JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 2, Summer 2011

ON THE ASSOCIATED GRADED RING OF A SEMIGROUP RING

M. D'ANNA, V. MICALE AND A. SAMMARTANO

ABSTRACT. Let $(R, \mathbf{m})$ be a numerical semigroup ring. In this paper we study the properties of its associated graded ring $G(\mathbf{m})$. In particular, we describe the $H^0_{\mathcal{M}}$ for $G(\mathbf{m})$ (where $\mathcal{M}$ is the homogeneous maximal ideal of $G(\mathbf{m})$) and we characterize when $G(\mathbf{m})$ is Buchsbaum. Furthermore, we find the length of $H^0_{\mathcal{M}}$ as a $G(\mathbf{m})$-module, when $G(\mathbf{m})$ is Buchsbaum. In the 3-generated numerical semigroup case, we describe the $H^0_{\mathcal{M}}$ in terms of the Apery set of the numerical semigroup associated to R. Finally, we improve two characterizations of the Cohen-Macaulayness and Gorensteinness of $G(\mathbf{m})$ given in [**2, 3**], respectively.

1. Introduction. Let $(R, \mathbf{m})$ be a Noetherian, one-dimensional, local ring with $|R/\mathbf{m}| = \infty$, and let $G(\mathbf{m}) = \oplus_{i \geq 0} \mathbf{m}^i/\mathbf{m}^{i+1}$ be the associated graded ring of R with respect to $\mathbf{m}$. The study of the properties of $G(\mathbf{m})$ is a classical subject in local algebra.

The concept of a Buchsbaum ring is the most important of all notions generalizing Cohen-Macaulay rings. While the property for $G(\mathbf{m})$ to be Cohen-Macaulay has been studied extensively (see, e.g., [**2, 14**], or, for the particular case of semigroup rings, [**7, 11**]), not much is known about the Buchsbaumness of $G(\mathbf{m})$, at least in the general case (see [**8, 9**]).

As for the semigroup ring case, Sapko, in [**15**], gives some necessary and sufficient conditions for $G(\mathbf{m})$ to be Buchsbaum, when R is associated to a 3-generated numerical semigroup; still in the 3-generated case Shen, in [**16**], studies the Buchsbaumness of $G(\mathbf{m})$ and gives positive answers to the conjectures proposed in [**15**]. If S is a general numerical semigroup, it is possible to find some results on the Buchsbaumness of $G(\mathbf{m})$ in [**5**] (where it is mainly studied the more general case of one dimensional rings) and in [**4**].

2010 AMS *Mathematics subject classification.* Primary 13A30, 13H10.
Received by the editors on March 4, 2010, and in revised form on March 31, 2010.

DOI:10.1216/JCA-2011-3-2-147

In this paper we mainly study the Buchsbaumness of $G(\mathbf{m})$, when $(R, \mathbf{m})$ is the semigroup ring associated to a numerical semigroup, but, applying our techniques, we also get some new results on its Cohen-Macaulayness and on its Gorensteinness.

In Section 2 we give some preliminaries about numerical semigroups and semigroup rings associated to a numerical semigroup, and we recall some results on the Buchsbaumness of one-dimensional graded rings proved in [**5**].

In Section 3 we give a description of $H^0_{\mathcal{M}} := (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ (where r is the reduction number of $\mathbf{m}$ and where $\mathcal{M}$ is the homogeneous maximal ideal of $G(\mathbf{m})$ (cf. Corollary 3.5) and we use it in order to characterize when $G(\mathbf{m})$ is Buchsbaum (cf. Proposition 3.6). Successively, we find the length of $H^0_{\mathcal{M}}$ as a $G(\mathbf{m})$-module when $G(\mathbf{m})$ is Buchsbaum (cf. Proposition 3.15). Finally, we relate the Buchsbaumness with a property of the Apery set of the associated numerical semigroup (cf. Proposition 3.19), using a partial ordering in S introduced in [**3**].

In Section 4, we restrict our attention to the semigroup ring associated to a 3-generated numerical semigroup S; we use the results of Section 3 in order to prove that, if $G(\mathbf{m})$ is Buchsbaum, then we can determine the $H^0_{\mathcal{M}}$ in terms of the Apery set of S (cf. Theorem 4.1 and Corollary 4.4). In particular, we completely solve [**15**, Conjecture 33] and [**15**, Conjecture 24]. Finally, we give a new proof of a result of Shen, which shows that $G(\mathbf{m})$ is Buchsbaum if and only if it is Cohen-Macaulay, for the 3-generated symmetric semigroup case (cf. Corollary 4.5).

Finally, in Section 5, using the techniques introduced in Section 3, we strengthen, for the semigroup ring case, a characterization of the Cohen-Macaulayness of $G(\mathbf{m})$ given in [**2**, Theorem 2.6] (cf. Proposition 5.1). Moreover, we prove that, assuming the hypotheses of M-purity and symmetry for S, $G(\mathbf{m})$ is Buchsbaum if and only if it is Cohen-Macaulay (cf. Proposition 5.5) and we use this result to give a characterization for $G(\mathbf{m})$ to be Gorenstein, improving an analogous result by [**3**] (cf. Corollary 5.6). Finally, Question 5.8, about a possible improvement of Proposition 5.5, received an affirmative answer by Shen that we publish with his permission (cf. Proposition 5.9).

The computations made for this paper are performed by using the GAP system [**6**] and, in particular, the NumericalSgps package [**13**].

2. Preliminaries. We start this section recalling some well-known facts on numerical semigroups and semigroup rings. For more details see, e.g., [**1**].

A subsemigroup S of the monoid of natural numbers $(\mathbf{N},+)$, such that $0 \in S$, is called a *numerical semigroup*. Each numerical semigroup S has a natural partial ordering $\leq_S$ where, for every s and t in S, $s \leq_S t$ if there is an $u \in S$ such that $t = s + u$. The set $\{g_i\}$ of the minimal elements in $S \setminus \{0\}$ in this ordering is called the *minimal set of generators* for S. In fact all elements of S are linear combinations of minimal elements, with non-negative integers coefficients. Note that the minimal set $\{g_i\}$ of generators is finite since, for any $s \in S$, $s \neq 0$, we have that g_i is not congruent to g_j modulo s.

A numerical semigroup generated by $g_1 < g_2 < \cdots < g_n$ is denoted by $\langle g_1, g_2, \ldots, g_n \rangle$. Since the semigroup $S = \langle g_1, g_2, \ldots, g_n \rangle$ is isomorphic to $\langle dg_1, dg_2, \ldots, dg_n \rangle$ for any $d \in \mathbf{N} \setminus \{0\}$, we assume, in the sequel, that $\gcd(g_1, g_2, \ldots, g_n) = 1$. It is well known that this condition is equivalent to $|\mathbf{N} \setminus S| < \infty$. Hence there is a well defined the integer $g = g(S) = \max\{x \in \mathbf{Z} \mid x \notin S\}$, called the *Frobenius number* of S.

Since the Frobenius number g does not belong to S, if $x \in S$, it is obvious that $g - x \notin S$. A numerical semigroup is called *symmetric* if the converse holds: let x be an integer, then $g - x \notin S$ implies that $x \in S$.

A *relative ideal* of a semigroup S is a nonempty subset H of $\mathbf{Z}$ such that $H + S \subseteq H$ and $H + s \subseteq S$ for some $s \in S$. A relative ideal of S which is contained in S is simply called an *ideal* of S. The ideal $M = \{s \in S \mid s \neq 0\}$ is called the *maximal ideal of* S. It is straightforward to see that, if H and L are relative ideals of S, then $H + L$, $kH (= H + \cdots + H$, k summands, for $k \geq 1)$ and $H -_{\mathbf{Z}} L := \{z \in \mathbf{Z} : z + L \subseteq H\}$ are also relative ideals of S.

The rings $R = k[[t^S]] = k[[t^{g_1}, \ldots, t^{g_n}]]$ and $R = k[t^S]_{\mathbf{m}}$ are called the *numerical semigroup rings* associated to S, where $\mathbf{m} = (t^{g_1}, \ldots, t^{g_n})$. R is a one-dimensional local domain with maximal ideal $\mathbf{m}$ and quotient field $Q(R) = k((t))$ and $Q(R) = k(t)$, respectively. In both cases the associated graded ring $G(\mathbf{m})$, which is the object of our investigation, is the same. From now on, we will assume that $R = k[[t^S]]$, but the other case is perfectly analogous.

We will denote by $v : k((t)) \to \mathbf{Z} \cup \infty$ the natural valuation (with associated (discrete) valuation ring $k[[t]]$), that is,

$$v\left(\sum_{h=i}^{\infty} r_h t^h\right) = i, \quad i \in \mathbf{Z},\ r_i \neq 0$$

(in the case $Q(R) = k(t)$, we have the valuation associated to the DVR $K[t]_{(t)}$). It is straightforward that $v(R) = \{v(r) \mid r \in R \setminus \{0\}\} = S$.

The relation between R and $S = v(R)$ is very tight, and we can translate many properties of R to the corresponding properties of S. In particular, if I and J are fractional ideals of R, then $v(I)$ and $v(J)$ are relative ideal of $S = v(R)$; moreover, if I and J are monomial ideals, it is not difficult to check that $v(I \cap J) = v(I) \cap v(J)$, $v(IJ) = v(I) + v(J)$ and $v(I :_{Q(R)} J) = v(I) -_{\mathbf{Z}} v(J)$. Furthermore, if $J \subseteq I$, then $\lambda_R(I/J) = |v(I) \setminus v(J)|$, where $\lambda_R(\cdot)$ is the length as R-module.

Following the notation in [**2**], we denote by $\mathrm{Ap}_{g_1}(S) = \{\omega_0, \dots, \omega_{g_1-1}\}$ the *Apery set* of S with respect of g_1, that is, the set of the smallest elements in S in each congruence class modulo g_1. More precisely, $\omega_0 = 0$ and $\omega_i = \min\{s \in S \mid s \equiv i \pmod{g_1}\}$. It is clear that the largest element in the Apery set is always $g + g_1$. Moreover, if S is symmetric, then, for every index j, there exists an index i such that $\omega_j + \omega_i = g + g_1$; hence, in the symmetric case, $g + g_1$ is the maximum of the Apery set with respect to $\leq_S$. Furthermore, it is easy to see that, if $\omega_h + \omega_t \equiv g + g_1$, then $\omega_h + \omega_t = g + g_1$.

By [**1**, formula I.2.4] we have that the blow up of S is the numerical semigroup $S' = \cup_i (iM -_{\mathbf{Z}} iM) = \langle g_1, g_2 - g_1, \dots, g_n - g_1 \rangle$. Note that the set of the generators $\{g_1, g_2 - g_1, \dots, g_n - g_1\}$ is not necessarily the minimal one for S'; moreover, g_1 might not be the smallest nonzero element in S'.

In [**2**] are defined two families of invariants of S, that give information on the Cohen-Macaulayness of $G(\mathbf{m})$. Let $\mathrm{Ap}_{g_1}(S') = \{\omega'_0, \dots, \omega'_{g_1-1}\}$. For each $i = 0, 1, \dots, g_1 - 1$, let a_i be the only integer such that $\omega'_i + a_i g_1 = \omega_i$, and let $b_i = \max\{l \mid \omega_i \in lM\}$. Clearly $b_0 = a_0 = 0$. Furthermore, $1 \leq b_i \leq a_i$ [**2**, Lemma 2.4]. The following result is proved in a more general setting, but we give the statement we will need in the sequel, that is for numerical semigroup rings; notice that under these hypotheses it could be deduced by Remark 2.4.

Theorem 2.1 [**2**, Theorem 2.6]. *If R is a semigroup ring, then $G(\mathbf{m})$ is Cohen-Macaulay (briefly,* C-M) *if and only if $a_i = b_i$, for every $i = 0, \dots, g_1 - 1$.*

A one-dimensional graded ring T with homogeneous maximal ideal $\mathcal{M}$ is called *Buchsbaum* if $\mathcal{M} \cdot H^0_{\mathcal{M}} = 0$ (cf. [**17**, Corollary 1.1]). Since $H^0_{\mathcal{M}} = (\cup_{k\geq 1}(0 :_T \mathcal{M}^k))$, the previous definition is equivalent to

$$\mathcal{M} \cdot (\cup_{k\geq 1}(0 :_T \mathcal{M}^k)) = 0.$$

Let R be a Noetherian, one-dimensional, local ring with maximal ideal $\mathbf{m}$ such that $|R/\mathbf{m}| = \infty$ and $\mathbf{m}$ contains a non-zerodivisor, and let r be the reduction number of $\mathbf{m}$, that is the minimal natural number such that $\mathbf{m}^{r+1} = x\mathbf{m}^r$, with x a superficial element of R (recall that such a number r exists by [**12**, Theorem 1, Section 2]). Then we have the following characterization.

Proposition 2.2 [**5**, Corollary 2.3]. *$G(\mathbf{m})$ is Buchsbaum if and only if $(0 :_{G(\mathbf{m})} \mathcal{M}) = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$.*

It is also possible to give the graded description of $(0 :_{G(\mathbf{m})} \mathcal{M}^r)$ as follows (cf. [**5**, Formula (2.3)]):

$$(0 :_{G(\mathbf{m})} \mathcal{M}^r) = \bigoplus_{h=1}^{r-2} \frac{(\mathbf{m}^{h+r+1} :_R \mathbf{m}^r) \cap \mathbf{m}^h}{\mathbf{m}^{h+1}}. \tag{2.1}$$

Furthermore, if we denote by R' the blow-up of R, that is, in our setting, $R' = \cup_i(\mathbf{m}^i :_{Q(R)} \mathbf{m}^i) = R[\mathbf{m}/x]$ (see, e.g., [**10**]), we have that $v(R') = S'$ and in [**5**, Proposition 2.5] it is proved that:

$$(0 :_{G(\mathbf{m})} \mathcal{M}^r) = \bigoplus_{h=1}^{r-2} \frac{x^{h+1}R' \cap \mathbf{m}^h}{\mathbf{m}^{h+1}}. \tag{2.2}$$

Remark 2.3. Since the valuation of any superficial element is $v(x) = g_1$, we can translate as follows the previous formula at the numerical semigroup level: let $G(\mathbf{m})$ not be C-M, and let $s \in hM \setminus (h+1)M$; then

$$\overline{t^s} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r) \Longleftrightarrow s - (h+1)g_1 \in S'.$$

Remark 2.4. Notice that $a_i > b_i$ if and only if $\overline{t^{\omega_i}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$. Indeed, if $a_i > b_i$, by definition, we have $\omega_i = \omega_i' + a_i g_1 = \omega_i' + (a_i - b_i - 1)g_1 + (b_i+1)g_1 \in b_i M \setminus (b_i+1)M$. Setting $\alpha = \omega_i' + (a_i - b_i - 1)g_1 \in S'$, we get $\alpha + (b_i+1)g_1 = \omega_i$. Hence, $\overline{t^{\omega_i}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$, by Remark 2.3. Conversely, if $a_i = b_i$, $\omega_i - (b_i+1)g_1 = \omega_i - a_i g_1 - g_1 = \omega_i' - g_1 \notin S'$; again by Remark 2.3, we get $\overline{t^{\omega_i}} \notin (0 :_{G(\mathbf{m})} \mathcal{M}^r)$.

3. Buchsbaumness in the general case. As in the preliminaries, $(R, \mathbf{m})$ is the numerical semigroup ring associated to a n-generated numerical semigroup S, $G(\mathbf{m})$ is its associated graded ring, $\mathcal{M}$ is the homogeneous maximal ideal of $G(\mathbf{m})$, M is the maximal ideal of S and r is the reduction number of $\mathbf{m}$.

For each i such that $a_i > b_i$, let $l_i = \max\{l \mid \overline{t^{\omega_i + l g_1}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)\}$.

Remark 3.1. We note that l_i is well defined because from formula (2.2) we have that $\overline{t^{\omega_i + l g_1}} \notin (0 :_{G(\mathbf{m})} \mathcal{M}^r)$, whenever $t^{\omega_i + l g_1} \in \mathbf{m}^{r-1}$.

By [**5**, Proposition 3.5], if $a_i > b_i$, then $r \geq b_i + 2$.

Remark 3.2. We note that $l_i \leq r - 2 - b_i$. Indeed, $\omega_i \in b_i M$; hence, $\omega_i + l_i g_1 \in (b_i + l_i)M$ and $b_i + l_i \leq r - 2$ by formula (2.2).

Lemma 3.3. *Let i and l_i be as above. Then $\overline{t^{\omega_i + l g_1}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ for every $l = 0, \dots, l_i$.*

Proof. By hypothesis $\overline{t^{\omega_i + l_i g_1}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$; therefore, by Remark 2.3, if $\omega_i + l_i g_1 \in hM \setminus (h+1)M$ then $\omega_i + l_i g_1 - (h+1)g_1 \in S'$. Let $l = l_i - 1$, and let us suppose $\omega_i + l g_1 \in nM \setminus (n+1)M$. If $\overline{t^{\omega_i + l g_1}} \notin (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ then $\omega_i + l g_1 - (n+1)g_1 \notin S'$; it follows that $\omega_i' > \omega_i + l g_1 - (n+1)g_1$. Since $n < h$ and $l = l_i - 1$, we get $\omega_i + l_i g_1 - (h+1)g_1 = \omega_i + l g_1 - h g_1 \leq \omega_i + l g_1 - (n+1)g_1 < \omega_i'$.

Hence $\omega_i + l_i g_1 - (h+1)g_1 \notin S'$, a contradiction. Using a decreasing induction we get the thesis. □

Lemma 3.4. *The only monomials in* $(0 :_{G(\mathbf{m})} \mathcal{M}^r)$ *are of the form* $\overline{t^{\omega_i + lg_1}}$, *with* i *such that* $a_i > b_i$.

Proof. Let $\overline{t^c} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$. Then $\overline{t^c} \in ((t^{g_1})^{h+1}R' \cap \mathbf{m}^h)/\mathbf{m}^{h+1}$, with h such that $t^c \in \mathbf{m}^h \setminus \mathbf{m}^{h+1}$. In particular $c \in S$; hence, $c = \omega_i + lg_1$, for some index i.

Let us show that the case $a_i = b_i$ is not possible. Since $\overline{t^{\omega_i + lg_1}} \in (0 :_{G(\mathbf{m})} \mathcal{M}^r)$, by Remark 2.3 it follows that $\omega_i + lg_1 = \alpha + (h+1)g_1 \in hM \setminus (h+1)M$ with $\alpha = \omega'_i + \mu g_1 \in S'$, $\mu \geq 0$, that is, $\omega_i + lg_1 = \omega'_i + (\mu + h + 1)g_1 \in hM \setminus (h+1)M$. In particular, it is in S and this implies $\mu + h + 1 \geq a_i = b_i$. Furthermore, $\omega'_i + (\mu + h + 1)g_1 = \omega'_i + b_i g_1 + (\mu + h + 1 - b_i)g_1 \in (b_i M + (\mu + h + 1 - b_i)M) \setminus (h+1)M = (\mu + h + 1)M \setminus (h+1)M$ and we get $\mu + h + 1 < h + 1$. Absurd. □

Corollary 3.5. *Let* $G(\mathbf{m})$ *not be* C-M. *Then*

$$(0 :_{G(\mathbf{m})} \mathcal{M}^r) = \left\langle \overline{t^{\omega_i + lg_1}} \mid a_i > b_i, \quad l = 0, \dots, l_i \right\rangle_k .$$

Proof. It follows by Lemmas 3.3 and 3.4. □

Furthermore, by the previous corollary and by Proposition 2.2, we get the following characterization.

Proposition 3.6. $G(\mathbf{m})$ *is Buchsbaum if and only if* $\overline{t^{\omega_i + lg_1}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$, *for every* i *such that* $a_i > b_i$ *and for every* $l = 0, \dots, l_i$.

The following proposition gives a bound for l_i when $G(\mathbf{m})$ is Buchsbaum.

Proposition 3.7. *Let* $G(\mathbf{m})$ *be Buchsbaum and* i *be such that* $a_i > b_i$. *Then* $l_i < a_i - b_i$.

Proof. By the definitions of a_i and b_i we have that $\omega_i = \omega'_i + (a_i - b_i - 1)g_1 + (b_i + 1)g_1 \in b_i M \setminus (b_i + 1)M$ with $\omega'_i + (a_i - b_i$

$-1)g_1 \in S'$. By hypothesis $(0:_{G(\mathbf{m})} \mathcal{M}) = (0:_{G(\mathbf{m})} \mathcal{M}^r)$; hence, $\omega_i + l_i g_1 \in hM \setminus (h+1)M$ with $h \geq b_i + 2l_i$. By definition of l_i and by Remark 2.3, we have that $\omega_i + l_i g_1 - (h+1)g_1 \in S'$; hence, $\omega_i + l_i g_1 - (h+1)g_1 \geq \omega_i' = \omega_i - a_i g_1$, that is, $l_i g_1 - (h+1)g_1 \geq -a_i g_1$. Finally, $l_i g_1 + a_i g_1 \geq (h+1)g_1 \geq (b_i + 2l_i + 1)g_1$ implies $l_i < a_i - b_i$. □

If $a_i - b_i = 1$, for every i such that $a_i > b_i$, then we can improve Proposition 3.6.

Proposition 3.8. *If $a_i - b_i = 1$ for every i such that $a_i > b_i$, then*

$G(\mathbf{m})$ is Buchsbaum if and only if $\overline{t^{\omega_i}} \in (0:_{G(\mathbf{m})} \mathcal{M})$
for every such index i.

Proof. By Proposition 3.6, we need only to prove the sufficient condition. By hypothesis and by definition of b_i and a_i, we have that $\omega_i + g_1 \in hM \setminus (h+1)M$ with $h \geq b_i + 2$ and $\omega_i + g_1 - (h+1)g_1 = \omega_i - hg_1 \notin S'$. Hence $\overline{t^{\omega_i + g_1}} \notin (0:_{G(\mathbf{m})} \mathcal{M}^r)$ by Remark 2.3 and, by Lemma 3.3, we get $\overline{t^{\omega_i + lg_1}} \notin (0:_{G(\mathbf{m})} \mathcal{M}^r)$ for every $l \geq 1$. Finally, by Proposition 3.7 we get the thesis. □

Remark 3.9. We note that the last proposition does not hold if there exists an i such that $a_i - b_i \geq 2$. Indeed, let $S = \langle 12, 19, 29, 104 \rangle$. The reduction number of S is $r = 8$. The only index for which $a_i > b_i$ is $i = 8$ with $a_8 = 4$ and $b_8 = 1$; moreover, $\omega_8 = g_4 = 104$. Since $\omega_8 + g_j \in 3M$ for each $g_j = 12, 19, 29, 104$, then $\overline{t^{\omega_8}} \in (0:_{G(\mathbf{m})} \mathcal{M})$. Anyway $G(\mathbf{m})$ is not Buchsbaum. Indeed $\overline{t^{\omega_8 + g_1}} \notin (0:_{G(\mathbf{m})} \mathcal{M})$ as $\omega_8 + g_1 = 116 \in 4M \setminus 5M$ and $116 + g_1 = 128 \in 5M$. On the other hand $\overline{t^{\omega_8 + g_1}} \in (0:_{G(\mathbf{m})} \mathcal{M}^8)$ as $116 + (8M \setminus 9M) \subseteq 13M$.

Example 3.10. Let R be the semigroup ring associated to the numerical semigroup $S = \langle 17, 18, 21, 28, 29, 32, 33 \rangle$. We use Proposition 3.8 in order to show that $G(\mathbf{m})$ is Buchsbaum. The only indices i such that $a_i > b_i$ are $i = 7, 10$ and in both cases we have that $a_i = 3 > 2 = b_i$. We need to check that $\omega_7 + g_j = 58 + g_j \in (b_7 + 2)M = 4M$ and $\omega_{10} + g_j = 61 + g_j \in (b_{10} + 2)M = 4M$, for each $g_j = 17, 18, 21, 28, 29, 32, 33$. Since $4M = \{68, \to\}$, this is clearly true and $G(\mathbf{m})$ is Buchsbaum.

Our next goal is to relate the Buchsbaumness of $G(\mathbf{m})$ to the length $\lambda = \lambda(H^0_{\mathcal{M}})$ of the $G(\mathbf{m})$-module $H^0_{\mathcal{M}} = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ (see the next Proposition 3.15).

Lemma 3.11. *We have $\lambda = 1$ if and only if $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = G(\mathbf{m})x$, with $x \in (0 :_{G(\mathbf{m})} \mathcal{M})$.*

Proof. Let $\lambda = 1$. Clearly $N := (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ must be principal as a $G(\mathbf{m})$-module. If $x \notin (0 :_{G(\mathbf{m})} \mathcal{M})$, then $(0) \subsetneq \mathcal{M}N$. Moreover, by the graded version of Nakayama's Lemma, $\mathcal{M}N \subsetneq N$. A contradiction to $\lambda = 1$.

Conversely, let $N := (0 :_{G(\mathbf{m})} \mathcal{M}^r) = G(\mathbf{m})x$, with $x \in (0 :_{G(\mathbf{m})} \mathcal{M})$, and let us suppose $(0) \subsetneq H \subsetneq N$, with H submodule of N. Then, every $\overline{h} \in H$, $\overline{h} \neq 0$, is of the form $\overline{h} = \overline{g}x$ with $\overline{g} \in G(\ m)$. Since $\overline{h} \neq 0$, we have that $\overline{g} \notin \mathcal{M}$. Then $\overline{g} = \overline{x}_0 + \overline{y} \in R/\mathbf{m} \oplus \mathcal{M}$ with $x_0 \neq 0$. Finally $\overline{h} = \overline{g}x = (\overline{x}_0 + \overline{y})x = \overline{x}_0 x + \overline{y}x = \overline{x}_0 x$ and, since $\overline{x}_0$ is a unit in $G(\mathbf{m})$, we get $x \in H$. By the choice of x, we get $N \subseteq H$; a contradiction. □

Corollary 3.12. *If $\lambda \leq 1$, then $G(\mathbf{m})$ is Buchsbaum.*

Proof. If $\lambda = 0$, then $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = (0)$, that is, $G(\mathbf{m})$ is C-M. If $\lambda = 1$, then, by Lemma 3.11, $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = G(\mathbf{m})x$, with $x \in (0 :_{G(\mathbf{m})} \mathcal{M})$. This implies $(0 :_{G(\mathbf{m})} \mathcal{M}^r) \subseteq (0 :_{G(\mathbf{m})} \mathcal{M})$. □

Remark 3.13. We note that the converse of Corollary 3.12 does not hold in general. Indeed, let R be as in the Example 3.10. We showed that $G(\mathbf{m})$ is Buchsbaum. Moreover, by Corollary 3.5 and Proposition 3.7, we have that $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = \langle \overline{t^{58}}, \overline{t^{61}} \rangle_{G(\mathbf{m})/\mathcal{M}}$; hence, $(0) \subsetneq G(\mathbf{m})\overline{t^{58}} \subsetneq (0 :_{G(\mathbf{m})} \mathcal{M}^r)$.

It is possible to relate λ with the l_i's when $G(\mathbf{m})$ is Buchsbaum.

Lemma 3.14. *Let $G(\mathbf{m})$ be Buchsbaum. Let i, j be such that $a_i > b_i$ and $a_j > b_j$; then $\overline{t^{\omega_i+lg_1}} \in G(\mathbf{m})\overline{t^{\omega_j}}$ if and only if $i = j$ and $l = 0$.*

Proof. If $\overline{t^{\omega_i+lg_1}} \in G(\mathbf{m})\overline{t^{\omega_j}}$, then $\overline{t^{\omega_i+lg_1}} = \overline{u}\overline{t^{\omega_j}}$. Since $\overline{t^{\omega_j}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$ and $\overline{t^{\omega_i+lg_1}} \neq \overline{0}$, then $\overline{u} \notin \mathcal{M}$ and $\overline{u} = u + \mathbf{m}$ with $u \in k$; hence $\overline{t^{\omega_i+lg_1}} = \overline{u}\overline{t^{\omega_j}} = \overline{ut^{\omega_j}}$. The last equality is equivalent to the fact that $t^{\omega_i+lg_1}, ut^{\omega_j} \in \mathbf{m}^{b_j} \setminus \mathbf{m}^{b_j+1}$ and $t^{\omega_i+lg_1} - ut^{\omega_j} \in \mathbf{m}^{b_j+1}$, that is, $\omega_i + lg_1 = \omega_j$, but this is true only for $l = 0$ and $i = j$. □

The next proposition immediately follows by Corollary 3.5 and Lemma 3.14.

Proposition 3.15. *If $G(\mathbf{m})$ is Buchsbaum, then $\lambda = \sum_{i \in I}(l_i + 1)$ with $I = \{i \mid a_i > b_i\}$.*

Corollary 3.16. *If $G(\mathbf{m})$ is Buchsbaum, then $\lambda \leq \sum_{i \in I}(a_i - b_i)$ with $I = \{i \mid a_i > b_i\}$. Moreover, if $a_i = b_i + 1$ for every $i \in I$, then $\lambda = |I|$.*

Proof. It follows by Propositions 3.7, 3.8 and 3.15. □

Our next aim is to study for which elements ω_i of the Apery set of S it is possible to have $a_i > b_i$; we get a necessary condition, in the case $G(\mathbf{m})$ is Buchsbaum.

Let $s \in S$, and define $\operatorname{ord}(s) := h$ if $s \in hM \setminus (h+1)M$. In particular we have $\operatorname{ord}(\omega_i) = b_i$. We now introduce a partial ordering on S as in [**3**]: given $u, u' \in S$, we say that $u \leq_M u'$ if $u + s = u'$ (hence $u \leq_S u'$) and $\operatorname{ord}(u) + \operatorname{ord}(s) = \operatorname{ord}(u')$ for some $s \in S$.

Remark 3.17. The set of maximal elements of $\operatorname{Ap}_{g_1}(S)$ with this partial ordering is denoted with $\max \operatorname{Ap}_M(S)$. We note that the set of maximal elements in $\operatorname{Ap}_{g_1}(S)$ with the usual ordering $\leq_S$ is contained in $\max \operatorname{Ap}_M(S)$ and the inclusion can be strict. For example, let $S = \langle 8, 9, 15 \rangle$. The only maximal element in $\operatorname{Ap}_{g_1}(S)$ with respect to $\leq_S$ is 45. Anyway $\max \operatorname{Ap}_M(S) = \{30, 45\}$. Note that $\operatorname{ord}(45) = 5 > 3 = \operatorname{ord}(30) + \operatorname{ord}(15)$.

We say that ω_i and ω_j are *comparable* if $\omega_i \leq_M \omega_j$ or vice versa.

Remark 3.18. Let $G(\mathbf{m})$ be Buchsbaum. If $a_i > b_i$ and $a_j > b_j$, then ω_i and ω_j are not comparable. Indeed, if there exists an $s \in S \setminus \{0\}$ such that $\omega_j = \omega_i + s \in S$ and $b_i + \operatorname{ord}(s) = b_j$, then $\overline{t^{\omega_i}} \notin (0 :_{G(\mathbf{m})} \mathcal{M}^{\operatorname{ord}(s)}) = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$. A contradiction to Remark 2.4.

Proposition 3.19. *Let $G(\mathbf{m})$ be Buchsbaum. Then $a_i > b_i$ implies $\omega_i \in \max \operatorname{Ap}_M(S)$.*

Proof. If $\omega_i \notin \max \operatorname{Ap}_M(S)$, then there exists ω_j such that $\omega_i <_M \omega_j$, that is, there exists an element $s \in S \setminus \{0\}$ such that $\omega_j = \omega_i + s$ and $b_j = b_i + \operatorname{ord}(s)$. By hypothesis $\overline{t^{\omega_i}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$; hence, $\overline{t^{\omega_i}} \cdot \overline{t^s} = \overline{0}$; this implies $\omega_j = \omega_i + s \in (b_i + \operatorname{ord}(s) + 1)M = (b_j + 1)M$. Absurd. □

Remark 3.20. We note that the converse of the last proposition does not hold in general. Indeed, let R be the semigroup ring associated to $S = \langle 12, 19, 29, 104\rangle$. In this case the unique index i such that $a_i > b_i$ is $i = 8$ and $\omega_8 = 104 \in \max \operatorname{Ap}_M(S)$; but $G(\mathbf{m})$ is not Buchsbaum as shown in Remark 3.9.

We end this section with a general remark that will be useful for the next sections.

Remark 3.21. Let $a_i = b_i$, and let $\omega_i = \alpha_2 g_2 + \cdots + \alpha_n g_n$ with $\sum_{k=2}^n \alpha_k = b_i$. By definition of a_i and by the equality $a_i = b_i$, we have that $\alpha_2(g_2 - g_1) + \cdots + \alpha_n(g_n - g_1) = \omega_i' \in \operatorname{Ap}_{g_1}(S')$.

On the other hand, let $a_i > b_i$ and $\omega_i = \alpha_2 g_2 + \cdots + \alpha_n g_n$ with $\sum_{k=2}^n \alpha_k = b_i$. In this case, $\alpha_2(g_2 - g_1) + \cdots + \alpha_n(g_n - g_1) \notin \operatorname{Ap}_{g_1}(S')$. Indeed, $\alpha_2 g_2 + \cdots + \alpha_n g_n - (\sum_{k=2}^n \alpha_k) g_1 = \alpha_2 g_2 + \cdots + \alpha_n g_n - b_i g_1 > \alpha_2 g_2 + \cdots + \alpha_n g_n - a_i g_1 = \omega_i'$, hence $\alpha_2(g_2 - g_1) + \cdots + \alpha_n(g_n - g_1) - g_1 \in S'$.

4. The 3-generated case. In this section we will apply and deepen our results, when the semigroup S is 3-generated. As a by-product, we will give a positive answer to two conjectures raised by Sapko in [**15**].

These two conjectures are also proved by Shen in [**16**] using completely different methods.

Let us fix our notation for this section: $S = \langle g_1, g_2, g_3 \rangle$, with $g_1 < g_2 < g_3$; the elements in $\mathrm{Ap}_{g_1}(S)$ are of the form $\omega_i = hg_2 + kg_3$ (with $h, k \in \mathbf{N}$).

With the symbol $x \equiv y$ we will always mean that x is congruent y modulo g_1; moreover, $x \lhd y$ (respectively $x \rhd y$) will always mean that $x \equiv y$ and that $x < y$ (respectively $x > y$). Finally $x \unlhd y$ (respectively $x \unrhd y$) will mean that $x \equiv y$ and that $x \leq y$ (respectively $x \geq y$).

If an element ω_i has more than one representation as a combination of g_2 and g_3, then the representation $hg_2 + kg_3$, where h is maximum, has the property that $h + k = b_i$ (this is not true if S has more than 3 generators).

We are ready to prove the main result of this section.

Theorem 4.1. *Assume that S is a 3 generated numerical semigroup, and assume that $G(\mathbf{m})$ is Buchsbaum. If $\omega_i = hg_2 + kg_3$ is an element of $\mathrm{Ap}_{g_1}(S)$ such that $a_i > b_i$, then $h = 0$.*

In particular, there is at most one element $\omega_i \in \mathrm{Ap}_{g_1}(S)$ such that $a_i > b_i$.

Proof. Since $G(\mathbf{m})$ is Buchsbaum, by Proposition 3.19 we have that, if $a_i > b_i$, then $\omega_i = hg_2 + kg_3 \in \max \mathrm{Ap}_M(S)$, for any representation of ω_i as a combination of g_2 and g_3. In particular, we consider the representation with h maximum, that is, $h + k = b_i$. If we prove that $h = 0$, then we get the first part of the theorem.

Assume, by contradiction, that $h > 0$, hence $(h-1)g_2 + kg_3 = \omega_j \in \mathrm{Ap}_{g_1}(S)$. We note that $\omega_j \leq_M \omega_i$ as $\omega_i = \omega_j + g_2$ and $b_i = b_j + 1$ (clearly $b_i \geq b_j + 1$ and, if $b_i > b_j + 1$, then $h + k > b_j + 1 \geq (h-1) + k + 1 = h + k$).

Since $\omega_j \notin \max \mathrm{Ap}_M(S)$ and $G(\mathbf{m})$ is Buchsbaum, we have $a_j = b_j$ again by Proposition 3.19. By Remark 3.21 we have

$$\omega_j' = (h-1)(g_2 - g_1) + k(g_3 - g_1).$$

By the same remark we also get $\omega_i' \lhd h(g_2 - g_1) + k(g_3 - g_1)$; recalling that $\omega_i' \in \mathrm{Ap}_{g_1}(S')$ and $S' = \langle g_1, g_2 - g_1, g_3 - g_1 \rangle$, we obtain $\omega_i' =$

$x(g_2 - g_1) + y(g_3 - g_1)$, for some nonnegative integers x and y. We collect this observation in the following formula

$$\omega_i' = x(g_2 - g_1) + y(g_3 - g_1) \triangleleft h(g_2 - g_1) + k(g_3 - g_1) \tag{4.1}$$

Moreover, by definition of a_i we obtain:

$$xg_2 + yg_3 - (x+y)g_1 + a_i g_1 = \omega_i \in \mathrm{Ap}_{g_1}(S);$$

it follows immediately that $x + y \geq a_i$. Hence, since $a_i > b_i = h + k$, we get $x + y > h + k$.

Now, if $x \leq h$, then $y > k$ and it would follow that $\omega_i' = x(g_2 - g_1) + y(g_3 - g_1) > h(g_2 - g_1) + k(g_3 - g_1)$, in contradiction to (4.1). Thus $x > h(> 0)$ and, in particular, $x > 0$.

It follows that $(x-1)(g_2 - g_1) + y(g_3 - g_1) \in S'$. But, again by (4.1), $(x-1)(g_2 - g_1) + y(g_3 - g_1) \triangleleft (h-1)(g_2 - g_1) + k(g_3 - g_1) = \omega_j'$. But this is a contradiction, since $\omega_j' \in \mathrm{Ap}_{g_1}(S')$.

Hence $h = 0$ and we have proved the first part of the theorem.

Let us prove the last assertion. By the first part, we have that the only elements ω_i for which it is possible to have $a_i > b_i$ are of the form jg_3 with $\mathrm{ord}\,(jg_3) = j$ and the set of this kind of elements is $\{0, g_3, \ldots, kg_3\}$, for some $k > 0$. Since this set is a subchain of $(\mathrm{Ap}(S), \leq_M)$, there is at most one maximal element. The thesis follows by Proposition 3.19. □

Remark 4.2. The integer k defined in the last part of the previous proof can be also defined in terms of the Apery set of S in the following way:

$$k = \min\{j \mid g_2 \text{ divides } (j+1)g_3 \text{ or } (j+1)g_3 - g_1 \in S\}.$$

As a consequence of the previous theorem we obtain a positive answer to two conjectures stated in [**15**] that we collect in the following statement.

Proposition 4.3. *Let S be a 3-generated numerical semigroup. Then the following conditions are equivalent*:

(i) $G(\mathbf{m})$ *is Buschsabum not* C-M;

(ii) $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = G(\mathbf{m})\overline{t^{kg_3}}$ *for some* $k \geq 1$, *with* $\overline{t^{kg_3}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$;

(iii) $\lambda = 1$.

Proof. The implication (ii) $\Rightarrow$ (iii) is obvious by Lemma 3.11 and the implication (iii) $\Rightarrow$ (i) is straightforward by Proposition 3.12. Let us prove the implication (i) $\Rightarrow$ (ii).

By Theorem 4.1, we know that there exists a unique ω_i such that $a_i > b_i$ and it is $\omega_i = kg_3$, with $k = \min\{j \mid g_2 \text{ divides } (j+1)g_3 \text{ or } (j+1)g_3 - g_1 \in S\}$. Hence $\omega_i = kg_3$ is the only element in the Apery set of S such that $\overline{t^{\omega_i}} \in (0 :_{G(\mathbf{m})} \mathcal{M}) = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$.

By Corollary 3.6 and Lemma 3.14, we need to prove that $\overline{t^{\omega_i + lg_1}} \notin (0 :_{G(\mathbf{m})} \mathcal{M})$, for every $l \geq 1$. By Lemma 3.3, it is enough to prove it for $l = 1$.

We note that, by definition of k, if $(k+1)g_3 \in \operatorname{Ap}(S)$, then g_2 divides $(k+1)g_3$; hence $kg_3 + g_3 = qg_2$, with $q > k+1$. Moreover, since $\overline{t^{kg_3}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$ and since $kg_3 \in \operatorname{Ap}(S)$, then $kg_3 + g_1 = \alpha g_2 + \beta g_3$, with $\operatorname{ord}(kg_3 + g_1) = \alpha + \beta > k + 1$ (and $\beta < k$). This implies that $g_3 - g_1 = (q-\alpha)g_2 - \beta g_3$, i.e., $(\beta+1)g_3 = (q-\alpha)g_2 + g_1$, with $\beta + 1 \leq k$. Contradiction against the assumption $(k+1)g_3 \in \operatorname{Ap}(S)$.

Hence we can assume $(k+1)g_3 \notin \operatorname{Ap}(S)$ and so there exists an integer $q \geq 0$, such that $kg_3 + qg_2$ is maximal in the Apery set of S. Now, if $kg_3 + qg_2 = ug_3 + vg_2$ with $v > q$, then $(k-u)g_3 = (v-q)g_2$ and this is a contradiction against the definition of k. Hence $\operatorname{ord}(kg_3 + qg_2) = k + q$ and necessarily $q = 0$ (if not $kg_3 \notin \max \operatorname{Ap}_M(S)$).

In order to show that $\overline{t^{\omega_i + g_1}} \notin (0 :_{G(\mathbf{m})} \mathcal{M})$, it is enough to prove that $kg_3 + 2g_1 \notin (\alpha + \beta + 2)M$ where, as above, $kg_3 + g_1 = \alpha g_2 + \beta g_3$ and $\operatorname{ord}(kg_3 + g_1) = \alpha + \beta > k + 1$ (with $\beta < k$, $\alpha > 0$).

If $kg_3 + 2g_1 = ag_1 + bg_2 + cg_3$ with $\operatorname{ord}(kg_3 + 2g_1) = a + b + c \geq \alpha + \beta + 2$, then, by definition of k and by $kg_3 \in \operatorname{Ap}(S)$, we have $a < 2$. The case $a = 1$ is not possible, as we would have $kg_3 + g_1 = bg_2 + cg_3$ with $b + c > \alpha + \beta = \operatorname{ord}(kg_3 + g_1)$. Absurd. Hence $a = 0$. If $c \geq k$, then $2g_1 \geq bg_2$ and so $b = 1$; but this is not possible since the case $c = k$ would give us $2g_1 = g_2$, and the case $c > k$ would give us $2g_1 = g_2 + (c-k)g_3$. Hence $c < k$ (and $b > 1$).

Since $a_i > b_i = k$, we have that $kg_3 - kg_1 \notin \mathrm{Ap}(S')$; hence $k(g_3 - g_1) \triangleright x(g_2 - g_1) + y(g_3 - g_1) \in \mathrm{Ap}(S')$, for some integers x and y. If $y > 0$, then $(k - y)(g_3 - g_1) \triangleright x(g_2 - g_1) \in S'$ and this is not possible since $(k - y)(g_3 - g_1) \in \mathrm{Ap}(S')$, by Remark 3.21. Hence $y = 0$ and $k(g_3 - g_1) \triangleright x(g_2 - g_1) \in \mathrm{Ap}(S')$ and so there exists a $z > 0$ such that $k(g_3 - g_1) = x(g_2 - g_1) + zg_1$, that is, $\alpha g_2 + \beta g_3 = kg_3 + g_1 = xg_2 - (x - k - z - 1)g_1$. Hence $xg_2 = kg_3 + \mu g_1 = \alpha g_2 + \beta g_3 + (\mu - 1)g_1$ with $\mu > 0$ and $\beta g_3 + (\mu - 1)g_1 = (x - \alpha)g_2$. Now $(x - \alpha)g_2 \in \mathrm{Ap}(S)$, since $(x - 1)g_2 \in \mathrm{Ap}(S)$; the last assertion follows by Remark 3.21 and by Theorem 4.1: the map

$$\varphi : \mathrm{Ap}(S) \setminus \{kg_3\} \longrightarrow \mathrm{Ap}(S') \setminus \{x(g_2 - g_1)\}$$

defined by $\varphi(\gamma g_2 + \delta g_3) = \gamma(g_2 - g_1) + \delta(g_3 - g_1)$ is bijective. Since $x(g_2 - g_1) \in \mathrm{Ap}(S')$, also $(x - 1)(g_2 - g_1) \in \mathrm{Ap}(S')$; the bijection implies that $(x - 1)g_2 \in \mathrm{Ap}(S)$.

By $\beta g_3 + (\mu - 1)g_1 = (x - \alpha)g_2 \in \mathrm{Ap}(S)$, we have that $\mu = 1$; moreover, since $\beta < k$, we get $x = \alpha$ and $\beta = 0$.

Hence $kg_3 + g_1 = \alpha g_2$ with $\alpha > k + 1$, and $kg_3 + 2g_1 = bg_2 + cg_3$ with $b + c > \alpha + 2$, $c < k$ and $b > 1$; hence $g_1 = (b - \alpha)g_2 + cg_3$, so, necessarily, $c \neq 0$ and $b < \alpha$. But this implies $g_1 + (\alpha - b)g_2 = cg_3$ and this is absurd by definition of k and by $c < k$. □

By the proof of the previous proposition, it is straightforward that the integer k of the statement (point (ii)) is the same integer defined in Remark 4.2; hence, it is determined in terms of the Apery set of S:

Corollary 4.4. *Let S be a 3-generated numerical semigroup. If $G(\mathbf{m})$ is Buchsbaum not* C-M*, then $(0 :_{G(\mathbf{m})} \mathcal{M}^r) = G(\mathbf{m})\overline{t^{kg_3}}$, where the integer k is determined as follows*:

$$k = \min\{j \mid g_2 \textit{ divides } (j + 1)g_3 \textit{ or } (j + 1)g_3 - g_1 \in S\}.$$

Using Theorem 4.1 we can also prove, in the case of 3 generators, that, if R is Gorenstein, then $G(\mathbf{m})$ is C-M if and only if it is Buchsbaum. This fact is also proved in [**16**] using different methods.

Corollary 4.5. *Let S be a 3-generated symmetric numerical semigroup. If $G(\mathbf{m})$ is Buchsbaum, then it is* C-M.

Proof. By the proof of Theorem 4.1 (and since $G(\mathbf{m})$ is Buchsbaum), it is possible to have $a_i > b_i$ only for $\omega_i = kg_3$, with $k = \min\{j \mid g_2$ does not divide $(j+1)g_3$ or $(j+1)g_3 - g_1 \in S\}$ (we underline that $b_i = \text{ord}\,(kg_3) = k$). So, by Theorem 2.1, we only need to show that $a_i = b_i$.

Since S is symmetric, there exists a unique maximal element $g + g_1$ in the Apery set of S (with the partial ordering $\leq_S$ as in the Preliminaries). Assume that $kg_3 + g_3 \notin \text{Ap}(S)$; it follows that $g + g_1 = qg_2 + kg_3$. Moreover, this representation is unique as, if $qg_2 + kg_3 = ug_2 + vg_3$, then $u > q$ and $v < k$ (if not $v > k$ and $ug_2 + vg_3 \notin \text{Ap}(S)$), and we get $(k - v)g_3 = (u - q)g_2$ that implies $\text{ord}\,(kg_3) > k$. The uniqueness of the representation implies that $\text{ord}\,(qg_2 + kg_3) = q + k$. It follows that $kg_3 \leq_M qg_2 + kg_3$ and so $kg_3 \notin \max \text{Ap}_M(S)$, unless $q = 0$. But $g_2 \in \text{Ap}(S)$ and $g_2 \leq_S g + g_1$, hence, if $q = 0$, then $\text{ord}\,(kg_3) > k$. Hence $q \neq 0$, $kg_3 \notin \max \text{Ap}_M(S)$ and, by Proposition 3.19, $a_i = b_i$.

Assume, now, that $kg_3 + g_3 \in \text{Ap}(S)$. By definition of k, it follows that $(*) kg_3 + g_3 = ug_2$. Let us suppose that $\text{ord}\,|(\omega_i + g_1) > b_i + 1 = k + 1$, that is, $kg_3 + g_1 = ag_1 + bg_2 + cg_3$, with $a + b + c > k + 1$. Since $\omega_i \in \text{Ap}(S)$, we get $a \leq 1$ and $c < k$. The case $a = 1$ is not possible as $\text{ord}\,(kg_3) = k$; hence, $a = 0$, that is, $(**) kg_3 + g_1 = bg_2 + cg_3$ (with $b + c > k + 1$). By $(*)$ and $(**)$ it follows that $g_3 - g_1 = (u - b)g_2 - cg_3$, i.e., $(c + 1)g_3 = (u - b)g_2 + g_1$. Since $c < k$, this is a contradiction against the definition of k; therefore, $\text{ord}\,(\omega_i + g_1) = b_i + 1 = k + 1$. Thus $\overline{t^{\omega_i}} \notin (0 :_{G(\mathbf{m})} \mathcal{M}) = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$ and, by Remark 2.4, $a_i = b_i$. □

5. Cohen-Macaulayness and Gorensteinness in the general case. In [**2**, Theorem 2.6] the authors proved, in particular, that $G(\mathbf{m})$ is C-M if and only if $a_i = b_i$ for every $i = 0, \dots, g_1 - 1$. As a consequence of this, there is an algorithm to check whether $G(\mathbf{m})$ is C-M or not:

1) compute hM for $h = 1, \dots, r$,

2) find $\text{Ap}_{g_1}(S)$ and $\text{Ap}_{g_1}(S')$,

3) determine a_i and b_i for $i = 0, \dots, g_1 - 1$

4) compare a_i and b_i. If an i exists such that $a_i > b_i$, then $G(\mathbf{m})$ is not C-M. If not, it is C-M.

In the next proposition we improve the characterization and the algorithm above, showing that in 3) it is sufficient to determine a_i and b_i, just for those i such that $\omega_i \in \max \mathrm{Ap}_M(S)$.

Proposition 5.1. *$G(\mathbf{m})$ is C-M if and only if $a_i = b_i$, for those i such that $\omega_i \in \max \mathrm{Ap}_M(S)$.*

Proof. By Theorem 2.1, we have only to prove the sufficient condition. Assume that $a_i = b_i$, for those i such that $\omega_i \in \max \mathrm{Ap}_M(S)$, and let $\omega_j = \alpha_2 g_2 + \cdots + \alpha_n g_n \notin \max \mathrm{Ap}_M(S)$. Then there exists $\omega_i = \beta_2 g_2 + \cdots + \beta_n g_n \in \max \mathrm{Ap}_M(S)$, with $\sum_{k=2}^n \beta_k = b_i = a_i$, such that $\omega_j + \eta_2 g_2 + \cdots + \eta_n g_n = \omega_i$ and $b_j + \mathrm{ord}\,(\eta_2 g_2 + \cdots + \eta_n g_n) = b_i$.

By Remark 3.21, if $a_j > b_j$, $\alpha_2(g_2 - g_1) + \cdots + \alpha_n(g_n - g_1) \notin \mathrm{Ap}_{g_1}(S')$; on the other hand, $a_i = b_i$ implies that $\beta_2(g_2 - g_1) + \cdots + \beta_n(g_n - g_1) = \omega_i' \in \mathrm{Ap}_{g_1}(S')$.

Finally, by $\sum_{k=2}^n (\alpha_k + \eta_k) g_k = \sum_{k=2}^n \beta_k g_k$ and $\sum_{k=2}^n \alpha_k + \sum_{k=2}^n \eta_k = \sum_{k=2}^n \beta_k$, we get $\sum_{k=2}^n \alpha_k(g_k - g_1) + \sum_{k=2}^n \eta_k(g_k - g_1) = \sum_{k=2}^n \beta_k(g_k - g_1) = \omega_i' \in \mathrm{Ap}_{g_1}(S')$. Contradiction. □

Example 5.2. Let $S = \langle 10, 13, 14 \rangle$. Then $G(\mathbf{m})$ is C-M as $\max \mathrm{Ap}_M(S) = \{\omega_5 = 55, \omega_9 = 39\}$ and $a_5 = b_5 = 4$ and $a_9 = b_9 = 3$.

Remark 5.3. Proposition 5.1 does not hold in general if we only consider the maximal elements in the Apery set of S with respect to $\leq_S$. Let R be the semigroup ring associated to $S = \langle 7, 8, 9, 19 \rangle$. The maximal elements in the Apery set of S are $\{\omega_3 = 17, \omega_6 = 27\}$ and $a_3 = b_3 = 2$ and $a_6 = b_6 = 3$. Anyway $G(\mathbf{m})$ is not C-M as $\omega_5 = 19$ and $a_5 = 2 > 1 = b_5$.

In Corollary 4.5, we showed that in the 3-generated case, if R is Gorenstein, then the properties for $G(\mathbf{m})$ to be C-M and Buchsbaum are equivalent.

Remark 5.4. We note that in the n-generated case, Corollary 4.5 is not true. Let us consider the symmetric numerical semigroup $S = \langle 8, 9, 12, 13, 19 \rangle$. The only index i for which $a_i > b_i$ is $i = 3$

(in particular $G(\mathbf{m})$ is not C-M); more precisely we have $a_3 = 2$, $b_3 = 1$ and $\omega_3 = 19$. Since $\overline{t^{19}} \in (0 :_{G(\mathbf{m})} \mathcal{M})$, then $G(\mathbf{m})$ is Buchsbaum by Proposition 3.8.

Anyway, if we force the elements of $\max \mathrm{Ap}_M(S)$ to have all the same order, then Corollary 4.5 is true in the n-generated case. A numerical semigroup S is called *M-pure* if every element in $\max \mathrm{Ap}_M(S)$ has the same order (cf. [**3**]). In this case it is clear that $\max \mathrm{Ap}_M(S)$ coincides with the set of the maximal elements of $\mathrm{Ap}(S)$ with respect to $\leq_S$.

Proposition 5.5. *Let S be a M-pure symmetric numerical semigroup. If $G(\mathbf{m})$ is Buchsbaum, then it is* C-M.

Proof. Let $\mathrm{Ap}(S) = \{\omega_0, \dots, \omega_{g_1-1}\} = \{0 < v_1 < \cdots < v_{g_1-1} = g + g_1\}$ (where $<$ is the natural ordering in $\mathbf{N}$). Since S is M-pure and symmetric, we get $\max \mathrm{Ap}_M(S) = \{v_{g_1-1}\}$. Since $G(\mathbf{m})$ is Buchsbaum, it is possible to have $a_i > b_i$ only for $\omega_i = v_{g_1-1}$. So, by Theorem 2.1, we only need to show that $a_i = b_i$.

Let us consider ω_i'. If it is a minimal generator of S' (different from g_1, because $\omega_i' \in \mathrm{Ap}(S')$), then $\omega_i' = g_t - g_1$, for some generator g_t of S $(t \neq 1)$; hence $\omega_i = g_t$ and this implies $a_i = 1 = b_i$.

On the other hand, if ω_i' is not a minimal generator of S', it can be written as a sum of two elements of S', that are necessarily elements of $\mathrm{Ap}(S')$. Hence we have an equality $\omega_i' = \omega_j' + \omega_h'$, for some $j, h \neq i$. It follows that $\omega_i = g + g_1 \equiv \omega_j + \omega_h$ and by the symmetry of S we immediately get $\omega_i = g + g_1 = \omega_j + \omega_h$. It follows that

$$\begin{aligned} \omega_i' + a_i g_1 = \omega_i = \omega_j + \omega_h &= \omega_j' + \omega_h' + (a_j + a_h) g_1 \\ &= \omega_j' + \omega_h' + (b_j + b_h) g_1 = \omega_i' + (b_j + b_h) g_1. \end{aligned}$$

Hence $a_i = b_j + b_h \leq b_i \leq a_i$ and so $a_i = b_i$. □

As an immediate corollary of the last proposition we can improve [**3**, Corollary 3.20] in which the author proved that:

$G(\mathbf{m})$ is Gorenstein $\iff$ S is symmetric, M-pure and $G(\mathbf{m})$ is C-M.

Corollary 5.6. *$G(\mathbf{m})$ is Gorenstein if and only if S is symmetric, M-pure and $G(\mathbf{m})$ is Buchsbaum.*

Let J be a parameter ideal of a Noetherian local ring $(R, \mathbf{m})$; the index of nilpotency of $\mathbf{m}$ with respect to J is defined to be the integer $s_J(\mathbf{m}) = \min\{n \mid \mathbf{m}^{n+1} \subseteq J\}$.

If $J = (t^{g_1})$, then J is a reduction of $\mathbf{m}$, $s_J(\mathbf{m}) = \max\{\mathrm{ord}\,(\omega_i) \mid \omega_i \in \mathrm{Ap}(S)\}$ and $s_J(\mathbf{m}) \leq r$, where r is the reduction number of R (see, e.g., [**3**]). In [**3**, Theorem 3,14], it is also proven that, with this choice of J,

$$\text{if } S \text{ is } M\text{-pure, then } G(m) \text{ is C-M} \Longleftrightarrow s_J(m) = r.$$

Hence, combining the previous results we immediately get

$$\begin{aligned} &S \text{ is } M\text{-pure, symmetric and } s_J(\mathbf{m}) = r \\ &\qquad \Longleftrightarrow S \text{ is } M\text{-pure, symmetric and } G(\mathbf{m}) \text{ is Buchsbaum.} \end{aligned}$$

It is natural to ask if, in the previous equivalence, one can skip one or both the condition S symmetric and S M-pure.

It is easy to see that the condition $G(\mathbf{m})$ Buchsbaum does not imply $s_J(\mathbf{m}) = r$: if $S = \langle 4, 5, 11 \rangle$ then $G(\mathbf{m})$ is Buchsbaum (since $r = 3$; cf. [**9**, Proposition 7.7]), but $s_J(\mathbf{m}) = 2 < r$. Also the implication $s_J(\mathbf{m}) = r \Rightarrow G(\mathbf{m})$ Buchsbaum is false: if $S = \langle 9, 10, 11, 23 \rangle$, $s_J(\mathbf{m}) = r = 4$, but $G(\mathbf{m})$ is not Buchsbaum, as follows by Proposition 3.19, since $2 = a_5 > b_5 = 1$ and $\omega_5 = 23 \notin \max \mathrm{Ap}_M(S)$.

Since $S = \langle 9, 10, 11, 23 \rangle$ is a symmetric numerical semigroup, the same example shows that

$$S \text{ symmetric and } s_J(\mathbf{m}) = r \not\Rightarrow S \text{ symmetric and } G(\mathbf{m}) \text{ Buchsbaum.}$$

As for the converse, we do not have counterexamples nor evidence that it should be false.

Question 5.7. Let $(R, \mathbf{m})$ be a numerical semigroup ring with associated semigroup S. Assume that S is symmetric and $G(\mathbf{m})$ is Buchsbaum; is it true that $s_J(\mathbf{m}) = r$?

Finally, by [**3**, Theorem 3.14] we know that, if S is M-pure, $s_J(\mathbf{m}) = r$ is equivalent to $G(\mathbf{m})$ C-M; hence, if S is M-pure and $s_J(\mathbf{m}) = r$, then $G(\mathbf{m})$ is Buchsbaum; conversely, if S is M-pure and $G(\mathbf{m})$ is Buchsbaum, we get that $s_J(\mathbf{m}) = r$ if and only if $G(\mathbf{m})$ is C-M. We do not have any example of an M-pure numerical semigroup such that $G(\mathbf{m})$ is Buchsbaum not C-M.

Question 5.8. Let $(R, \mathbf{m})$ be a numerical semigroup ring with associated semigroup S. Assume that S is M-pure and $G(\mathbf{m})$ is Buchsbaum; is it true that $G(\mathbf{m})$ is C-M?

After the paper was accepted for publication, we received from Y.H. Shen the following affirmative answer to Question 5.8, that we publish with his permission.

Proposition 5.9 (Shen). *Let $(R, \mathbf{m})$ be a numerical semigroup ring with M-pure associated semigroup S. If $G(\mathbf{m})$ is Buchsbaum, then $s_J(\mathbf{m}) = r$.*

Proof. Since S is M-pure, for every $\omega_i \in \max Ap_M(S)$, we have $\operatorname{ord}(\omega_i) = s_J(\mathbf{m}) \leq r$. For simplicity of notation let us denote $s_J(\mathbf{m}) = s$. If $s < r$, then $\mathbf{m}^{s+1} \neq t^{g_1}\mathbf{m}^s$. Indeed, we will have $(t^{g_1}) \supseteq \mathbf{m}^{s+1} \supsetneq t^{g_1}\mathbf{m}^s$. Hence, there exists a monomial $x \in \mathbf{m}^{s+1}$, so that $x = (t^{g_1})y$, but $y \notin \mathbf{m}^s$. Thus, $\overline{t}^{g_1}\overline{y} = 0 \in G(\mathbf{m})$ and, since $(t^{g_1}) \supseteq \mathbf{m}^r$ (by $r > s$), $y \in H^0_{\mathcal{M}} = (0 :_{G(\mathbf{m})} \mathcal{M}^r)$.

Now, by Lemma 3.4, $y = t^{\omega_i + l g_1}$ for some i, such that $a_i > b_i$, and some l, with $0 \leq l \leq l_i$. Meanwhile, $G(\mathbf{m})$ is Buchsbaum; hence, by Proposition 3.19, for this index i we have $\omega_i \in \max Ap_M(S)$; moreover, by M-purity we get $\operatorname{ord}(\omega_i) = s$. But then $\operatorname{ord}(\omega_i) + l \leq \operatorname{ord}(y) < s$, a contradiction. Thus we have $s = r$, as expected. □

We can collect the previous results and discussion in the following corollary.

Corollary 5.10. *Let $(R, \mathbf{m})$ be a numerical semigroup ring with M-pure associated semigroup S. Then the following conditions are equivalent:*

(i) $G(\mathbf{m})$ *is Buchsbaum;*

(ii) $G(\mathbf{m})$ *is* C-M;

(iii) $s_J(\mathbf{m}) = r$.

REFERENCES

1. V. Barucci, D.E. Dobbs and M. Fontana, *Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains*, Mem. Amer. Math. Soc. **598** (1997).

2. V. Barucci and R. Fröberg, *Associated graded rings of one dimensional analytically irreducible rings*, J. Algebra **304** (2006), 349–358.

3. L. Bryant, *Goto numbers of a numerical semigroup ring and the Gorensteiness of associated graded rings*, arXiv:0809.0476.

4. T. Cortadellas Benítez and S. Zarzuela Armengou, *Tangent cones of numerical semigroup rings*, arXiv:0906.0911v1.

5. M. D'Anna, M. Mezzasalma and V. Micale, *On the Buchsbaumness of the associated graded ring of a one-dimensional local ring*, Commun. Algebra **37** (2009), 1594–1603.

6. The GAP Group, GAP–*Groups, algorithms, and programming–Version* 4.4.10, (2007) (http://www.gap-system.org).

7. A. Garcia, *Cohen-Macaulayness of the associated graded of a semigroup ring*, Commun. Algebra **10** (1982), 393–415.

8. S. Goto, *Buchsbaum rings of maximal embedding dimension*, J. Algebra **76** (1982), 383–399.

9. ———, *Noetherian local rings with Buchsbaum associated graded rings*, J. Algebra **86** (1984), 336–384.

10. J. Lipman, *Stable ideals and Arf rings*, Amer. J. Math. **93** (1971), 649–685.

11. S. Molinelli, D.P. Patil and G. Tamone, *On the Cohen-Macaulayness of the associated graded ring of certain monomial curves*, Contrib. Algebra Geometry **39** (1998), 433–446.

12. D.G. Northcott and D. Rees, *Reduction of ideals in local rings*, Proc. Camb. Philos. Soc. **50** (1954), 145–158.

13. M. Delgado, P.A. Garcia-Sanchez and J. Morais, *NumericalSgps–A GAP package*, 0.96 (2005), (http://www.gap-system.org/Packages/numericalsgps).

14. L. Robbiano and G. Valla, *On the equations defining tangent cones*, Proc. Camb. Phil. Soc. **88** (1980), 281–297.

15. V.A. Sapko, *Associated graded rings of numerical semigroup rings*, Comm. Algebra **29** (2001), 4759–4773.

16. Y.H. Shen, *Tangent cone of numerical semigroup rings with small embedding dimension*, arXiv:0808.2162v1.

17. J. Stückrad-W. Vogel, *Toward a theory of Buchsbaum singularities*, Amer. J. Math. **100** (1978), 727–746.

Dipartimento di Matematica e Informatica, Catania University, Viale A. Doria, 6, 95125 Catania, Italy
Email address: mdanna@dmi.unict.it

Dipartimento di Matematica e Informatica, Catania University, Viale A. Doria, 6, 95125 Catania, Italy
Email address: vmicale@dmi.unict.it

Dipartimento di Matematica e Informatica, Catania University, Viale A. Doria, 6, 95125 Catania, Italy
Email address: alessiosammartano@gmail.com