

CONFORMAL EMBEDDINGS OF AFFINE VERTEX ALGEBRAS IN MINIMAL W -ALGEBRAS I: STRUCTURAL RESULTS

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ABSTRACT. We find all values of $k \in \mathbb{C}$, for which the embedding of the maximal affine vertex algebra in a simple minimal W -algebra $W_k(\mathfrak{g}, \theta)$ is conformal, where \mathfrak{g} is a basic simple Lie superalgebra and $-\theta$ its minimal root. In particular, it turns out that if $W_k(\mathfrak{g}, \theta)$ does not collapse to its affine part, then the possible values of these k are either $-\frac{2}{3}h^\vee$ or $-\frac{h^\vee-1}{2}$, where h^\vee is the dual Coxeter number of \mathfrak{g} for the normalization $(\theta, \theta) = 2$. As an application of our results, we present a realization of simple affine vertex algebra $V_{-\frac{n+1}{2}}(sl(n+1))$ inside of the tensor product of the vertex algebra $W_{\frac{n-1}{2}}(sl(2|n), \theta)$ (also called the Bershadsky-Knizhnik algebra) with a lattice vertex algebra.

To Efim Zelmanov, on the occasion of his 60th birthday

1. INTRODUCTION

Let V be a vertex algebra with a Virasoro (= conformal) vector ω_V and let W be a vertex subalgebra of V endowed with a Virasoro vector ω_W . The embedding $W \subset V$ is called *conformal* if $\omega_W = \omega_V$ [6].

This notion is a generalization of the notion of conformal embedding of a reductive Lie algebra \mathfrak{g}^0 in a simple Lie algebra \mathfrak{g} , studied extensively in the mathematics and physics literature in the 1980's, due to its relevance to string compactifications. Namely, the embedding $\mathfrak{g}^0 \subset \mathfrak{g}$ was called conformal, if there exists a nontrivial vacuum integrable highest weight module over the affine Lie algebra $\widehat{\mathfrak{g}}$, such that the Sugawara constructions of the Virasoro algebra for $\widehat{\mathfrak{g}}$ and for $\widehat{\mathfrak{g}}^0$ coincide. It is not difficult to show that this property is equivalent to 1) the equality of Sugawara central charges, and also equivalent to 2) the finiteness of the decomposition of each eigenspace of the center of \mathfrak{g}^0 of an irreducible level 1 highest weight $\widehat{\mathfrak{g}}$ -module (equivalently, all of them) with respect to $\widehat{\mathfrak{g}}^0$. Maximal such conformal embeddings were classified in [31] (see also [8]), and the corresponding decompositions of level 1 $\widehat{\mathfrak{g}}$ -modules with respect to $\widehat{\mathfrak{g}}^0$ were completely described in [23], [22], [15], [26].

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More recently, some new examples of conformal embeddings for affine vertex algebras have been found [4], [5], [29]. Of course, the corresponding $\widehat{\mathfrak{g}}$ -modules are not integrable, but still the equality of Sugawara conformal vectors turned out to be equivalent to the equality of their central charges. Moreover, it turned out that the decompositions of these modules (when known) are always completely reducible, though not necessarily finite (at each eigenspace of the center of \mathfrak{g}^0). In almost all cases they are described in [6].

Recall that for an affine vertex (super)algebra V of level k , associated to a simple Lie (super)algebra \mathfrak{g} with a (super)symmetric non-zero invariant bilinear form, the Sugawara Virasoro vector is given by the formula

$$\omega_V = \frac{1}{2(k + h^\vee)} \sum_i b_i a_i,$$

where $\{a_i\}$, $\{b_i\}$ are dual bases of \mathfrak{g} , h^\vee is the dual Coxeter number (i.e., half of the eigenvalue of the Casimir element of \mathfrak{g}), and $k + h^\vee$ is assumed to be nonzero.

In the present paper we study conformal embeddings of the maximal affine vertex subalgebra of a simple minimal W-algebra $W_k(\mathfrak{g}, \theta)$, attached to a minimal grading of a simple basic Lie superalgebra \mathfrak{g} . Recall that a minimal grading of \mathfrak{g} is the $\frac{1}{2}\mathbb{Z}$ -grading of the form

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$ are the root spaces of \mathfrak{g} attached to the even roots $\pm\theta$. It is well known that there exists a unique up to isomorphism such grading for any simple Lie algebra, but for a simple basic Lie superalgebra there are several (see [24]).

It is proven in [24] that, for $k \neq -h^\vee$, the universal minimal W-algebra $W^k(\mathfrak{g}, \theta)$ has a unique simple quotient, denoted by $W_k(\mathfrak{g}, \theta)$, and that $W^k(\mathfrak{g}, \theta)$ is freely generated by the elements $J^{\{a\}}$ (a runs over a basis of \mathfrak{g}^\natural), $G^{\{u\}}$ (u runs over a basis of $\mathfrak{g}_{-1/2}$), and the Virasoro vector ω , where \mathfrak{g}^\natural is the centralizer in \mathfrak{g}_0 of $e_{-\theta}$ (hence of e_θ). Furthermore the elements $J^{\{a\}}$ (resp. $G^{\{u\}}$) are primary of conformal weight 1 (resp. $3/2$), with respect to ω , and they satisfy the following λ -brackets:

$$\begin{aligned} [J^{\{a\}}{}_\lambda J^{\{b\}}] &= J^{\{[a,b]\}} + \lambda \left((k + h^\vee/2)(a|b) - \frac{1}{4}\kappa_0(a, b) \right), \quad a, b \in \mathfrak{g}^\natural, \\ [J^{\{a\}}{}_\lambda G^{\{u\}}] &= G^{\{[a,u]\}}, \quad a \in \mathfrak{g}^\natural, u \in \mathfrak{g}_{-1/2}, \end{aligned}$$

where $(\cdot|\cdot)$ is the invariant bilinear form of \mathfrak{g} , normalized by the condition $(\theta|\theta) = 2$ and κ_0 is the Killing form of \mathfrak{g}_0 . The λ -brackets between elements $G^{\{u\}}$ and $G^{\{v\}}$ given in [25] (see (2.7)) are much more complicated.

Our first key result is an important simplification of this formula. It reads (see Lemma 3.1):

$$(1.1) \quad [G^{\{u\}}{}_{\lambda} G^{\{v\}}] = -2(k + h^{\vee})(e_{\theta}|[u, v])\omega + (e_{\theta}|[u, v]) \sum_{\alpha=1}^{\dim \mathfrak{g}^{\natural}} : J^{\{a^{\alpha}\}} J^{\{a_{\alpha}\}} : + \\ \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} : J^{\{[u, u^{\gamma}]^{\natural}\}} J^{\{[u_{\gamma}, v]^{\natural}\}} : + 2(k+1)\partial J^{\{[e_{\theta}, u, v]^{\natural}\}} \\ + 4\lambda \sum_{i \in I} \frac{p(k)}{k_i} J^{\{[e_{\theta}, u, v]_i^{\natural}\}} + 2\lambda^2 (e_{\theta}|[u, v])p(k)\mathbf{1}.$$

Here $\{a_{\alpha}\}$ (resp. $\{u_{\gamma}\}$) is a basis of \mathfrak{g}^{\natural} (resp. $\mathfrak{g}_{1/2}$) and $\{a^{\alpha}\}$ (resp. $\{u^{\gamma}\}$) is the corresponding dual basis w.r.t. $(\cdot|\cdot)$ (resp. w.r.t. $\langle \cdot, \cdot \rangle_{\text{ne}} = (e_{-\theta}|[\cdot, \cdot])$), a^{\natural} is the orthogonal projection of $a \in \mathfrak{g}_0$ on \mathfrak{g}^{\natural} , a_i^{\natural} is the projection of a^{\natural} on the i th minimal ideal $\mathfrak{g}_i^{\natural}$ of \mathfrak{g}^{\natural} , $k_i = k + \frac{1}{2}(h^{\vee} - h_{0,i}^{\vee})$, where $h_{0,i}^{\vee}$ is the dual Coxeter number of $\mathfrak{g}_i^{\natural}$ with respect to the restriction of the form $(\cdot|\cdot)$.

The important new ingredient here is the monic quadratic polynomial $p(k)$, which is given in Table 4 of the present paper. (We show that all linear polynomials k_i divide $p(k)$). Note that in all cases \mathfrak{g}^{\natural} is a direct sum of all its minimal ideals, with the exception of $\mathfrak{g} = sl(2+n|n)$ ($n > 0$), which we exclude from consideration.

This result allows us to determine all non-critical *collapsing levels*, i.e. those k with $k \neq -h^{\vee}$ for which the image in $W_k(\mathfrak{g}, \theta)$ of the universal affine vertex algebra generated by $J^{\{a\}}$, $a \in \mathfrak{g}^{\natural}$, coincides with the whole vertex algebra $W_k(\mathfrak{g}, \theta)$ (see Section 3). Namely, we show that a non-critical level k is collapsing iff $p(k) = 0$ (Theorem 3.3).

Special collapsing levels are those k for which $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$. It turns out that this happens precisely in the following two situations (see Proposition 3.4):

- (1) $k = -\frac{h^{\vee}}{6} - 1$ and \mathfrak{g} is one of the Lie algebras of exceptional Deligne's series [16]: $A_2, G_2, D_4, F_4, E_6, E_7, E_8$, or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$), $osp(n+8|8)$ ($n \geq 2$), $spo(2|1), F(4), G(3)$ (for both choices of θ); in all these cases

$$p(k) = \left(k + \frac{h^{\vee}}{6} + 1\right)\left(k + \frac{h^{\vee}}{3}\right).$$

- (2) $k = -\frac{1}{2}$ and $\mathfrak{g} = spo(n|m)$ ($n \geq 1$); in this case

$$p(k) = \left(k + \frac{1}{2}\right)\left(k + \frac{h^{\vee} + 1}{6}\right).$$

Note that a connection of minimal simple W -algebras to Deligne's series was first observed in [27], and the collapsing levels k for which $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$ when \mathfrak{g} is Lie algebra have been already found in [11]. Note also that, as

an application, we obtain an (almost) unified proof of the formula for the superdimension when \mathfrak{g} appears in (1) above (cf. [16], [32]):

$$\text{sdim } \mathfrak{g} = \frac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6}.$$

We then proceed to determine the *conformal levels*, i.e. those k for which the image in $W_k(\mathfrak{g}, \theta)$ of the universal affine vertex algebra generated by $J^{\{a\}}$, $a \in \mathfrak{g}^\natural$, is conformally embedded in $W_k(\mathfrak{g}, \theta)$. Obviously, the collapsing levels are conformal. A remarkable fact, discovered in this paper, is that a non-collapsing conformal embedding may happen only if $k = -\frac{2}{3}h^\vee$ or $k = \frac{1-h^\vee}{2}$. We do not have a uniform explanation for this; we discovered this fact after listing all non-collapsing conformal levels by a case-wise verification (see Section 4). Another striking fact follows from this: the equality of central charges of the affine part and of $W_k(\mathfrak{g}, \theta)$ is not only (obviously) a necessary, but also a sufficient condition for k to be a conformal level!

A further application of our simplified λ -bracket is given in Section 5. It is the generalization to higher rank of the realization, discovered in [3], of $\widehat{sl(3)}$ at level $-\frac{3}{2}$ in the $N = 4$ superconformal algebra tensored with a lattice vertex algebra. More precisely we realize $\widehat{sl(n+1)}$ at level $-\frac{n+1}{2}$ as the zero charge component of the tensor product of $W_{\frac{n-1}{2}}(sl(2|n), \theta)$ with a rank one lattice vertex algebra. The W -algebras $W_k(\widehat{sl(2|n)}, \theta)$ have been very much investigated in physics literature under the name Bershadsky-Knizhnik superconformal algebra [13], [18], [28], [30]. Our result show that these algebras at certain central charges are closely related with affine vertex algebras at level $-\frac{n}{2}$.

In our subsequent paper [7] we describe in most of the cases how $W_k(\mathfrak{g}, \theta)$ decomposes with respect to its affine part when k is a conformal (non-collapsing) level. In all these cases the decomposition turns out to be completely reducible, though not necessarily finite.

Notation. The base field is \mathbb{C} . As usual, tensor product of a family of vector spaces ranging over the empty set is meant to be \mathbb{C} . For a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ we set $\text{Dim}V = \dim V_{\bar{0}} | \dim V_{\bar{1}}$, $\dim V = \dim V_{\bar{0}} + \dim V_{\bar{1}}$ and $\text{sdim}V = \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

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2. MINIMAL QUANTUM AFFINE W -ALGEBRAS

Let \mathfrak{g} be a basic simple Lie superalgebra. Recall that among all simple finite-dimensional Lie superalgebras it is characterized by the properties that its even part \mathfrak{g}_0 is reductive and that it admits a non-degenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$. A complete list of basic simple Lie superalgebras consists of simple finite-dimensional Lie algebras and the Lie superalgebras $sl(m|n)$ ($m, n \geq 1, m \neq n$), $psl(m|m)$ ($m \geq 2$), $osp(m|n) = spo(n|m)$ ($m \geq 1, n \geq 2$ even), $D(2, 1; a)$ ($a \in \mathbb{C}, a \neq 0, -1$), $F(4)$, $G(3)$ [19]. Recall that $sl(2|1)$ and $spo(2|2)$ are isomorphic. Also, the Lie superalgebras $D(2, 1; a)$ and $D(2, 1; a')$ are isomorphic if and only if a, a' lie on the same orbit of the group generated by the transformations $a \mapsto a^{-1}$ and $a \mapsto -1 - a$, and $D(2, 1; 1) = osp(4|2)$. See [19] for details.

Choose a Cartan subalgebra \mathfrak{h} for \mathfrak{g}_0 and let Δ be the set of roots. A root $-\theta$ is called *minimal* if it is even and there exists an additive function $\varphi : \Delta \rightarrow \mathbb{R}$ such that $\varphi|_{\Delta} \neq 0$ and $\varphi(\theta) > \varphi(\eta)$, $\forall \eta \in \Delta \setminus \{\theta\}$. Obviously, a minimal root $-\theta$ is the lowest root of one of the simple components of \mathfrak{g}_0 (in the ordering defined by φ). Conversely, it is easy to see, using the description of Δ given in [19], that a long root of any simple component of \mathfrak{g}_0 (with respect to its Killing form) is minimal except when $\mathfrak{g} = osp(3|n)$ and the simple component of \mathfrak{g}_0 is $so(3)$.

Fix a minimal root $-\theta$ of \mathfrak{g} . We may choose root vectors e_θ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \quad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of adx defines a *minimal* $\frac{1}{2}\mathbb{Z}$ -grading ([24, (5.1)]):

$$(2.1) \quad \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm\theta}$. We thus have a bijective correspondence between minimal gradings (up to an automorphism of \mathfrak{g}) and minimal roots (up to the action of the Weyl group). Furthermore, one has

$$(2.2) \quad \mathfrak{g}_0 = \mathfrak{g}^{\natural} \oplus \mathbb{C}x, \quad \mathfrak{g}^{\natural} = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

Note that \mathfrak{g}^{\natural} is the centralizer of the triple $\{f_\theta, x, e_\theta\}$. We can choose $\mathfrak{h}^{\natural} = \{h \in \mathfrak{h} \mid (h|x) = 0\}$, as a Cartan subalgebra of the Lie superalgebra \mathfrak{g}^{\natural} , so that $\mathfrak{h} = \mathfrak{h}^{\natural} \oplus \mathbb{C}x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on \mathfrak{g} by the condition

$$(2.3) \quad (\theta|\theta) = 2.$$

The dual Coxeter number h^\vee of the pair (\mathfrak{g}, θ) (equivalently, of the minimal gradation (2.1)) is defined to be half the eigenvalue of the Casimir operator of \mathfrak{g} corresponding to $(\cdot|\cdot)$, normalized by (2.3).

The complete list of the Lie superalgebras \mathfrak{g}^{\natural} , the \mathfrak{g}^{\natural} -modules $\mathfrak{g}_{\pm 1/2}$ (they are isomorphic and self-dual), and h^\vee for all possible choices of \mathfrak{g} and of θ (up to isomorphism) is given in Tables 1,2,3 of [24]. For the reader's convenience

we reproduce them below. Note that in these tables $\mathfrak{g} = osp(m|n)$ (resp. $\mathfrak{g} = spo(n|m)$) means that θ is the highest root of the simple component $so(m)$ (resp. $sp(n)$) of $\mathfrak{g}_{\bar{0}}$. Also, for $\mathfrak{g} = sl(m|n)$ or $psl(m|m)$ we always take θ to be the highest root of the simple component $sl(m)$ of $\mathfrak{g}_{\bar{0}}$ (for $m = 4$ we take one of the simple roots). Note that the exceptional Lie superalgebras $\mathfrak{g} = F(4)$ and $\mathfrak{g} = G(3)$ appear in both Tables 2 and 3, which corresponds to the two inequivalent choices of θ , the first one being a root of the simple component $sl(2)$ of $\mathfrak{g}_{\bar{0}}$.

Table 1

\mathfrak{g} is a simple Lie algebra.

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}	\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(n), n \geq 3$	$gl(n-2)$	$\mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$	n	F_4	$sp(6)$	$\bigwedge^3 \mathbb{C}^6$	9
$so(n), n \geq 5$	$sl(2) \oplus so(n-4)$	$\mathbb{C}^2 \otimes \mathbb{C}^{n-4}$	$n-2$	E_6	$sl(6)$	$\bigwedge^3 \mathbb{C}^6$	12
$sp(n), n \geq 2$	$sp(n-2)$	\mathbb{C}^{n-2}	$n/2 + 1$	E_7	$so(12)$	$spin_{12}$	18
G_2	$sl(2)$	$S^3 \mathbb{C}^2$	4	E_8	E_7	$\dim = 56$	30

Table 2

\mathfrak{g} is not a Lie algebra but \mathfrak{g}^{\natural} is and $\mathfrak{g}_{\pm 1/2}$ is purely odd ($m \geq 1$).

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}	\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(2 m), m \geq 2$	$gl(m)$	$\mathbb{C}^m \oplus (\mathbb{C}^m)^*$	$2-m$	$D(2, 1; a)$	$sl(2) \oplus sl(2)$	$\mathbb{C}^2 \otimes \mathbb{C}^2$	0
$psl(2 2)$	$sl(2)$	$\mathbb{C}^2 \oplus \mathbb{C}^2$	0	$F(4)$	$so(7)$	$spin_7$	-2
$spo(2 m)$	$so(m)$	\mathbb{C}^m	$2-m/2$	$G(3)$	G_2	$\text{Dim} = 0 7$	-3/2
$osp(4 m)$	$sl(2) \oplus sp(m)$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$2-m$				

Table 3

Both \mathfrak{g} and \mathfrak{g}^{\natural} are not Lie algebras. ($m, n \geq 1$)

\mathfrak{g}	\mathfrak{g}^{\natural}	$\mathfrak{g}_{1/2}$	h^{\vee}
$sl(m n), n \neq m > 2$	$gl(m-2 n)$	$\mathbb{C}^{m-2 n} \oplus (\mathbb{C}^{m-2 n})^*$	$m-n$
$psl(m m), m > 2$	$sl(m-2 m)$	$\mathbb{C}^{m-2 m} \oplus (\mathbb{C}^{m-2 m})^*$	0
$spo(n m), n \geq 4$	$spo(n-2 m)$	$\mathbb{C}^{n-2 m}$	$1/2(n-m) + 1$
$osp(m n), m \geq 5$	$osp(m-4 n) \oplus sl(2)$	$\mathbb{C}^{m-4 n} \otimes \mathbb{C}^2$	$m-n-2$
$F(4)$	$D(2, 1; 2)$	$\text{Dim} = 6 4$	3
$G(3)$	$osp(3 2)$	$\text{Dim} = 4 4$	2

In this paper we shall exclude the case of $\mathfrak{g} = sl(n+2|n)$, $n > 0$. In all other cases the Lie superalgebra \mathfrak{g}^{\natural} decomposes in a direct sum of all its minimal ideals, called components of \mathfrak{g}^{\natural} :

$$\mathfrak{g}^{\natural} = \bigoplus_{i \in I} \mathfrak{g}_i^{\natural},$$

where each summand is either the (at most 1-dimensional) center of \mathfrak{g}^{\natural} or is a basic classical simple Lie superalgebra different from $psl(n|n)$. We will also exclude $\mathfrak{g} = sl(2)$ for reasons which will be explained at the very beginning of Section 3.

It follows from the tables that the index set I has cardinality $r = 0, 1, 2$, or 3 (the latter case happens only for $\mathfrak{g} = so(8)$). The case $r = 0$, i.e. $\mathfrak{g}^{\natural} = \{0\}$, happens if and only if $\mathfrak{g} = spo(2|1)$. In the case when the center

is non-zero (resp. zero) we use $I = \{0, 1, \dots, r-1\}$ (resp. $I = \{1, \dots, r\}$) as the index set, and denote the center of \mathfrak{g}^{\natural} by $\mathfrak{g}_0^{\natural}$.

Let $C_{\mathfrak{g}_i^{\natural}}$ be the Casimir operator of $\mathfrak{g}_i^{\natural}$ corresponding to $(\cdot|\cdot)_{|\mathfrak{g}_i^{\natural} \times \mathfrak{g}_i^{\natural}}$. We define the dual Coxeter number $h_{0,i}^{\vee}$ of $\mathfrak{g}_i^{\natural}$ as half of the eigenvalue of $C_{\mathfrak{g}_i^{\natural}}$ acting on $\mathfrak{g}_i^{\natural}$ (which is 0 if $\mathfrak{g}_i^{\natural}$ is abelian). Their values are given in Table 4 of [24]. If all $h_{0,i}, i \in I$, are equal, we denote this number by h_0^{\vee} .

In [21], [24] a vertex algebra $W^k(\mathfrak{g}, f)$, called a *universal W -algebra*, has been associated to each triple (\mathfrak{g}, f, k) , where \mathfrak{g} is a basic simple Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form, f is a nilpotent element of \mathfrak{g}_0 , and $k \in \mathbb{C}$, by applying the quantum Hamiltonian reduction functor H_f to the affine vertex algebra $V^k(\mathfrak{g})$ (see below for the definition of $V^k(\mathfrak{g})$ and [21, Section 2] for the definition of H_f). In particular, it was shown that, for k non-critical, i.e., $k \neq -h^{\vee}$, $W^k(\mathfrak{g}, f)$ has a Virasoro vector ω , [24, (2.2)], making it a conformal vertex algebra, and a set of free generators was constructed. For k non-critical the vertex algebra $W^k(\mathfrak{g}, f)$ has a unique simple quotient, denoted by $W_k(\mathfrak{g}, f)$.

In greater detail, paper [24] studies the universal minimal W -algebras of level k , which correspond to $f = e_{-\theta}$ where $-\theta$ is a minimal root. These W -algebras and their simple quotients will be of primary interest for the present paper. To simplify notation, we set

$$W^k(\mathfrak{g}, \theta) = W^k(\mathfrak{g}, e_{-\theta}), \quad W_k(\mathfrak{g}, \theta) = W_k(\mathfrak{g}, e_{-\theta}).$$

Throughout the paper we shall assume that $(\theta|\theta) = 2$ and that $k \neq -h^{\vee}$. Then the Virasoro vector ω of $W^k(\mathfrak{g}, \theta)$ has central charge [24, (5.7)]

$$(2.4) \quad c(\mathfrak{g}, k) = \frac{k \operatorname{sdim} \mathfrak{g}}{k + h^{\vee}} - 6k + h^{\vee} - 4.$$

Thereafter, we adopt the following notation for the vertex operator corresponding to the state a :

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

We frequently use the notation of the λ -bracket and of the normally ordered product:

$$[a_{\lambda} b] = \sum_{n \geq 0} \frac{\lambda^n}{n!} (a_{(n)} b), \quad : ab := a_{(-1)} b.$$

A defining set of axioms, satisfied by these operations, is described in [14]. We frequently use these axioms in our calculations.

If the vertex algebra admits a Virasoro vector and Δ_a is the conformal weight of a state a , then we also write the corresponding vertex operator as

$$Y(a, z) = \sum_{m \in \mathbb{Z} - \Delta_a} a_m z^{-m - \Delta_a},$$

so that

$$a_{(n)} = a_{n-\Delta_a+1}, \quad n \in \mathbb{Z}, \quad a_m = a_{(m+\Delta_a-1)}, \quad m \in \mathbb{Z} - \Delta_a.$$

The vacuum vector will be denoted by $\mathbf{1}$.

The following result is Theorem 5.1 of [24] and gives the structure of $W^k(\mathfrak{g}, \theta)$. Notice that formula (2.7) below is taken from [25]. Notation is as in the Introduction.

Theorem 2.1. (a) *The vertex algebra $W^k(\mathfrak{g}, \theta)$ is strongly and freely generated by elements $J^{\{a\}}$, where a runs over a basis of \mathfrak{g}^{\natural} , $G^{\{v\}}$, where v runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector ω .*

(b) *The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and 3/2, respectively, with respect to ω .*

(c) *The following λ -brackets hold:*

$$(2.5) \quad [J^{\{a\}} \lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda \left((k + h^\vee/2)(a|b) - \frac{1}{4}\kappa_0(a, b) \right), \quad a, b \in \mathfrak{g}^{\natural},$$

where κ_0 is the Killing form of \mathfrak{g}_0 , and

$$(2.6) \quad [J^{\{a\}} \lambda G^{\{u\}}] = G^{\{[a,u]\}}, \quad a \in \mathfrak{g}^{\natural}, \quad u \in \mathfrak{g}_{-1/2}.$$

(d) *The following λ -brackets hold for $u, v \in \mathfrak{g}_{-1/2}$:*

(2.7)

$$\begin{aligned} [G^{\{u\}} \lambda G^{\{v\}}] &= -2(k + h^\vee)(e_\theta|[u, v])\omega + (e_\theta|[u, v]) \sum_{\alpha=1}^{\dim \mathfrak{g}^{\natural}} : J^{\{u^\alpha\}} J^{\{u_\alpha\}} : \\ &+ \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} : J^{\{[u, u^\gamma]^{\natural}\}} J^{\{[u_\gamma, v]^{\natural}\}} : + \lambda \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} J^{\{[[u, u^\gamma], [u_\gamma, v]]^{\natural}\}} \\ &+ 2(k+1)(\partial + 2\lambda)J^{\{[e_\theta, u, v]^{\natural}\}} + \frac{\lambda^2}{3}v \sum_{\gamma=1}^{\dim \mathfrak{g}_{1/2}} c'([u, u^\gamma]^{\natural}, [u_\gamma, v]^{\natural}). \\ &+ \frac{\lambda^2}{3}(e_\theta|[u, v]) \left(-(k + h^\vee)c(\mathfrak{g}, k) + (k + h^\vee/2)\text{sdim} \mathfrak{g}^{\natural} - \frac{1}{2} \sum_{i \in I} h_{0,i}^\vee \text{sdim} \mathfrak{g}_i^{\natural} \right). \end{aligned}$$

Here $\{u_\alpha\}$ (resp. $\{u_\gamma\}$) is a basis of \mathfrak{g}^{\natural} (resp. $\mathfrak{g}_{1/2}$) and $\{u^\alpha\}$ (resp. $\{u^\gamma\}$) is the corresponding dual basis w.r.t. $(\cdot|\cdot)$ (resp w.r.t. $\langle \cdot, \cdot \rangle_{\text{ne}} = (e_{-\theta}[[\cdot, \cdot]])$), a^{\natural} is the orthogonal projection of $a \in \mathfrak{g}_0$ to \mathfrak{g}^{\natural} , and

$$c'(a, b) = k(a|b) + \frac{1}{2} \text{str}_{\bigoplus_{i>0} \mathfrak{g}_i} (ad(a)ad(b)).$$

Let \mathfrak{a} be a Lie superalgebra equipped with a nondegenerate invariant supersymmetric bilinear form B . The universal affine vertex algebra $V^B(\mathfrak{a})$

is the universal enveloping vertex algebra of the Lie conformal superalgebra $R = (\mathbb{C}[T] \otimes \mathfrak{a}) \oplus \mathbb{C}$ with λ -bracket given by

$$[a_\lambda b] = [a, b] + \lambda B(a, b), \quad a, b \in \mathfrak{a}.$$

In the following, we shall say that a vertex algebra V is an affine vertex algebra if it is a quotient of some $V^B(\mathfrak{a})$. If \mathfrak{a} is equipped with a fixed bilinear form $(\cdot|\cdot)$ and $B = m(\cdot|\cdot)$, then we denote $V^B(\mathfrak{a})$ by $V^m(\mathfrak{a})$. Set $\mathfrak{a}' = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affinization of \mathfrak{a} . This is the Lie superalgebra with K a central element and Lie bracket defined, for $a, a' \in \mathfrak{a}$,

$$[a \otimes t^k, a' \otimes t^h] = [a, a'] \otimes t^{h+k} + \delta_{h,-k}(a|a')K.$$

Set $\mathfrak{a}^+ = \text{span}(a \otimes t^k \mid a \in \mathfrak{a}, k \geq 0)$. Let \mathbb{C}_m be the one-dimensional representation of $\mathfrak{a}^+ \oplus \mathbb{C}K$ such that $a \otimes t^k \cdot 1 = 0$ for all $a \in \mathfrak{a}$ and $k \geq 0$ and $K \cdot 1 = m$. Recall that $V^m(\mathfrak{a})$ can be realized as $U(\mathfrak{a}') \otimes_{U(\mathfrak{a}^+ + \mathbb{C}K)} \mathbb{C}_m$. Regard $d = t \frac{d}{dt}$ as a derivation of \mathfrak{a}' . Then $-d$ induces a Hamiltonian operator H whose eigenspace decomposition defines a \mathbb{Z} -grading of $V^m(\mathfrak{a})$. Since the eigenvalues of H are positive and the zero eigenspace is \mathbb{C}_m , there is a unique proper maximal graded ideal in $V^m(\mathfrak{a})$. We let $V_m(\mathfrak{a})$ denote the corresponding graded simple quotient.

It is known that $V^m(\mathfrak{a})$ admits a unique graded irreducible quotient, which we denote by $V_m(\mathfrak{a})$.

Let $\mathcal{V}^k(\mathfrak{g}^\natural)$ be the subalgebra of the vertex algebra $W^k(\mathfrak{g}, \theta)$, generated by $\{J^{\{a\}} \mid a \in \mathfrak{g}^\natural\}$. The vertex algebra $\mathcal{V}^k(\mathfrak{g}^\natural)$ is isomorphic to a universal affine vertex algebra. More precisely, letting

$$(2.8) \quad k_i = k + \frac{1}{2}(h^\vee - h_{0,i}^\vee), \quad i \in I,$$

the map $a \mapsto J^{\{a\}}$ extends, by (2.5), to an isomorphism $\mathcal{V}^k(\mathfrak{g}^\natural) \simeq V^B(\mathfrak{g}^\natural)$ where $B(a, b) = \delta_{ij}k_i(a|b)$, $a \in \mathfrak{g}_i^\natural, b \in \mathfrak{g}_j^\natural$. Note that

$$(2.9) \quad \mathcal{V}^k(\mathfrak{g}^\natural) \simeq \bigotimes_{i \in I} V^{k_i}(\mathfrak{g}_i^\natural).$$

We also set $\mathcal{V}_k(\mathfrak{g}^\natural)$ to be the image of $\mathcal{V}^k(\mathfrak{g}^\natural)$ in $W_k(\mathfrak{g}, \theta)$. Clearly we can write

$$\mathcal{V}_k(\mathfrak{g}^\natural) \simeq \bigotimes_{i \in I} \mathcal{V}_{k_i}(\mathfrak{g}_i^\natural),$$

where $\mathcal{V}_{k_i}(\mathfrak{g}_i^\natural)$ is some quotient (not necessarily simple) of $V^{k_i}(\mathfrak{g}_i^\natural)$.

Recall that the fusion product of two subsets A, B of a vertex algebra V is

$$A \cdot B = \text{span}(a_{(n)}b \mid n \in \mathbb{Z}, a \in A, b \in B).$$

This product is associative due to the Borcherds identity (cf. [14]).

Recall that a vector v in a representation of a vertex algebra $V^B(\mathfrak{a})$ is *singular* if $a_n v = 0$ for any $a \in \mathfrak{a}$ and $n > 0$. We introduce a similar notion for quantum affine W -algebras.

Definition 2.1. We will say that a vector $v \in W^k(\mathfrak{g}, \theta)$ is singular if $a_n v = 0$ for $n > 0$ and a in the set of generators $J^{\{b\}}$ ($b \in \mathfrak{g}^{\natural}$), $G^{\{u\}}$ ($u \in \mathfrak{g}_{-1/2}$), ω .

Note that if $v \notin \mathbb{C}\mathbf{1}$ is singular, then $W^k(\mathfrak{g}, \theta) \cdot v$ is a proper ideal of $W^k(\mathfrak{g}, \theta)$. (It is a left ideal due to the associativity of the fusion product, and is also a right ideal since $W^k(\mathfrak{g}, \theta)$ admits a Virasoro vector).

3. COLLAPSING LEVELS

Definition 3.1. If $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural})$, we say that k a *collapsing level*.

In this Section we give a complete classification of collapsing levels. Our main tool will be a certain degree two polynomial $p(k)$, which turns out to govern the first and the second product of the elements $G^{\{u\}}$, $u \in \mathfrak{g}_{-1/2}$, of $W^k(\mathfrak{g}, \theta)$.

Remark that if $\mathfrak{g} = sl(2)$, then $\mathfrak{g}_{-1/2} = \mathfrak{g}_{1/2} = \{0\}$, and $W^k(\mathfrak{g}, \theta)$ is the universal Virasoro vertex algebra with central charge $\frac{3k}{k+2} - 6k - 2$, see (2.4). Hence k is a collapsing level in this case if and only if $k = -\frac{1}{2}$ or $-\frac{4}{3}$, in which case $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$; also, only these k are conformal. We shall therefore exclude this case from further considerations.

We introduce the aforementioned polynomial $p(k)$ through the following structural lemma. For $a \in \mathfrak{g}$, we denote by $P(a) = \bar{0}$ or $\bar{1}$ the parity of a .

Lemma 3.1. *Let $u, v \in \mathfrak{g}_{-1/2}$. There is a monic polynomial $p(k)$ of degree two, independent of u and v , such that*

$$(3.1) \quad G^{\{u\}}_{(2)} G^{\{v\}} = 4(e_{\theta}|[u, v])p(k)\mathbf{1}.$$

Moreover, the linear polynomial k_i , $i \in I$, defined by (2.8), divides $p(k)$ and

$$(3.2) \quad G^{\{u\}}_{(1)} G^{\{v\}} = 4 \sum_{i \in I} \frac{p(k)}{k_i} J^{\{([e_{\theta}, u], v])_i^{\natural}\}}$$

where $(a)_i^{\natural}$ denotes the orthogonal projection of $a \in \mathfrak{g}_0$ onto $\mathfrak{g}_i^{\natural}$.

Proof. Recall formal (2.7). If \mathfrak{g}^{\natural} is simple then (1) follows directly from the Remark in [25]. In the other cases, let $f_k(u, v)$ be the constant such that $G^{\{u\}}_{(2)} G^{\{v\}} = f_k(u, v)\mathbf{1}$. We observe that, by skew-symmetry of λ -product, $f_k(\cdot, \cdot)$ is an anti-symmetric form and, by Jacobi identity, $f_k(\cdot, \cdot)$ is \mathfrak{g}^{\natural} -invariant. Since $\mathfrak{g}_{1/2}$, as a \mathfrak{g}^{\natural} -module, is either irreducible or a sum $U \oplus U^*$ with U irreducible and U inequivalent to U^* , we see that, up to a constant, there is a unique anti-symmetric \mathfrak{g}^{\natural} -invariant nondegenerate bilinear form on $\mathfrak{g}_{-1/2}$. Since $(e_{\theta}|[u, v])$ is such a form, we have that $f_k(\cdot, \cdot) = 4p(k)(e_{\theta}|[\cdot, \cdot])$. The fact that $p(k)$ is a monic polynomial of degree two follows immediately from (2.4) and (2.7).

We now prove (3.2). If $\mathfrak{g}^{\natural} = \{0\}$, i.e. if $\mathfrak{g} = spo(2|1)$, the left hand side of (3.2) is zero by (2.7), whereas the right hand side is obviously zero. Hence we can assume $\mathfrak{g}^{\natural} \neq \{0\}$. Let $\{a_i^j\}$ be a basis of $\mathfrak{g}_i^{\natural}$ and $\{b_i^j\}$ its dual basis with respect to $(\cdot|\cdot)_{|\mathfrak{g}_i^{\natural} \times \mathfrak{g}_i^{\natural}}$. From (2.7), we see that

$$(3.3) \quad G^{\{u\}}(1)G^{\{v\}} = \sum_{i,j} \left(4([e_{\theta}, u], v|b_i^j)k + c_{i,j}(u, v) \right) J^{\{a_i^j\}}$$

with $c_{i,j}(u, v)$ a constant independent of k . By (2.6), $J^{\{b_i^j\}}(1)(G^{\{u\}}(1)G^{\{v\}}) = G^{\{b_i^j, u\}}(2)G^{\{v\}} = 4(e_{\theta}|[[b_i^j, u], v])p(k)\mathbf{1}$.

On the other hand, by (3.3),

$$\begin{aligned} J^{\{b_i^j\}}(1)(G^{\{u\}}(1)G^{\{v\}}) &= \sum_{r,s} \left(4([e_{\theta}, u], v|b_s^r)k + c_{r,s}(u, v) \right) J^{\{b_i^j\}}(1)J^{\{a_s^r\}} \\ &= \left(4([e_{\theta}, u], v|b_i^j)k + c_{i,j}(u, v) \right) (-1)^{P(b_i^j)}k_i\mathbf{1}, \end{aligned}$$

Hence

$$(3.4) \quad 4(e_{\theta}|[[b_i^j, u], v])p(k)\mathbf{1} = \left(4([e_{\theta}, u], v|b_i^j)k + c_{i,j}(u, v) \right) (-1)^{P(b_i^j)}k_i\mathbf{1}.$$

It is easily checked browsing Tables 1–3 that no component of \mathfrak{g}^{\natural} acts trivially on $\mathfrak{g}_{-1/2}$, so for each $i \in I$ there is j such that $[b_i^j, u] \neq 0$. Since the form $(e_{\theta}|[\cdot, \cdot])$ is nondegenerate, we can choose v so that $(e_{\theta}|[[b_i^j, u], v]) \neq 0$. It follows then at once from (3.4) that k_i divides $p(k)$.

Write explicitly $p(k) = (k+a)k_i$. It follows from (3.4) that, for all i, j and all $u, v \in \mathfrak{g}_{-1/2}$,

$$\left(4([e_{\theta}, u], v|b_i^j)k + c_{i,j}(u, v) \right) (-1)^{P(b_i^j)} = 4(e_{\theta}|[[b_i^j, u], v])(k+a).$$

It follows that $(e_{\theta}|[[b_i^j, u], v]) = ([e_{\theta}, u], v|b_i^j)(-1)^{b_i^j}$ and that

$$c_{i,j}(u, v) = 4(e_{\theta}|[[b_i^j, u], v])(-1)^{P(b_i^j)}a = 4([e_{\theta}, u], v|b_i^j).$$

Substituting in (3.3) we get (3.2). \square

The polynomial $p(k)$ can be easily calculated for all \mathfrak{g} . If $\mathfrak{g}^{\natural} = \{0\}$ then, by (2.7), $p(k) = -\frac{1}{6}(k+h^{\vee})c(\mathfrak{g}, k)$. If $\mathfrak{g}^{\natural} \neq \{0\}$ and the set $\{k_i \mid i \in I\}$ has only one element, then we can use formula 2 in [25, Remark]. If $\{k_i \mid i \in I\}$ has more than one element then, by Lemma 3.1, it has only two elements and $p(k)$ is their product. The outcome is listed in the following table.

Table 4

\mathfrak{g}	$p(k)$	\mathfrak{g}	$p(k)$
$sl(m n), n \neq m$	$(k+1)(k+(m-n)/2)$	E_6	$(k+3)(k+4)$
$psl(m m)$	$k(k+1)$	E_7	$(k+4)(k+6)$
$osp(m n)$	$(k+2)(k+(m-n-4)/2)$	E_8	$(k+6)(k+10)$
$spo(n m)$	$(k+1/2)(k+(n-m+4)/4)$	F_4	$(k+5/2)(k+3)$
$D(2, 1; a)$	$(k-a)(k+1+a)$	G_2	$(k+4/3)(k+5/3)$
$F(4), \mathfrak{g}^{\natural} = so(7)$	$(k+2/3)(k-2/3)$	$G(3), \mathfrak{g}^{\natural} = G_2$	$(k-1/2)(k+3/4)$
$F(4), \mathfrak{g}^{\natural} = D(2, 1; 2)$	$(k+3/2)(k+1)$	$G(3), \mathfrak{g}^{\natural} = osp(3 2)$	$(k+2/3)(k+4/3)$

In Proposition 3.5 we will give a uniform formula for $p(k)$ for a certain class of pairs (\mathfrak{g}, θ) .

Proposition 3.2. *The following are equivalent:*

- (1) *There is $v \in \mathfrak{g}_{-1/2}$, $v \neq 0$, such that $G^{\{v\}} = 0$ in $W_k(\mathfrak{g}, \theta)$.*
- (2) *There is $v \in \mathfrak{g}_{-1/2}$, $v \neq 0$, such that $G^{\{u\}}_{(2)}G^{\{v\}} = 0$ in $W^k(\mathfrak{g}, \theta)$ for all $u \in \mathfrak{g}_{-1/2}$.*
- (3) *$G^{\{v\}}_{(2)}G^{\{u\}} = 0$ in $W^k(\mathfrak{g}, \theta)$ for all $v, u \in \mathfrak{g}_{-1/2}$.*
- (4) *$G^{\{v\}} = 0$ in $W_k(\mathfrak{g}, \theta)$ for all $v \in \mathfrak{g}_{-1/2}$.*

In particular, $G^{\{v\}} = 0$ in $W_k(\mathfrak{g}, \theta)$ for all $v \in \mathfrak{g}_{-1/2}$ if and only if $p(k) = 0$.

Proof. (1) \Rightarrow (2): if $G^{\{v\}} = 0$ in $W_k(\mathfrak{g}, \theta)$ for some $v \in \mathfrak{g}_{-1/2}$, then, by Lemma 3.1, $(e_\theta|[u, v])p(k) = 0$ for all $u \in \mathfrak{g}_{-1/2}$, thus $G^{\{u\}}_{(2)}G^{\{v\}} = 0$ in $W^k(\mathfrak{g}, \theta)$ for all $u \in \mathfrak{g}_{-1/2}$.

(2) \Rightarrow (3): since $(e_\theta|[\cdot, \cdot])$ is nondegenerate, if (2) holds, then $p(k) = 0$ so $G^{\{v\}}_{(2)}G^{\{u\}} = 0$ in $W^k(\mathfrak{g}, \theta)$ for all $v, u \in \mathfrak{g}_{-1/2}$.

(3) \Rightarrow (4): assume first $G^{\{v\}}_{(i)}G^{\{u\}} = 0$ for all $v \in \mathfrak{g}_{-1/2}$ and $i > 0$. Then, since $J^{\{a\}}_{(i)}G^{\{u\}} = 0$ for all $a \in \mathfrak{g}^{\natural}$ and $i > 0$, we have that $G^{\{u\}}$ is a singular vector in $W^k(\mathfrak{g}, \theta)$, so $G^{\{u\}}$ is in the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$.

If, otherwise, there is $v \in \mathfrak{g}_{-1/2}$ such that $G^{\{v\}}_{(i)}G^{\{u\}} \neq 0$ for some $i > 0$, then, since we are assuming (3), we have $G^{\{v\}}_{(1)}G^{\{u\}} \neq 0$. By (1.1) we know that $G^{\{v\}}_{(1)}G^{\{u\}} = \sum_i J^{\{a_i\}}$ with $a_i \in \mathfrak{g}_i^{\natural}$. In particular $\mathfrak{g}^{\natural} \neq \{0\}$. Assume $a_j \neq 0$. Observe that $J^{\{a\}}_{(1)}(G^{\{v\}}_{(1)}G^{\{u\}}) = G^{\{[a, v]\}}_{(2)}G^{\{u\}} = 0$ for all $a \in \mathfrak{g}^{\natural}$. Choosing $b_j \in \mathfrak{g}_j^{\natural}$ such that $(b_j|a_j) = 1$ and arguing as in the proof of (3.2) in Lemma 3.1 we see that

$$0 = J^{\{b_j\}}_{(1)}(G^{\{v\}}_{(1)}G^{\{u\}}) = J^{\{b_j\}}_{(1)} \left(\sum_i J^{\{a_i\}} \right) = k_j \mathbf{1},$$

hence $J^{\{a\}}_{(1)}J^{\{a_j\}} = 0$ for all $a \in \mathfrak{g}^{\natural}$. Since, by skew-symmetry,

$$G^{\{w\}}_{(i)}J^{\{a_j\}} = -\delta_{i,0}J^{\{a_j\}}_{(0)}G^{\{w\}}, \quad (i \geq 0),$$

we see that $J^{\{a_j\}}$ is a singular vector in $W^k(\mathfrak{g}, \theta)$, thus $J^{\{a_j\}}$ belongs to the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$. Since $\mathfrak{g}_j^{\natural}$ is simple if $j > 0$ and $\mathfrak{g}_0^{\natural}$ has at most dimension one, we see that, if $a_j \neq 0$ then $J^{\{a\}}$ is in the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$ for all $a \in \mathfrak{g}_j^{\natural}$.

Assume that u is a highest weight vector of weight α for an irreducible component of $\mathfrak{g}_{-1/2}$. By looking at Tables 1–3, we see that \mathfrak{g}^{\natural} does not act trivially on $\mathfrak{g}_{1/2}$, hence there is $h \in \mathfrak{h}^{\natural}$ such that $[h, u] = \alpha(h)u$ with $\alpha(h) \neq 0$. Since $J^{\{h\}}_{(0)}G^{\{u\}} = \alpha(h)G^{\{u\}}$ is in the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$, it follows that $G^{\{u\}}$ is zero in $W_k(\mathfrak{g}, \theta)$. By the same argument it is clear that, if the highest weight vector of a component of $\mathfrak{g}_{-1/2}$ is in a

proper ideal, then the whole component is, thus $G^{\{u\}} = 0$ in $W_k(\mathfrak{g}, \theta)$ for all $u \in \mathfrak{g}_{-1/2}$.

(4) \Rightarrow (1): obvious. \square

Theorem 3.3. $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural})$, i.e., k is a collapsing level, if and only if $p(k) = 0$. In such cases,

$$(3.5) \quad W_k(\mathfrak{g}, \theta) = \bigotimes_{i \in I: k_i \neq 0} V_{k_i}(\mathfrak{g}_i^{\natural}).$$

Remark 3.2. Notice that for some i , it may hold $k_i + h_i^{\vee} = 0$. Since we are assuming $k + h^{\vee} \neq 0$, we still have the Virasoro vector ω acting on $\mathcal{V}^k(\mathfrak{g}^{\natural})$. Since $a \in \mathfrak{g}^{\natural}$ has conformal weight 1, ω_0 acts on $\mathcal{V}^k(\mathfrak{g}^{\natural})$ as the Hamiltonian operator H .

Proof of Theorem 3.3. Assume $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural})$. Then, for $v \in \mathfrak{g}_{-1/2}$, $G^{\{v\}}$ belongs to $\mathcal{V}_k(\mathfrak{g}^{\natural})$. Since conformal weights in $\mathcal{V}_k(\mathfrak{g}^{\natural})$ are integral and the conformal weight of $G^{\{v\}}$ is $3/2$, we have $G^{\{v\}} = 0$ and, by Proposition 3.2, $p(k) = 0$. Conversely, if $p(k) = 0$, by Proposition 3.2 all $G^{\{v\}}$ are 0. By (1.1), we have that $\omega \in \mathcal{V}_k(\mathfrak{g}^{\natural})$, hence $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^{\natural})$.

We now prove (3.5). Let \mathcal{I} be the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$. Since ω_0 acts as H on $\mathcal{V}^k(\mathfrak{g}^{\natural})$, we have that $\mathcal{I} \cap \mathcal{V}^k(\mathfrak{g}^{\natural})$ is a graded proper ideal of $\mathcal{V}^k(\mathfrak{g}^{\natural})$. If $p(k) = 0$, since $W_k(\mathfrak{g}, \theta)$ is simple, we have that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ is the simple graded quotient of $\mathcal{V}^k(\mathfrak{g}^{\natural})$, hence

$$\mathcal{V}_k(\mathfrak{g}^{\natural}) \simeq \bigotimes_{i \in I} V_{k_i}(\mathfrak{g}_i^{\natural}).$$

If $k_i = 0$, then $V_{k_i}(\mathfrak{g}_i^{\natural}) = \mathbb{C}\mathbf{1}$, so the right hand side is the same as in (3.5). \square

Remark 3.3. We note that in the case when \mathfrak{g} is of type D_n , Theorem 3.3 implies that

$$W_{-2}(\mathfrak{g}) = V_{n-4}(sl(2)).$$

Since it is well-known that $V_{n-4}(sl(2))$ is a rational vertex algebra for $n \geq 5$, we slightly refine [11, Theorem 1.2 (2)] in this case.

In the following result we determine the levels k for which $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$. The Lie algebra case has been already discussed in [11, Theorem 7.2].

Proposition 3.4. $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$ if and only if

- (1) $k = -\frac{h^{\vee}}{6} - 1$ and either \mathfrak{g} is a Lie algebra belonging to the Deligne's series (i.e. \mathfrak{g} is of type $A_2, G_2, D_4, F_4, E_6, E_7, E_8$, [16]) or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$), $\mathfrak{g} = osp(n+8|8)$ ($n \geq 2$), \mathfrak{g} is of type $F(4)$, $G(3)$ (for both choices of θ), or $\mathfrak{g} = spo(2|1)$;
- (2) $k = -\frac{1}{2}$ and $\mathfrak{g} = spo(n|m)$ ($n \geq 1$).

Proof. By Theorem 3.3, $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$ if and only if k is a collapsing level and $k_i = 0$ for all i . If $\mathfrak{g} = \mathit{spo}(2|1)$, then the latter condition is empty ($\mathfrak{g}^\natural = \{0\}$) so, in this special case, $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$ if and only if $p(k) = 0$. This proves that both $-\frac{h^\vee}{6} - 1$ and $-\frac{1}{2}$ are collapsing levels for $\mathit{spo}(2|1)$.

If $\mathfrak{g} \neq \{0\}$, then $k_i = 0$ for all i if and only if $h_{0,i}^\vee$ have the same value for all i (denoted by h_0^\vee). A direct inspection, using Table 4 from [24], shows that this happens only in the cases listed in (1), (2) above. Moreover, we must have $k_i = k + \frac{h^\vee - h_0^\vee}{2} = 0$, so

$$(3.6) \quad k = -\frac{h^\vee - h_0^\vee}{2}.$$

For the cases listed in (1), we checked case by case that

$$(3.7) \quad h_0^\vee = \frac{2}{3}h^\vee - 2$$

while for $\mathfrak{g} = \mathit{spo}(n|m)$,

$$(3.8) \quad h_0^\vee = h^\vee - 1.$$

Plugging in (3.6), we obtain the values of k in (1) and (2). \square

In the following Proposition we obtain certain uniform formulas for $\mathit{sdim} \mathfrak{g}$ and $p(k)$ (see [16], [32], [27]).

Proposition 3.5. *If (\mathfrak{g}, θ) is as in Proposition 3.4 (1), then*

$$(3.9) \quad \mathit{sdim} \mathfrak{g} = \frac{2(h^\vee + 1)(5h^\vee - 6)}{h^\vee + 6}, \quad p(k) = \left(k + \frac{h^\vee}{6} + 1\right)\left(k + \frac{h^\vee}{3}\right).$$

If $\mathfrak{g} = \mathit{spo}(n|m)$, then

$$(3.10) \quad \mathit{sdim} \mathfrak{g} = (2h^\vee - 1)(h^\vee - 1), \quad p(k) = \left(k + \frac{1}{2}\right)\left(k + \frac{h^\vee + 1}{2}\right).$$

Proof. If $W_k(\mathfrak{g}, \theta) = \mathbb{C}\mathbf{1}$, then $c(\mathfrak{g}, k) = 0$. Substituting in (2.4) the values of k given in Proposition 3.4 and then solving for $\mathit{sdim} \mathfrak{g}$ in the equation $c(\mathfrak{g}, k) = 0$, one obtains the formula for $\mathit{sdim} \mathfrak{g}$ given in (3.9) for the Lie superalgebras \mathfrak{g} listed in Proposition 3.4 (1) and the formula for $\mathit{sdim} \mathfrak{g}$ given in (3.10) for $\mathfrak{g} = \mathit{spo}(n|m)$.

We have $\kappa_0(a, b) = 2h_0^\vee(a|b)$ for $a, b \in \mathfrak{g}^\natural$, hence we can apply [25, Remark] and obtain that

$$\begin{aligned} G^{\{u\}}{}_{(2)}G^{\{v\}} &= \\ &= -\frac{2}{3}(e_\theta[[u, v]])\left\{(k + h^\vee)c(\mathfrak{g}, k) - \left(k + \frac{h^\vee - h_0^\vee}{2}\right)(\mathit{sdim} \mathfrak{g}_0 + \mathit{sdim} \mathfrak{g}_{1/2})\right\}. \end{aligned}$$

Hence, by (3.1),

$$(3.11) \quad p(k) = -\frac{1}{6}\left\{(k + h^\vee)c(\mathfrak{g}, k) - \left(k + \frac{h^\vee - h_0^\vee}{2}\right)(\mathit{sdim} \mathfrak{g}_0 + \mathit{sdim} \mathfrak{g}_{1/2})\right\}.$$

Since $\mathit{sdim} \mathfrak{g}_0 = \mathit{sdim} \mathfrak{g} - 2\mathit{sdim} \mathfrak{g}_{1/2} - 2$ and $\mathit{sdim} \mathfrak{g}_{1/2} = 2h^\vee - 4$ [24], plugging into (3.11) either (3.7) or (3.8) one gets also our formulas for $p(k)$. \square

Remark 3.4. Note also that for $\mathfrak{g} = sl(m|n)$ ($m \neq n$) and for $\mathfrak{g} = osp(m|n)$ one has respectively

$$\begin{aligned} \text{sdim } \mathfrak{g} &= h^{\vee 2} - 1, & p(k) &= (k+1)\left(k + \frac{h^{\vee}}{2}\right), \\ \text{sdim } \mathfrak{g} &= \frac{(h^{\vee} + 1)(h^{\vee} + 2)}{2}, & p(k) &= (k+2)\left(k + \frac{h^{\vee} - 2}{2}\right). \end{aligned}$$

Let us conclude this Section with one application of our results on collapsing levels. Let V_c denote the simple Virasoro vertex algebra of central charge c . Let $M(1)$ be the Heisenberg vertex algebra of central charge 1.

Recall from Section 2 that $H_{e_{-\theta}}$ denotes the quantum reduction functor from the category of $V^k(\mathfrak{g})$ -modules to the category of $W^k(\mathfrak{g}, \theta)$ -modules, and that

$$(3.12) \quad H_{e_{-\theta}}(V^k(\mathfrak{g})) = W^k(\mathfrak{g}, \theta).$$

It was proved in [9] that $H_{e_{-\theta}}$ is an exact functor and it maps an irreducible highest weight $V^k(\mathfrak{g})$ -modules M either to zero or to an irreducible highest weight $W^k(\mathfrak{g}, \theta)$ -module; furthermore, one has that, if M is not integrable, then $H_{e_{-\theta}}(M)$ is nonzero. This implies that, if \mathfrak{g} is a Lie algebra and $k \notin \mathbb{Z}_{\geq 0}$, then

$$H_{e_{-\theta}}(V_k(\mathfrak{g})) = W_k(\mathfrak{g}, \theta).$$

Proposition 3.6.

- (1) Let $\mathfrak{g} = sl(2m)$. Then there is an exact functor from the category of $V_{-m}(\mathfrak{g})$ -modules to the category of $V_{c=1}$ -modules which maps $V_{-m}(\mathfrak{g})$ to $V_{c=1}$.
- (2) Let $\mathfrak{g} = sp(2n)$. Then there is an exact functor from the category of $V_{-1-\frac{n}{2}}(\mathfrak{g})$ -modules to the category of $V_{c=-2}$ -modules which maps $V_{-1-\frac{n}{2}}(\mathfrak{g})$ to $V_{c=-2}$.
- (3) Let $\mathfrak{g} = so(2n)$ with n odd, $n \geq 3$. Then there is an exact functor from the category of $V_{-(n-2)}(\mathfrak{g})$ -modules to the category of $M(1)$ -modules which maps $V_{-(n-2)}(\mathfrak{g})$ to $M(1)$.

Proof. (1) The case $m = 1$ is clear. Assume that $m \geq 2$. First we notice that $k = -m$ is a collapsing level and that $W_k(\mathfrak{g}, \theta) = V_{-m+1}(sl(2m - 2))$. Set $H_1 = H_{e_{-\theta}}$. By (3.12), H_1 is an exact functor from the category of $V_{-m}(sl(2m))$ -modules to the category of $V_{-m+1}(sl(2m - 2))$ -modules. By the same argument we get an exact functor H_2 from the category of $V_{-m+1}(sl(2m - 2))$ -modules to the category of $V_{-m+2}(sl(2m - 4))$ -modules. By repeating this procedure we get exact functors H_i , $i = 1, \dots, m - 1$ from the category of $V_{-m-1+i}(sl(2m + 2 - 2i))$ -modules to the category of $V_{-m+i}(sl(2m - 2i))$ -modules. Finally we use the functor H_{Vir} from the category of $V_{-1}(sl_2)$ -modules to the category of $V_{c=1}$ -modules. Now the composition $H_{Vir} \circ H_{m-1} \circ \dots \circ H_1$ gives the required exact functor.

(2) Since k is a collapsing level, we have $W_k(sp(2n), \theta) = V_{k+1/2}(sp(2n - 2))$. But $k + 1/2$ is a collapsing level for $sp(2n - 2)$ and we finish the proof as in case (1).

(3) The proof is similar to (1) and (2) and uses the fact that $W_{-1}(so(6), \theta) = W_{-1}(sl(4), \theta) = M(1)$. \square

Part (1) has been obtained by different methods in Arakawa-Moreau [12].

Remark 3.5. There is a different method for proving that a certain level is collapsing. One can take a non-trivial ideal I in the universal affine vertex algebra $V^k(\mathfrak{g})$ (usually generated by singular vector of conformal weight 2), apply the BRST functor $H_{e_{-\theta}}$, and show that all generators $G^{\{u\}}$ of conformal weight $3/2$ of $W^k(\mathfrak{g}, \theta)$ lie in the non-trivial ideal $H_{e_{-\theta}}(I)$ of $W^k(\mathfrak{g}, \theta)$. This will imply that $G^{\{u\}} = 0$ in $W_k(\mathfrak{g}, \theta)$. Such approach was applied in [12] for W -algebras of type A , but can be also applied for all collapsing levels classified in our paper. In particular, our results show that for every collapsing level there exists a non-trivial ideal I in $V^k(\mathfrak{g})$ such that generators $G^{\{u\}}$ lie in $H_{e_{-\theta}}(I)$.

4. CONFORMAL EMBEDDINGS OF AFFINE VERTEX ALGEBRAS IN W -ALGEBRAS

In this Section we introduce a notion of a conformal embedding for $\mathcal{V}_k(\mathfrak{g}^{\natural})$ into $W_k(\mathfrak{g}, \theta)$. Our first task is to endow $\mathcal{V}_k(\mathfrak{g}^{\natural})$ with a Virasoro vector. In most cases this Virasoro vector is the image in $W_k(\mathfrak{g}, \theta)$ of the Virasoro vector for $\mathcal{V}^k(\mathfrak{g}^{\natural})$ given by the Sugawara construction. On the other hand it might happen that $\mathcal{V}_k(\mathfrak{g}^{\natural})$ still has a Virasoro vector though Sugawara construction for $\mathcal{V}^k(\mathfrak{g}^{\natural})$ does not make sense. This motivates the following definitions.

If $k_i + h_{0,i}^{\vee} \neq 0$, then $\mathcal{V}^k(\mathfrak{g}_i^{\natural})$ is equipped with a Virasoro vector

$$(4.1) \quad \omega_{sug}^i = \frac{1}{2(k_i + h_{0,i}^{\vee})} \sum_{j=1}^{\dim \mathfrak{g}_i^{\natural}} : J^{\{b_i^j\}} J^{\{a_i^j\}} :,$$

where $\{a_i^j\}$ is a basis of $\mathfrak{g}_i^{\natural}$ and $\{b_i^j\}$ is its dual basis with respect to $(\cdot | \cdot)_{|\mathfrak{g}_i^{\natural} \times \mathfrak{g}_i^{\natural}}$. If $k_i + h_{0,i}^{\vee} \neq 0$ for all i , we set

$$\omega_{sug} = \sum_{i \in I} \omega_{sug}^i.$$

Define

$$\mathcal{K} = \{k \in \mathbb{C} \mid k + h^{\vee} \neq 0, k_i + h_{0,i}^{\vee} \neq 0 \text{ whenever } k_i \neq 0\}.$$

If $k \in \mathcal{K}$ we also set

$$\omega'_{sug} = \sum_{i \in I: k_i \neq 0} \omega_{sug}^i.$$

Clearly, if $k_i \neq 0$ for all $i \in I$ then $\omega_{sug} = \omega'_{sug}$. Note that $\omega_{sug} = \omega'_{sug}$ in $W_k(\mathfrak{g}, \theta)$ also when $k_i + h_{0,i}^\vee \neq 0$ for all $i \in I$. Indeed, if there is $j \in I$ such that $k_j = 0$, by Theorem 3.3, $\mathcal{V}_{k_j}(\mathfrak{g}_j^\natural)$ collapses to $\mathbb{C}\mathbf{1}$ so ω'_{sug} collapses to zero. If instead $k_i = k_i + h_{0,i}^\vee = 0$ then, by Theorem 3.3, ω'_{sug} still defines a Virasoro vector for $\mathcal{V}_k(\mathfrak{g}^\natural)$, but ω_{sug} is not well defined.

Finally, we define

$$c_{sug} = \text{central charge of } \omega'_{sug}.$$

Definition 4.1. Assume $k \in \mathcal{K}$. We say that $\mathcal{V}_k(\mathfrak{g}^\natural)$ is conformally embedded in $W_k(\mathfrak{g}, \theta)$ if $\omega'_{sug} = \omega$. The level k is called a *conformal level*.

The goal of this Section is to determine the conformal levels. An easy result in this direction is the following Proposition which explains why we replace ω_{sug} by ω'_{sug} .

Proposition 4.1. *Assume that k is a collapsing level in \mathcal{K} . Then k is conformal.*

Proof. If k is a collapsing level, then $\omega \in \mathcal{V}_k(\mathfrak{g}^\natural)$. Since $J^{\{a\}}, a \in \mathfrak{g}^\natural$ is primary of conformal weight 1 for ω , we see that $[(\omega - \omega'_{sug})_\lambda J^{\{a\}}] = 0$ for all $a \in \mathfrak{g}^\natural$. This implies that $J^{\{a\}}_{(n)}(\omega - \omega'_{sug}) = 0$ for $n > 0$, so $\omega - \omega'_{sug}$ is a singular vector. Since $\omega - \omega'_{sug}$ has conformal weight 2 for ω , we see that $\omega - \omega'_{sug} \neq \mathbf{1}$, hence, since $\mathcal{V}_k(\mathfrak{g}^\natural)$ is simple, $\omega - \omega'_{sug} = 0$. \square

We now start looking for conformal non-collapsing levels. Set

$$(4.2) \quad \mathcal{J}^k(\mathfrak{g}) = W^k(\mathfrak{g}, \theta) \cdot (\omega - \omega_{sug}), \quad \mathcal{W}^k(\mathfrak{g}, \theta) = W^k(\mathfrak{g}, \theta) / \mathcal{J}^k(\mathfrak{g}).$$

We have:

Proposition 4.2. *Assume $k_i + h_{0,i}^\vee \neq 0$ for all $i \in I$. The following conditions are equivalent:*

- (1) $(\omega_{sug})_0 G^{\{u\}} = \frac{3}{2} G^{\{u\}}$ in $W^k(\mathfrak{g}, \theta)$ for all $u \in \mathfrak{g}_{-1/2}$.
- (2) The vector $\omega - \omega_{sug}$ is a singular vector in $W^k(\mathfrak{g}, \theta)$.
- (3) $G^{\{v\}} \neq 0$ in $\mathcal{W}^k(\mathfrak{g}, \theta)$ for all $v \in \mathfrak{g}_{-1/2}$, hence $\mathcal{J}^k(\mathfrak{g})$ is a proper ideal in $W^k(\mathfrak{g}, \theta)$.

Proof. (1) \implies (2). Recall that $J^{\{a\}}$ is primary of conformal weight 1 for ω for all $a \in \mathfrak{g}^\natural$. This, by (4.1), also implies that ω_{sug} is primary of conformal weight 2 for ω . By (2.9), if $n \geq 0$, $(\omega_{sug})_n J^{\{a\}} = \delta_{n,0} J^{\{a\}}$. By the right Wick formula, $(\omega_{sug})_n G^{\{v\}} = 0$ for $n > 0$ and, by hypothesis, $(\omega_{sug})_0 G^{\{v\}} = \omega_0 G^{\{v\}} = \frac{3}{2} G^{\{v\}}$. It follows that $(\omega - \omega_{sug})_{(n)} X = 0$ for $X = J^{\{a\}}$ or $X = G^{\{v\}}$ and $n > 0$. Thus, by skewsymmetry of the λ -bracket, $X_{(n)}(\omega - \omega_{sug}) = 0$, for $n > 0$ so $Z_n(\omega - \omega_{sug}) = 0$ for $n > 0$ and Z any generator.

(2) \implies (3). By (2), since $J^{\{b\}}(b \in \mathfrak{g}^{\natural}), G^{\{u\}}(u \in \mathfrak{g}_{-1/2}), \omega$ strongly generate $W^k(\mathfrak{g}, \theta)$, we have that any element of $W^k(\mathfrak{g}, \theta) \cdot (\omega - \omega_{sug})$ can be written as a sum of elements of type

$$a_{s_1}^1 \cdots a_{s_t}^t \cdot (\omega - \omega_{sug})$$

with a^i in the given set of generators and $s_i \leq 0$ for all i .

Since $a_{s_1}^1 \cdots a_{s_t}^t \cdot (\omega - \omega_{sug})$ has conformal weight

$$-\sum_i s_i + 2 \geq 2,$$

we see that $W^k(\mathfrak{g}, \theta) \cdot (\omega - \omega_{sug})$ cannot contain $G^{\{v\}}$.

(3) \implies (1). Recall from (4.1) the definition of ω_{sug}^i . From (2.6), we see that $\sum_{j=1}^{\dim \mathfrak{g}_i^{\natural}} : J^{\{b_i^j\}} J^{\{a_i^j\}} :_0$ acts on $G^{\{v\}}$ as the Casimir operator of $\mathfrak{g}_i^{\natural}$ acting on v . Since $\mathfrak{g}_{-1/2}$ is either irreducible or is a direct sum of two contragredient \mathfrak{g}^{\natural} -modules, we have that $G^{\{v\}}$ is an eigenvector for $\sum_{j=1}^{\dim \mathfrak{g}_i^{\natural}} : J^{\{b_i^j\}} J^{\{a_i^j\}} :_0$ hence also for $(\omega_{sug})_0 = \sum_i (\omega_{sug}^i)_0$. Since $G^{\{v\}} \neq 0$ and $\omega = \omega_{sug}$ in $W^k(\mathfrak{g}, \theta)$, the corresponding eigenvalue must be $\frac{3}{2}$. \square

The next result gives a classification of conformal embedding at non-collapsing levels. If $V(\mu)$, $\mu \in (\mathfrak{h}^{\natural})^*$, is an irreducible \mathfrak{g}^{\natural} -module, then we can write

$$(4.3) \quad V(\mu) = \bigotimes_{j \in I} V_{\mathfrak{g}_j^{\natural}}(\mu^j),$$

where $V_{\mathfrak{g}_j^{\natural}}(\mu^j)$ is an irreducible $\mathfrak{g}_j^{\natural}$ -module. Let ρ_0^j be the Weyl vector in $\mathfrak{g}_j^{\natural}$ (with respect to the positive system induced by the choice of positive roots for \mathfrak{g}).

Theorem 4.3. *Let $k \in \mathcal{K}$ be a non-collapsing level. Then k is a conformal level if and only if $k_i \neq 0$ for all i and*

$$(4.4) \quad \sum_{i \in I} \frac{(\mu^i | \mu^i + 2\rho_0^i)}{2(k_i + h_{0,i}^{\vee})} = \frac{3}{2},$$

where $V(\mu)$ is an irreducible component of the \mathfrak{g}^{\natural} -module $\mathfrak{g}_{-1/2}$ and the μ^j are defined in (4.3).

Proof. First of all we note that, if $k \in \mathcal{K}$ is a non-collapsing level and $k_i \neq 0$ for all i , then (4.4) is equivalent to condition (1) of Proposition 4.2. Indeed, by (2.6), the map $u \mapsto G^{\{u\}}$ from $\mathfrak{g}_{-1/2}$ to $\text{span}(G^{\{u\}} \mid u \in \mathfrak{g}_{-1/2})$ is a map of \mathfrak{g}^{\natural} -modules and, since the level is non-collapsing, it is an isomorphism. By (2.6) again, $J^{\{a\}}_{(n)} G^{\{v\}} = 0$ for all $a \in \mathfrak{g}^{\natural}$, $v \in \mathfrak{g}_{-1/2}$, and $n > 0$. It follows that $(\omega_{sug})_0$ acts on $G^{\{v\}}$ as $\sum_{i \in I} \frac{1}{k_i + h_{0,i}^{\vee}} C_{\mathfrak{g}_i^{\natural}}$. Thus, if v belongs to the irreducible component $V(\mu)$ of $\mathfrak{g}_{-1/2}$, then the left hand side of (4.4) gives the eigenvalue for the action of $(\omega_{sug})_0$ on $G^{\{v\}}$. Since $\mathfrak{g}_{-1/2}$ is either

irreducible or the sum of two contragredient irreducible \mathfrak{g}^{\natural} -modules, the left hand side of (4.4) gives the eigenvalue of $(\omega_{sug})_0$ on $G^{\{v\}}$ for all $v \in \mathfrak{g}_{-1/2}$.

Assume that k is a conformal non-collapsing level. By formula (3.2), if $k_i = 0$ for some i , then k is a collapsing level, thus $k_i \neq 0$ for all i . Since the level is non-collapsing, we see from Theorem 3.3 that $p(k) \neq 0$. It follows from Proposition 3.2 that, for all $v \in \mathfrak{g}_{-1/2}$, $G^{\{v\}}$ is nonzero in $W_k(\mathfrak{g}, \theta)$. Since k is a conformal level, $\mathcal{J}^k(\mathfrak{g})$ is contained in the maximal proper ideal of $W^k(\mathfrak{g}, \theta)$. Hence there is an onto map $\mathcal{W}^k(\mathfrak{g}, \theta) \rightarrow W_k(\mathfrak{g}, \theta)$, and in particular $G^{\{v\}}$ is also nonzero in $\mathcal{W}^k(\mathfrak{g}, \theta)$. From Proposition 4.2, we deduce that $(\omega_{sug})_0 G^{\{v\}} = \frac{3}{2} G^{\{v\}}$ hence, for any irreducible component $V(\mu)$ of the \mathfrak{g}^{\natural} -module $\mathfrak{g}_{-1/2}$, we have that (4.4) holds.

Conversely, if $k_i \neq 0$ for all i and (4.4) holds for a component $V(\mu)$ of $\mathfrak{g}_{-1/2}$, then, as already observed, (4.4) holds for all components of $\mathfrak{g}_{-1/2}$. Thus (1) of Proposition 4.2 holds. It follows that $\mathcal{J}^k(\mathfrak{g})$ is a proper ideal, hence $\omega = \omega_{sug}$ in $W_k(\mathfrak{g}, \theta)$. \square

Corollary 4.4. *Assume that k is a collapsing level such that relation (4.4) holds. Then $\mathcal{W}^k(\mathfrak{g})$ is not simple.*

Proof. Since k is a collapsing level we have that $G^{\{u\}} = 0$ in $W_k(\mathfrak{g}, \theta)$. But Proposition 4.2 gives that $G^{\{u\}} \neq 0$ in $W^k(\mathfrak{g}, \theta)$. The proof follows. \square

Solving (4.4) case-by-case, one finds that, if k is a solution, then $k \in \{-\frac{2}{3}h^\vee, -\frac{h^\vee-1}{2}\}$. We do not have a uniform explanation for this striking coincidence, but we can provide a proof for broad classes of examples with a limited amount of case-by-case checking. In the next Propositions we perform these calculations and list all conformal non-collapsing levels in all cases.

Proposition 4.5. *Assume that \mathfrak{g}^{\natural} is either zero or simple or 1-dimensional.*

If $\mathfrak{g} = sl(3)$, or $\mathfrak{g} = spo(n|n+2)$ with $n \geq 2$, $\mathfrak{g} = spo(n|n-1)$ with $n \geq 2$, $\mathfrak{g} = spo(n|n-4)$ with $n \geq 4$, then there are no non-collapsing conformal levels.

In all other cases the non-collapsing conformal levels are

- (1) $k = -\frac{h^\vee-1}{2}$ if \mathfrak{g} is of type $G_2, F_4, E_6, E_7, E_8, F(4), G(3)$ (for both choices of θ), or $\mathfrak{g} = psl(m|m)$ ($m \geq 2$);
- (2) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = sp(n)$ ($n \geq 6$), or $\mathfrak{g} = spo(2|m)$ ($m \geq 2$), or $\mathfrak{g} = spo(n|m)$ ($n \geq 4$).

Proof. If $\mathfrak{g}^{\natural} = \{0\}$ (i.e. $\mathfrak{g} = spo(2|1)$) then, clearly, (4.4) is never satisfied. In all other cases $\mathfrak{g}_{-1/2}$, as a \mathfrak{g}^{\natural} -module, is either irreducible or the sum of two contragredient modules. Let $V(\mu)$, $\mu \in (\mathfrak{h}^{\natural})^*$, be an irreducible component of $\mathfrak{g}_{-1/2}$ as a \mathfrak{g}^{\natural} -module. Let ρ_0 be the Weyl vector in \mathfrak{g}^{\natural} (with respect to the positive system induced by the choice of positive roots for \mathfrak{g}). Then (4.4)

reduces to

$$(4.5) \quad \frac{(\mu|\mu + 2\rho_0)}{2(k + \frac{1}{2}(h^\vee + h_0^\vee))} = \frac{3}{2}.$$

Let $C_{\mathfrak{g}_0}, C_{\mathfrak{g}^\natural}$ be the Casimir elements of $\mathfrak{g}_0, \mathfrak{g}^\natural$ respectively. Clearly

$$C_{\mathfrak{g}_0} = \frac{x^2}{(x|x)} + C_{\mathfrak{g}^\natural} = 2x^2 + C_{\mathfrak{g}^\natural}.$$

By Lemma 5.1 of [24], we see that the eigenvalue for the action of $C_{\mathfrak{g}_0}$ on $\mathfrak{g}_{-1/2}$ is $h^\vee - 1$. On the other hand, since $2ad(x)^2(v) = \frac{1}{2}v$ for all $v \in \mathfrak{g}_{-1/2}$, we see that the eigenvalue of $C_{\mathfrak{g}_0}$ is also $\frac{1}{2} + (\mu|\mu + 2\rho_0)$, hence

$$(\mu|\mu + 2\rho_0) = h^\vee - \frac{3}{2}.$$

Plugging (3.7) into (4.5) we obtain that $k = -\frac{h^\vee - 1}{2}$. $\mathfrak{g} = sl(3)$ has to be dropped from the list because in that case $k = -1$ is a collapsing level.

A case-by-case check shows that in cases $\mathfrak{g} = spo(n|m)$ ($m \geq 0$) we have that $h_0^\vee = h^\vee - 1$. Plugging the data into (4.5) we obtain $k = -\frac{2}{3}h^\vee$. The cases $\mathfrak{g} = spo(n|n+2)$ ($n \geq 2$) have to be discarded because $k + h^\vee = 0$, while the cases $\mathfrak{g} = spo(n|n-1)$ with $n > 2$ have to be discarded because $k_1 + h_0^\vee = k + \frac{1}{2}(h^\vee + h_0^\vee) = 0$. Finally the cases $spo(n|n-4)$ ($n \geq 4$) have to be discarded because $k = -2$ is a collapsing level. \square

Proposition 4.6. *Assume that $\mathfrak{g}^\natural = \mathfrak{g}_0^\natural \oplus \mathfrak{g}_1^\natural$ with $\mathfrak{g}_0^\natural \simeq \mathbb{C}$ and \mathfrak{g}_1^\natural simple.*

If $\mathfrak{g} = sl(m|m-3)$ with $m \geq 4$, then there are no non-collapsing conformal levels.

In other cases the non-collapsing conformal levels are

- (1) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = sl(m|m+1)$ ($m \geq 2$), and $\mathfrak{g} = sl(m|m-1)$ ($m \geq 3$);
- (2) $k = -\frac{2}{3}h^\vee$ and $k = -\frac{h^\vee - 1}{2}$ in all other cases.

Proof. Write $\mathfrak{g}^\natural = \mathbb{C}c \oplus \mathfrak{g}_1^\natural$ with c central in \mathfrak{g}^\natural and \mathfrak{g}_1^\natural simple. Since $\theta(c) = 0$ we see that c acts nontrivially on $\mathfrak{g}_{\pm 1/2}$, otherwise c would be central in \mathfrak{g} . We choose c so that one of the eigenvalues of the action of c on $\mathfrak{g}_{-1/2}$ is 1. Since $\mathfrak{g}_{-1/2}$ is self-dual as \mathfrak{g}^\natural -module, we see that also -1 is an eigenvalue. Let U^\pm be the corresponding eigenspaces. Since U^\pm are \mathfrak{g}^\natural -modules, we see that $\mathfrak{g}_{-1/2}$ is the direct sum of at least two submodules. Browsing Tables 1–3, we see that $\mathfrak{g}_{-1/2}$ breaks in at most two irreducible components, so we have $\mathfrak{g}_{-1/2} = U^+ \oplus U^-$, with U^\pm irreducible \mathfrak{g}^\natural -modules.

Let $V(\mu)$ be one of these two irreducible components. According to (4.3), $V(\mu) = V_{\mathbb{C}c}(\mu^0) \otimes V_{\mathfrak{g}_1^\natural}(\mu^1)$. We have to solve

$$(4.6) \quad \frac{(\mu^0|\mu^0)}{2(k + h^\vee/2)} + \frac{(\mu^1|\mu^1 + 2\rho_0^1)}{2(k + \frac{1}{2}(h^\vee + h_{0,1}^\vee))} = \frac{3}{2}.$$

First we compute $(\mu^0|\mu^0)$: this is the eigenvalue for the action of $\frac{c^2}{(c|c)}$ on U^+ , so $(\mu^0|\mu^0) = \frac{1}{(c|c)}$. On the other hand, letting $\kappa(\cdot, \cdot)$ denote the Killing form of \mathfrak{g} , we have

$$2h^\vee(c|c) = \kappa(c, c) = \text{str}(ad(c)^2) = 2\text{sdim}\mathfrak{g}_{-1/2}$$

where, in the last equality, we used the fact that c^2 acts trivially on $\mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_{-1}$ and as the identity on $\mathfrak{g}_{\pm 1/2}$. Using (5.6) of [24], it follows that

$$(4.7) \quad (\mu^0|\mu^0) = \frac{h^\vee}{2h^\vee - 4}.$$

Arguing as in the proof of Proposition 4.5, we see that $(\mu^1|\mu^1 + 2\rho_0^1) + (\mu^0|\mu^0) = h^\vee - \frac{3}{2}$, hence

$$(\mu^1|\mu^1 + 2\rho_0^1) = h^\vee - \frac{3}{2} - \frac{h^\vee}{2h^\vee - 4}.$$

Finally a case-by-case check shows that $h_{0,1}^\vee = h^\vee - 2$. Plugging these data in (4.6), we obtain the result. The cases $\mathfrak{g} = sl(m|m+1)$ with $k = -\frac{h^\vee-1}{2}$ and $m \geq 2$ have to be discarded because $k + h^\vee = 0$. The cases $\mathfrak{g} = sl(m|m-1)$ with $k = -\frac{h^\vee-1}{2}$ and $m \geq 3$ have to be discarded because $k_1 + h_{0,1}^\vee = 0$. The cases $\mathfrak{g} = sl(m|m-3)$ with $k = -\frac{h^\vee-1}{2}$ and $m \geq 4$ have to be discarded because k is a collapsing level. The cases $\mathfrak{g} = sl(m|m-3)$ with $k = -\frac{2}{3}h^\vee$ and $m \geq 4$ have to be discarded because $k_1 + h_{0,1}^\vee = 0$. \square

Proposition 4.7. *Assume that $\mathfrak{g}^{\natural} = \sum_{i=1}^r \mathfrak{g}_i^{\natural}$ with $\mathfrak{g}_1^{\natural} \simeq sl(2)$ and $r \geq 2$.*

If $\mathfrak{g} = osp(n+5|n)$ with $n \geq 2$ or $\mathfrak{g} = D(2, 1; a)$ with $a = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$, then there are no non-collapsing conformal levels.

In the other cases the non-collapsing conformal levels are

- (1) $k = -\frac{h^\vee-1}{2}$ if $\mathfrak{g} = D(2, 1; a)$ ($a \notin \{\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}$), $\mathfrak{g} = osp(n+8|n)$ ($n \geq 0$), $\mathfrak{g} = osp(n+2|n)$ ($n \geq 2$), $\mathfrak{g} = osp(n-4|n)$ ($n \geq 8$);
- (2) $k = -\frac{2}{3}h^\vee$ if $\mathfrak{g} = osp(n+7|n)$ ($n \geq 0$), $\mathfrak{g} = osp(n+1|n)$ ($n \geq 4$);
- (3) $k = -\frac{2}{3}h^\vee$ and $k = -\frac{h^\vee-1}{2}$ in all other cases.

Proof. Let α be the positive root for $\mathfrak{g}_1^{\natural}$. By browsing Tables 1–3 we see that $\mathfrak{g}_{-1/2}$ is irreducible and that, if we write $\mathfrak{g}_{-1/2} \simeq V_{sl(2)}(\mu^1) \otimes (\otimes_{i \geq 2} V_{\mathfrak{g}_i^{\natural}}(\mu^i))$, then $\mu^1 = \alpha/2$, where α is the positive root of $sl(2)$.

Then (4.4) becomes

$$(4.8) \quad \frac{3(\alpha|\alpha)}{8(k + \frac{1}{2}(h^\vee + (\alpha|\alpha)))} + \sum_{i=2}^r \frac{(\mu^i|\mu^i + 2\rho_0^i)}{2(k + \frac{1}{2}(h^\vee + h_{0,i}^\vee))} = \frac{3}{2}.$$

Let us assume first that $(\alpha|\alpha) = 2$ and that $r = 2$. This assumption leaves out only $D(2, 1; a)$ and D_4 . Arguing as in Proposition 4.5, we see that

$$\frac{3(\alpha|\alpha)}{4} + (\mu^2|\mu^2 + 2\rho_0^2) = \frac{3}{2} + (\mu^2|\mu^2 + 2\rho_0^2) = h^\vee - \frac{3}{2}$$

so $(\mu^2|\mu^2 + 2\rho_0^2) = h^\vee - 3$. Moreover one checks case-by-case that $h_{0,2} = h^\vee - 4$. Plugging these data in (4.8), we obtain that the solutions are $k = -\frac{h^\vee - 1}{2}$ and $k = -\frac{2}{3}h^\vee$. The case $\mathfrak{g} = osp(n+1|n)$ with $n \geq 4$ and $k = -\frac{h^\vee - 1}{2}$ has to be discarded because $k + h^\vee = 0$. This happens also in case $\mathfrak{g} = osp(n+2|n)$ with $n \geq 2$ and $k = -\frac{2}{3}h^\vee$. In case $\mathfrak{g} = osp(n+7|n)$ with $n \geq 0$ the level $k = -\frac{h^\vee - 1}{2}$ is collapsing. This also happens at level $k = -\frac{2}{3}h^\vee$ when $\mathfrak{g} = osp(n-4|n)$ with $n \geq 8$ or $\mathfrak{g} = osp(n+5|n)$ with $n \geq 2$. The case $\mathfrak{g} = osp(n+5|n)$ with $n \geq 2$ and $k = -\frac{h^\vee - 1}{2}$ has to be dropped because $k_i + h_{0,i}^\vee = 0$ for some i . The same happens in case $\mathfrak{g} = osp(n+8|n)$ with $n \geq 2$ and $k = -\frac{2}{3}h^\vee$.

Let us now discuss $\mathfrak{g} = D(2,1;a)$: in this case $\mathfrak{g}_2^{\natural} \simeq sl(2)$. Let β be the positive root of $\mathfrak{g}_2^{\natural}$. From Table 2 we see that $\mathfrak{g}_{-1/2} = V_{sl(2)}(\alpha/2) \otimes V_{sl(2)}(\beta/2)$. Then (4.4) becomes

$$(4.9) \quad \frac{3(\alpha|\alpha)}{8(k + \frac{1}{2}(h^\vee + (\alpha|\alpha)))} + \frac{3(\beta|\beta)}{8(k + \frac{1}{2}(h^\vee + (\beta|\beta)))} = \frac{3}{2}.$$

Plugging $h^\vee = 0$, $(\alpha|\alpha) = -2(a+1)$, $(\beta|\beta) = 2a$ into (4.9) and solving for k , one gets that $k = 0 = -\frac{2}{3}h^\vee$ or $k = \frac{1}{2} = -\frac{h^\vee - 1}{2}$. The value $k = 0$ has to be discarded because this level is critical. If $a = \frac{1}{2}$ or $a = -\frac{3}{2}$ then $k = \frac{1}{2}$ is a collapsing level. If $a = -\frac{1}{2}$ and $k = \frac{1}{2}$ then $k_i + h_{0,i}^\vee = 0$ for all i .

Finally, if \mathfrak{g} is of Type $D_4 (= osp(8|0))$, then $r = 3$ and $\mathfrak{g}_i^{\natural} \simeq sl(2)$ for all i . Furthermore, if α_i is the positive root of $\mathfrak{g}_i^{\natural}$, then $\mathfrak{g}_{-1/2} = \otimes_{i=1}^3 V_{sl(2)}(\alpha_i/2)$. Since $(\alpha_i|\alpha_i) = 2$ for all i , it follows that (4.4) becomes

$$(4.10) \quad \frac{9}{4(k + \frac{1}{2}(h^\vee + 2))} = \frac{3}{2}.$$

Since $h^\vee = 6$, we obtain that $k = -\frac{h^\vee - 1}{2}$ is the only solution of (4.10). One checks directly that $k + h^\vee \neq 0$, k is a non-collapsing level, and that $k_i + h_{0,i}^\vee \neq 0$ for all i . \square

Corollary 4.8. $\mathcal{V}_k(\mathfrak{g}^{\natural})$ embeds conformally in $W_k(\mathfrak{g}, \theta)$ if and only if $c(\mathfrak{g}, k) = c_{sug}$.

Proof. The set of collapsing levels k , such that $k + h^\vee \neq 0$ and $k_i + h_{0,i}^\vee \neq 0$ whenever $k_i \neq 0$, together with the set of conformal non-collapsing levels turns out to be exactly the set of solutions of the equation $c(\mathfrak{g}, k) = c_{sug}$. \square

Remark 4.2. It is worthwhile to note that, in general, equality of central charges does not imply conformal embedding. A counterexample is the following: take $W = \mathbb{C}\mathbf{1} \subset V = W_{-1/3}(sl(3))$. Then $c(sl(3), -1/3) = 0$, but, since $-1/3$ is a non-collapsing level, $\omega_V \neq 0$.

Example 4.3. Consider the case of $\mathfrak{g} = sl(n)$, $n \geq 3$.

If $n \geq 4$ then $\mathfrak{g}^{\natural} = \mathfrak{g}_0^{\natural} \oplus \mathfrak{g}_1^{\natural}$ with $\mathfrak{g}_0^{\natural} = \mathbb{C}c$ the center of \mathfrak{g}^{\natural} and $\mathfrak{g}_1^{\natural} \simeq sl(n-2)$. Thus $\mathfrak{g}^{\natural} \simeq gl(n-2)$. In this case $h^{\vee} = n$, $h_{0,0}^{\vee} = 0$, $h_{0,1}^{\vee} = n-2$. Combining Theorem 3.3 and Proposition 4.6 we obtain

- (1) If $k = -1$ then $W_k(sl(n), \theta)$ is the Heisenberg vertex algebra $V_{\frac{n-2}{2}}(\mathbb{C}c)$; (this result is also independently obtained in [12] by using singular vectors from $V^{-1}(sl(n))$).
- (2) If $k = -\frac{h^{\vee}}{2} = -\frac{n}{2}$ then $W_k(sl(n), \theta) \cong V_{k+1}(sl(n-2))$;
- (3) Assume that $k \notin \{-1, -h^{\vee}/2\}$. The embedding $\mathcal{V}_k(gl(n-2)) \hookrightarrow W_k(sl(n), \theta)$ is conformal if and only if $k \in \{\frac{1-n}{2}, -\frac{2}{3}n\}$.

The equation $c(\mathfrak{g}, k) = c_{sug}$ in this case reads

$$\begin{cases} \frac{k(n^2-1)}{k+n} - 6k + n - 4 = 1 + \frac{(k+1)((n-2)^2-1)}{k+n-1} & \text{if } k \neq -\frac{n}{2} \\ \frac{k(n^2-1)}{k+n} - 6k + n - 4 = \frac{(k+1)((n-2)^2-1)}{k+n-1} & \text{if } k = -\frac{n}{2} \end{cases}$$

whose solutions are precisely $-1, -\frac{2}{3}n, \frac{1-n}{2}, -\frac{n}{2}$.

The case $n = 3$ is of particular interest. In this case \mathfrak{g}^{\natural} is one dimensional abelian: $\mathfrak{g}^{\natural} = \mathbb{C}c$, $h^{\vee} = 3$, $h_{0,0}^{\vee} = 0$. We have that $W^k(sl(3), \theta)$ is the Bershadsky-Polyakov vertex algebra, freely generated by even vectors G^{\pm} of conformal weight $3/2$, the Virasoro vector ω , and the Heisenberg vector $J = J^{\{c\}}$ (cf. [13], [10]). By Theorem 3.3, we obtain $W_k(sl(3), \theta) = \mathbb{C}\mathbf{1}$ if $k = -\frac{h^{\vee}}{2} = -\frac{3}{2}$ while it is the Heisenberg vertex algebra $V_{\frac{1}{2}}(\mathbb{C}c)$ if $k = -1$.

Since $k = -1 = -\frac{h^{\vee}-1}{2}$, by the proof of Proposition 4.5, there are no other conformal levels, but condition (1) of Proposition 4.2 still holds. Since k is a collapsing level, we have that $G^{\{u\}} = 0$ in $W_k(\mathfrak{g})$. But Proposition 4.2 gives that $G^{\{u\}} \neq 0$ in $\mathcal{W}^{-1}(sl(3), \theta)$, hence the vertex algebra $\mathcal{W}^{-1}(sl(3), \theta)$ is intermediate between $W^{-1}(sl(3), \theta)$ and $W_{-1}(sl(3), \theta) = V_{\frac{1}{2}}(\mathbb{C}c)$.

By (1.1) with $k = -1$ we get

$$G^+_{(0)}G^- = \nu(\omega - \omega_{sug}) \quad (\nu \neq 0).$$

This fact and (3) in Proposition 4.2 imply that $\mathcal{W}^{-1}(sl(3), \theta)$ is an extension of a Heisenberg vertex algebra by elements G^{\pm} with λ -brackets

$$[G^+_{\lambda}G^+] = [G^+_{\lambda}G^-] = [G^-_{\lambda}G^-] = 0.$$

One can also show that for every $k \in \mathbb{Z}_{>0}$, $(G^+_{(-1)})^k \mathbf{1}$ is a non-trivial singular vector in $\mathcal{W}^{-1}(sl(3), \theta)$ and therefore $\mathcal{W}^{-1}(sl(3), \theta)$ contains infinitely many non-trivial ideals.

Another occurrence of this phenomenon happens when $\mathfrak{g} = sp(4)$ at level $k = -2$. In this case $\mathfrak{g}^{\natural} \simeq sl(2)$, the level k is collapsing but condition (4.4) still holds. We have that $\mathcal{W}^{-2}(\mathfrak{g}, \theta)$ is a non-simple extension of $\mathcal{V}_{-2}(sl(2))$ with $[G^{\{u\}}_{\lambda}G^{\{v\}}] = 0$ for every $u, v \in \mathfrak{g}_{-1/2}$. It is shown in [7] that also relation $G^{\{u\}}G^{\{v\}} := 0$ holds. As in the $\mathfrak{g} = sl(3)$ example above, one can construct infinitely many ideals in the vertex algebra $\mathcal{W}^{-2}(sp(4), \theta)$.

5. AFFINE VERTEX ALGEBRA $V_k(sl(n))$ AND W -ALGEBRAS

In this Section we apply our simplified formula (1.1) for $[G^{\{u\}} \lambda G^{\{v\}}]$ and the conformal embedding of $\mathcal{V}_k(gl(n))$ in $W_k(sl(2|n), \theta)$ at level $k = -\frac{h^\vee - 1}{2} = \frac{n-1}{2}$ to construct a realization of $V_{-\frac{n+1}{2}}(sl(n+1))$ in the tensor product of $W_k(sl(2|n), \theta)$ with a lattice vertex algebra for $n \geq 4$, $n \neq 5$: see Theorem 5.6.

Since $W_k(psl(2|2), \theta)$ is the $N = 4$ superconformal algebra, our result can be considered a higher rank generalization of the realization, given in [3], of $V_{-\frac{3}{2}}(sl(3))$ as a vertex subalgebra of the tensor product of a $N = 4$ superconformal algebra with a lattice vertex algebra.

Let $\mathfrak{g} = sl(2|n)$ and $k = -\frac{h^\vee - 1}{2} = \frac{n-1}{2}$. In this case $\mathfrak{g}^{\natural} \simeq gl(n)$, $k_0 = k + \frac{1}{2}h^\vee = \frac{1}{2}$ and $k_1 = k + \frac{1}{2}(h^\vee - h_{0,1}^\vee) = \frac{n+1}{2}$. By Proposition 4.6, $\mathcal{V}_{\frac{n-1}{2}}(\mathfrak{g}^{\natural})$ embeds conformally in $W_k(\mathfrak{g}, \theta)$. This embedding is a generalization of the correspondence between affine vertex algebra $\mathcal{V}_k(sl(2))$ and superconformal vertex algebras from [1] and [17] (see also [2] for the critical level version of this correspondence).

First we start with the following useful criterion for establishing that some elements of $W^k(sl(2|n), \theta)$ are in the ideal $\mathcal{J}^k(sl(2|n))$ (cf. (4.2)).

Lemma 5.1. *Assume that $v \in W^k(sl(2|n), \theta)$ satisfies*

$$(\omega_{sug})_0 v = av, \quad \omega_0 v = bv, \quad a \neq b,$$

where $a, b \in \mathbb{C}$. Then $v \in \mathcal{J}^k(sl(2|n))$.

Proof. Recall that $\omega_{sug} = \omega$ in $\mathcal{W}^k(sl(2|n), \theta) = W^k(sl(2|n), \theta) / \mathcal{J}^k(sl(2|n))$. If $v \neq 0$ in $\mathcal{W}^k(sl(2|n), \theta)$ then it would be an eigenvector for both $(\omega_{sug})_0$ and ω_0 , but then the corresponding eigenvalues must coincide. \square

It follows that in order to construct elements of the ideal $\mathcal{J}^k(sl(2|n))$ it is enough to construct common eigenvectors v for ω_0 and $(\omega_{sug})_0$ which have different eigenvalues.

To simplify notation let us introduce

$$G_i^+ = G^{\{e_{2,2+i}\}}, \quad G_i^- = G^{\{e_{2+i,1}\}}.$$

We shall consider the elements

$$v_{i,j}^+ =: G_i^+ G_j^+ \quad ; \quad v_{i,j}^- =: G_i^- G_j^- \quad : \quad (i, j = 1, \dots, n).$$

We shall now give some structural formulas. Recall from the proof of Proposition 4.6 that $\mathfrak{g}^{\natural} = \mathfrak{g}_0^{\natural} \oplus \mathfrak{g}_1^{\natural}$ with $\mathfrak{g}_0^{\natural} = \mathbb{C}c$ and $\mathfrak{g}_1^{\natural} \simeq sl(n)$, $\mathfrak{g}_{-1/2} = U^+ \oplus U^-$, and $(c|c) = 2 - \frac{4}{h^\vee}$. Furthermore, note that $span(e_{2,2+i} \mid 1 \leq i \leq n)$ and $span(e_{2+i,1} \mid 1 \leq i \leq n)$ are both stable under the action of c and $sl(n)$. It follows that we can choose c so that $U^+ = span(e_{2,2+i} \mid 1 \leq i \leq n)$ and $U^- = span(e_{2+i,1} \mid 1 \leq i \leq n)$. With this setting we have

Lemma 5.2.

$$(5.1) \quad G_i^+ G_j^+ = G_i^- G_j^- = 0 \quad (s \geq 0),$$

$$(5.2) \quad : G_i^+ G_i^+ :=: G_i^- G_i^- := 0,$$

$$(5.3) \quad G_i^+ G_j^- = \delta_{i,j} 2(k+1) \left(k + \frac{h^\vee}{2}\right) \mathbf{1},$$

$$(5.4) \quad G_i^+ G_j^- = 2 \left(k + \frac{h^\vee}{2}\right) J^{\{e_{2+j, 2+i}\}} \quad (i \neq j),$$

$$(5.5) \quad G_i^+ G_i^- = \frac{n-2}{n} (k+1) J^{\{e\}} + 2 \left(k + \frac{h^\vee}{2}\right) J^{\{a\}}.$$

where $a = (e_{1,1} + e_{2+i, 2+i})_1^\natural = (e_{2,2} + e_{2+i, 2+i})_1^\natural \in \mathfrak{g}_1^\natural \simeq sl(n)$.

Proof. Formula (5.1) follows by observing that, when $u = e_{2,2+i}, v = e_{2,2+j}$ or $u = e_{2+i,1}, v = e_{2+j,1}$, all brackets in the r.h.s. of (1.1) vanish.

Recall that, since $\mathfrak{g}_{-1/2}$ is purely odd in this case,

$$: G_i^\pm G_j^\pm := - : G_j^\pm G_i^\pm : + \int_{-\partial}^0 [G_i^\pm G_j^\pm] d\mu.$$

By (5.1), $[G_i^\pm G_j^\pm] = [G_i^\pm G_j^\pm] = 0$, so (5.2) follows.

Formulas (5.3), (5.4), and (5.5) follow from Lemma 3.1 and the explicit expression for $p(k)$ given in Table 4. \square

Proposition 5.3. *Assume that $1 \leq i, j \leq n, i \neq j$. Then we have*

(1)

$$(\omega_{sug})_0 v_{i,j}^\pm = \frac{4(n-2)}{n-1} v_{i,j}^\pm.$$

(2) *Assume that $n \neq 5$. Then $v_{i,j}^\pm \in \mathcal{J}^k(sl(2|n))$, hence*

$$(5.6) \quad : G_i^+ G_j^+ :=: G_i^- G_j^- := 0 \quad \text{in } W_k(sl(2|n), \theta).$$

Proof. Set

$$\widetilde{U}^\pm(2) = \text{span}_{\mathbb{C}}(v_{i,j}^\pm \mid 1 \leq i < j \leq n).$$

Formula (2.6) shows that $J^{\{a\}}(s)u = 0$ for all $u \in \widetilde{U}^\pm(2)$ and $s > 0$. As a $gl(n)$ -module $\widetilde{U}^+(2)$ is isomorphic to $\wedge^2 \mathbb{C}^n$ and $\widetilde{U}^-(2)$ is isomorphic to $\wedge^2(\mathbb{C}^n)^*$. It follows that the eigenvalue of $(\omega_{sug})_0$ on $\widetilde{U}^+(2)$ is the same as the eigenvalue of $(\omega_{sug})_0$ on $\widetilde{U}^-(2)$ and the latter is

$$\frac{(2\mu_0 | 2\mu_0)}{2(k + \frac{1}{2}h^\vee)} + \frac{(\omega_2 | \omega_2 + 2\rho_1)}{2(k + \frac{1}{2}(h^\vee + h_{0,1}^\vee))}.$$

Using (4.7) and substituting $k = \frac{n-1}{2}$, $h^\vee = 2 - n$, and $h_{0,1}^\vee = -n$, we find assertion (1).

Since $\omega_0 v_{i,j}^\pm = 3v_{i,j}^\pm$, Lemma 5.1 implies that, if $n \neq 5$, then $v_{i,j}^\pm \in \mathcal{J}^k(sl(2|n))$ hence (2) follows. \square

From now on we assume that $n \neq 5$. Recall that we assumed $n \geq 4$. We have chosen c so that $J^{\{c\}}_{(0)}G_i^\pm = \pm G_i^\pm$. It follows that we can define a \mathbb{Z} -gradation on $W_k(sl(2|n), \theta)$

$$W_k(sl(2|n), \theta) = \bigoplus_{i \in \mathbb{Z}} W_k(sl(2|n), \theta)^{(i)},$$

setting

$$W_k(sl(2|n), \theta)^{(i)} = \{v \in W_k(sl(2|n), \theta) \mid J^{\{c\}}_{(0)}v = iv\}.$$

Clearly $\mathcal{V}_k(sl(2|n)^\natural) \subset W_k(sl(2|n), \theta)^{(0)}$, and $G_i^\pm \in W_k(sl(2|n), \theta)^{(\pm 1)}$.

Let F_{-1} be the simple lattice vertex superalgebra $V_{\mathbb{Z}\varphi}$ associated to the lattice $\mathbb{Z}\varphi$ where $\langle \varphi, \varphi \rangle = -1$. As a vector space, $F_{-1} = M_\varphi(1) \otimes \mathbb{C}[\mathbb{Z}\varphi]$ where $M_\varphi(1)$ is the Heisenberg vertex algebra generated by the field $\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_{(n)} z^{-n-1}$, and $\mathbb{C}[\mathbb{Z}\varphi]$ is the group algebra with generator e^φ . F_{-1} admits the following \mathbb{Z} -gradation:

$$F_{-1} = \bigoplus_{m \in \mathbb{Z}} F_{-1}^{(m)}, \quad F_{-1}^{(m)} := M_\varphi(1) \otimes e^{m\varphi}.$$

Consider now the simple vertex superalgebra $W_k(sl(2|n), \theta) \otimes F_{-1}$. Set

$$\bar{\varphi} = J^{\{c\}} \otimes \mathbf{1} + \mathbf{1} \otimes \varphi_{(-1)} \mathbf{1}.$$

Note that $\bar{\varphi}_{(0)}$ acts semisimply on $W_k(sl(2|n), \theta) \otimes F_{-1}$. We consider the following vertex subalgebra

$$(5.7) \quad (W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)} = \text{Ker } \bar{\varphi}_{(0)}.$$

Explicitly,

$$(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)} = \bigoplus_{i \in \mathbb{Z}} W_k(sl(2|n), \theta)^{(i)} \otimes F_{-1}^{(i)}.$$

Define

$$\begin{aligned} \bar{U}^\pm &= \text{span}_{\mathbb{C}} \{G_i^\pm \otimes e^{\pm\varphi} \mid 1 \leq i \leq n\}, \\ \bar{U}^0 &= \{J^{\{a\}} \otimes \mathbf{1} \mid a \in sl(n)\} \oplus \mathbb{C}J^{\{c\}} \otimes \mathbf{1} \oplus \mathbb{C}\mathbf{1} \otimes \varphi_{(-1)} \mathbf{1}. \end{aligned}$$

Clearly, \bar{U}^0 generates the vertex subalgebra of $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ which is equal to $\mathcal{V}_k(sl(2|n)^\natural) \otimes M_\varphi(1)$.

Proposition 5.4.

- (1) *The vertex algebra $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ is simple.*
- (2) *The vertex algebra $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ is generated by the vector space $\bar{U}^- \oplus \bar{U}^0 \oplus \bar{U}^+$.*

Proof. The proof is similar to that of Theorem 10.1. in [3].

It is a general fact that the degree zero component of a \mathbb{Z} -graded simple vertex algebra is simple. The simplicity of $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ follows.

Let S be the vertex subalgebra of $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ generated by the space $\bar{U}^- \oplus \bar{U}^0 \oplus \bar{U}^+$. To prove (2), we show that $W_k(sl(2|n), \theta)^{(\ell)} \otimes F_{-1}^{(\ell)} \subset$

S for every $\ell \in \mathbb{Z}$. Let $\mathcal{U}^\pm = \text{span}_{\mathbb{C}}\{G_i^\pm \mid 1 \leq i \leq n\}$. Note that, since $\omega = \omega_{\text{sug}}$, $W_k(\mathfrak{sl}(2|n))^{(\ell)} \otimes F_{-1}^{(\ell)}$ is linearly spanned by the vectors of the form

$$(5.8) \quad G_{i_1}^+ \cdots G_{i_r}^+ G_{j_1}^- \cdots G_{j_s}^- w \otimes w^{(1)},$$

where $\ell = r - s$, $w \in \mathcal{V}(\mathfrak{sl}(2|n)^\natural)$, $w^{(1)} \in F_{-1}^{(\ell)}$, thus $W_k(\mathfrak{sl}(2|n))^{(\ell)} \otimes F_{-1}^{(\ell)}$ is contained in the linear span of

$$\underbrace{\mathcal{U}^+ \cdots \mathcal{U}^+}_{r \text{ times}} \cdot \underbrace{\mathcal{U}^- \cdots \mathcal{U}^-}_{s \text{ times}} \cdot \mathcal{V}_k(\mathfrak{sl}(2|n)^\natural) \cdot F_{-1}^{(\ell)}.$$

By using relations

$$G_i^\pm \otimes \mathbf{1} = (G_i^\pm \otimes e^{\pm\varphi})_{(-2)} (\mathbf{1} \otimes e^{\mp\varphi}) \quad (1 \leq i \leq n)$$

one sees that $\mathcal{U}^\pm \subset \overline{\mathcal{U}}^\pm \cdot F_{-1}^{(\mp 1)}$ so $W_k(\mathfrak{sl}(2|n), \theta)^{(\ell)} \otimes F_{-1}^{(\ell)}$ is contained in the linear span of

$$\begin{aligned} & \underbrace{\overline{\mathcal{U}}^+ \cdots \overline{\mathcal{U}}^+}_{r \text{ times}} \cdot \underbrace{\overline{\mathcal{U}}^- \cdots \overline{\mathcal{U}}^-}_{s \text{ times}} \cdot \mathcal{V}_k(\mathfrak{sl}(2|n)^\natural) \cdot F_{-1}^{(s-r)} \cdot F_{-1}^{(\ell)} \\ & \subseteq \underbrace{\overline{\mathcal{U}}^+ \cdots \overline{\mathcal{U}}^+}_{r \text{ times}} \cdot \underbrace{\overline{\mathcal{U}}^- \cdots \overline{\mathcal{U}}^-}_{s \text{ times}} \cdot \mathcal{V}_k(\mathfrak{sl}(2|n)^\natural) \cdot F_{-1}^{(0)} \end{aligned}$$

The claim follows. \square

Lemma 5.5. *We have:*

- (1) $(G_i^+ \otimes e^{+\varphi})_{(r)} (G_j^+ \otimes e^{+\varphi}) = (G_i^- \otimes e^{-\varphi})_{(r)} (G_j^- \otimes e^{-\varphi}) = 0$ for $r \geq 0$;
- (2) $(G_i^+ \otimes e^{+\varphi})_{(0)} (G_j^- \otimes e^{-\varphi}) = -J^{\{e_{2+j, 2+i}\}} \otimes \mathbf{1}$ if $i \neq j$;
- (3) $(G_i^+ \otimes e^{+\varphi})_{(0)} (G_i^- \otimes e^{-\varphi}) = -\frac{(n-2)(n+1)}{2n} J^{\{c\}} - \frac{n+1}{2} \mathbf{1} \otimes \varphi_{(-1)} \mathbf{1} - J^{\{a\}}$
with $a = (e_{1,1} + e_{2+i, 2+i})_1^\natural = (e_{2,2} + e_{2+i, 2+i})_1^\natural \in \mathfrak{g}_1^\natural \simeq \mathfrak{sl}(n)$;
- (4) $(G_i^+ \otimes e^{+\varphi})_{(1)} (G_j^- \otimes e^{-\varphi}) = -\delta_{i,j} \frac{n+1}{2} \mathbf{1} \otimes \mathbf{1}$;
- (5) $(G_i^+ \otimes e^{+\varphi})_{(m)} (G_j^- \otimes e^{-\varphi}) = 0$ for $m \geq 2$.

Proof. Since, by [20, (4.3.2)], we have

$$(G_i^+ \otimes e^\varphi)_{(r)} (G_j^+ \otimes e^{+\varphi}) = - \sum_{m \geq r-1} G_i^+{}_{(m)} G_j^+ \otimes e^\varphi{}_{(-m+r-1)} e^\varphi,$$

and, likewise,

$$(G_i^- \otimes e^{-\varphi})_{(r)} (G_j^- \otimes e^{-\varphi}) = - \sum_{m \geq r-1} G_i^-{}_{(m)} G_j^- \otimes e^{-\varphi}{}_{(-m+r-1)} e^{-\varphi},$$

assertion (1) follows from relations (5.6), (5.1), and (5.2).

Since

$$(G_i^+ \otimes e^\varphi)_{(r)} (G_j^- \otimes e^{-\varphi}) = - \sum_{m \geq r+1} G_i^+{}_{(m)} G_j^- \otimes e^\varphi{}_{(-m+r-1)} e^{-\varphi},$$

formulas (2)–(5) follow from Lemma 5.2 and the fact that $e^\varphi{}_{(-2)} e^{-\varphi} = \mathbf{1}$ while $e^\varphi{}_{(-3)} e^{-\varphi} = \varphi_{(-1)} \mathbf{1}$. \square

With notation as in Section 2, we set $V^{k'}(sl(n+1)) = V^B(sl(n+1))$ where $B = k'(\cdot, \cdot)$ and (\cdot, \cdot) is the invariant form of $sl(n+1)$, normalized by the condition $(\alpha, \alpha) = 2$ for any root α . We also let, if $k' + n + 1 \neq 0$, $V_{k'}(sl(n+1))$ be its unique simple quotient.

Theorem 5.6. *We have*

$$(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)} \cong V_{-\frac{n+1}{2}}(sl(n+1)) \otimes M_{\bar{\varphi}}(1),$$

where $M_{\bar{\varphi}}(1)$ is the Heisenberg vertex algebra generated by $\bar{\varphi}$.

Proof. Set

$$\begin{aligned} \bar{U}^{0,0} &= \{v \in \bar{U}^0 \mid \bar{\varphi}_{(n)}v = 0 \text{ for } n \geq 0\} \\ &= \{J^{\{a\}} \otimes \mathbf{1} \mid a \in \mathfrak{g}_1^{\natural}\} \oplus \mathbb{C}w, \end{aligned}$$

where $w = J^{\{c\}} \otimes \mathbf{1} + \frac{n}{n-2}\mathbf{1} \otimes \varphi_{(-1)}\mathbf{1}$.

Set

$$U^{ext} = \bar{U}^+ \oplus \bar{U}^{0,0} \oplus \bar{U}^-.$$

Identify $\mathfrak{g}_1^{\natural}$ with $sl(n)$ by identifying $a \in sl(n)$ with $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$ in $sl(2|n)$.

Embed $gl(n)$ in $sl(n+1)$ by mapping a to $\iota(a) = \begin{pmatrix} -tr(a) & 0 \\ 0 & a \end{pmatrix}$. Define $\gamma : sl(n+1) \rightarrow U^{ext}$ by mapping:

$$\begin{aligned} \iota(a) &\mapsto J^{\{a\}} \otimes \mathbf{1}, \quad \iota(I_n) \mapsto \frac{(n+1)(n-2)}{2}w, \\ e_{11+i} &\mapsto G_i^+ \otimes e^{\varphi}, \quad e_{1+i1} \mapsto G_i^- \otimes e^{-\varphi}, \quad 1 \leq i \leq n, \end{aligned}$$

We claim that, if $a, b \in sl(n+1)$, then

$$(5.9) \quad [\gamma(a)_{\lambda}\gamma(b)] = \gamma([a, b]) - \frac{n+1}{2}\lambda(a, b),$$

Indeed if $a, b \in sl(n)$, (5.9) follows from (2.5) observing that $(a|b) = -(a, b)$. Next, assume $a \in sl(n)$, $b = e_{11+i}$. Then, by (2.6), we have

$$[J^{\{a\}} \otimes \mathbf{1}_{\lambda} G_i^+ \otimes e^{\varphi}] = [J^{\{a\}}_{\lambda} G_i^+] \otimes e^{\varphi} = G^{\{[a, e_{22+i}]\}} \otimes e^{\varphi},$$

which is (5.9) in the present case, since $G^{\{[a, e_{22+i}]\}} \otimes e^{\varphi} = \gamma([\iota(a), e_{11+i}])$. The same argument proves (5.9) when $a \in sl(n)$ and $b = e_{1+i1}$. If $a = e_{11+i}$, $b = e_{11+j}$ or $b = e_{1+j1}$, then (5.9) follows from Lemma 5.5. If $a = I_n$ and $b \in sl(n)$ then $[\gamma(\iota(I_n))_{\lambda}\gamma(\iota(b))] = \frac{(n+1)(n-2)}{2}[w_{\lambda}(J^{\{b\}} \otimes \mathbf{1})] = \frac{(n+1)(n-2)}{2}[J^{\{c\}}_{\lambda} J^{\{b\}}] \otimes \mathbf{1} = 0$ as desired. If $a = I_n$ and $b = e_{11+i}$ then

$$\begin{aligned} [\gamma(\iota(I_n))_{\lambda}\gamma(b)] &= \frac{(n+1)(n-2)}{2}[(J^{\{c\}}_{\lambda} G_i^+) \otimes e^{\varphi}] + \frac{(n+1)n}{2}[G_i^+ \otimes [\varphi_{\lambda} e^{\varphi}]] \\ &= -(n+1)G_i^+ \otimes e^{\varphi} = \gamma([I_n, e_{11+i}]). \end{aligned}$$

The case with $a = I_n$ and $b = e_{1+i1}$ is treated similarly. Finally, if $a = b = I_n$, then

$$[\gamma(\iota(I_n))_{\lambda}\gamma(\iota(I_n))] = \left(\frac{(n+1)(n-2)}{2}\right)^2([(J^{\{c\}}_{\lambda} J^{\{c\}}) \otimes \mathbf{1}] + \left(\frac{n}{n-2}\right)^2[\mathbf{1} \otimes [\varphi_{\lambda}\varphi]]).$$

It follows that $\gamma(\iota(I_n))_{(0)}\gamma(\iota(I_n)) = 0$ and

$$\gamma(\iota(I_n))_{(1)}\gamma(\iota(I_n)) = -\frac{(n+1)^2}{2}n = -\frac{n+1}{2}(I_n, I_n),$$

as desired. Having shown that (5.9) holds, we furthermore observe that

$$\overline{\varphi}_{(1)}w = (J^{\{c\}}_{(1)}J^{\{c\}}) \otimes \mathbf{1} + \frac{n}{n-2}\mathbf{1} \otimes \varphi_{(1)}\varphi = \frac{1}{2}(c|c) - \frac{n}{n-2} = 0.$$

This, together with (5.7), shows that $[\overline{\varphi}_\lambda a] = 0$ for all $a \in U^{ext}$. It follows that there is a vertex algebra homomorphism

$$V^{-\frac{n+1}{2}}(sl(n+1)) \otimes M_{\overline{\varphi}}(1) \rightarrow (W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}.$$

Proposition 5.4 implies that this homomorphism is surjective. Since $(W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}$ is a simple vertex algebra, we get that

$$V_{-\frac{n+1}{2}}(sl(n+1)) \otimes M_{\overline{\varphi}}(1) \cong (W_k(sl(2|n), \theta) \otimes F_{-1})^{(0)}.$$

□

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