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Robust constrained control of piecewise affine systems through set-based reachability computations

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Summary

This paper addresses finite-horizon robust control of a piecewise affine system affected by uncertainty and characterized by different affine dynamics (modes) associated with a polyhedral partition of the state space. The goal is to design a static state-feedback control law that maintains the state of the system within given – possibly time-varying – sets, subject to actuation constraints.

The proposed approach rests on two phases: a reference mode sequence with a sufficiently large robustness level is determined first, and then a tracking state-feedback control law defined on the reach sets of the controlled system is designed to counteract uncertainty and maintain the reach sets within the reference sequence. If this is not possible and the reach sets split over different modes, then, further reference mode sequences and tracking controllers are computed.

The designed state-feedback control law is represented through a collection of controllers defined on pre-computed reach sets of the closed-loop control system.

Performance of the approach is shown on some numerical examples.

1 | INTRODUCTION

PieceWise Affine (PWA) systems are a class of nonlinear dynamical models whose evolution is described by a finite collection of affine dynamics associated with a polyhedral partition of the state space. Despite the simplicity of their mathematical description, they are characterized by significant modelling capabilities. They naturally arise as model of certain classes of systems with a phased behaviour, and they can be used to approximate smooth nonlinear systems with a given accuracy [1] to the purpose of formal verification and control design (see, e.g., [2,3,4,5]). Furthermore, in [6] PWA systems have been proven to be equivalent to a class of hybrid models called Mixed Logical Dynamical (MLD) that can be used to describe a wide range of systems [7].

PWA models have been adopted in several domains of applications, including the robotics and automotive fields. For instance, a recent work on hybrid model predictive control of a humanoid robot has been presented in [8], where the piecewise affinity property appears spontaneously when modelling the occurrence of contact phenomena affecting the dynamics of the robot. In the automotive context, uncertain PWA models have been adopted to design a robust lateral stability controller for ground vehicles in [9,10].

In this paper, we address robust control of uncertain discrete-time PWA systems along a finite time horizon. More precisely, the goal is to design a static state-feedback control law that makes the system satisfy some specifications given in terms of constraints on the admissible value of the state for every instance of the uncertainty, subject to input actuation limits. Uncertainty enters the system dynamics through an additive disturbance and the initial state, that are only known to take values in some compact sets.

Robust control of PWA systems has been tackled in the literature according to different paradigms. In [11] backward reachability analysis and symbolic model checking are applied to check attainability of a sequence of state regions, that is, the property of a controller to drive the state to a (safe) target set along a given sequence of sets for all possible realizations of the system uncertainty. Approaches based on polyhedral computations and dynamic programming are also adopted, [12,13], where [13] includes also reachability analysis. An alternative approach not relying neither on reachability analysis nor on polyhedral computations and dynamic programming is presented in [14], where a computational procedure for the class of MLD systems that is based on Mixed Integer Programming (MIP) is described. Also, works are available in the literature dealing with discrete-time PWA systems within a model predictive control framework, [15,16,17,18].

A key feature of the method that we propose in this paper is that it rests on the computation of reachability sets of the PWA system, i.e., the sets of states that can be reached through the PWA system dynamics starting from the set of admissible initial conditions and applying all admissible input values. Using polyhedral sets for reachability analysis as in [11] and [13] may become computationally challenging as the computation horizon length and the system dimension increase. A possible strategy to mitigate this problem is to employ lower complexity outer-approximating sets, as done for example in [19], where hyper-rectangular sets are adopted for outer-approximating polytopes. A computationally convenient way to perform reachability computations is to introduce zonotopes, i.e., a class of convex polytopes whose points are expressed as a linear combination of a center and of a finite collection of vectors called generators, suitably weighted with coefficients ranging in $[-1, 1]$.

Performing reachability computations on affine systems affected by inputs taking values in a set reduces to applying affine maps and computing Minkowski sums. Zonotopes are close under these operations and, differently from generic polytopes, present better scalability properties as the system dimension grows, [20]. Moreover, a higher degree of tunability is provided by zonotopes while computing outer-approximations, since techniques are available to outer-approximate complex zonotopes with simpler ones by suitably setting the number of desired generators, [21].

A set-based reachability method using zonotopes has been recently proposed in [22] for addressing a finite horizon control problem where the goal is driving the state of a system affected by uncertainty to some target value, while satisfying some safety specifications and actuation constraints. To achieve this, the effect of the uncertainty on the initial state and of an additive bounded disturbance entering the system dynamics has to be counteracted via state feedback. The approach is proposed for linear systems and extended to nonlinear systems via linearization along a nominal trajectory. The key idea in [22] is to use zonotopes to represent specifications, actuation constraints, disturbance and initial state sets, possibly via suitable under/over approximations, and then express the control input at each time step as a linear combination of center and generators of the actuation constraints set according to coefficients that are optimized so as to minimize the extent of the reach set at next step while enforcing the specifications. The input applied on-line will depend on the actual value taken by the state in the reach set, via coefficients re-scaling factors.

The control design method proposed in this paper relies on reachability computations, but differently from [11], analysis is performed on a look-ahead horizon and, similarly to [22], we adopt zonotopes to represent the reach sets of the system. A static state-feedback control law, whose structure is inspired by the one adopted in [22], is introduced. However, differently from [22], center and generators of the zonotopic control policy are taken as design parameters and, in particular, generators providing the state feedback component of the control law are not constrained to be a linear combination of the generators of the (zonotopic) set representing the input constraints, but they are optimized. Moreover, the considered PWA dynamics in our setup does not require any kind of regularity, while applicability of [22] to nonlinear dynamics relies on linearization, thus requiring differentiability of the nonlinear function.

To obtain a computationally convenient procedure for the *offline controller design phase*, the concept of robust mode control introduced in [14] is here applied in the context of reachability computations by fixing a-priori a mode sequence to be tracked by the reach sets of the closed-loop system at each time instant. In this way, the PWA system can be reformulated as a time-varying affine system, and the controller design procedure reduces to the solution of convex quadratic programs, for which efficient solvers are available, [23]. However, differently from [14], a feedback policy is here adopted, and in case the controller fails to keep some reach set inside a single mode, computations at subsequent time instants are performed independently for each non-empty intersection of the reach set with the modes by adopting suitable zonotopic outer-approximations.

The designed feedback controller is described through two collections of zonotopes, one in the state and the other in the control input space. Locating the current state value in the appropriate state space zonotope (to then determine the corresponding control action in the associated input zonotope) is usually easier than locating it within the state-space partition associated with an explicit MPC controller as in[18]. More precisely, the *online implementation* of the designed controller requires solving a convex quadratic program in order to determine the control action to be applied at the current state and, hence, it is computationally low demanding.

The problem of choosing among the admissible switching sequences the one to be tracked in closed-loop is here addressed by assessing their robustness level through a procedure inspired by the one presented in[24] but for a different problem and class of systems. In[24] discrete time linear systems subject to an additive disturbance are considered, and the objective is to determine the maximal amount of uncertainty that can enter the system dynamics while guaranteeing the existence of a control policy ensuring the satisfaction of polytopic state and input constraints. To this aim, parameterized uncertainty sets and a suitable metric are introduced to quantify the amount of uncertainty. Differently from[24], where a closed-loop affine policy is considered together with several families of sets (ellipsoidal, rectangular, polyhedral) to model uncertainty and recover computational tractability, we choose the control inputs according to an open-loop strategy and parametrize uncertainty sets with zonotopes whose generators are rescaled by suitable factors representing the decision variables to be chosen so as to optimize the metric quantifying the uncertainty amount.

The rest of the paper is organized as follows. We first introduce some basic notions and notations and formulate the addressed robust control problem for uncertain piecewise affine systems. We then describe in Section 2 the control design method based on reachability computations proposed in this paper. More in detail, we first address the problem of generating an admissibile switching sequence for a PWA system with no uncertainty in Section 2.1, then we describe in Section 2.2 a procedure to assess the robustness level of a computed switching sequence, and in Section 2.3 we present the state-feedback controller design procedure for uncertain PWA systems. Section 2.4 concludes Section 2 with a description of the online implementation of the designed state-feedback controller. In Section 3, we perform a complexity analysis of the proposed design methodology, and show in Section 4 some numerical results. We conclude the paper with some remarks in Section 5.

Basic notions and notations

Given two positive integers m and n, the symbol $\mathbb{R}^{m\times n}$ denotes the space of the $m \times n$ real matrices and $\mathbb{R}^m$ stands for $\mathbb{R}^{m\times 1}$. The symbol I_m denotes the identity matrix of order m, while 0_m and 1_m are the elements of $\mathbb{R}^m$ with all zero and unitary entries, respectively.
Given two sets $A, B \subseteq \mathbb{R}^h$, the symbol $A \oplus B$ denotes the Minkowski sum of A and B. Given two matrices $M_1 \in \mathbb{R}^{m,n}$ and $M_2 \in \mathbb{R}^{p,q}$, the symbol $M_1 \otimes M_2$ denotes the Kronecker product of M_1 and M_2.
A (convex) polyhedron $\mathcal{P} \subseteq \mathbb{R}^h$ is defined as the intersection of q half-spaces (H-representation[25]), and can be expressed through $P_A \in \mathbb{R}^{q\times h}$ and $p_B \in \mathbb{R}^q$ as $\mathcal{P} = \{z \in \mathbb{R}^h | P_A z \leq p_B\}$ or $\mathcal{P} = (P_A, p_B)$ for ease of notation. A polytope is a (convex) bounded polyhedron. Given a finite set $\{x_i\}_{i=1}^p$ in $\mathbb{R}^h$, the symbol $\text{conv}(\{x_i\}_{i=1}^p)$ indicates the convex hull of $\{x_1, \dots, x_p\}$. Zonotopes are centrally symmetric convex polytopes. More precisely, a convex polytope in $\mathbb{R}^h$ is called a zonotope if it can be written as $\mathcal{Z} = \{z \in \mathbb{R}^h | z = c + \sum_{i=1}^r \beta_i g_i, \beta_i \in [-1, 1]\}$, where $c \in \mathbb{R}^h$ is the center and $g_i \in \mathbb{R}^h$, $i = 1, \dots, r$, are the generators. We shall then use $\langle c, G\rangle$ as a more concise notation of $\mathcal{Z}$ (G-representation), where $G \in \mathbb{R}^{h\times r}$ is the generator matrix, which contains the generators as its columns. A parallelotope is a zonotope with an invertible generator matrix. Given a generator matrix G, $G^{[k]}$ denotes its k-th generator and $G^{[k,l]}$ denotes the submatrix composed by the generators of G from column k to column l, $l \geq k$. Note that an interval in $\mathbb{R}^h$ is a zonotope whose generators are parallel to the coordinate axes.

Robust constrained control problem formulation

We consider a discrete-time PieceWise Affine (PWA) dynamical system whose state space is partitioned into s polyhedral regions $\mathcal{M}_i$, $i = 1, \dots, s$. Each region is called a mode and activates a different affine dynamics as follows:

$$x_{t+1} = A_i x_t + B_{u,i} u_t + B_{w,i} w_t + f_i, \quad x_t \in \mathcal{M}_i,\ i = 1, \dots, s, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is a control input, and $w \in \mathbb{R}^{n_w}$ is a disturbance input taking values in a polytope $\mathcal{W} \subset \mathbb{R}^{n_w}$.

Uncertainty affects the initial state, which is only known to belong to a polytope $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$.

The control input is subject to actuation limits expressed via a polytope $\mathcal{U} \subset \mathbb{R}^{n_u}$, whereas the state is subject to specifications given in terms of a sequence of polyhedral sets $\mathcal{X}_{sp,t} \subseteq \mathbb{R}^{n_x}$, $t = 1, \dots, N$, along the look-ahead time horizon $[1, N]$ which are named *specs* in the sequel.

Our goal is designing a finite-horizon static state-feedback policy $\psi = (\psi_t)_{t=0}^{N-1}$, with $\psi_t : \mathbb{R}^{n_x} \to \mathcal{U}$, such that the closed-loop system

$$\begin{cases} x_{t+1} = A_i x_t + B_{u,i} u_t + B_{w,i} w_t + f_i, & x_t \in \mathcal{M}_i,\ i = 1, \dots, s \\ u_t = \psi_t(x_t) \end{cases} \tag{2}$$

satisfies the state specifications $x_{t+1} \in \mathcal{X}_{sp,t+1}$, for any $x_0 \in \mathcal{X}_0$ and any $w_t \in \mathcal{W}$, $t = 0, \dots, N-1$. Note that a particular instance of the introduced problem is obtained when $\mathcal{X}_{sp,t} = \mathbb{R}^{n_x}$, $t = 1, \dots, N-1$, and $\mathcal{X}_{sp,N} = \mathcal{X}_f$, which corresponds to state regulation to the target set $\mathcal{X}_f$.

2 | CONTROL DESIGN BASED ON REACHABILITY COMPUTATIONS

We introduce a method for robust constrained control design for the uncertain PWA system (1) that rests on set-based reachability computations. The basic idea of the proposed method is to design a controller that acts on position and shape of the reach sets of the controlled system, i.e., the sets of states that can be reached by the system under the designed control law and subject to all admissible realizations of uncertainty, with the aim of maintaining them inside the specs. The effectiveness of any approach to control design that is based on reach set computations obviously depends on the class of sets that are used to model the reach sets of the system. Here, we represent reach sets by means of zonotopes because zonotopes are easy to describe, have favorable scalability properties with the system dimension, and they are closed under affine transformations and the Minkowski sum, which are the two main operations involved in set propagation according to the PWA system dynamics.

The method is particularly efficient if, at each time instant, the reach set of the closed-loop system is contained within a single mode. In such a case, the PWA system can be described as a time-varying affine system and, by over-approximating the polytopic sets $\mathcal{W}$ and $\mathcal{X}_0$ with outer-zonotopes $\mathcal{Z}_w$ and $\mathcal{Z}_{x,0}$, respectively, we can exploit the property of closeness with respect to affine transformation and Minkowski sum of zonotopic sets to obtain a computationally convenient procedure for reach set propagation. If instead some reach set of the closed-loop system covers multiple modes, then each mode intersection is approximated by an outer-zonotope and propagated independently, thus generating multiple branches.

The design of the control law in the horizon $[0, N-1]$ is structured in two phases: in the first phase, we neglect uncertainty and determine an admissible sequence of modes for the nominal system associated with (1), and, in the second phase, we consider uncertainty and design a static state-feedback control law that best compensates it while maintaining the reach sets within the specs. To avoid splitting, we select the mode sequence that has the largest degree of robustness in open-loop (first phase) and then reduce the size of the reach sets in closed-loop (second phase). If a reach set split is unavoidable at some time instant $t < N$, then the two phases described above are performed over the residual time horizon $[t, N-1]$ starting from the zonotopic outer approximation of each mode fragment of the split set.

The resulting control law is defined through a collection of closed-loop reach sets along the time horizon $[0, N-1]$ and of the associated static state-feedback control laws. Each reach set is approximated via an outer-zonotope and the control law is zonotopic too, with given center and generator matrix, while the coefficients weighting the generators depend on the state value and are set equal to the ones identifying the state position in the (zonotopic) reach set to which it belongs.

The described control design procedure is sketched in Algorithm 1. Note that for short-hand notation, we use the switching sequence $S_{[0,N-1]} = (i_0, i_1, \dots, i_{N-1})$ to denote the mode sequence $\mathcal{M}_{i_0}, \mathcal{M}_{i_1}, \dots, \mathcal{M}_{i_{N-1}}$.

The procedures involved in the implementation of Algorithm 2 are thoroughly described in Sections 2.1 – 2.3, whereas the online control law implementation is described in Section 2.4.

2.1 | Generating an admissible switching sequence for the nominal PWA

Assume without loss of generality that the zonotope $\mathcal{Z}_w$ outer approximating the disturbance set $\mathcal{W}$ is centered at 0, so that $\mathcal{Z}_w = \langle 0_{n_w}, G_w \rangle$. Then, the nominal system associated with (1) is given by

$$\bar{x}_{t+1} = A_i \bar{x}_t + B_{u,i} u_t + f_i, \quad \bar{x}_t \in \mathcal{M}_i,\ i = 1, \dots, s \tag{3}$$

and is initialized at $\bar{x}_0 = c_{x,0}$, where $c_{x,0}$ denotes the center of the zonotope $\mathcal{Z}_{x,0}$ outer approximating $\mathcal{X}_0$, i.e., $\mathcal{Z}_{x,0} = \langle c_{x,0}, G_0 \rangle$. We now introduce the notion of admissible switching sequence for the nominal system (3).

Algorithm 1 Finite-horizon robust constrained control design

Require: N, $\mathcal{Z}_{x,0}$, $\mathcal{Z}_w$, PWA dynamics (1), $\{\mathcal{X}_{sp,j}\}_{j=1}^N$, $\mathcal{U}$
Apply Algorithm 2 by setting $t \leftarrow 0$ and $\mathcal{Z}_{x,t} \leftarrow \mathcal{Z}_{x,0}$
if k provided by Algorithm 2 satisfies $k = N$ **then**
Algorithm 2 has computed the control law along $[0, N-1]$ without any branching
else branching has occurred
while the output k of Algorithm 2 satisfies $k < N$ **do**
Apply Algorithm 2 to all branches originated by the $\{\mathcal{Z}_{x,k}^{(i)}\}$ splits of the reach set by setting $t \leftarrow k$ and $\mathcal{Z}_{x,t} \leftarrow \mathcal{Z}_{x,k}^{(i)}$
end while
end if
Provide as output the control laws computed along all branches

Algorithm 2 Control law design via reachability computations along a branch

Require: t, N, $\mathcal{Z}_{x,t}$, $\mathcal{Z}_w$, PWA dynamics (1), $\{\mathcal{X}_{sp,j}\}_{j=t+1}^N$, $\mathcal{U}$, $\varrho_{\min}$
$\varrho \leftarrow 0$
while $\varrho < \varrho_{\min}$ **do**
Compute a switching sequence $S_{[t,N-1]}$ over $[t, N-1]$ for the nominal system initialized at the center of $\mathcal{Z}_{x,t}$ (§ 2.1)
Evaluate admissibility and robustness level ϱ of $S_{[t,N-1]}$ (§2.2)
end while
continue $\leftarrow 1$; $k \leftarrow t$
while continue $\wedge$ $k < N$ **do**
Design a static state-feedback law $\psi_k : \mathbb{R}^{n_x} \to \mathcal{U}$ at time k that makes the control system robustly satisfy the specs when the system evolves according to $S_{[k,N-1]}$ starting from $\mathcal{Z}_{x,k}$ (§2.3)
if $k < N-1$ **then**
Compute the reach set $\mathcal{R}_{x,k+1}$ of system (2) at time $k+1$ by applying ψ_k starting from $\mathcal{Z}_{x,k}$
if $\mathcal{R}_{x,k+1}$ does not split among modes **then**
$\mathcal{Z}_{x,k+1} \leftarrow \mathcal{R}_{x,k+1}$
else
Determine an outer-zonotope $\mathcal{Z}_{x,k+1}^{(i)}$ for each non-empty intersection of $\mathcal{R}_{x,k+1}$ with the system modes
continue $\leftarrow 0$
end if
end if
$k \leftarrow k+1$;
end while
Provide as output: k, the control law within $[t, k-1]$ along the branch starting from $\mathcal{Z}_{x,t}$ and, if $k < N$, also $\{\mathcal{Z}_{x,k}^{(i)}\}$

Definition 1 (Admissible switching sequence for the nominal system)**.** A switching sequence $S^*_{[0,N-1]} = (i_0^*, i_1^*, \dots, i_{N-1}^*)$ is admissible for the nominal system (3), if there exists a sequence of input values $u_t \in \mathcal{U}$, $t = 0, 1, \dots, N-1$, that makes (3) initialized at $\bar{x}_0 = c_{x,0}$ evolve inside the specs through the corresponding mode sequence $\mathcal{M}_{i_0^*}, \mathcal{M}_{i_1^*}, \dots, \mathcal{M}_{i_{N-1}^*}$.

In this section we present three methods for generating an admissible switching sequence for system (3), highlighting their relevant features. A comparison in terms of complexity is postponed to Sections 3 and 4.

2.1.1 | A mixed integer linear programming approach

In this method, the problem of determining an admissible switching sequence for (3) is formulated as a mixed integer linear programming feasibility test for finding an input sequence that guarantees the specs satisfaction in nominal conditions. The corresponding switching sequence will be an admissible switching sequence for (3).

The Mixed Integer Linear Program (MILP) reformulation is achieved by first rewriting the PWA system (3) as a Mixed Logical Dynamical (MLD) system given by linear equalities and inequalities involving also binary variables, and then adding the specs and actuation constraints.

Converting the PWA system to an equivalent MLD model

An MLD systems is a dynamical model of the following form:

$$\begin{aligned} x_{t+1} &= Ax_t + B_u u_t + B_\delta \delta_t + B_z z_t + f \\ E_x x_t &+ E_u u_t + E_\delta \delta_t + E_z z_t \leq e \end{aligned} \tag{4}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the input, $\delta \in \{0,1\}^{n_\delta}$ is a vector of binary auxiliary variables and $z \in \mathbb{R}^{n_z}$ is a vector of continuous auxiliary variables.

Suppose, without loss of generality, that the modes $\mathcal{M}_i$, $i = 1, \dots, s$, of the PWA system are defined through n_h half-spaces $\{\mathcal{H}_j\}_{j=1}^{n_h}$, so that each $\mathcal{M}_i$ is obtained as the intersection of a subset of them and of the complements of the remaining ones. This can be encoded via a $\delta^{(i)}$ vector with n_h binary elements where the j-th element is 1 if $\mathcal{M}_i$ is contained within $\mathcal{H}_j$, and 0 if it is within its complement. We can then introduce a binary auxiliary vector $\delta_t \in \{0,1\}^{n_h}$ such that:

$$x_t \in \mathcal{M}_i \quad \Longleftrightarrow \quad \delta_t = \delta^{(i)}.$$

In order to translate the definition of δ_t into a set of mixed-integer linear constraints, we consider the H-representations (H_{aj}, h_{bj}), $j = 1, \dots, n_h$, of the half-spaces $\{\mathcal{H}_j\}_{j=1}^{n_h}$, and apply the big-$M$ technique (see[7]):

$$\begin{cases} H_{aj} x_t \leq h_{bj} + M(1 - \delta_{tj}) \\ -H_{aj} x_t \leq -h_{bj} - m\delta_{tj} - \epsilon_m (1 - \delta_{tj})|h_{bj}| \end{cases} \quad j = 1, \dots, n_h \tag{5}$$

with $M = \max_{j=1,\dots,n_h} \max_{x \in [x_{\min}, x_{\max}]} (H_{aj}x - h_{bj})$ and $m = \min_{j=1,\dots,n_h} \min_{x \in [x_{\min}, x_{\max}]} (H_{aj}x - h_{bj})$, where $\epsilon_m > 0$ is the machine precision and $[x_{\min}, x_{\max}] \subseteq \mathbb{R}^{n_x}$ is an interval large enough to contain all the reachable states of (3) in the horizon $[0, N]$. Furthermore, a H-representation (M_{ia}, m_{ib}) of mode $\mathcal{M}_i$ is given by:

$$\begin{aligned} M_{ia} &= (2\,\mathrm{diag}(\delta_1^{(i)}, \dots, \delta_{n_h}^{(i)}) - I_{n_h})\; H_a \\ m_{ib} &= (2\,\mathrm{diag}(\delta_1^{(i)}, \dots, \delta_{n_h}^{(i)}) - I_{n_h})\; (h_b + (I_{n_h} - \epsilon_m\,\mathrm{diag}(\delta_1^{(i)}, \dots, \delta_{n_h}^{(i)}))|h_b|) \end{aligned} \tag{6}$$

where the operator $|\cdot|$ is applied elementwise and we set

$$H_a = \begin{pmatrix} H_{a1} \\ \vdots \\ H_{an_h} \end{pmatrix}, \qquad h_b = \begin{pmatrix} h_{b1} \\ \vdots \\ h_{bn_h} \end{pmatrix}.$$

By setting $z_t = x_{t+1}$ and using the big-M technique, the PWA dynamics

$$x_{t+1} = A_i x_t + B_{u,i} u_t + f_i \quad \Longleftrightarrow \quad x_t \in \mathcal{M}_i \quad \Longleftrightarrow \quad \delta_t = \delta^{(i)}$$

can be expressed as follows:

$$\begin{cases} z_t \leq A_i x_t + B_{u,i} u_t + f_i - \Gamma(2\delta^{(i)} - 1_{n_h})^T \delta_t + \Gamma \|\delta^{(i)}\|_1 \\ -z_t \leq -A_i x_t - B_{u,i} u_t - f_i - \Gamma(2\delta^{(i)} - 1_{n_h})^T \delta_t + \Gamma \|\delta^{(i)}\|_1 \end{cases} \quad i = 1, \dots, s, \tag{7}$$

where

$$\Gamma = \max_{i,j=1,\dots,s} \max_{\substack{x \in [x_{\min}, x_{\max}] \\ u \in [u_{\min}, u_{\max}]}} |(A_j - A_i)x + (B_{u,j} - B_{u,i})u + f_j - f_i|,$$

$[u_{\min}, u_{\max}] \subseteq \mathbb{R}^{n_u}$ being an interval including $\mathcal{U}$. Note that equality $(2\delta^{(i)} - 1_{n_h})^T \delta_t = \|\delta^{(i)}\|_1$ holds if and only if $\delta_t = \delta^{(i)}$. Summarizing, the inequality constraints in (4) are obtained by stacking constraints (5) and (7), while the equality is simply $x_{t+1} = z_t$.

MILP formulation

We now describe a procedure to compute an admissible switching sequence for system (3) starting from reformulation (4), which consists in solving a feasibility problem with $(u_t)_{t=0}^{N-1}$, $(\delta_t)_{t=0}^{N-1}$ and $(z_t)_{t=0}^{N-1}$ as decision variables. The aim is to choose them so as to make the state of system (4) evolve inside the specs for $t = 1, \dots, N$, while the input is constrained to $\mathcal{U}$. If such an assignment of the decision variables exists, then the selected sequence of binary vectors $(\delta_t^*)_{t=0}^{N-1}$ induces an admissible switching sequence $S^*_{[0,N-1]} = (i^*_0, i^*_1, \dots, i^*_{N-1})$ for system (3) according to Definition 1, being each binary vector δ^*_t associated with a mode index i^*_t of system (3).

More formally, we need to test if there exists an assignment for the decision variables $(u_t)_{t=0}^{N-1}$, $(\delta_t)_{t=0}^{N-1}$ and $(z_t)_{t=0}^{N-1}$ such that the constraints in (4) with $t = 0, \dots, N-1$, and the additional constraints

$$\begin{aligned} x_t &\in \mathcal{X}_{sp,t}, \quad t = 1, \dots, N \\ u_t &\in \mathcal{U}, \quad t = 0, \dots, N-1 \end{aligned}$$

are satisfied. Since $\mathcal{X}_{sp,t}$ and $\mathcal{U}$ are polyhedral sets and $x_t = z_{t-1}$, all constraints are linear in the decision variables. The problem at hand is then a MILP that is feasible if and only if there exists an admissible switching sequence for system (3). In case of feasibility, such a sequence is retrieved by recovering the mode indices $(i^*_t)_{t=0}^{N-1}$ associated to the selected binary vectors $(\delta^*_t)_{t=0}^{N-1}$.

If we need to compute multiple admissible switching sequences with this method, we have to perform multiple feasibility tests, where every time we eliminate the previously computed switching sequences by adding appropriate constraints.

More precisely, suppose that n_s switching sequences S^*_h, $h = 1, \dots, n_s$, have been already computed, each one associated with a set of binary vectors δ^*_{ht}, $t = 0, \dots, N-1$. In order to express in terms of mixed integer linear conditions the constraint that the switching sequence coded through the decision variables δ_t, $t = 0, \dots, N-1$, is different from S^*_h, we associate to S^*_h a set of binary variables b_{hti}, $t = 0, \dots, N-1$, $i = 1, \dots, n_h$, defined as:

$$b_{hti} = 1 \iff \delta_{ti} \neq \delta^*_{hti} \iff \delta_{ti} + (2\delta^*_{hti} - 1)b_{hti} = \delta^*_{hti} \quad t = 0, \dots, N-1, \; i = 1, \dots, n_h \tag{8}$$

and impose the constraint:

$$\sum_{t=0}^{N-1} \sum_{i=1}^{n_h} b_{hti} \geq 1, \tag{9}$$

which expresses the condition that at least one component of at least one binary vector associated to the new switching sequence must change. By embedding (8) and (9) with $h = 1, \dots, n_s$ into the original feasibility problem, we obtain a MILP that is infeasible only if no further admissible switching sequence for system (3) exists besides the n_s computed ones.

2.1.2 | A reachability analysis based approach

The second method relies on reachability analysis through efficient set-based computations to determine admissible switching sequences for the nominal system (3). Differently from the first method, where the generation of an admissible switching sequence is reduced to the solution of a MILP by suitably embedding specs inside the problem as hard constraints, here the idea is to compute the reach sets of system (3) (i.e., the sets of the states that can be reached along the time horizon $[1, N]$ starting from $\bar{x}_0$ by applying at each time step all possible input values within $\mathcal{U}$), and then to check if they intersect the spec sets so as to determine admissible switching sequences.

Efficiency is obtained by over-approximating the polytopic input set $\mathcal{U}$ with a zonotope $\mathcal{Z}_u$ and representing the reach sets of system (3) as zonotopes through the procedure described next.

We start at time $t = 0$ by considering the nominal initial state $\bar{x}_0$. Such state belongs to a mode $\mathcal{M}_{i^*_0}$, which activates an affine dynamics $(A_{i^*_0}, B_{i^*_0}, f_{i^*_0})$. The first element of the currently generated switching sequence is then i^*_0. We apply to $\bar{x}_0$ the dynamics of $\mathcal{M}_{i^*_0}$ for all input values in $\mathcal{Z}_u$, i.e., we compute the set $\mathcal{Z}_1 = B_{u,i^*_0}\mathcal{Z}_u + (A_{i^*_0}\bar{x}_0 + f_{i^*_0})$. Note that $\mathcal{Z}_1$ is a zonotope, because it is the image of $\mathcal{Z}_u$ through an affine map and zonotopes are closed with respect to such maps. According to Definition 1, a necessary condition to obtain an admissible switching sequence is that the system state at time $t = 1$ enters the spec set $\mathcal{X}_{sp,1}$, and so we cut from the propagation the states of $\mathcal{Z}_1$ that are outside $\mathcal{X}_{sp,1}$. Since zonotopes are not closed under intersection, the set $\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}$ is a polytope but not necessarily a zonotope. To keep performing reachability computations efficiently, we then have to outer-approximate it with a zonotope. Before doing this, in order to minimize the introduced level of conservativeness, we first check if $\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}$ covers multiple modes. If it is included into a single mode, say $\mathcal{M}_{i^*_1}$, then we set i^*_1 as the second element of the generated switching sequence. We then outer-approximate $\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}$ with a zonotope and apply to it the dynamics of $\mathcal{M}_{i^*_1}$, i.e., we compute the set $\mathcal{Z}_2 = A_{i^*_1} OZ(\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}) \oplus B_{u,i^*_1}\mathcal{Z}_u \oplus f_{i^*_1}$, $OZ(\mathcal{P})$ denoting an outer-zonotope of $\mathcal{P}$. Set $\mathcal{Z}_2$ is still a zonotope, due to the closure property of zonotopes with respect to Minkowski sum. Consider now the case where $\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}$ covers multiple modes. Then, each part of $\mathcal{Z}_1 \cap \mathcal{X}_{sp,1}$ is associated in general with a different dynamics, and so we have to propagate them independently, thus generating multiple branches of sets. At this point, in order to fix the next index of the currently generated sequence, one has to select a single branch. Computations are then carried out along the selected branch only, while the other ones are stored in memory. This allows to rapidly generate a potentially admissible switching sequence.

By iterating the above procedure until the end of the reference time horizon, a switching sequence $S^*_{[0,N-1]} = (i^*_0, i^*_1, \dots, i^*_{N-1})$ is finally generated.

Due to the introduced over-approximation, some a-posteriori verification procedure is needed to test the actual admissibility of any computed switching sequence. In order to verify whether the generated sequence is admissible or not, the procedure described in Section 2.2 can be adopted. If the admissibility test fails, another sequence has to be generated, which can be done easily by choosing a different branch to be propagated after splitting. Note also that, if no further sequence can be generated, then no additional admissible switching sequence exists for system (3), because reachability analysis is performed by using an input set $\mathcal{Z}_u$ that includes the original one $\mathcal{U}$ and the computed zonotopes include the actual reach sets of the system.

Since the proposed method strongly relies on a zonotopic outer-approximation of polytopes, we now describe how to tightly enclose a given polytope $\mathcal{P}$ within a parallelotope $\mathcal{Z}_{oa} = \langle c_{oa}, G_{oa} \rangle$.

Given a coordinate transformation matrix T, the center and the generator matrix are computed as follows:

$$c_{oa} = \frac{T}{2}\left(w_{\max} + w_{\min}\right), \quad G_{oa} = \frac{T}{2}\,\mathrm{diag}\left(w_{\max} - w_{\min}\right),$$

with $w_{\max} = \max_{j=1,\dots,n_v} w_j$ and $w_{\min} = \min_{j=1,\dots,n_v} w_j$, where w_j, $j = 1, \dots, n_v$, is the j-th vertex of the polytope obtained by applying the coordinate transformation matrix T^{-1} to $\mathcal{P}$. T needs to be chosen so as to get a tight over-approximation. To this purpose, one can adopt two methods:

Principal Component Analysis (PCA): the set of the vertices of $\mathcal{P}$ is interpreted as a set of data and T provides the transformation to a new orthogonal coordinate system where the greatest variance of the data is along the first axis, the second greatest variance of the data is along the second axis and so on and so forth. The procedure described in[21] can be adopted. The drawback associated to this method is the necessity of computing the vertices of $\mathcal{P}$, which can be costly.

Maximum volume inner ellipsoid: the largest ellipsoidal inner approximation of $\mathcal{P}$ is computed and its axes are taken as directions of the new coordinate system. The algorithm presented in[26] can be adopted to determine the ellipsoidal inner approximation of a full-dimensional polyhedron by solving a convex optimization program. We extended such an algorithm so as to deal with lower-dimensional polyhedra too. Differently from PCA, ellipsoid computation does not require the vertices of $\mathcal{P}$.

2.1.3 | A randomized approach

We finally propose a randomized approach for determining an admissible switching sequence $S^*_{[0,N-1]} = (i^*_t)_{t=0}^{N-1}$ for the nominal PWA system (3), which rests on extracting at random a sequence $u_t \in \mathcal{U}$, $t = 0, \dots, N-1$, and checking if it makes system (3) satisfy the specs along the time interval $[1, N]$. If this is the case, the mode sequence, say $\mathcal{M}_{i^*_0}, \mathcal{M}_{i^*_1}, \dots, \mathcal{M}_{i^*_{N-1}}$, associated with the obtained state trajectory $(x^*_t)_{t=1}^N$ identifies an admissible switching sequence $S^*_{[0,N-1]} = (i^*_t)_{t=0}^{N-1}$ for system (3). We next provide an estimate on the number of extractions to be performed before giving up in looking at an admissible switching sequence.

Let us assume that $u_0, \dots, u_{N-1}$ are independent random variables uniformly distributed over $\mathcal{U}$. We can associate a probability measure p to the event, say E, that the multi-sample $u_0, \dots, u_{N-1}$ provides an admissible switching sequence. Our goal is to determine a number n of independent extractions for $u_0, \dots, u_{N-1}$ such that, if no extracted multi-sample provides an admissible switching sequence, then, p is smaller than or equal to an a-priori defined threshold probability $p_T \in (0, 1)$. The threshold probability p_T can in turn be set equal to ϵ^N, with $\epsilon \in (0, 1)$, so as to account for the horizon length N. Indeed, the Lebesgue measure of the set within $\mathcal{U}^N$ corresponding to probability p_T is $p_T V_{\mathcal{U}}^N$ where $V_{\mathcal{U}}$ denotes the volume of $\mathcal{U}$.

Suppose that we run n independent experiments to extract n input sequences. Then, the random variable n_E representing the number of experiments where event E has occurred is a binomial random variable with parameters p and n. Its cumulative distribution is given by:

$$f_n(m; p) = \mathbb{P}(n_E \leq m) = \sum_{h=0}^{m} \binom{n}{h} p^h (1-p)^{n-h}, \; m = 0, 1, \dots, n, \tag{10}$$

and depends on the unknown probability p of event E and on the number of experiments n.

We now choose n so that the probability of extracting no multi-sample in E while having $p > p_T$ is smaller or equal to some predefined (small) $\beta \in (0, 1)$:

$$f_n(0; p) = (1-p)^n \leq \beta, \; \text{if } p > p_T.$$

Given that $f_n(0; p)$ is decreasing as a function of p, the condition above is satisfied if the following one holds:

$$f_n(0; p_T) = (1 - p_T)^n \leq \beta,$$

which entails

$$n \geq -\frac{1}{\log(1 - p_T)} \log \frac{1}{\beta} \overset{p_T \to 0}{\sim} \frac{1}{\epsilon^N} \log \frac{1}{\beta}.$$

If none of the n independent extractions is in E, then, the probability that there exists an admissible switching sequence for system (3) is smaller than ϵ^N with probability larger than or equal to $1 - \beta$.

If we set $\beta = 10^{-6}$ and $\epsilon = 0.01$, then, we obtain $n \geq 14 \cdot 100^N$.

Note that, differently from the previous two approaches, the randomized one does not guarantee that all admissible switching sequences are eventually found. Also, as the time horizon length grows, the number of input sequences to be extracted grows exponentially.

2.2 | Evaluating the robustness level of a switching sequence

In this section, we assess the robustness level of a given switching sequence for system (3) when re-introducing uncertainty as in the original system (1) where an additive disturbance is present and the initial state is uncertain.

To this purpose, the rescaling factors $\beta_{0i} \in [0, 1]$, $i = 1, \dots, p_0$, and $\beta_{wj} \in [0, 1]$, $j = 1, \dots, p_w$, are associated to the generators of the uncertainty zonotopic sets $\mathcal{Z}_{x,0}$ and $\mathcal{Z}_w$, and system (1) is considered with the following parametrized sets, which describe the uncertainty on the initial state x_0 and the disturbance w:

$$\mathcal{Z}_{0\beta_0} = \left\langle c_{x,0}, G_{0\beta_0} \right\rangle = \left\langle c_{x,0}, \left[\beta_{01} G_0^{[1]} \ \dots \ \beta_{0p_0} G_0^{[p_0]}\right] \right\rangle$$
$$\mathcal{Z}_{w\beta_w} = \left\langle 0_{n_w}, G_{w\beta_w} \right\rangle = \left\langle 0_{n_w}, \left[\beta_{w1} G_w^{[1]} \ \dots \ \beta_{wp_w} G_w^{[p_w]}\right] \right\rangle$$

Note that, since β_0 and β_w have components in $[0, 1]$, it holds that $\mathcal{Z}_{0\beta_0} \subseteq \mathcal{Z}_{x,0}$ and $\mathcal{Z}_{w\beta_w} \subseteq \mathcal{Z}_w$, and the sets coincide if all rescaling factors are unitary.

In order to assess the robustness level of the switching sequence $S^*_{[0,N-1]} = (i^*_0, \dots, i^*_{N-1})$, we look for an input sequence $u_0, \dots, u_{N-1}$ for system (1) that satisfies specs and input constraints under $S^*_{[0,N-1]}$ while maximizing the level of uncertainty given by the zonotopes $\mathcal{Z}_{0\beta_0}$ and $\mathcal{Z}_{w\beta_w}$. More precisely, we maximize

$$\varrho(\beta_0, \beta_w) = \frac{1}{p_0 + p_w} \left(\sum_{i=1}^{p_0} \beta_{0i} + \sum_{j=1}^{p_w} \beta_{wj} \right)$$

subject to the constraints:

$$\beta_{0i}, \beta_{wj} \in [0, 1], \quad i = 1, \dots, p_0, \ j = 1, \dots, p_w$$
$$\mathcal{R}_{x,t}(\beta_0, \beta_w) \subseteq \mathcal{X}_{sp,t} \cap \mathcal{M}_{i^*_t}, \quad t = 1, \dots, N$$
$$u_t \in \mathcal{U}, \quad t = 0, \dots, N - 1$$

where $\mathcal{M}_{i^*_N} = \mathbb{R}^{n_x}$ and $\mathcal{R}_{x,t}(\beta_0, \beta_w)$ is the reach set of system (1) at time t when sequence $S^*_{[0,N-1]}$ is applied and the uncertainty affecting the system is represented by the zonotopes $\mathcal{Z}_{0\beta_0}$ and $\mathcal{Z}_{w\beta_w}$.

We now show that the reach set $\mathcal{R}_{x,t}(\beta_0, \beta_w)$ is a zonotope.

Consider the expression of the state at time t when sequence $S^*_{[0,N-1]}$ is applied:

$$x_t = \prod_{j=1}^{t} A_{i^*_{t-j}} x_0 + \sum_{j=0}^{t-1} \prod_{h=j+1}^{t-1} A_{i^*_{t+j-h}} (B_{ui^*_j} u_j + B_{wi^*_j} w_j + f_{i^*_j}). \tag{11}$$

Since x_0 and w_j, $j = 0, \dots, t - 1$, belong to $\mathcal{Z}_{0\beta_0}$ and $\mathcal{Z}_{w\beta_w}$ respectively, then, there exist α_0 and $\alpha_{w,j}$ with $\|\alpha_0\|_\infty \leq 1$ and $\|\alpha_{w,j}\|_\infty \leq 1$ such that $x_0 = c_{x,0} + G_{0\beta_0}\alpha_0$ and $w_j = G_{w\beta_w}\alpha_{w,j}$, so that expression (11) rewrites as:

$$x_t = \prod_{j=1}^{t} A_{i^*_{t-j}} (c_{x,0} + G_{0\beta_0}\alpha_0) + \sum_{j=0}^{t-1} \prod_{h=j+1}^{t-1} A_{i^*_{t+j-h}} (B_{ui^*_j} u_j + B_{wi^*_j} G_{w\beta_w}\alpha_{w,j} + f_{i^*_j}).$$

By defining

$$c_{x,t} = \prod_{j=1}^{t} A_{i^*_{t-j}} c_{x,0} + \sum_{j=0}^{t-1} \prod_{h=j+1}^{t-1} A_{i^*_{t+j-h}} (B_{ui^*_j} u_j + f_{i^*_j})$$

$$G_{x,t}(\beta_0, \beta_w) = \left[\prod_{j=1}^{t} A_{i^*_{t-j}} G_{0\beta_0} \; \prod_{h=1}^{t-1} A_{i^*_{t-h}} B_{wi^*_0} G_{w\beta_w} \; \dots \; B_{wi^*_{t-1}} G_{w\beta_w} \right]$$

we obtain

$$x_t = c_{x,t} + G_{x,t}(\beta_0, \beta_w)\alpha_{x,t}, \tag{12}$$

where

$$\alpha_{x,t} = \begin{bmatrix} \alpha_0 \\ \alpha_{w,0} \\ \vdots \\ \alpha_{w,t-1} \end{bmatrix}, \quad \|\alpha_{x,t}\|_\infty \le 1.$$

The expression in (12) shows that $\mathcal{R}_{x,t}(\beta_0, \beta_w)$ is a zonotope with center $c_{x,t}$ and generator matrix $G_{x,t}(\beta_0, \beta_w)$. By exploiting the zonotopic structure of $\mathcal{R}_{x,t}(\beta_0, \beta_w)$, we can express the state constraints as:

$$X^*_{a,t} c_{x,t} + \max_{\alpha \in [-1,1]^{p_t}} X^*_{a,t} G_{x,t}(\beta_0, \beta_w)\alpha \le x^*_{b,t}, \quad t = 1, \dots, N, \tag{13}$$

where $(X^*_{a,t}, x^*_{b,t})$ is a H-representation of $\mathcal{X}_{sp,t} \cap \mathcal{M}_{i^*_t}$, $p_t = p_0 + tp_w$ and the operator $\max(\cdot)$ is applied elementwise. Condition (13) is equivalent to:

$$X^*_{a,t} c_{x,t} + \|X^*_{a,t} G_{x,t}(\beta_0, \beta_w)\|_1 \le x^*_{b,t}, \quad t = 1, \dots, N, \tag{14}$$

where the operator $\|\cdot\|_1$ is applied row-wise. Now, observe that

$$\|X^*_{a,t} G_{x,t}(\beta_0, \beta_w)\|_1 = \|X^*_{a,t} G_{x,t} \operatorname{diag}(\beta_{01}, \dots, \beta_{0p_0}, \beta_{w1}, \dots, \beta_{wp_w})\|_1 = |X^*_{a,t} G_{x,t}| \begin{bmatrix} \beta_0 \\ 1_t \otimes \beta_w \end{bmatrix} = |X^*_{a,t} G_{x,t}| D_t \begin{bmatrix} \beta_0 \\ \beta_w \end{bmatrix},$$

where the operator $|\cdot|$ is applied elementwise and we set

$$G_{x,t} = \left[\prod_{j=1}^{t} A_{i^*_{t-j}} G_0 \; \prod_{h=1}^{t-1} A_{i^*_{t-h}} B_{wi^*_0} G_w \; \dots \; B_{wi^*_{t-1}} G_w \right], \quad D_t = \operatorname{diag}(I_{p_0}, 1_t \otimes I_{p_w}).$$

Constraints (14) finally rewrite as

$$X^*_{a,t} c_{x,t} + |X^*_{a,t} G_{x,t}| D_t \begin{bmatrix} \beta_0 \\ \beta_w \end{bmatrix} \le x^*_{b,t} \quad t = 1, \dots, N,$$

and are linear in the decision variables $u_0, \dots, u_{N-1}$, β_0 and β_w since $c_{x,t}$ is affine as a function of $u_0, \dots, u_{t-1}$.

The actuation constraints can be written as

$$U_a u_t \le u_b \quad t = 0, \dots, N-1,$$

where (U_a, u_b) is a H-representation of $\mathcal{U}$ and, hence, they are linear in $u_0, \dots, u_{N-1}$.

Summarizing, to assess the robustness level of a given switching sequence for system (3), we have to solve the following Linear Program (LP):

$$\max_{u_0, \dots, u_{N-1}, \beta_0, \beta_w} \varrho(\beta_0, \beta_w) \tag{15}$$

subject to:

$$\begin{cases} \beta_{0i}, \beta_{wj} \in [0,1], \quad i = 1, \dots, p_0, \; j = 1, \dots, p_w \\ X^*_{a,t} c_{x,t} + |X^*_{a,t} G_{x,t}| D_t \begin{bmatrix} \beta_0 \\ \beta_w \end{bmatrix} \le x^*_{b,t}, \quad t = 1, \dots, N \\ U_a u_t \le u_b, \quad t = 0, \dots, N-1. \end{cases}$$

Note that problem (15) is feasible if and only if $S^*_{[0,N-1]}$ is admissible for system (3). Admissibility evaluation of $S^*_{[0,N-1]}$ can then be embedded in this phase, setting the robustness level to -1 in case of infeasibility. In case of feasibility, given an optimal solution β_0^*, β_w^*, the optimal value $\varrho(\beta_0^*, \beta_w^*)$ is compared with some user-chosen threshold $\varrho_{\min} > 0$: sequence $S^*_{[0,N-1]}$ is then accepted if $\varrho(\beta_0^*, \beta_w^*) \geq \varrho_{\min}$, otherwise it is discarded (see Algorithm 2).

2.3 | Static state-feedback design based on reach-set computations

We address the problem of determining a state-feedback control law $\psi_k : \mathbb{R}^{n_x} \to \mathcal{U}$ at time $k \in [0, N-1]$ that makes the control system (2) initialized at time k with $x_k \in \mathcal{Z}_{x,k} = \langle c_{x,k}, G_{x,k} \rangle$ satisfy the spec at time $k+1$, when evolving according to a given switching sequence $S_{[k,N-1]} = (i_k, \dots, i_{N-1})$.

To avoid a blind selection of ψ_k, we fix a prediction horizon length M and design a static state-feedback policy $(\tilde{\psi}_t)_{t=k}^{\min\{k+M-1,N-1\}}$ along the look-ahead horizon $[k, k+M-1]$, truncated to $[k, N-1]$ if $k > N-M$, satisfying the specs for system (1) constrained to the switching sequence $S_{[k,N-1]}$, i.e.:

$$x_{t+1} = A_t x_t + B_{u,t} u_t + B_{w,t} w_t + f_t \tag{16}$$

where $A_t = A_{i_t}$, $B_{u,t} = B_{u,i_t}$, $B_{w,t} = B_{w,i_t}$, $f_t = f_{i_t}$, $t = k, \dots, \min\{k+M-1, N-1\}$.

We then set $\psi_k = \tilde{\psi}_k$ and apply it to $\mathcal{Z}_{x,k}$ to obtain the zonotopic reach set $\mathcal{R}_{x,k+1}$ at time $k+1$, which becomes the initial set for computations at time $k+1$ (see Algorithm 2.)

The structure chosen for the static state feedback policy $(\tilde{\psi}_t)_{t=k}^{\min\{k+M-1,N-1\}}$ rests on the following result.

Proposition 1. Consider system (16) initialized at time k with $x_k \in \langle c_k, G_k \rangle$, $c_k \in \mathbb{R}^{n_x}$ and $G_k \in \mathbb{R}^{n_x \times p_k}$, and subject to $w_t \in \langle 0_{n_w}, G_w \rangle$ with generator matrix $G_w \in \mathbb{R}^{n_w \times p_w}$.

Then, for all $t \geq k$, based on x_t, we can choose $\alpha_t \in \mathbb{R}^{p_t}$, $p_t = p_k + (t-k)p_w$, with $\|\alpha_t\|_\infty \leq 1$, so that by setting

$$u_t = c_{u,t} + G_{u,t}\alpha_t,$$

where $c_{u,t} \in \mathbb{R}^{n_u}$ and $G_{u,t} \in \mathbb{R}^{n_u \times p_t}$, the reach set at time $t+1$ is a zonotope $\mathcal{Z}_{x,t+1} = \langle c_{x,t+1}, G_{x,t+1} \rangle$ with center $c_{x,t}$ and generator matrix $G_{x,t}$ given by:

$$c_{x,t} = \left(\prod_{j=1}^{t-k} A_{t-j}\right) c_k + \sum_{j=k}^{t-1} \left(\prod_{l=j+1}^{t-1} A_{t+j-l}\right)(B_{u,j}c_{u,j} + f_j) \tag{17}$$

$$\begin{aligned} G_{x,t} = \Bigg[& \left(\prod_{j=1}^{t-k} A_{t-j}\right) G_k + \sum_{j=k}^{t-1} \left(\prod_{l=j+1}^{t-1} A_{t+j-l}\right) B_{u,j} G_{u,j}^{[1,p_k]} \quad \left(\prod_{j=2}^{t-k} A_{t+1-j}\right) B_{w,k} G_w + \sum_{j=k}^{t-2} \left(\prod_{l=j+2}^{t-1} A_{t+j+1-l}\right) B_{u,j+1} G_{u,j+1}^{[p_k+1,p_{k+1}]} \\ & \left(\prod_{j=3}^{t-k} A_{t+2-j}\right) B_{w,k+1} G_w + \sum_{j=k}^{t-3} \left(\prod_{l=j+3}^{t-1} A_{t+j+2-l}\right) B_{u,j+2} G_{u,j+2}^{[p_{k+1}+1,p_{k+2}]} \quad \dots \quad B_{w,t-1} G_w \Bigg] \end{aligned} \tag{18}$$

with the understanding that summations and products ranging from h_1 to h_2 with $h_1 > h_2$ correspond, respectively, to the zero and the identity matrix. Specifically, α_t should satisfy:

$$x_t = c_{x,t} + G_{x,t}\alpha_t, \ \|\alpha_t\|_\infty \leq 1. \tag{19}$$

Proof. We prove the statement by induction on t: for $t = k$, since $x_k \in \langle c_k, G_k \rangle$, then, there exists $\alpha_k \in \mathbb{R}^{p_k}$ such that

$$x_k = c_k + G_k \alpha_k, \quad \|\alpha_k\|_\infty \leq 1 \tag{20}$$

Now, by applying:

$$u_k = c_{u,k} + G_{u,k}\alpha_k,$$

since $w_k \in \langle 0_{n_w}, G_w \rangle$, we obtain:

$$x_{k+1} = A_k x_k + B_{u,k}u_k + B_{w,k}w_k + f_k = A_k c_k + B_{u,k}c_{u,k} + f_k + \begin{bmatrix} A_k G_k + B_{u,k}G_{u,k} & B_{w,k}G_w \end{bmatrix} \begin{bmatrix} \alpha_k \\ \alpha_{w,k} \end{bmatrix}, \quad \left\| \begin{matrix} \alpha_k \\ \alpha_{w,k} \end{matrix} \right\|_\infty \leq 1$$

that is, $x_{k+1} \in \langle c_{x,k+1}, G_{x,k+1} \rangle$ where

$$\begin{aligned} c_{x,k+1} &= A_k c_k + B_{u,k}c_{u,k} + f_k \\ G_{x,k+1} &= \begin{bmatrix} A_k G_k + B_{u,k}G_{u,k} & B_{w,k}G_w \end{bmatrix} \end{aligned}$$

are given in (17) and (18) with $t = k+1$.

Now we prove that if the statement holds for $h-1$, then it holds also for h. By the induction hypothesis, $x_h \in \langle c_{x,h}, G_{x,h} \rangle$ where the expressions of $c_{x,h}$ and $G_{x,h}$ are given in (17) and (18) by posing $t = h$. Now, by applying:

$$u_h = c_{u,h} + G_{u,h}\alpha_h,$$

where α_h satisfies (19) with $t = h$, we obtain:

$$x_{h+1} = A_h x_h + B_{u,h}u_h + B_{w,h}w_h + f_h = A_h c_{x,h} + B_{u,h}c_{u,h} + f_h + \begin{bmatrix} A_h G_{x,h} + B_{u,h}G_{u,h} & B_{w,h}G_w \end{bmatrix} \begin{bmatrix} \alpha_h \\ \alpha_{w,h} \end{bmatrix}, \quad \left\| \begin{matrix} \alpha_h \\ \alpha_{w,h} \end{matrix} \right\|_\infty \leq 1$$

By substituting $c_{x,h}$ and $G_{x,h}$ with their expressions obtained by applying the induction hypothesis, we get:

$$\begin{aligned}
A_h c_{x,h} + B_{u,h}c_{u,h} + f_h &= A_h \left(\prod_{j=1}^{h-k} A_{h-j} \right) c_k + A_h \sum_{j=k}^{h-1} \left(\prod_{l=j+1}^{h-1} A_{h+j-l} \right) (B_{u,j}c_{u,j} + f_j) + B_{u,h}c_{u,h} + f_h = \\
&= \left(\prod_{j=1}^{h+1-k} A_{h+1-j} \right) c_k + \sum_{j=k}^{h-1} \left(\prod_{l=j+1}^{h} A_{h+1+j-l} \right) (B_{u,j}c_{u,j} + f_j) + B_{u,h}c_{u,h} + f_h = \\
&= \left(\prod_{j=1}^{h+1-k} A_{h+1-j} \right) c_k + \sum_{j=k}^{h} \left(\prod_{l=j+1}^{h} A_{h+1+j-l} \right) (B_{u,j}c_{u,j} + f_j),
\end{aligned}$$

$$\begin{aligned}
&\begin{bmatrix} A_h G_{x,h} + B_{u,h}G_{u,h} & B_{w,h}G_w \end{bmatrix} = \\
&\Bigg[A_h \Bigg[\left(\prod_{j=1}^{h-k} A_{h-j} \right) G_k + \sum_{j=k}^{h-1} \left(\prod_{l=j+1}^{h-1} A_{h+j-l} \right) B_{u,j} G_{u,j}^{[1,p_k]} \quad \left(\prod_{j=2}^{h-k} A_{h+1-j} \right) B_{w,k}G_w + \sum_{j=k}^{h-2} \left(\prod_{l=j+2}^{h-1} A_{h+j+1-l} \right) B_{u,j+1} G_{u,j+1}^{[p_k+1,p_{k+1}]} \\
&\left(\prod_{j=3}^{h-k} A_{h+2-j} \right) B_{w,k+1}G_w + \sum_{j=k}^{h-3} \left(\prod_{l=j+3}^{h-1} A_{h+j+2-l} \right) B_{u,j+2} G_{u,j+2}^{[p_{k+1}+1,p_{k+2}]} \quad \dots \quad B_{w,h-1}G_w \Bigg] + B_{u,h}G_{u,h} \quad B_{w,h}G_w \Bigg] = \\
&\Bigg[\left(\prod_{j=1}^{h+1-k} A_{h+1-j} \right) G_k + \sum_{j=k}^{h} \left(\prod_{l=j+1}^{h} A_{h+j+1-l} \right) B_{u,j} G_{u,j}^{[1,p_k]} \quad \left(\prod_{j=2}^{h+1-k} A_{h+2-j} \right) B_{w,k}G_w + \sum_{j=k}^{h-1} \left(\prod_{l=j+2}^{h} A_{h+j+2-l} \right) B_{u,j+1} G_{u,j+1}^{[p_k+1,p_{k+1}]} \\
&\left(\prod_{j=3}^{h+1-k} A_{h+3-j} \right) B_{w,k+1}G_w + \sum_{j=k}^{h-2} \left(\prod_{l=j+3}^{h} A_{h+j+3-l} \right) B_{u,j+2} G_{u,j+2}^{[p_{k+1}+1,p_{k+2}]} \quad \dots \quad A_h B_{w,h-1}G_w + B_{u,h} G_{u,h}^{[p_{h-1}+1,p_h]} \quad B_{w,h}G_w \Bigg],
\end{aligned}$$

that is, $x_{h+1} \in \langle c_{x,h+1}, G_{x,h+1} \rangle$, where $c_{x,h+1}$ and $G_{x,h+1}$ are given in (17) and (18) with $t = h+1$. □

By exploiting Proposition 1, we can parametrize the static state feedback control law $\tilde{\psi}_t : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$, $t = k, \dots, k+H-1$, where $H = \min\{M, N-k\}$, as follows:

$$\tilde{\psi}_t(x) = c_{u,t} + G_{u,t}\alpha_t^*(x), \tag{21}$$

where

$$\alpha_t^*(x) = \underset{\alpha \in \{ \beta \in \mathbb{R}^{p_t} : x = c_{x,t} + G_{x,t}\beta, \ \|\beta\|_\infty \leq 1 \}}{\arg\min} \|\alpha\|_2^2$$

and $c_{x,t}$ and $G_{x,t}$ are respectively the center and the generators of the zonotopic set $\mathcal{Z}_{x,t}$ in Proposition 1.[1]

By using the parametrization (21), control design at time k reduces to determine $\{c_{u,k+j}\}_{j=0}^{H-1}$ and $\{G_{u,k+j}\}_{j=0}^{H-1}$ so as to satisfy the following constraints:

$$\begin{aligned}
&\mathcal{Z}_{x,k+j} \subseteq \mathcal{X}_{sp,k+j} \cap \mathcal{M}_{i_{k+j}} \quad j = 1, \dots, H \\
&\mathcal{Z}_{u,k+j} \subseteq \mathcal{U} \quad j = 0, \dots, H-1
\end{aligned}$$

where $\mathcal{M}_{i_N} = \mathbb{R}^{n_x}$, $\mathcal{Z}_{u,k+j-1} = \langle c_{u,k+j-1}, G_{u,k+j-1} \rangle$, $\mathcal{Z}_{x,k+j} = \langle c_{x,k+j}, G_{x,k+j} \rangle$, $j = 1, \dots, H$, and the expressions of $c_{x,k+j}$ and $G_{x,k+j}$ are given in (17) and (18) by setting $c_k = c_{x,k}$, $G_k = G_{x,k}$ and $t = k+j$.

In order to avoid that the design problem turns out to be infeasible because the reach sets split over multiple modes and not because of the specs, we introduce H (nonnegative) auxiliary decision variables ρ_j, $j = 1, \dots, H$, to relax the modes constraints.

[1]We adopt the minimum Euclidean norm as a tie break rule for choosing a vector α_t that satisfies (19).

More precisely, by exploiting the H-representation (6) of $\mathcal{M}_i$, we introduce a non-negative scalar $\rho \geq 0$ and define

$$\mathcal{M}_{i,\rho} = \{x \in \mathbb{R}^{n_x} \,:\, M_{ia}x \leq m_{ib} + \rho 1_{n_h}\},$$

which satisfies $\mathcal{M}_i \subseteq \mathcal{M}_{i,\rho}$. We then consider the modified constraints:

$$\begin{aligned} &\mathcal{Z}_{x,k+j} \subseteq \mathcal{X}_{sp,k+j} \cap \mathcal{M}_{i_{k+j},\rho_j} \quad j = 1, \dots, H \\ &\mathcal{Z}_{u,k+j} \subseteq \mathcal{U} \quad j = 0, \dots, H-1 \\ &\rho_j \geq 0 \quad j = 1, \dots, H \end{aligned} \tag{22}$$

In this way, infeasibility occurs only when the state specifications and the actuation constraints cannot be satisfied, and splitting occurs at time $k+1$ if and only if $\rho_1 > 0$. The idea is then to minimize the cost

$$J_\rho = \sum_{j=1}^{H} \gamma_{\rho j} \rho_j,$$

where $\gamma_{\rho j} \geq 0$, so as to obtain a state feedback that keeps the reach sets inside the modes defined by the given switching sequence $S_{[k,N-1]}$. Indeed, if $\rho_j = 0$, $j = 1, \dots, H$ are feasible values for constraints (22), then splitting does not occur in $[k+1, k+H]$.

In order to reduce the possible splits at the subsequent time instants, we minimize the size of the reach sets through the following cost:

$$J_g = \sum_{j=1}^{H} \gamma_{gj} \sum_{l=1}^{p_{k+j-1}} \left\| G_{x,k+j}^{[l]} \right\|_2^2$$

where $\gamma_{gj} \geq 0$ and the last p_w generators of $G_{x,k+j}$ are neglected because they provide a constant contribution.

To favor recursive feasibility of the constraints (22), we introduce the following cost:

$$J_c = \sum_{j=1}^{H} \gamma_{cj} \left\| c_{x,k+j} - c_{x,k+j}^{OL} \right\|_2^2$$

where $\gamma_{cj} \geq 0$ and $c_{x,k+j}^{OL}$, $j = 1, \dots, H$, are the centers of the open-loop reach sets obtained during the evaluation of the robustness level of $S_{[k,N-1]}$. If the robustness level were maximal, by minimizing J_c (i.e., the distance between the centers of the zonotopic reach sets of the closed-loop system and the corresponding ones of the open-loop system), we aim at imposing a containment condition that guarantees the specs satisfaction over the whole horizon $[k+1, N]$ and not only over the limited time window $[k+1, k+H]$.

Summarizing, at time k we need to solve the following optimization problem:

$$\min_{c_{u,k+j-1}, G_{u,k+j-1}, \rho_j, j=1,\dots,H} (J_c + J_g + J_\rho) \tag{23}$$

subject to:

$$\begin{cases} \mathcal{Z}_{x,k+j} \subseteq \mathcal{X}_{sp,k+j} \cap \mathcal{M}_{i_{k+j},\rho_j}, \quad j = 1, \dots, H \\ \mathcal{Z}_{u,k+j} \subseteq \mathcal{U}, \quad j = 0, \dots, H-1 \\ \rho_j \geq 0, \quad j = 1, \dots, H \end{cases}$$

that is next shown to be a convex Quadratic Program (cQP).

As for the cost, it is clearly convex and quadratic, given the expressions of J_c, J_g and J_ρ, jointly with the fact that $c_{x,k+j}$ and the columns of $G_{x,k+j}$ are affine functions of $c_{u,k}, \dots, c_{u,k+j-1}$ and the columns of $G_{u,k}, \dots, G_{u,k+j-1}$ respectively (see (17) and (18)).

As for the state constraints, by exploiting the zonotopic structure of $\mathcal{Z}_{x,k+j}$, we can express them as:

$$\begin{aligned} X_{a,k+j}^{sp} c_{x,k+j} + \|X_{a,k+j}^{sp} G_{x,k+j}\|_1 \leq x_{b,k+j}^{sp}, \quad j = 1, \dots, H \\ M_{i_{k+j}a} c_{x,k+j} + \|M_{i_{k+j}a} G_{x,k+j}\|_1 \leq m_{i_{k+j}b} + \rho_j 1_{n_h}, \quad j = 1, \dots, H \end{aligned} \tag{24}$$

where $(X_{a,k+j}^{sp}, x_{b,k+j}^{sp})$ is a H-representation of $\mathcal{X}_{sp,k+j}$, the expressions of $M_{i_{k+j}a}$ and $m_{i_{k+j}b}$ are obtained from (6) by posing $i = i_{k+j}$, and the operator $\|\cdot\|_1$ has to be applied row-wise.

In order to obtain linear constraints in the decision variables, we introduce auxiliary variables, each one upper bounding an element of the absolute value matrices $|X_{a,k+j}^{sp} G_{x,k+j}|$ and $|M_{i_{k+j}a} G_{x,k+j}|$, and then plug in these variables in (24). More explicitly, we introduce vectors $h_{sp,jl} \in \mathbb{R}^{q_{sp,j}}$ and $h_{m,jl} \in \mathbb{R}^{n_h}$ and write:

$$\begin{aligned}
&X^{sp}_{a,k+j} c_{x,k+j} + \sum_{l=1}^{p_{k+j-1}} h_{sp,jl} + \|X^{sp}_{a,k+j} B_{w,k+j-1} G_w\|_1 \le x^{sp}_{b,k+j}, \quad j = 1, \dots, H \\
&M_{i_{k+j}a} c_{x,k+j} + \sum_{l=1}^{p_{k+j-1}} h_{m,jl} + \|M_{i_{k+j}a} B_{w,k+j-1} G_w\|_1 \le m_{i_{k+j}b} + \rho_j 1_{n_h}, \quad j = 1, \dots, H \\
&- h_{sp,jl} \le X^{sp}_{a,k+j} G^{[l]}_{x,k+j} \le h_{sp,jl}, \quad l = 1, \dots, p_{k+j-1} \\
&- h_{m,jl} \le M_{i_{k+j}a} G^{[l]}_{x,k+j} \le h_{m,jl}, \quad l = 1, \dots, p_{k+j-1}.
\end{aligned}$$

The same procedure is applied also to the actuation constraints, which become:

$$\begin{aligned}
&U_a c_{u,k+j} + \sum_{l=1}^{p_{k+j}} h_{u,jl} \le u_b, \quad j = 0, \dots, H-1 \\
&- h_{u,jl} \le U_a G^{[l]}_{u,k+j} \le h_{u,jl}, \quad l = 1, \dots, p_{k+j},
\end{aligned}$$

where (U_a, u_b) is a H-representation of $\mathcal{U}$ and $h_{u,jl} \in \mathbb{R}^{q_u}$ are suitable auxiliary vectors.

The state-feedback controller design procedure presented in this section is performed offline, prior to the actual implementation of the controller on the operating system. In the next section, we illustrate the computational effort involved in the online implementation of the designed controller, which consists in retrieving at each time step the control action to be applied to system (1), given the current value of the state.

2.4 | Online control law implementation

The outcome of Algorithm 1 is a time-varying state feedback control law along the time horizon $[0, N-1]$, expressed as a collection of zonotopes $\{\mathcal{Z}^{(i)}_{x,t}, i = 1, \dots, n_t\}_{t=0}^{N-1}$ jointly with the associated static maps $\{\psi^{(i)}_t : \mathcal{Z}^{(i)}_{x,t} \to \mathcal{U}, i = 1, \dots, n_t\}_{t=0}^{N-1}$. The zonotopes represent the (possibly over-approximated) initial uncertainty set if $t = 0$ and the reach sets of the controlled system if $t \in [1, N-1]$. The images of each control law $\psi^{(i)}_t : \mathcal{Z}^{(i)}_{x,t} \to \mathcal{U}$ form a collection of control input zonotopes $\{\mathcal{Z}^{(i)}_{u,t},\ i = 1, \dots, n_t\}_{t=0}^{N-1}$, each zonotope $\mathcal{Z}^{(i)}_{u,t} = \langle c^{(i)}_{u,t}, G^{(i)}_{u,t} \rangle$ being associated with a reach set $\mathcal{Z}^{(i)}_{x,t} \subset \mathbb{R}^{n_x}$, and with center and generator matrix obtained via (23).

If x_t is the state value at time t and belongs to $\mathcal{Z}^{(i)}_{x,t} = \langle c^{(i)}_{x,t}, G^{(i)}_{x,t} \rangle$, then, the associated input is given by:

$$u_t = c^{(i)}_{u,t} + G^{(i)}_{u,t} \alpha^*_t(x_t),$$

where the computation of $\alpha^*_t(x_t)$ requires solving the following cQP problem:

$$\alpha^*_t(x_t) = \underset{\alpha \in \{\beta \in \mathbb{R}^{p_t} : x = c^{(i)}_{x,t} + G^{(i)}_{x,t}\beta,\ \|\beta\|_\infty \le 1\}}{\arg\min} \|\alpha\|_2^2.$$

Identifying the set $\mathcal{Z}^{(i)}_{x,t}$ to which x_t belongs is easy: one has to consider the previous set $\mathcal{Z}^{(j)}_{x,t-1}$ to which state x_{t-1} belongs, consider its one-step successor reach sets, and identify the one to which x_t belongs by determining its mode.

In conclusion, the complexity of the online control law implementation reduces to solving a single cQP with a number of decision variables and constraints that in the worst case grow linearly in time, since $p_t = p_0 + tp_w$.

3 | COMPLEXITY ANALYSIS OF THE OFFLINE CONTROL DESIGN PROCEDURE

In this section we assess the complexity of Algorithm 1 proposed in Section 2 to address the control problem introduced in Section 1. The proposed algorithm rests on some optimization problems as building blocks. Since these problems finally reduce to LP, cQP, MILP, we shall express their complexity in terms of the LP, cQP, MILP complexity.

Linear and convex quadratic programs can be solved in polynomial time, while MILP is NP-hard. In the following, we shall denote with MILP(n_c, n_b, m) an upper bound on the time required to solve a mixed integer linear program with n_c continuous decision variables, n_b binary decision variables, and m inequality constraints, while the symbols LP(n, m) and cQP(n, m) indicate the worst-case computation time required to solve respectively a linear program and a convex quadratic program with n decision variables and m inequality constraints.

We start by considering the problem of generating an admissible switching sequence for system (3) and assessing its robustness level. In particular, we first consider the effort associated to the first switching sequence generation and then we extend the reasoning to the case where multiple switching sequences have to be computed.

If we adopt the MILP method to compute the first switching sequence, a feasibility problem is solved whose complexity is bounded by MILP($N(n_u + n_x)$, $N n_h$, $m_{mld} + m_{sc} + q_u N$), where m_{mld} denotes the number of inequalities of system (4) and m_{sc} and $q_u N$ are respectively the number of the additional state and actuation constraints introduced in Section 2.1. In particular:

$$m_{mld} = 2\,N(n_h + n_x s), \quad m_{sc} = \sum_{t=1}^{N} q_{sp,t} \leq N q_{sp,max},$$

where $q_{sp,max} = \max_{t=1,\dots,N} q_{sp,t}$ and $q_{sp,t}$ is the number of half-spaces used to describe $\mathcal{X}_{sp,t}$.
If, instead, we adopt the method based on reachability analysis, the first sequence is found when the reach sets computation along a branch is not prematurely interrupted for infeasibility. Clearly, the more the number of prematurely interrupted branches, the higher the computational burden. We now quantify the cost of exploring a single branch with no interruptions: at time $t = 0, \dots, N-1$, given a zonotope $\mathcal{Z}_t$, a zonotope $\mathcal{Z}_{t+1} = \langle c_{x,t+1}, G_{x,t+1} \rangle$ is computed with negligible cost, since its G-representation is simply obtained from the ones of $\mathcal{Z}_t$ and $\mathcal{Z}_u$ through matrix multiplications and sums. To assess if the intersection $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}$ is empty, a feasibility test is performed, where a vector α has to be chosen so that $\|\alpha\|_\infty \leq 1$ and:

$$X^{sp}_{a,t+1} c_{x,t+1} + X^{sp}_{a,t+1} G_{x,t+1} \alpha \leq x^{sp}_{b,t+1}$$

where $(X^{sp}_{a,t+1}, x^{sp}_{b,t+1})$ is a H-representation of $\mathcal{X}_{sp,t+1}$. The complexity of this test is bounded by LP($n_{g,t+1}$,$2n_{g,t+1} + q_{sp,t+1}$), $n_{g,t+1}$ being the number of generators of $\mathcal{Z}_{t+1}$. Then, a H-representation of $\mathcal{Z}_{t+1}$ has to be computed and stacked into the one of $\mathcal{X}_{sp,t+1}$. The conversion of a G-representation to a H-representation can be performed by means of the algorithm in [21], whose complexity is $O\left(n_{g,t+1}\binom{n_{g,t+1}}{n_x-1}\right)$. The next step is to check if $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}$ covers multiple modes. To this aim, we check if such a set crosses at least one boundary of the mode partition and thus we need to verify for each $j = 1, \dots, n_h$ if some of the two set-containment conditions $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1} \subseteq \mathcal{H}_j$ and $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1} \subseteq \mathcal{H}^c_j$ is met, $\mathcal{H}^c_j$ being the complement of the half-space $\mathcal{H}_j$. The introduced set-containment conditions are equivalent to $\max_{x \in \mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}} H_{aj} x \leq h_{bj}$ and $\max_{x \in \mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}} -H_{aj} x \leq -h_{bj} - |h_{bj}|\epsilon_m$ respectively, and so we need to solve $2\,n_h$ LP with n_x decision variables and up to $2\binom{n_{g,t+1}}{n_x-1} + q_{sp,t+1}$ inequality constraints, $2\binom{n_{g,t+1}}{n_x-1}$ being a tight upper bound on the number of facets of $\mathcal{Z}_{t+1}$,[21]. After solving such programs, we are able to obtain a (possibly conservative) subset of indices associated to modes covering $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}$. If $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}$ does not violate any half-space boundary, then no splitting occurs and thus the set is outer-approximated with a zonotope, which is used as starting set for computations at time $t + 1$. Otherwise, we have to determine which modes actually cover the set $\mathcal{Z}_{t+1} \cap \mathcal{X}_{sp,t+1}$ by searching for each possibly intersecting mode $\mathcal{M}_i$ a vector α such that $\|\alpha\|_\infty \leq 1$ and:

$$\begin{aligned} M_{ia} c_{x,t+1} + M_{ia} G_{x,t+1} \alpha &\leq m_{ib} \\ X^{sp}_{a,t+1} c_{x,t+1} + X^{sp}_{a,t+1} G_{x,t+1} \alpha &\leq x^{sp}_{b,t+1} \end{aligned}$$

where M_{ia} and m_{ib} assume the expressions in (6). The complexity of this test is bounded by LP($n_{g,t+1}$, $2n_{g,t+1} + n_h + q_{sp,t+1}$) and in the worst-case we have to repeat it s times. Once the actually intersecting modes have been detected, the non-empty intersections with them are easily retrieved by suitably stacking the H-representations of $\mathcal{Z}_{t+1}$, $\mathcal{X}_{sp,t+1}$ and $\mathcal{M}_i$, and zonotopic outer-approximations are computed for each intersection. The complexity of over-approximating a polytope with a zonotope depends on the adopted method. In the following we indicate with $OA(\mathcal{P})$ an upper bound on the computation time required to compute a zonotopic outer-approximation of a polytope $\mathcal{P}$ with one of the two methods presented in Section 2.1.

Summarizing, the cost of reachability analysis along a single branch is bounded by:

$$\sum_{t=1}^{N}\left[LP(n_{g,t},2\,n_{g,t}+q_{sp,t})+Cn_{g,t}\binom{n_{g,t}}{n_x-1}+2\,n_h LP\left(n_x,2\binom{n_{g,t}}{n_x-1}+q_{sp,t}\right)\right]+$$
$$+\sum_{t\in I_{ns}} OA(\mathcal{Z}_t\cap\mathcal{X}_{sp,t})+\sum_{t\in I_s}\left[|I_{cm,t}|LP(n_{g,t},2\,n_{g,t}+n_h+q_{sp,t})+\sum_{i\in I^*_{cm,t}} OA(\mathcal{Z}_t\cap\mathcal{X}_{sp,t}\cap\mathcal{M}_i)\right]\leq$$
$$\leq N\left[LP(n_{g,max},2\,n_{g,max}+q_{sp,max})+Cn_{g,max}\binom{n_{g,max}}{n_x-1}+2\,n_h LP\left(n_x,2\binom{n_{g,max}}{n_x-1}+q_{sp,max}\right)\right]+$$
$$+|I_{ns}|\max_{t\in I_{ns}} OA(\mathcal{Z}_t\cap\mathcal{X}_{sp,t})+|I_s|sLP(n_{g,max},2\,n_{g,max}+n_h+q_{sp,max})+$$
$$+|I_s|s\max_{t\in I_s}\max_{i=1,\dots,s} OA(\mathcal{Z}_t\cap\mathcal{X}_{sp,t}\cap\mathcal{M}_i),$$

where I_s and I_{ns} form a partition of $\{1,\dots,N\}$ and their elements are the time instants where the current set respectively splits among modes and covers a single mode, $I_{cm,t}\subseteq\{1,\dots,s\}$ is the set of the indices associated to modes possibly intersecting $\mathcal{Z}_t\cap\mathcal{X}_{sp,t}$, $I^*_{cm,t}\subseteq I_{cm,t}$ is the set of the indices of the modes actually intersecting $\mathcal{Z}_t\cap\mathcal{X}_{sp,t}$, $n_{g,max}=\max_{t=1,\dots,N} n_{g,t}$ and C is a positive constant.

Finally, if we apply the randomized method, the first switching sequence generation is performed at negligible cost with respect to the other two methods.

Consider now the problem of evaluating the first computed switching sequence: the proposed procedure rests on linear programming to determine the maximal amount of the uncertainty entering system (1) such that specifications can be met by designing an open-loop controller. The complexity associated to this stage is given by the complexity of problem (15), which is bounded by LP($n_uN+p_0+p_w$, $m_\beta+m_{sc}+q_uN$), where

$$m_\beta=2\,(p_0+p_w),\quad m_{sc}=\sum_{j=1}^{N} q_{sp,j}+(N-1)n_h\leq Nq_{sp,max}+(N-1)n_h.$$

If the first computed sequence is not admissible or does not obtain a sufficiently high value of ϱ, then it is necessary to recompute a switching sequence for system (3). Note that, although the evaluation procedure has the same complexity bound for each generated sequence, the generation process may become more or less expensive according to the adopted method.
Consider the MILP method: each time a sequence has to be recomputed, we have to add Nn_h extra binary variables and $1+2Nn_h$ extra inequality constraints, and so its complexity grows at each recomputation step. Differently from MILP, the complexity associated to reachability analysis along a single branch does not depend on the number of already computed sequences and thus the derived bound for the first sequence computation remains unchanged. Similarly, complexity associated to a new switching sequence generation grows in the randomized method, since the greater the number of generated sequences, the smaller is the chance to obtain a new one.

We finally consider the state-feedback control design procedure of Section 2.3. Here, the aim is to satisfy the specifications while at the same time minimizing the number of sets that split: in fact, each time splitting occurs, it is required to outer-approximate each part of the split set with a zonotope and then to branch the parts by generating a new admissible switching sequence for each one of them. A new switching sequence has to be recomputed also when recursive feasibility does not hold. If such situations occur too frequently (splitting and recursive feasibility violation), the computational effort may grow too much and the design procedure may become computationally hard.

We now focus the attention on the case in which splitting never occurs and recursive feasibility holds while considering a given switching sequence. In this case, designing the controller requires only to compute at each time $t=0,\dots,N-1$ a state-feedback control law in the horizon $[t,t+H_t-1]$, $H_t=\min\{M,N-t\}$ by solving a convex quadratic program whose complexity is bounded by cQP($n_{cp,t}+H_t+n_{aux,t}$, $m_{sc,t}+H_t+m_{ac,t}$), where $n_{cp,t}$ is the number of control law parameters at time t, $n_{aux,t}$ is the total number of auxiliaries variables introduced at time t, $m_{sc,t}$ is the number of state constraints applied at time t and $m_{ac,t}$

is the number of actuation constraints applied at time t. Moreover:

$$n_{cp,t} = n_u \left(H_t + \sum_{j=0}^{H_t-1} p_{t+j} \right) = n_u \left(H_t + p_t H_t + \sum_{j=0}^{H_t-1} j p_w \right) = n_u \left[(1+p_t) H_t + \frac{1}{2} p_w H_t (H_t - 1) \right],$$

$$\begin{aligned} n_{aux,t} &= \sum_{j=1}^{H_t} (q_{sp,t+j} + n_h + q_u)\, p_{t+j-1} = \sum_{j=1}^{H_t} (q_{sp,t+j} + n_h + q_u)\, p_t + \sum_{j=1}^{H_t} (q_{sp,t+j} + n_h + q_u)\,(j-1) p_w \leq \\ &\leq \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h + q_u \right) p_t H_t + \frac{1}{2} \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h + q_u \right) p_w H_t (H_t - 1) = \\ &= \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h + q_u \right) \left[p_t H_t + \frac{1}{2} p_w H_t (H_t - 1) \right], \end{aligned}$$

$$\begin{aligned} m_{sc,t} &= \sum_{j=1}^{H_t} q_{sp,t+j} + n_h H_t + \sum_{j=1}^{H_t} 2\,(q_{sp,t+j} + n_h) p_{t+j-1} = \\ &= \sum_{j=1}^{H_t} q_{sp,t+j} + n_h H_t + \sum_{j=1}^{H_t} 2\,(q_{sp,t+j} + n_h) p_t + \sum_{j=1}^{H_t} 2\,(q_{sp,t+j} + n_h)(j-1) p_w \leq \\ &\leq \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) H_t + 2 \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) p_t H_t + \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) p_w H_t (H_t - 1) = \\ &= \left(\max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) \left[(1 + 2\,p_t) H_t + p_w H_t (H_t - 1) \right], \\ m_{ac,t} &= q_u H_t + \sum_{j=0}^{H_t-1} 2\, q_u\, p_{t+j} = q_u H_t + \sum_{j=0}^{H_t-1} 2\, q_u\, p_t + \sum_{j=0}^{H_t-1} 2\, q_u\, j p_w = q_u \left[(1 + 2 p_t) H_t + p_w H_t (H_t - 1) \right]. \end{aligned}$$

Note that both the number of decision variables and the number of constraints are not the same at each time t, since the number of the initial set generators, the prediction horizon length and the number of state specifications are functions of t. An upper-bound to the overall complexity of controller computation in $[0, N-1]$ when no splitting occurs and a feasible controller exists at each time t is then obtained by summing up the bounds on the required computational effort at each time t, that is:

$$\begin{aligned} &\sum_{t=0}^{N-1} \mathrm{cQP}(n_{cp,t} + H_t + n_{aux,t}, m_{sc,t} + H_t + m_{ac,t}) \leq N \cdot \max_{t=0,\dots,N-1} \mathrm{cQP}(n_{cp,t} + H_t + n_{aux,t}, m_{sc,t} + H_t + m_{ac,t}) \leq \\ &\leq N \cdot \mathrm{cQP} \left(\max_{t=0,\dots,N-1} (n_{cp,t} + H_t + n_{aux,t}), \max_{t=0,\dots,N-1} (m_{sc,t} + H_t + m_{ac,t}) \right), \end{aligned}$$

where

$$\begin{aligned} &\max_{t=0,\dots,N-1} (n_{cp,t} + H_t + n_{aux,t}) \leq \max_{t=0,\dots,N-1} \left[(1 + n_u) H_t + \left(n_u + \max_{j=1,\dots,H_t} q_{sp,t+j} + n_h + q_u \right) \left[p_t H_t + \frac{1}{2} p_w H_t (H_t - 1) \right] \right] \leq \\ &\leq (1 + n_u) \max_{t=0,\dots,N-1} H_t + \left(n_u + \max_{t=0,\dots,N-1} \max_{j=1,\dots,H_t} q_{sp,t+j} + n_h + q_u \right) \max_{t=0,\dots,N-1} \left[p_t H_t + \frac{1}{2} p_w H_t (H_t - 1) \right] \leq \\ &\leq (1 + n_u) M + \left(n_u + q_{sp,max} + n_h + q_u \right) \left[(p_0 + (N-1) p_w) M + \frac{1}{2} p_w M (M - 1) \right], \\ &\max_{t=0,\dots,N-1} (m_{sc,t} + H_t + m_{ac,t}) \leq \max_{t=0,\dots,N-1} \left(q_u + \max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) \left[(1 + 2\,p_t) H_t + p_w H_t (H_t - 1) \right] \leq \\ &\leq \left(q_u + \max_{t=0,\dots,N-1} \max_{j=1,\dots,H_t} q_{sp,t+j} + n_h \right) \max_{t=0,\dots,N-1} \left[(1 + 2\,p_t) H_t + p_w H_t (H_t - 1) \right] \leq \\ &\leq \left(q_u + q_{sp,max} + n_h \right) \left[(1 + 2\,(p_0 + (N-1) p_w)) M + p_w M (M - 1) \right]. \end{aligned}$$

This means that the overall computation time is not greater than the one required to solve N convex quadratic programs whose number of decision variables and inequality constraints scale linearly with N and quadratically with M.

4 | NUMERICAL EXAMPLES

We present some numerical examples to illustrate the control design methodology introduced in this paper.

More precisely, our goal is threefold: (i) compare the three approaches presented in Section 2.1 for the generation of an admissible switching sequence for the nominal PWA system, (ii) show the performance of the static state-feedback controller designed via Algorithm 1, and (iii) compare the proposed state-feedback controller with a tube-based controller from the literature.

Computations were performed on a personal computer equipped with a 2.8 GHz Intel Core i7 processor and 16 GB of RAM. All the optimization problems (LP, cQP and MILP) were solved with CPLEX,[23].

4.1 | Generation of an admissible switching sequence and assessment of its robustness level

We consider a discrete-time PWA model obtained through suitable linearization and discretization of the equations of the quadruple-tank process presented in[27]. The resulting system has a four-dimensional state x, a two-dimensional control input u and a four-dimensional disturbance input w accounting for the linearization error. The control input is constrained to the set $\mathcal{U} = [0, 12]^2$ and the disturbance input ranges in $\mathcal{W} = [-1.5, 1.5]^4$. The system has $s = 16$ modes, associated with the state partition generated by the half-spaces:

$$\mathcal{HS}_j : x_j \geq 3 \quad j = 1, \dots, 4.$$

The PWA system dynamics is described by:

$$x_{t+1} = A_i x_t + B_{u,i} u_t + B_{w,i} w_t + f_i, \quad x_t \in \mathcal{M}_i,\ i = 1, \dots, s, \tag{25}$$

where

$$A_1 = \begin{bmatrix} 0.9836 & 0 & 0.0163 & 0 \\ 0 & 0.9884 & 0 & 0.0115 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.9836 & 0 & 0.0163 & 0 \\ 0 & 0.9884 & 0 & 0.0315 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix} \quad A_3 = \begin{bmatrix} 0.9836 & 0 & 0.0444 & 0 \\ 0 & 0.9884 & 0 & 0.0115 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0.9836 & 0 & 0.0444 & 0 \\ 0 & 0.9884 & 0 & 0.0315 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0.9836 & 0 & 0.0163 & 0 \\ 0 & 0.9683 & 0 & 0.0114 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_6 = \begin{bmatrix} 0.9836 & 0 & 0.0163 & 0 \\ 0 & 0.9683 & 0 & 0.0312 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0.9836 & 0 & 0.0444 & 0 \\ 0 & 0.9683 & 0 & 0.0114 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_8 = \begin{bmatrix} 0.9836 & 0 & 0.0444 & 0 \\ 0 & 0.9683 & 0 & 0.0312 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0.9552 & 0 & 0.0161 & 0 \\ 0 & 0.9884 & 0 & 0.0115 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_{10} = \begin{bmatrix} 0.9552 & 0 & 0.0161 & 0 \\ 0 & 0.9884 & 0 & 0.0315 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix} \quad A_{11} = \begin{bmatrix} 0.9552 & 0 & 0.0438 & 0 \\ 0 & 0.9884 & 0 & 0.0115 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 0.9552 & 0 & 0.0438 & 0 \\ 0 & 0.9884 & 0 & 0.0315 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 0.9552 & 0 & 0.0161 & 0 \\ 0 & 0.9683 & 0 & 0.0114 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_{14} = \begin{bmatrix} 0.9552 & 0 & 0.0161 & 0 \\ 0 & 0.9683 & 0 & 0.0312 \\ 0 & 0 & 0.9836 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix} \quad A_{15} = \begin{bmatrix} 0.9552 & 0 & 0.0438 & 0 \\ 0 & 0.9683 & 0 & 0.0114 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9884 \end{bmatrix} \quad A_{16} = \begin{bmatrix} 0.9552 & 0 & 0.0438 & 0 \\ 0 & 0.9683 & 0 & 0.0312 \\ 0 & 0 & 0.9552 & 0 \\ 0 & 0 & 0 & 0.9683 \end{bmatrix}$$

$$B_{u,1} = \begin{bmatrix} 0.0584 & 0.0005 \\ 0.0003 & 0.0513 \\ 0 & 0.0584 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,2} = \begin{bmatrix} 0.0584 & 0.0005 \\ 0.0008 & 0.0513 \\ 0 & 0.0584 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,3} = \begin{bmatrix} 0.0584 & 0.0013 \\ 0.0003 & 0.0513 \\ 0 & 0.0576 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,4} = \begin{bmatrix} 0.0584 & 0.0013 \\ 0.0008 & 0.0513 \\ 0 & 0.0576 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,5} = \begin{bmatrix} 0.0584 & 0.0005 \\ 0.0003 & 0.0507 \\ 0 & 0.0584 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,6} = \begin{bmatrix} 0.0584 & 0.0005 \\ 0.0008 & 0.0507 \\ 0 & 0.0584 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,7} = \begin{bmatrix} 0.0584 & 0.0013 \\ 0.0003 & 0.0507 \\ 0 & 0.0576 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,8} = \begin{bmatrix} 0.0584 & 0.0013 \\ 0.0008 & 0.0507 \\ 0 & 0.0576 \\ 0.0507 & 0 \end{bmatrix}$$

$$B_{u,9} = \begin{bmatrix} 0.0576 & 0.0005 \\ 0.0003 & 0.0513 \\ 0 & 0.0584 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,10} = \begin{bmatrix} 0.0576 & 0.0005 \\ 0.0008 & 0.0513 \\ 0 & 0.0584 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,11} = \begin{bmatrix} 0.0576 & 0.0013 \\ 0.0003 & 0.0513 \\ 0 & 0.0576 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,12} = \begin{bmatrix} 0.0576 & 0.0013 \\ 0.0008 & 0.0513 \\ 0 & 0.0576 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,13} = \begin{bmatrix} 0.0576 & 0.0005 \\ 0.0003 & 0.0507 \\ 0 & 0.0584 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,14} = \begin{bmatrix} 0.0576 & 0.0005 \\ 0.0008 & 0.0507 \\ 0 & 0.0584 \\ 0.0507 & 0 \end{bmatrix} \quad B_{u,15} = \begin{bmatrix} 0.0576 & 0.0013 \\ 0.0003 & 0.0507 \\ 0 & 0.0576 \\ 0.0513 & 0 \end{bmatrix} \quad B_{u,16} = \begin{bmatrix} 0.0576 & 0.0013 \\ 0.0008 & 0.0507 \\ 0 & 0.0576 \\ 0.0507 & 0 \end{bmatrix}$$

$$B_{w,1} = \begin{bmatrix} 0.0438 & 0 & 0.0002 & 0 \\ 0 & 0.0316 & 0 & 0.0001 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,2} = \begin{bmatrix} 0.0447 & 0 & 0.0002 & 0 \\ 0 & 0.0377 & 0 & 0.0004 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix} \quad B_{w,3} = \begin{bmatrix} 0.0541 & 0 & 0.0008 & 0 \\ 0 & 0.0291 & 0 & 0.0001 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,4} = \begin{bmatrix} 0.0521 & 0 & 0.0008 & 0 \\ 0 & 0.0375 & 0 & 0.0004 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix}$$

$$B_{w,5} = \begin{bmatrix} 0.0430 & 0 & 0.0002 & 0 \\ 0 & 0.0366 & 0 & 0.0001 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,6} = \begin{bmatrix} 0.0437 & 0 & 0.0002 & 0 \\ 0 & 0.0430 & 0 & 0.0004 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix} \quad B_{w,7} = \begin{bmatrix} 0.0557 & 0 & 0.0008 & 0 \\ 0 & 0.0372 & 0 & 0.0001 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,8} = \begin{bmatrix} 0.0484 & 0 & 0.0008 & 0 \\ 0 & 0.0402 & 0 & 0.0004 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix}$$

$$B_{w,9} = \begin{bmatrix} 0.0489 & 0 & 0.0002 & 0 \\ 0 & 0.0302 & 0 & 0.0001 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,10} = \begin{bmatrix} 0.0486 & 0 & 0.0002 & 0 \\ 0 & 0.0377 & 0 & 0.0004 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix} \quad B_{w,11} = \begin{bmatrix} 0.0617 & 0 & 0.0008 & 0 \\ 0 & 0.0308 & 0 & 0.0001 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,12} = \begin{bmatrix} 0.0629 & 0 & 0.0008 & 0 \\ 0 & 0.0362 & 0 & 0.0004 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix}$$

$$B_{w,13} = \begin{bmatrix} 0.0501 & 0 & 0.0002 & 0 \\ 0 & 0.0370 & 0 & 0.0001 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,14} = \begin{bmatrix} 0.0495 & 0 & 0.0002 & 0 \\ 0 & 0.0393 & 0 & 0.0004 \\ 0 & 0 & 0.0226 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix} \quad B_{w,15} = \begin{bmatrix} 0.0589 & 0 & 0.0008 & 0 \\ 0 & 0.0346 & 0 & 0.0001 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0159 \end{bmatrix} \quad B_{w,16} = \begin{bmatrix} 0.0575 & 0 & 0.0008 & 0 \\ 0 & 0.0449 & 0 & 0.0004 \\ 0 & 0 & 0.0336 & 0 \\ 0 & 0 & 0 & 0.0238 \end{bmatrix}$$

$$f_1 = \begin{bmatrix} 0.0213 \\ 0.0152 \\ -0.1550 \\ -0.1091 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0.0210 \\ -0.0727 \\ -0.1550 \\ -0.0119 \end{bmatrix} \quad f_3 = \begin{bmatrix} -0.1043 \\ 0.0135 \\ -0.0168 \\ -0.1091 \end{bmatrix} \quad f_4 = \begin{bmatrix} -0.1023 \\ -0.0737 \\ -0.0168 \\ -0.0119 \end{bmatrix} \quad f_5 = \begin{bmatrix} 0.0201 \\ 0.1081 \\ -0.1550 \\ -0.1091 \end{bmatrix} \quad f_6 = \begin{bmatrix} 0.0209 \\ 0.0218 \\ -0.1550 \\ -0.0119 \end{bmatrix} \quad f_7 = \begin{bmatrix} -0.1045 \\ 0.1091 \\ -0.0168 \\ -0.1091 \end{bmatrix} \quad f_8 = \begin{bmatrix} -0.1010 \\ 0.0169 \\ -0.0168 \\ -0.0119 \end{bmatrix}$$

$$f_9 = \begin{bmatrix} 0.1467 \\ 0.0149 \\ -0.1550 \\ -0.1091 \end{bmatrix} \quad f_{10} = \begin{bmatrix} 0.1467 \\ -0.0738 \\ -0.1550 \\ -0.0119 \end{bmatrix} \quad f_{11} = \begin{bmatrix} 0.0287 \\ 0.0152 \\ -0.0168 \\ -0.1091 \end{bmatrix} \quad f_{12} = \begin{bmatrix} 0.0322 \\ -0.0722 \\ -0.0168 \\ -0.0119 \end{bmatrix} \quad f_{13} = \begin{bmatrix} 0.1492 \\ 0.1079 \\ -0.1550 \\ -0.1091 \end{bmatrix} \quad f_{14} = \begin{bmatrix} 0.1496 \\ 0.0175 \\ -0.1550 \\ -0.0119 \end{bmatrix} \quad f_{15} = \begin{bmatrix} 0.0280 \\ 0.1043 \\ -0.0168 \\ -0.1091 \end{bmatrix} \quad f_{16} = \begin{bmatrix} 0.0278 \\ 0.0227 \\ -0.0168 \\ -0.0119 \end{bmatrix}.$$

Specs are defined by the sets $\{\mathcal{X}_{sp,t}\}_{t=1}^{N}$ given by:

$$\mathcal{X}_{sp,t} = \{(x_1, x_2, x_3, x_4) : 0.7t \leq x_1 \leq 5 + t,\ 0.15t \leq x_2 \leq 2 + \frac{t^2}{25},\ 4 \leq x_3 \leq 13,\ 4 \leq x_4 \leq 11\}, \quad t = 1, \ldots, 10,$$
$$\mathcal{X}_{sp,t} = [3,\ 10]^4, \quad t = 11, \ldots, N.$$

Our aim is to estimate the time required for the three methods presented in Section 2.1 to generate an admissible switching sequence for the nominal system associated to (25) initialized at $c_{x,0}$ and to evaluate the robustness level of the obtained sequence with respect to the uncertainty sets $\mathcal{X}_0 = c_{x,0} \oplus [-0.3, 0.3]^2 \times [-0.1, 0.1]^2$ and $\mathcal{W}$ as illustrated in Section 2.2. Note that the reachability analysis based approach and the randomized method embed the admissibility and robustness level evaluation in a single stage, while in MILP approach the two computations are performed separately. However, the computation of the robustness level with MILP is performed only once, on an admissible switching sequence, while the other two methods may require multiple switching sequences evaluations until an admissible one is found.

We consider $N_s = 100$ values of $c_{x,0}$ extracted at random according to a uniform distribution from the set $\mathcal{X}_0 = [0.7, 1.3] \times [0.7, 1.3] \times [11.9, 12.1] \times [6.9, 7.1]$, and determine an admissible switching sequence and its robustness level with the three methods. Tables 1-3 report the maximal and minimal computing time, its average and standard deviation, for increasing values of the time-horizon length N. The problem was feasible for each extracted initial nominal state $c_{x,0}$.

TABLE 1 MILP method: computing time (in seconds) as a function of the time horizon length N.

N	10	50	100	500	1000
$t_{\max}$	0.0268	0.2507	1.7545	203.25	1976.5
$t_{\min}$	0.0188	0.2213	1.0825	87.882	682.17
t_{mean}	0.0212	0.2341	1.1879	113.53	982.88
σ	0.0011	0.0050	0.1131	20.777	236.73

TABLE 2 Set-based reachability analysis method: computing time (in seconds) as a function of the time horizon length N.

N	10	50	100	500	1000
$t_{\max}$	0.3565	1.0647	4.1210	154.57	1745.8
$t_{\min}$	0.2160	0.8716	2.0225	69.266	500.02
t_{mean}	0.2482	0.9460	2.3086	87.351	639.39
σ	0.0227	0.0377	0.2778	15.866	185.07

TABLE 3 Randomized method: computing time (in seconds) as a function of the time horizon length N.

N	10	50	100	500	1000
$t_{\max}$	0.0446	0.5650	1.5409	303.37	2446.7
$t_{\min}$	0.0083	0.1129	0.6087	63.290	491.53
t_{mean}	0.0137	0.1538	0.8088	98.078	827.09
σ	0.0060	0.0703	0.2695	43.858	392.93

By comparing the values in Tables 1-3, one can easily see that, as long as small values of N are adopted (up to $N = 100$), the randomized method provides the smallest computation times. The MILP method exhibits good performance, providing an admissible switching sequence with its robustness level in about a second. However, when larger values of N are considered ($N \geq 500$), the method based on reachability analysis dominates the MILP method and shows better performance also with respect to the randomized method. The randomized method performs better than the MILP one on average, but exhibits a higher standard deviation. This is because as N increases, the probability of getting an admissible switching sequence decreases, and thus robustness level evaluation is repeated multiple times, increasing the computational load. The method based on reachability analysis, instead, takes into account the specs while searching for a sequence, leading to a higher chance of obtaining an admissible sequence with respect to the randomized method.

4.2 | State-feedback controller design

We consider a PWA system with a two-dimensional state x and a two-dimensional control input u, The system is subject to an additive two-dimensional disturbance w taking values within $\mathcal{W} = [-d_w, d_w]^2$.

The modes of the system are associated with the elements of the state partition shown in Figure 1, which is generated by the half-spaces

$$\mathcal{HS}_1 : 0.5\, x_1 - x_2 \leq -2, \quad \mathcal{HS}_2 : x_1 + x_2 \leq 10, \quad \mathcal{HS}_3 : 3\, x_1 - x_2 \leq -15.$$

Since the intersection $\mathcal{HS}_1^c \cap \mathcal{HS}_2^c \cap \mathcal{HS}_3$ is empty, the actual number of modes is $s = 7$.

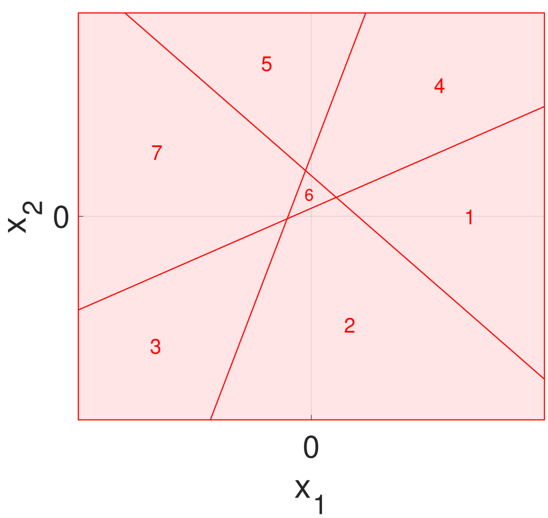

FIGURE 1 PWA modes partition of Example 4.2.

The system dynamics is described by:

$$x_{t+1} = A_i x_t + B_{u,i} u_t + B_{w,i} w_t + f_i, \quad x_t \in \mathcal{M}_i,\ i = 1, \dots, s,$$

where

$$A_1 = O_2, \quad A_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix}, \quad A_3 = -I_2, \quad A_4 = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, \quad A_5 = I_2, \quad A_6 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{4}{5} & -\frac{1}{2} \end{bmatrix}, \quad A_7 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix},$$

$$B_{u,i} = \begin{bmatrix} 1 & b_u \\ b_u & 1 \end{bmatrix}, \quad B_{w,i} = I_2, \quad f_i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i = 1, \dots, s.$$

Note that the control input matrix $B_{u,i}$ depends on some parameter b_u.

The initial state of the system is uncertain and belongs to $\mathcal{X}_0 = c_{x,0} \oplus [-1, 1]^2$. The control input u is constrained within $\mathcal{U} = [-d_u, d_u]^2$.

The goal is to design a static state-feedback control law in the horizon $[0, N-1]$ with $N = 8$ so as to robustly keep the closed-loop system trajectories inside the polyhedral set $\mathcal{X}_{sp}$ defined by the H-representation (X_a^{sp}, x_b^{sp}) with

$$X_a^{sp} = \begin{bmatrix} 0.0990 & -0.0990 \\ 0.0305 & -0.1221 \\ -0.0919 & -0.0306 \\ -0.1246 & 0.1039 \\ -0.0632 & 0.1896 \\ 0.1161 & 0.1161 \\ 0.1322 & 0 \end{bmatrix}, \quad x_b^{sp} = \begin{bmatrix} 0.7426 \\ 0.6867 \\ 0.9188 \\ 1.2466 \\ 1.4538 \\ 1.7767 \\ 0.9912 \end{bmatrix}.$$

Let $c_{x,0} = \begin{bmatrix} -6 & 5 \end{bmatrix}^T$ be the nominal initial state, and $d_w = 0.3$.

By setting $b_u = 1$, the two-dimensional control input is reduced to a scalar input, in that each component of the term $\boldsymbol{B}_{u,i}\, u$ equals $u_1 + u_2$, taking values in $[-2d_u, 2d_u]$. If we set $d_u = 5$, an admissible switching sequence over $[0, N-1]$ for the nominal system is found to be $S_{[0,7]} = (7, 6, 2, 2, 3, 2, 6, 2)$ through the randomized method. Its robusteness level, evaluated by solving problem (15) in Section 2.2, is $\varrho(\beta_0^*, \beta_w^*) = 1$ and corresponds to $\beta_0^* = \beta_w^* = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

This reveals that an open-loop controller (which acts only on the centers of the controlled system zonotopic state sets and not on its generators) can be designed to keep the state evolution inside the intersection of $\mathcal{X}_{sp}$ with the modes of $S_{[0,7]}$ for the full amount of uncertainty entering the system. Consequently, we expect the state-feedback controller (which acts also on the generators of the zonotopic reach sets of the controlled system) to be able to keep the state evolution inside the intersection of $\mathcal{X}_{sp}$ with the modes of the switching sequence $S_{[0,7]}$, as long as recursive feasibility is guaranteed. This is actually achieved by applying the control design procedure of Section 2.3, setting as prediction horizon length $M = 3$ and as cost weights $\gamma_c = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, $\gamma_\rho = \gamma_g = \begin{bmatrix} 0.45 & 0.025 & 0.025 \end{bmatrix}$. The resulting closed-loop reach sets are depicted in Figure 2 together with the open-loop ones. One can see that the designed state-feedback controller is actually able to keep the closed-loop system trajectories inside $\mathcal{X}_{sp}$ and the modes of the switching sequence $S_{[0,7]}$ for each $t = 1, \dots, 7$, thus avoiding splitting.

However, if we reduce the actuation capabilities of the controller by setting e.g. $d_u = 3$, we have the same admissible switching sequence $S_{[0,7]}$ but the closed-loop system reach sets cannot be contained anymore inside all the modes of $S_{[0,7]}$, since a splitting occurs at time $t = 6$ (see Figure 3). Indeed, the switching sequence $S_{[0,7]}$ has a robustness level $\varrho(\beta_0^*, \beta_w^*) = 0.7266$, which shows a reduced capability of the open-loop controller to deal with uncertainty. A new switching sequence is computed at $t = 6$ for the time horizon $[6, 7]$ for each zonotopic outer-approximation of the split set parts. If we set $\varrho_{min} = 0.7$, we obtain an admissible switching sequence with unitary value of ϱ for each branch, and thus the state-feedback controller is able to robustly satisfy the specifications without generating additional splits. This, however, does not occur if we set $\varrho_{min} = 0.5$, since a different admissible switching sequence is computed for the second branch, leading to another split at time $t = 7$ along the second branch. The two situations are depicted in Figure 4.

We finally show the feedback controller performances when full actuation capabilities are available and splitting occurs at some time instants. To this aim, we set $b_u = 0$ and $d_u = 9$. We set $c_{x,0} = \begin{bmatrix} 0 & 5 \end{bmatrix}^T$ and $d_w = 1$. The applied switching sequence is now $S_{[0,7]}^{(1)} = (6, 3, 2, 2, 2, 3, 6, 2)$, whose robustness level is $\varrho(\beta_0^*, \beta_w^*) = 0.6696$ and corresponds to $\beta_0^* = \begin{bmatrix} 0.8468 & 1 \end{bmatrix}^T$ and $\beta_w^* = \begin{bmatrix} 0 & 0.8315 \end{bmatrix}^T$. By applying the control design procedure of Section 2.3 (with the same values of M, γ_c, γ_ρ and γ_g), we obtain the sets depicted in Figure 5, where it can be seen that although multiple splitting occurs along the considered time horizon, all the generated branches of sets can be kept by the controller inside the specifications set $\mathcal{X}_{sp}$. Note that the computed sets at each time instant coincide with $\mathcal{W}$, which means that the controller completely compensates the effect of past uncertainty. Note also that the generated splits are due to bad choices of the switching sequences, which force the controller to steer sets towards regions where avoiding splitting is complicated by the small size of their intersection with $\mathcal{X}_{sp}$. In fact, if we apply the sequence $S_{[0,7]}^{(2)} = (6, 6, 2, 2, 2, 3, 6, 2)$, then splitting is avoided for all modes of $S_{[0,7]}^{(2)}$, as shown in Figure 6. This is because $S_{[0,7]}^{(2)}$ does not require the controller to steer the state inside mode 3 starting from $\mathcal{X}_0$. We finally highlight that the robustness level of $S_{[0,7]}^{(2)}$ is $\varrho(\beta_0^*, \beta_w^*) = 0.7093$ and corresponds to $\beta_0^* = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\beta_w^* = \begin{bmatrix} 0 & 0.8373 \end{bmatrix}^T$, and thus is not very different from $S_{[0,7]}^{(1)}$.

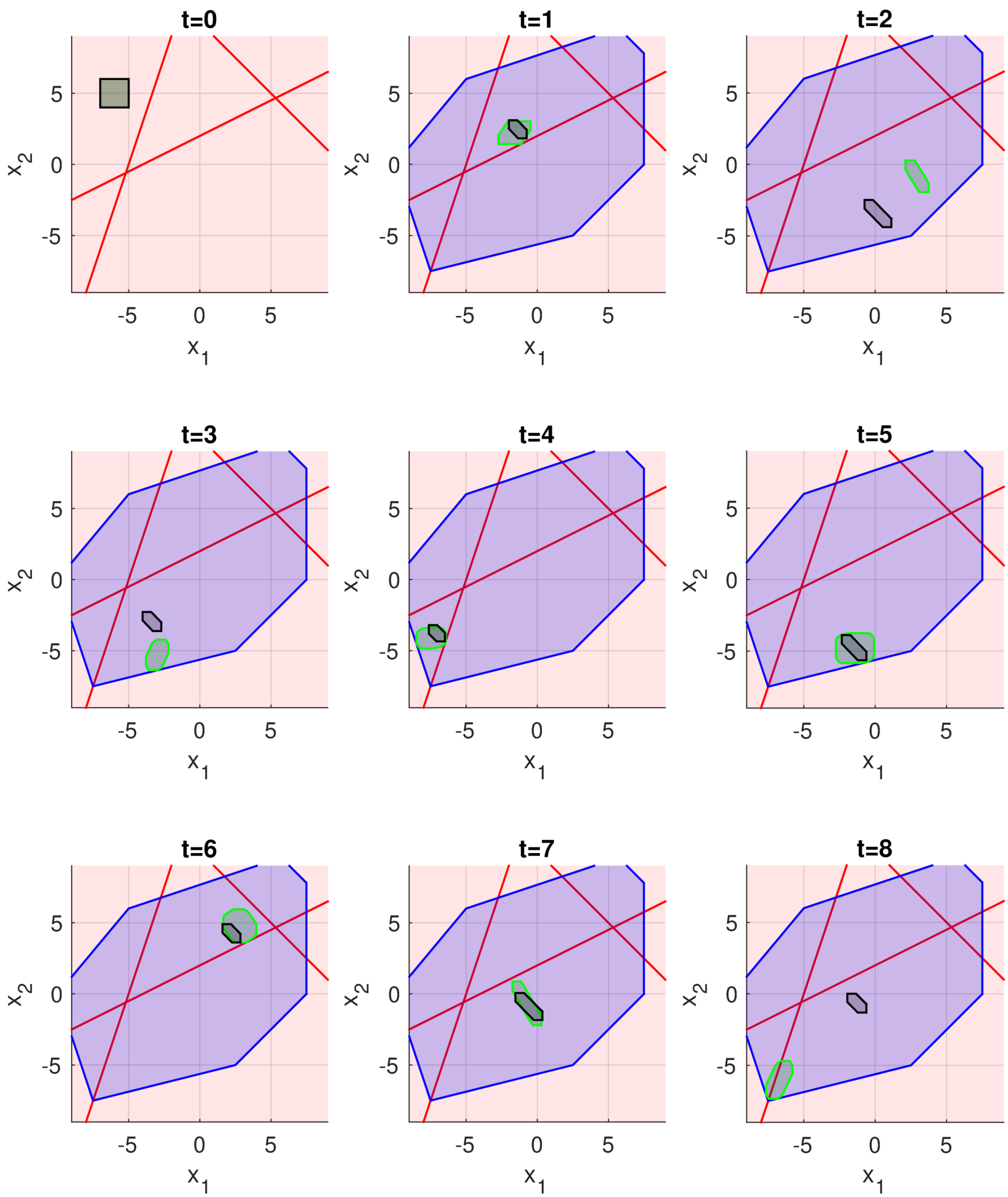

FIGURE 2 State-space sets of the open-loop control system and of the closed-loop control system of Example 4.2 with reduced actuation capabilities when splitting is avoided: in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in black the closed-loop sets and in green the open-loop sets.

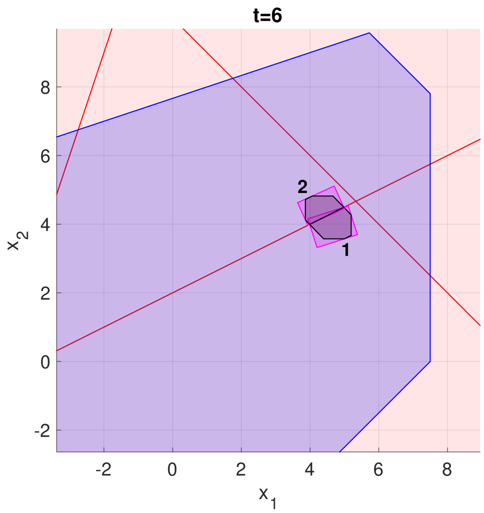

FIGURE 3 State-space set at time $t = 6$ of the closed-loop system of Example 4.2 with reduced actuation capabilities when splitting occurs: in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in black the closed-loop set and in magenta the zonotopic outer-approximations of the split set parts. The numbers near the sets indicate the branches to which they belong.

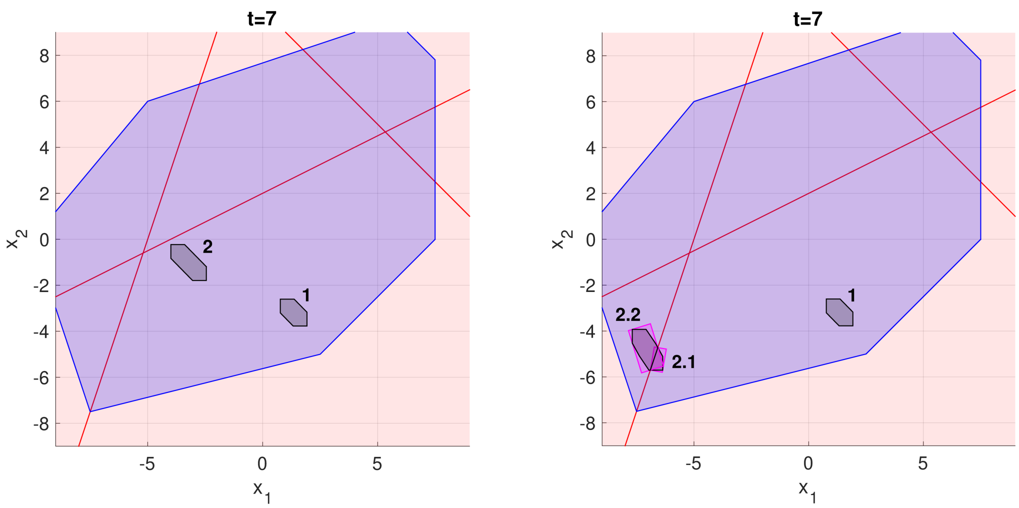

FIGURE 4 State-space sets at time $t = 7$ of the closed-loop system of Example 4.2 with reduced actuation capabilities when splitting occurs and we set $\varrho_{min} = 0.7$ (left) and $\varrho_{min} = 0.5$ (right): in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in black the closed-loop sets and in magenta the zonotopic outer-approximations of the split set parts. The numbers near the sets indicate the branches to which they belong.

4.3 | A comparison with a tube-based controller

In this section we describe an alternative static state-feedback controller and compare its performance with that of the zonotopic law proposed in Section 2.3. The comparative analysis is performed with reference to the problem of making the state of the

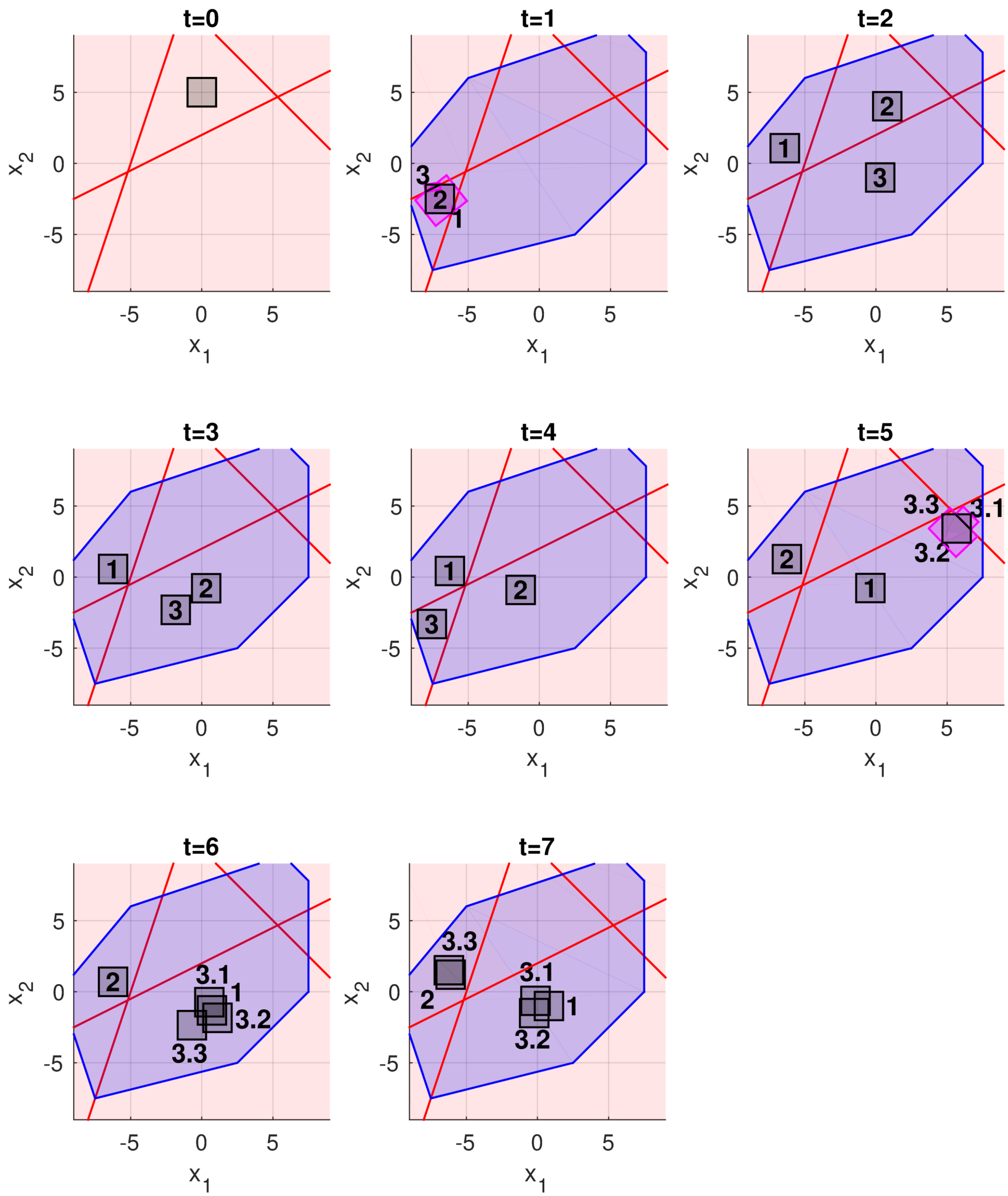

FIGURE 5 State-space sets along $S^{(1)}_{[0,7]}$ of the closed-loop system of Example 4.2 with full actuation capabilities: in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in black the closed-loop sets and in magenta the zonotopic outer-approximations of the split sets parts. The numbers near the sets indicate the branches to which they belong.

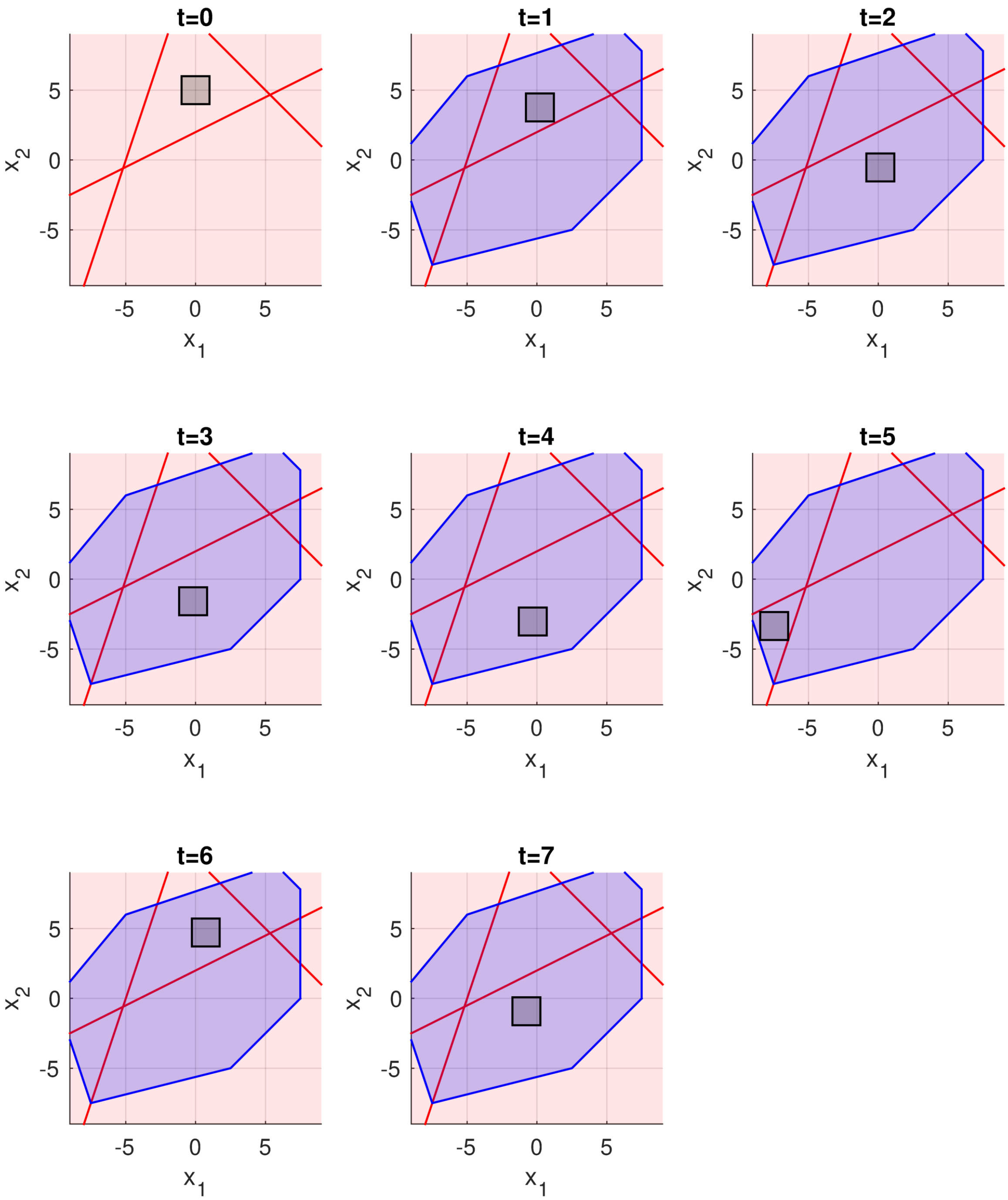

FIGURE 6 State-space sets along $S^{(2)}_{[0,7]}$ of the closed-loop system of Example 4.2 with full actuation capabilities: in red the modes partition, in blue the set $\mathcal{X}_{sp}$ and in black the closed-loop sets.

system (1) evolve inside a given set $\mathcal{X}_{sp}$ along the time horizon $[0, N]$, while accounting for the actuation limits $\mathcal{U}$. We assume that the switching sequence $S_{[0,N-1]} = (i_0, \dots, i_{N-1})$ along the time horizon $[0, N-1]$ is assigned, and that the system is initialized within $\mathcal{X}_0 \subset \mathcal{M}_{i_0}$.

4.3.1 | Tube-based controller design

The considered state-feedback controller[17] is built starting from a parametrized tube of polytopes $\{\mathcal{T}_{x,t}\}_{t=0}^N$ in the state space bounding the uncertain dynamics of (1). More precisely, the section of the tube at each time t depends on a vector z_t and a rescaling factor $\eta_t \geq 0$ as follows:

$$\mathcal{T}_{x,t} = z_t \oplus \eta_t \mathcal{T}_{sh,t},$$

where $\{\mathcal{T}_{sh,t}\}_{t=0}^N$ are predefined polytopes setting the tube geometry. At each time $t = 0, \dots, N-1$, the controller applies to the current state value x_t a convex combination of the control actions $\{u_{tj}\}_{j=1}^{n_{vt}}$ assigned to the n_{vt} vertices $\{x_{tj}\}_{j=1}^{n_{vt}}$ of the tube section $\mathcal{T}_{x,t}$. The coefficients of such a convex combination are those defining the position of the current state inside the current tube section. In symbols:

$$u_t = \sum_{j=1}^{n_{vt}} \lambda_j(x_t) u_{tj}, \quad x_t = \sum_{j=1}^{n_{vt}} \lambda_j(x_t) x_{tj}.$$

Coefficients $\lambda^*(x_t)$ are computed online based on the current state value x_t, as the (unique) solution of the following convex quadratic program:

$$\lambda^*(x_t) = \underset{\lambda \in \left\{\zeta \in \mathbb{R}^{n_{vt}} : x_t = \sum_{j=1}^{n_{vt}} \zeta_j x_{tj},\ \zeta_j \geq 0,\ \sum_{j=1}^{n_{vt}} \zeta_j = 1\right\}}{\arg\min} \|\lambda\|_2^2.$$

We start from $t = 0$ and fix a look-ahead time horizon length $M \leq N$. At each time $t = 0, \dots, N-1$, we design the controllers in the horizon $[t, t+H-1]$, where $H = \min\{M, N-t\}$, and obtain both the control input zonotopes $\{\mathcal{Z}_{u,t+k}\}_{k=0}^{H-1}$ together with the (zonotopic) reach sets $\{\mathcal{Z}_{x,t+k}\}_{k=1}^{H}$ of the control system, and the control input polytopes $\{\mathcal{T}_{u,t+k}\}_{k=0}^{H-1}$ together with the tube sections $\{\mathcal{T}_{x,t+k}\}_{k=1}^{H}$. Then, we retain only the designed controllers at t and move computations at time $t+1$, where controllers are designed in the horizon $[t+1, t+H]$ starting from sets $\mathcal{Z}_{x,t+1}$ and $\mathcal{T}_{x,t+1}$. Note that, only at time $t = 0$, we also have to compute sets $\mathcal{Z}_{x,0}$ and $\mathcal{T}_{x,0}$ so as to include $\mathcal{X}_0$.

During control design at time t, the control actions $\{u_{t+k,j}\}_{j=1}^{n_{v,t+k}}$ applied to the vertices $\{x_{t+k,j}\}_{j=1}^{n_{v,t+k}}$ of $\mathcal{T}_{x,t+k}$, $k = 0, \dots, H-1$, jointly with parameters z_{t+k+1} and η_{t+k+1}, are decision variables to be suitably chosen so as to minimize the size of $\mathcal{T}_{x,t+k+1}$ while robustly satisfying the constraints on state and input. More precisely, starting for $t = 0$ from $\mathcal{X}_0$ and for $t \geq 1$ from the tube section $\mathcal{T}_{x,t}$ computed at time $t-1$, we minimize the cost:

$$J_{tb,t} = \sum_{k=1}^{H} \gamma_{\eta k} \eta_{t+k}$$

where $\gamma_{\eta k} \geq 0$, and impose the following constraints:

$$\begin{aligned}
&\mathcal{T}_{x,t+k} \subseteq \mathcal{X}_{sp} \cap \mathcal{M}_{i_{t+k}}, \quad k = 1, \dots, H \\
&\mathcal{T}_{u,t+k} \subseteq \mathcal{U}, \quad k = 0, \dots, H-1 \\
&(A_{i_{t+k}} x_{t+k,j} + B_{u,i_{t+k}} u_{t+k,j} + f_{i_{t+k}}) \oplus B_{w,i_{t+k}} \mathcal{W} \subseteq \mathcal{T}_{x,t+k+1}, \quad j = 1, \dots, n_{v,t+k},\ k = 0, \dots, H-1 \\
&x_{t+k,j} = z_{t+k} + \eta_{t+k} v_{t+k,j}, \quad j = 1, \dots, n_{v,t+k},\ k = 1, \dots, H,
\end{aligned}$$

where $\mathcal{M}_{i_N} = \mathbb{R}^n$, $\mathcal{T}_{u,t+k} = \mathrm{conv}(\{u_{t+k,j}\}_{j=1}^{n_{v,t+k}})$, and $\mathcal{T}_{sh,t+k} = \mathrm{conv}(\{v_{t+k,j}\}_{j=1}^{n_{v,t+k}})$. At time $t = 0$, since variables z_0 and η_0 are additional degrees of freedom, we impose the additional constraint $\mathcal{X}_0 \subseteq \mathcal{T}_{x,0}$ and add to cost $J_{tb,0}$ a term $\gamma_{\eta 0}\eta_0$, with $\gamma_{\eta 0} \geq 0$. Since both the switching sequence and the tube geometry are fixed, computation of the tube sections with the corresponding feedback laws can be reduced to the solution of a Linear Program (LP).

4.3.2 | Comparative analysis

Consider now the problem instance of Section 4.2 with the following parameter settings:

$$N = 5, \quad M = 3, \quad b_u = 1, \quad d_u = 5, \quad c_{x,0} = \begin{bmatrix} -6 & 5 \end{bmatrix}^T.$$

The aim is to track the modes associated to the switching sequence $S_{[0,4]} = (7, 6, 2, 3, 2)$, while robustly keeping the state inside $\mathcal{X}_{sp}$ and the input inside $\mathcal{U}$. The geometry of the tube sections is defined by the sets:

$$\mathcal{T}_{sh,0} = [-1, 1]^2, \quad \mathcal{T}_{sh,t} = \left\langle 0_2, \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \right\rangle, \quad t = 1, \dots, 5,$$

which resemble the structure of the zonotopic reach sets as in Section 4.2. The set $\mathcal{X}_{sp}$ and the weights γ_ρ, γ_g and γ_c are the same as in the previous section, while for the tube-based controller we set γ_η with all unitary components. In Figures 7-9 we represented (when available) the state sets computed with the two methods: for $d_w = 0.4$ and $d_w = 0.45$, both approaches provide a feasible solution, while for $d_w = 0.5$ only the zonotopic control law provides feasibility. This is because of the lack of flexibility of the tube-based approach, which imposes a fixed structure to the over-approximation of the reach sets. As a result, when constraints become tighter and tighter, this leads prematurely to infeasibility (see Figure 10).

5 | CONCLUSIONS

We proposed a reachability-based solution to the problem of robustly controlling an uncertain PWA system over a finite horizon so as to satisfy state specifications and actuation limits. The designed controller is easy to store and apply, since it is described as a collection of control input zonotopes defined over zonotopic reach sets, and the control input can be determined by solving online a convex quadratic program.

The proposed control design method can be combined with some hybridization technique so as to cope with nonlinear dynamics by space gridding and a PWA approximation of the nonlinear function describing the system dynamics. In this case, however, some adaptation is required since the additive disturbance represents the unmodeled dynamics and takes values in a mode-dependent set. More importantly, some guided refinement of the PWA approximation is needed to obtain a solution to the control design problem while avoiding a too fine state space gridding leading to an exponential growth of the number of modes.

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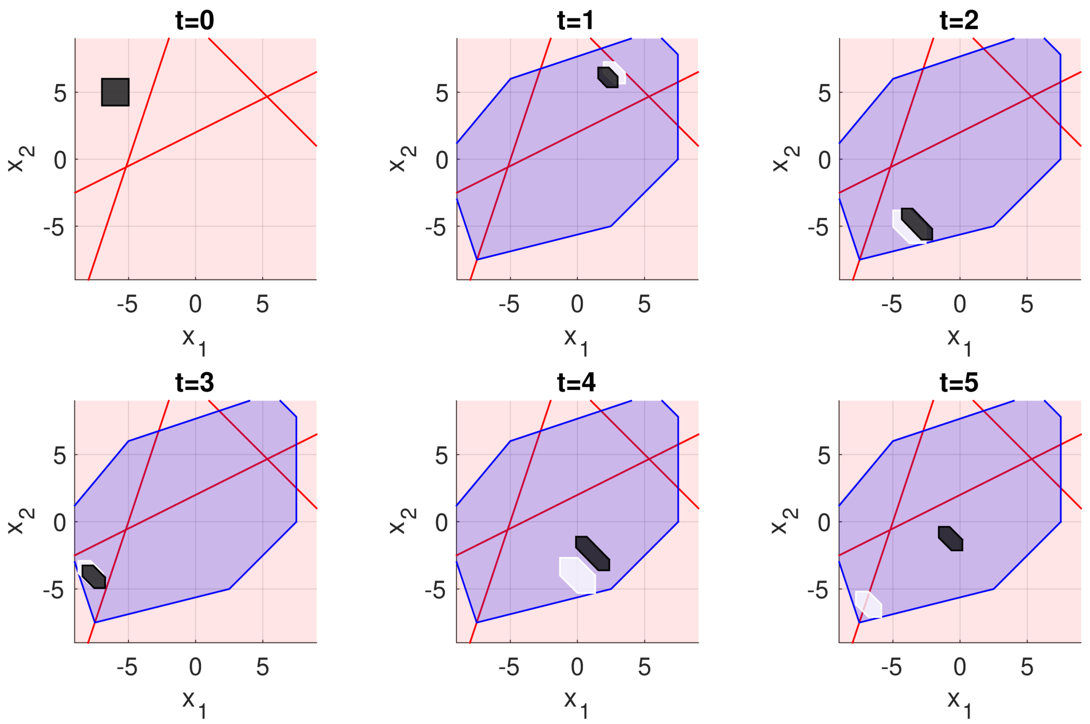

FIGURE 7 State-space sets along $S_{[0,4]}$ of the control systems of Example 4.3 for $d_w = 0.4$: in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in white the tube sections and in black the zonotopic reach sets.

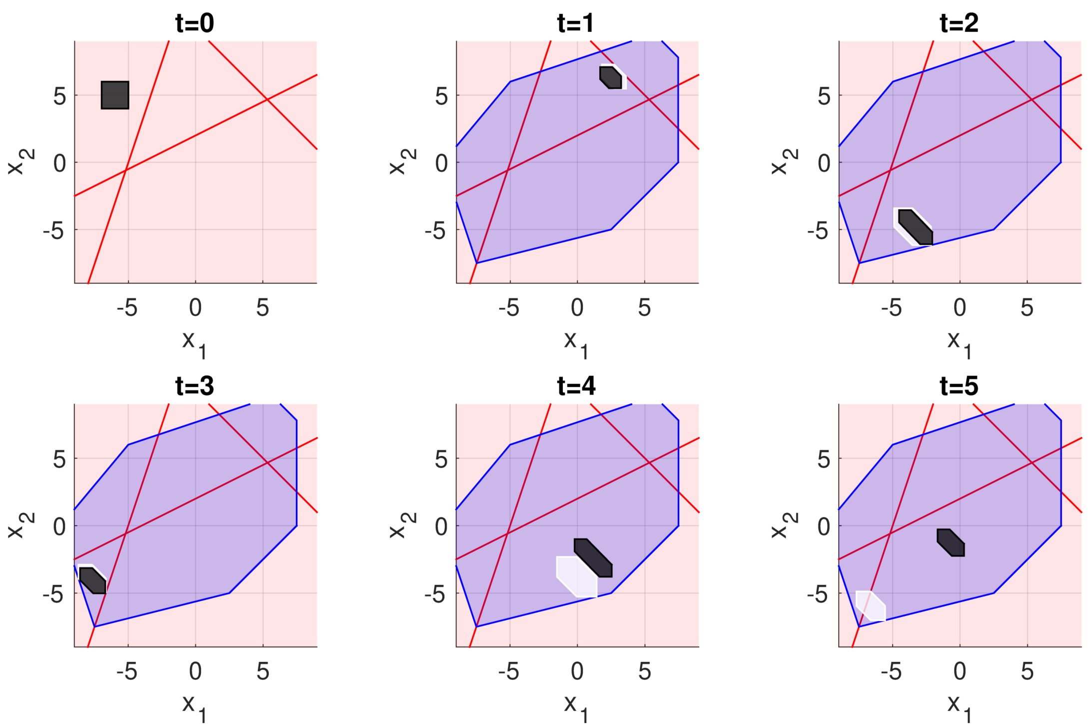

FIGURE 8 State-space sets along $S_{[0,4]}$ of the control systems of Example 4.3 for $d_w = 0.45$: in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in white the tube sections and in black the zonotopic reach sets.

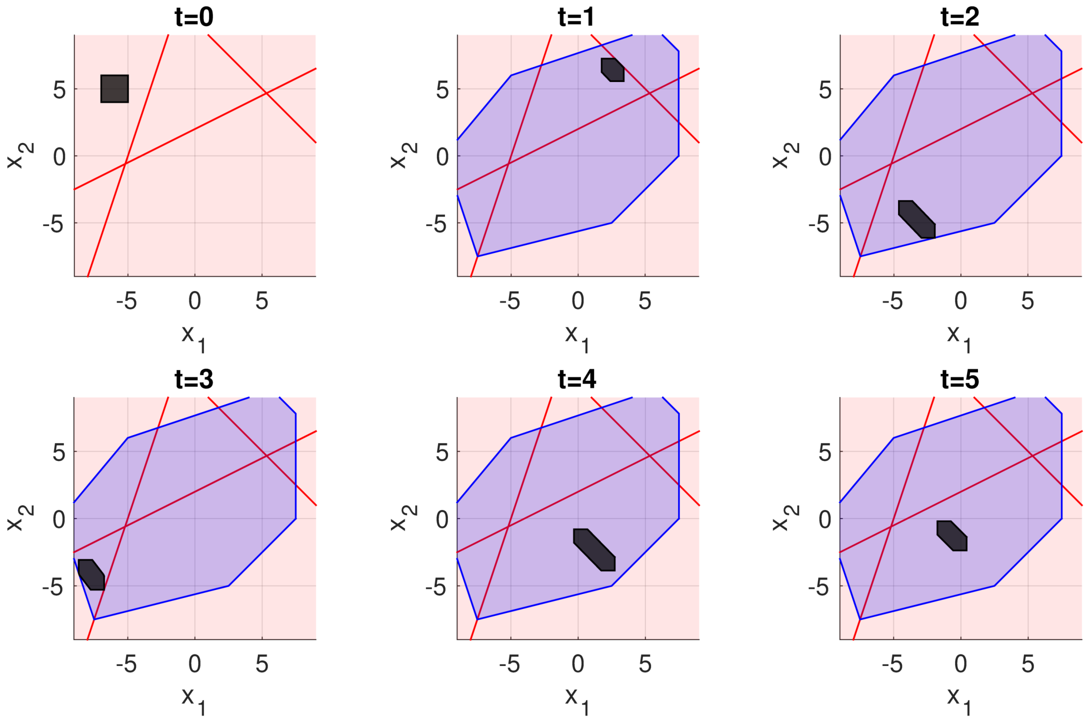

FIGURE 9 State-space sets along $S_{[0,4]}$ of the control system of Example 4.3 for $d_w = 0.5$: in red the modes partition, in blue the set $\mathcal{X}_{sp}$ and in black the zonotopic reach sets.

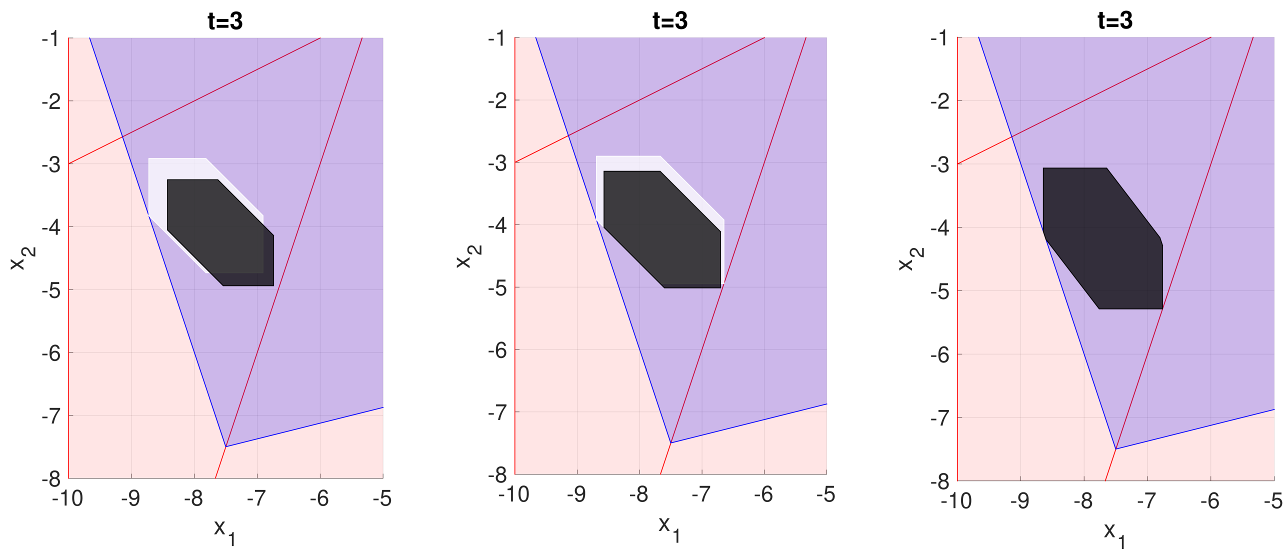

FIGURE 10 State-space sets at time $t = 3$ of the control systems of Example 4.3 for $d_w = 0.4$ (left), $d_w = 0.45$ (center) and $d_w = 0.5$ (right): in red the modes partition, in blue the set $\mathcal{X}_{sp}$, in white the tube sections and in black the zonotopic reach sets. Note that for $d_w = 0.5$ we could only compute the zonotopic reach set.