I|$ is a finitely additive measure. In this case, by construction it will be absolutely continuous with respect to the Lebesgue measure hence there will exist a locally integrable almost everywhere positive function $p(\,\cdot\,)$ satisfying

$$a_I|I| := \int_I p(s)\mathrm{d}s; \qquad \forall \text{ Borel } I \subseteq \mathbb{R}$$

$p(\,\cdot\,)$ must be almost everywhere strictly positive because, by Lemma 3, $a_I > 0$ for any Borel set $I \subseteq \mathbb{R}$. This proves the first statement of the lemma. The second one follows because the Lebesgue measure is the unique translation invariant positive measure on $\mathbb{R}$. □

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