On the Third Critical Speed for Rotating Bose-Einstein Condensates

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Abstract

We study a two-dimensional rotating Bose-Einstein condensate confined by an anharmonic trap in the framework of the Gross-Pitaevksii theory. We consider a rapid rotation regime close to the transition to a giant vortex state. It was proven in [CPRY3] that such a transition occurs when the angular velocity is of order ε^{-4}, with ε^{-2} denoting the coefficient of the nonlinear term in the Gross-Pitaevskii functional and $\varepsilon \ll 1$ (Thomas-Fermi regime). In this paper we identify a finite value Ω_c such that, if $\Omega = \Omega_0/\varepsilon^4$ with $\Omega_0 > \Omega_c$, the condensate is in the giant vortex phase. Under the same condition we prove a refined energy asymptotics and an estimate of the winding number of any Gross-Pitaevskii minimizer.

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1 Introduction

Since the first experimental realization of *Bose-Einstein (BE) condensation* in the 90's, BE condensates and cold atoms in general have been extensively studied to investigate quantum properties on almost macroscopic scales. Among the typical features of BE condensates, one of the most striking is certainly *superfluidity*, which has been studied in several experiments in the last years by putting the quantum system under rotation and observing its response (see, e.g., the reviews [Co, Fe1]). Because of the quantum nature of BE condensates, the only possible change to the condensate profile due to the imposed rotation is the nucleation of isolated defects, i.e., quantum *vortices*. The generation of vortices has been observed in various experiments as well as the growth of their number when the angular velocity increases [MCWD, RAVXK, CHES]. For even larger angular velocities the number of vortices becomes so large that they fill the bulk of the system and arrange in a typical Abrikosov lattice [ARVK]. In presence of harmonic trapping the rotation can not be arbitrarily fast, otherwise the centrifugal forces would break down the trapping and system would eventually fly apart. On the opposite when the trapping contains some stronger confinement, e.g., some anharmonic potential growing faster that $|\mathbf{r}|^2$ for large $|\mathbf{r}|$, regimes with much more rapid rotation can in principle be reached. Unfortunately so far a loss of coherence of the system has prevented the exploration of such regimes in the experiments [BSSD], although a depression at the center of the trap has been observed for large angular velocities.

However it has been predicted [CD, CDY2, Fe2, FJS, FB, KTU, KB, KF, R1] that, besides the nucleation of vortices, other phase transitions should be observed in rapid rotating condensates in case of anharmonic confinement, with the occurrence of macroscopic defects or the transition to *giant vortex* states: when the rotational velocity gets very large, the centrifugal forces constrain the condensate in some thin annular region around a macroscopic hole and, if the rotation gets even more rapid, vortices disappear from the bulk of the system, which seems then to carry a huge circulation centered at the origin.

Although BE condensates are many-body quantum system composed of a number of atoms ranging from few thousands to many millions, all the physical prediction about them are made by using an effective theory, the *Gross-Pitaevskii (GP) theory*, namely a one-particle approximation in which the energy of the system is given by a suitable nonlinear functional (see below). In spite of its simplicity the agreement with experimental observations is quite good, specially in the so called Thomas-Fermi regime, i.e., when the effective coupling becomes large. One of the major advantages of GP theory is the possibility of run very sophisticated and accurate numerical simulations [Dan, FJS, KTU]. See also the webpage `http://gpelab.math.cnrs.fr/`, where one can find an efficient free code for simulations of the GP energy or dynamics developed by X. Antoine and R. Duboscq [AD1, AD2].

In the framework of the GP theory the energy of a two-dimensional rotating BE condensate in physical units on the plane orthogonal to the rotational axis reads

$$\mathcal{E}^{\mathrm{GP}}_{\mathrm{phys}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\mathbf{r} \, \left\{ \tfrac{1}{2} \left| (\nabla - i\mathbf{A}_{\mathrm{rot}}) \Psi \right|^2 + \left(V(r) - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2 \right) |\Psi|^2 + \tfrac{|\Psi|^4}{\varepsilon^2} \right\}, \tag{1.1}$$

where Ω_{rot} is the angular velocity, $\mathbf{A}_{\mathrm{rot}} := \Omega_{\mathrm{rot}} r \mathbf{e}_\vartheta$, $r = |\mathbf{r}|$, with $\mathbf{r} = (x, y) \in \mathbb{R}^2$, and $\mathbf{e}_\vartheta = (-y, x)/|\mathbf{r}|$ is the unit vector in the transverse direction. The trapping potential is assumed to be of the form

$$V(r) := kr^s + \tfrac{1}{2}\Omega_{\mathrm{osc}}^2 r^2 \tag{1.2}$$

with $k > 0$ and

$$2 < s < \infty, \tag{1.3}$$

i.e., the harmonic trapping is corrected by some anharmonic perturbation. Finally, we will focus on the study of the Thomas-Fermi (TF) regime $\varepsilon \to 0$. The ground state energy of the system is thus obtained by minimizing the functional (1.1) under the normalization constraint $\|\Psi\|_2 = 1$, which amounts to require conservation of the particle number. Any minimizer is called condensate wave function and its modulus square, i.e., the associated probability distribution, is what can be observed experimentally.

The range of validity of the GP description as well the derivation of the GP effective theory from the quantum mechanical description of a condensed Bose gas is an interesting topic on its own, which has been completely solved in the non-rotating case [LSY1, LSSY]. In presence of rotation on the other hand [LS] contains a derivation of the GP functional, which is however restricted to bounded angular velocities and therefore not directly applicable to the case under discussion (see also [BCPY] for further results). However we will not investigate further such questions and take as a starting point the GP theory.

The mathematical physics literature contains now a large number of works studying the behavior of the GP minimization problem in different asymptotic regimes of the angular velocity. If we restrict the discussion to trapping potentials of the form (1.2), three *phase transitions* have been identified (see [CPRY3] for an extensive discussion or [CPRY4] for a more concise exposition), corresponding to three critical values of the rotational velocity. Here we briefly sum up the most relevant features of the physics of rotating condensates in anharmonic traps:

- for small angular velocities Ω_{rot}, the rotation has no effect on the condensate wave function, i.e., the minimizer of $\mathcal{E}^{\mathrm{GP}}_{\mathrm{phys}}$ coincides with the one in absence of rotation [AJR];
- when the first critical speed $\Omega_{\mathrm{c}_1} \propto \varepsilon^{\frac{4}{s+2}} |\log \varepsilon|$ is crossed, one observes the nucleation of quantum vortices, i.e., isolated zero of the condensate wave function [CR1];
- if Ω_{rot} stays far from a second critical speed $\Omega_{\mathrm{c}_2} \propto \varepsilon^{-\frac{s-2}{s+2}}$, the number of vortices might increase but the profile of the condensate wave function is still close to the non-rotating one. Close to Ω_{c_1} it is possible to derive the explicit distribution of vortices [CR1], which eventually cover the whole bulk of the condensate. In this regime one expects that they arrange in a regular (Abrikosov) lattice to minimize the interaction energy. This remains an open question although it has been proven that the vorticity is uniformly distributed [CPRY3];
- for $\Omega_{\mathrm{rot}} \propto \varepsilon^{-\frac{s-2}{s+2}}$ a first change of the macroscopic profile of the condensate is observed, due to the effect of the centrifugal forces. When the second critical speed Ω_{c_2} is crossed this change has a dramatic effect since a macroscopic hole is created at the center of the trap. However the vorticity remains uniform in the bulk of the system [CPRY3];
- for very rapid rotations above Ω_{c_2}, the bulk of the condensate becomes essentially annular and its width shrinks as $\varepsilon \to 0$. No further changes are however observed until a third critical speed $\Omega_{\mathrm{c}_3} \propto \varepsilon^{-\frac{4(s-2)}{s+2}}$ is crossed. Then vortices are expelled from the bulk and the condensate behaves as if the whole vorticity was concentrated at the origin of the trap [CPRY3, CPRY5]. This is the giant vortex state that we plan to study in this paper.

So far we have only discussed condensates in anharmonic traps of the type (1.2) but a lot of results are also available for other classes of trapping potentials. First of all the harmonic case has been extensively studied both in the physics and mathematical literature and, while there exists a first critical value of the angular velocity [IM1, IM2] corresponding to the occurrence of vortices and the behavior of the condensate for not too rapid rotation is similar to the one described above (vortex lattice, uniform distribution of vorticity, etc.), when the angular velocity approaches the harmonic frequency of the trap, some new physical features come into play and fractional quantum Hall states emerge [ABD, ABN, LSY2]. As we have already mentioned larger angular velocity are not allowed because the system would otherwise be no longer trapped. See however [Ka] for an alternative setting in which the trapping potential is suitably rescaled to reach fast rotation regimes.

Even if we restrict to the anharmonic traps (1.2) there is an extreme case which is of certain interest, namely $s = \infty$. Formally this corresponds to a confinement of the system to a two-dimensional disc of unit radius. Naturally one has then to provide suitable boundary conditions and both the Neumann [CDY1, CY, CRY] and Dirichlet [CPRY1] cases have been deeply studied.

Indeed phase transitions analogous to the one described above has been found out even in this extreme case, although the nature of the third one is much more subtle.

Let us now go back to the functional (1.1) and introduce more convenient parameters: if we set

$$\Omega_{\mathrm{phys}} := \sqrt{\Omega_{\mathrm{rot}}^2 - \Omega_{\mathrm{osc}}^2}, \tag{1.4}$$

and obviously assume that $\Omega_{\mathrm{osc}} < \Omega_{\mathrm{rot}}$, the trapping potential can be cast in the form

$$V(r) = kr^s - \tfrac{1}{2}\Omega_{\mathrm{phys}}^2 r^2. \tag{1.5}$$

Since we are interested in exploring a regime in which both $\varepsilon \to 0$ and $\Omega_{\mathrm{phys}} \to \infty$ (or, equivalently, $\Omega_{\mathrm{rot}} \to \infty$), it is convenient to rescale units in the GP functional, in order to observe a non-trivial behavior [CPRY3, Sect. I.A]: if one would naively minimize $\mathcal{E}_{\mathrm{phys}}^{\mathrm{GP}}$ under the mass constraint $\|\Psi\| = 1$, one would get trivially that the ground state energy diverges and the corresponding minimizer tends to 0 pointwise. The appropriate rescaling depends however on the asymptotics of Ω_{phys} and, in the regime we want to explore (very fast rotation),

$$\Omega_{\mathrm{phys}} \gg \varepsilon^{-\frac{s-2}{s+2}}, \tag{1.6}$$

which leads to the rescaling (see [CPRY3, Sect. I.A])

$$\mathbf{r} = R_{\mathrm{m}}\mathbf{x}, \qquad \Psi(\mathbf{r}) = R_{\mathrm{m}}^{-1}\psi(\mathbf{x}), \qquad \Omega_{\mathrm{phys}} = R_{\mathrm{m}}^{-2}\Omega, \qquad \mathbf{A}_{\Omega} = \Omega x \mathbf{e}_{\vartheta}, \tag{1.7}$$

where R_{m} stands for the unique minimum point of the potential (1.5), i.e., explicitly

$$R_{\mathrm{m}} := \left(\frac{\Omega_{\mathrm{phys}}^2}{sk} \right)^{\frac{1}{s-2}}. \tag{1.8}$$

Under the scaling (1.7), the GP functional (1.1) becomes

$$\mathcal{E}_{\mathrm{phys}}^{\mathrm{GP}}[\Psi] = R_{\mathrm{m}}^{-2} \left[\mathcal{E}^{\mathrm{GP}}[\psi] + \left(\tfrac{s}{2} - \tfrac{1}{2} \right) \Omega^2 \right], \tag{1.9}$$

with

$$\boxed{\mathcal{E}^{\mathrm{GP}}[\psi] := \int_{\mathbb{R}^2} \mathrm{d}\mathbf{x} \left\{ \tfrac{1}{2} \left| \left(\nabla - i\mathbf{A}_{\Omega} \right) \psi \right|^2 + \Omega^2 W(x) |\psi|^2 + \varepsilon^{-2} |\psi|^4 \right\}}. \tag{1.10}$$

The rescaled potential

$$W(x) := \tfrac{1}{s}\left(x^s - 1 \right) - \tfrac{1}{2}\left(x^2 - 1 \right) \tag{1.11}$$

is positive and has a unique minimum at $x = 1$, i.e., $\inf_{x\in\mathbb{R}^+} W(x) = W(1) = 0$. The rescaled angular velocity Ω is related to the original physical quantities via

$$\Omega = (sk)^{-\frac{2}{s-2}} \Omega_{\mathrm{phys}}^{\frac{s+2}{s-2}}, \tag{1.12}$$

and condition (1.6) becomes

$$\Omega \gg \varepsilon^{-1}. \tag{1.13}$$

From now on we will focus on the analysis of the minimization of the functional (1.10) on the domain

$$\mathscr{D}^{\mathrm{GP}} := \left\{ \psi \in H^1(\mathbb{R}^2) \,\middle|\, x^{s/2}\psi \in L^2(\mathbb{R}^2), \|\psi\|_2 = 1 \right\}. \tag{1.14}$$

We also set

$$E^{\mathrm{GP}} := \inf_{\psi \in \mathscr{D}^{\mathrm{GP}}} \mathcal{E}^{\mathrm{GP}}[\psi], \tag{1.15}$$

and denote by ψ^{GP} any minimizer, which is known to exist by standard arguments. In addition any ψ^{GP}, which might be non-unique due to a breaking of the rotational symmetry and the occurrence of isolated vortices, solves the variational equation

$$-\tfrac{1}{2}\left(\nabla - i\mathbf{A}_\Omega\right)^2 \psi^{\mathrm{GP}} + \Omega^2 W(x)\psi^{\mathrm{GP}} + 2\varepsilon^{-2}\left|\psi^{\mathrm{GP}}\right|^2 \psi^{\mathrm{GP}} = \mu^{\mathrm{GP}}\psi^{\mathrm{GP}}, \tag{1.16}$$

where the chemical potential (Lagrange multiplier) is fixed by imposing the L^2-normalization of ψ^{GP}:

$$\mu^{\mathrm{GP}} = E^{\mathrm{GP}} + \varepsilon^{-2}\int_{\mathbb{R}^2} \mathrm{d}\mathbf{x}\,\left|\psi^{\mathrm{GP}}\right|^4. \tag{1.17}$$

As discussed in details in [CPRY3, Sect. I.B], when $\Omega \gg \varepsilon^{-1}$ the condensate has already crossed the second critical speed, i.e., its profile approaches a density supported on an annulus centered in the origin, whose inner and outer radii tend to 1 as $\varepsilon \to 0$. More precisely $|\psi^{\mathrm{GP}}|^2$ is close in L^p, $p < \infty$, to the TF profile

$$\rho^{\mathrm{TF}}(x) = \tfrac{1}{2}\left[\mu^{\mathrm{TF}} - \varepsilon^2\Omega^2 W(x)\right]_+, \tag{1.18}$$

with μ^{TF} the chemical potential fixed by the L^1-normalization of the function. A straightforward analysis shows indeed that ρ^{TF} is compactly supported and $\mathrm{supp}(\rho^{\mathrm{TF}}) = [x_{\mathrm{in}}, x_{\mathrm{out}}]$ with [CPRY3, Eq. (2.7)]

$$x_{\mathrm{out}} - x_{\mathrm{in}} = C\,(\varepsilon\Omega)^{-2/3} \ll 1, \qquad x_{\mathrm{in/out}} = 1 + \mathcal{O}\big((\varepsilon\Omega)^{-2/3}\big), \tag{1.19}$$

as it can be proven by taking a Taylor expansion of W around $x = 1$ in (1.18) and imposing the L^1 normalization.

The vortex structure of ψ^{GP} is richer: being well above the first critical speed $\Omega_{\mathrm{c}_1} \sim |\log\varepsilon|$ for the nucleation of vortices, the GP minimizer contains a very large number of vortices distributed all over its support. More precisely one can prove that the vorticity is uniformly distributed in the bulk of the condensate. As in [CPRY3, Eq. (1.42)], we denote by $\mathcal{R}_{\mathrm{bulk}} \subset \mathrm{supp}(\rho^{\mathrm{TF}})$ a suitable annulus $\{\mathbf{x} \mid x_< \le x \le x_>\}$ with $x_{>/<} = x_{\mathrm{out/in}} + o((\varepsilon\Omega)^{-2/3})$.

Theorem 1.1 ([CPRY3, Theorem 1.2]).
If $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ as $\varepsilon \to 0$, there exists a finite family of disjoint balls $\{\mathcal{B}_i\} := \{\mathcal{B}(\mathbf{x}_i, \varrho_i)\} \subset \mathcal{R}_{\mathrm{bulk}}$, $i = 1, \ldots, N$, such that

1. $\varrho_i \le \mathcal{O}\left(\Omega^{-1/2}\right)$, $\sum \varrho_i^2 \le (1 + (\varepsilon\Omega)^{2/3})^{-1}$;
2. $\left|\psi^{\mathrm{GP}}\right| > 0$ *on* $\partial\mathcal{B}_i$, $i = 1, \ldots, N$.

Moreover, setting $d_i := \deg\{\psi^{\mathrm{GP}}, \partial\mathcal{B}_i\}$ and defining the vorticity measure as $\mu := 2\pi\sum_{i=1}^N d_i\delta(\mathbf{x} - \mathbf{x}_i)$, then, for any set $\mathcal{S} \subset \mathcal{R}_{\mathrm{bulk}}$ such that $|\partial\mathcal{S}| = 0$ and $|\mathcal{S}| \gg \Omega^{-1}|\log(\varepsilon^4\Omega)|^2$ as $\varepsilon \to 0$,

$$\frac{\mu(\mathcal{S})}{\Omega|\mathcal{S}|} \underset{\varepsilon\to 0}{\longrightarrow} 1. \tag{1.20}$$

The inner region $\left\{\mathbf{x} \in \mathbb{R}^2 \mid x \le x_{\mathrm{in}}\right\}$ is presumably also filled with vortices but, because of the exponential smallness of ψ^{GP} there, the vortex structure in that region is practically inaccessible. An important condition contained in Theorem 1.1 is the request

$$\Omega \ll \varepsilon^{-4}.$$

The reason is that at angular velocities of that order the proof of Theorem 1.1 might fail due to the occurrence of a further phase transition, i.e., the transition to a giant vortex state. This paper is precisely devoted to the investigation of such a transition.

From the heuristic point of view it is quite simple to explain why one should expect a change in the vortex structure when $\Omega \sim \varepsilon^{-4}$: from energy considerations it is easy to see that the average size of the vortex core, i.e., the radius of the region around a vortex when $|\psi^{\mathrm{GP}}|^2$ is substantially far from ρ^{TF}, is of order $\varepsilon^{2/3}\Omega^{-1/3}$. The width of $\mathrm{supp}(\rho^{\mathrm{TF}})$ is on the other hand of order $(\varepsilon\Omega)^{-2/3}$ and the two quantities are clearly of the same order when $\Omega \sim \varepsilon^{-4}$. Hence it must happen that for $\Omega = \Omega_0 \varepsilon^{-4}$ with Ω_0 a large enough constant, the vortex core becomes larger that the bulk of the condensate, i.e., vortices can no longer be accommodated in $\mathrm{supp}(\rho^{\mathrm{TF}})$. A non trivial phase factor of ψ^{GP} is however needed in order to compensate the effect of the rotation but, because no vortex can occur in the bulk of the condensate, all the vorticity should get concentrated in the inner region where ψ^{GP} is exponentially small. In fact when this occurs it is impossible to distinguish from the energetic point of view such a state with vortices distributed in the inner hole from a giant vortex state of the form $f(x)e^{in\vartheta}$, $n \in \mathbb{Z}$.

Notice that although this might seem to suggest that the rotational symmetry is restored, such a phenomenon never occurs as proven in [CPRY3, Theorem 1.6]. However the GP energy is expected to be well approximated above the critical speed for the transition to a giant vortex state by a one-dimensional energy functional obtained by evaluating $\mathcal{E}^{\mathrm{GP}}$ on functions of the form $f(x)e^{in\vartheta}$. In fact by some very simple observations one can show that $n = \lfloor \Omega \rfloor (1 + o(1))$, where $\lfloor \cdot \rfloor$ stands for the integer part. Let us now fix the angular velocity to be

$$\Omega = \frac{\Omega_0}{\varepsilon^4}, \tag{1.21}$$

with Ω_0 a positive constant. Concerning the giant vortex regime, the main results proven in [CPRY3] are stated below. We denote by $\mathcal{A}_{\mathrm{bulk}}$ a suitable annular layer around $x = 1$ containing the bulk of the condensate (see next (2.8) for a precise definition).

Theorem 1.2 ([CPRY3, Theorem 1.3]).
If Ω is given by (1.21), there exists a finite constant $\bar{\Omega}_0$ such that for any $\Omega_0 > \bar{\Omega}_0$, no minimizer ψ^{GP} has a zero inside $\mathcal{A}_{\mathrm{bulk}}$ if ε is sufficiently small.

Theorem 1.3 ([CPRY3, Theorem 1.4]).
If Ω is given by (1.21) with $\Omega_0 > \bar{\Omega}_0$ as in Theorem 1.2, then as $\varepsilon \to 0$[1]

$$E^{\mathrm{GP}} = \min_{\|f\|_2 = 1} \mathcal{E}^{\mathrm{GP}} \left[f(x) e^{i \lfloor \Omega \rfloor \vartheta} \right] + \mathcal{O}(|\log \varepsilon|^{9/2}). \tag{1.22}$$

The first result, although being a consequence of the energy asymptotics (1.22), is the most relevant one, since it shows the occurrence of the giant vortex transition for angular velocities of order ε^{-4}. The precise mathematical statement is a pointwise estimate in the bulk region of $|\psi^{\mathrm{GP}}|$ in terms of a strictly positive function, i.e., the minimizer of the functional appearing on the r.h.s. of (1.22): since the latter is bounded from below by a positive constant in the bulk and the difference is pointwise small in ε, also ψ^{GP} can not vanish there.

For the analysis of the present paper it is very important to remark that both results hold true if the angular velocity is expressed by (1.21) with Ω_0 *large enough*, namely no precise estimate is derived there on the sharp value for the transition (see also Remark 2.3). We indeed expect that the giant vortex structure appears as soon as Ω becomes (asymptotically) larger than

$$\Omega_{\mathrm{c}_3} = \frac{\Omega_{\mathrm{c}}}{\varepsilon^4}, \tag{1.23}$$

for some explicit value Ω_{c}. In this paper we will indeed investigate such a question and exhibit a finite value Ω_{c} which is a good candidate for the sharp constant. Actually we are going to see that such a constant is a solution of some algebraic equation (see (2.7)) involving quantities relative to

[1] We use here polar coordinates $(x, \vartheta) \in \mathbb{R}^+ \times [0, \pi)$ on the plane.

a limit problem independent of ε. Although we have not proven it yet, we do expect that next (2.7) has a unique solution, thus providing the sharp value of the critical velocity[2].

We outline here the structure of the paper. Next Section contains the main results, i.e., the identification of the explicit value of the angular velocity for the transition to the giant vortex state, together with an asymptotic expansion of the GP ground state energy which is actually on the main ingredients of the proof of the above mentioned result. We also show that, as in [CPRY3, Theorem 1.5], one can deduce a (better) estimate of the total winding number of any GP minimizer.

Sections 3 contains some preliminary estimates and a detailed analysis of the effective functionals that will play a significant role throughout the proofs. In Section 3.4 we prove the main properties of the cost function and in particular its positivity, which is the main mathematical tool used in the proof of the giant vortex transition as in several other works [CPRY1, CPRY2, CPRY3, CR1, CRY].

Sections 4 and 5 are devoted to the proofs of the main results: we first (Section 4) obtain the asymptotic expansion of the GP energy by comparing suitable upper and lower bounds and then (Section 5) use such a result to deduce the pointwise estimate of $|\psi^{\mathrm{GP}}|$ showing the absence of vortices in the bulk.

Notation: In the asymptotic analysis $\varepsilon \to 0$ we will often use the Landau symbols: given a positive function g, we say that

- $f = \mathcal{O}(g)$ (resp. $= o(g)$), if $\lim_{\varepsilon\to 0} |f|/g \leq C < \infty$ (resp. $= 0$);
- $f \propto g$, whenever $\lim_{\varepsilon\to 0} |f|/g = C$, with $0 < C < \infty$;
- if $f \geq 0$, $f \ll g$ is synonimous of $f = o(g)$ and $f \gg g$ simply means that $g \ll f$.

Sometimes we will use the notation $\mathcal{O}(|\log\varepsilon|^{\infty})$ to indicate a quantity of order $|\log\varepsilon|^a$ for some finite but possibly large a. Since such a quantity will typically appear multiplied by powers of ε, the explicit value of a will be irrelevant.
We denote by $\mathcal{B}_\varrho(\mathbf{x})$ any two-dimensional ball centered in $\mathbf{x}$ and with radius ϱ and by $\lfloor x \rfloor$ the integer part of the real number x. The symbol C will stand for a finite constant independent of ε, whose value might change from line to line.

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2 Main Results

The first non trivial observation to improve the results proven in [CPRY3] is that instead of making a special choice of the giant vortex winding number ($\lfloor\Omega\rfloor$ in [CPRY3]), one might try and optimize w.r.t. such a parameter, so obtaining a better candidate for the giant vortex state. This leads to consider the functional obtained evaluating $\mathcal{E}^{\mathrm{GP}}$ on a giant vortex ansatz $f(x)e^{in\vartheta}$ and minimize w.r.t. both f and n to find out the optimal giant vortex phase, i.e., explicitly

$$\mathcal{E}^{\mathrm{gv}}_{\beta}[g] = \int_{-\eta}^{\eta} \mathrm{d}y\, (1+\varepsilon^2 y) \left\{ \tfrac{1}{2} |\nabla g(y)|^2 + U_\beta(y) g^2(y) + \varepsilon^2 y^3 v(y) g^2(y) + \tfrac{1}{2\pi} g^4(y) \right\}, \tag{2.1}$$

where we have set for convenience $n = \Omega + \beta$ and exploited the exponential fall off of ψ^{GP} to cut the tails $|y| \geq \eta \propto |\log\varepsilon|$. The spatial coordinate has also been rescaled around $|\mathbf{x}| = 1$ by setting

[2] Strictly speaking in order to show that Ω_{c} is the sharp value for the transition one should also prove that, below Ω_{c}, vortices are still present in the bulk of the condensate, as done in [R2] for hard anharmonic traps. We will come back to this question later.

$x = 1 + \varepsilon^2 y$. The potentials U_β and v are obtained via a Taylor expansion of $W(x)$ around $x = 1$ and to the leading order in ε are simply given by a shifted quadratic potential (see (3.8) and (3.9) for their explicit expressions). We remark however that in U_β, the parameter β always appears multiplied (at least) by ε^2, so showing that the correction is only lower order.

Setting $E^{\mathrm{gv}}_\beta := \inf_{\|f\|=1} \mathcal{E}^{\mathrm{gv}}_\beta[f]$ and denoting by g_β the corresponding minimizer, which can be proven to exist and be unique (up to multiplication by a phase factor) (see Proposition 3.1), one can subsequently minimize w.r.t. $\beta \in \mathbb{R}$, obtaining the energy $E^{\mathrm{gv}}_\star$, an optimal phase $\beta_\star$ and a density $g_\star$, i.e.,

$$E^{\mathrm{gv}}_\star := \min_{\beta\in\mathbb{R}} E^{\mathrm{gv}}_\beta = E^{\mathrm{gv}}_{\beta_\star} = \mathcal{E}^{\mathrm{gv}}_{\beta_\star}[g_\star]. \tag{2.2}$$

In Subsection 3.3 we will prove that $\beta_\star = \mathcal{O}(1)$, so that, by the above argument, one expects the functional $\mathcal{E}^{\mathrm{gv}}_\beta$ to be close in the limit $\varepsilon \to 0$ to the following simplified giant vortex functional

$$\mathcal{E}^{\mathrm{gv}}[g] = \int_{\mathbb{R}} \mathrm{d}y \left\{ \tfrac{1}{2} (g')^2 + \tfrac{\alpha^2}{2} y^2 g^2 + \tfrac{1}{2\pi} g^4 \right\}, \tag{2.3}$$

with ground state energy E^{gv} and minimizer g_{gv}, i.e.,

$$E^{\mathrm{gv}} := \inf_{g\in\mathscr{D}^{\mathrm{gv}}} \mathcal{E}^{\mathrm{gv}}[g] = \mathcal{E}^{\mathrm{gv}}[g_{\mathrm{gv}}], \tag{2.4}$$

where

$$\mathscr{D}^{\mathrm{gv}} := \left\{ g \in H^1(\mathbb{R}) \;\middle|\; yg \in L^2(\mathbb{R}), \|g\|_{L^2(\mathbb{R})} = 1 \right\}.$$

Here we have denoted for short

$$\alpha := \Omega_0 \sqrt{s+2}. \tag{2.5}$$

The minimizer g_{gv} solves the variational equation

$$-\tfrac{1}{2} g'' + \tfrac{1}{2}\alpha^2 y^2 g + \tfrac{1}{\pi} g^3 = \mu^{\mathrm{gv}} g, \tag{2.6}$$

where $\mu^{\mathrm{gv}} = E^{\mathrm{gv}} + \frac{1}{2\pi} \|g_{\mathrm{gv}}\|_4^4$.

We are now in position to introduce the explicit value of the constant Ω_{c} appearing in the critical value of the angular velocity Ω_{c_3}, which can be expressed in terms of the critical quantities associated with the effective one-dimensional functional $\mathcal{E}^{\mathrm{gv}}$ and, specifically, g_{gv} and μ^{gv}: we denote by Ω_{c} the *largest* solution of the equation

$$\boxed{\Omega_0 = \frac{4}{s+2} \left[\mu^{\mathrm{gv}} - \frac{1}{2\pi} g^2_{\mathrm{gv}}(0) \right],} \tag{2.7}$$

where the r.h.s. depends on Ω_0 through μ^{gv} and g_{gv}. The existence of such a solution is proven in Proposition 3.11. Note that thanks to the estimate $\|g_{\mathrm{gv}}\|^2_\infty \leq \pi\mu^{\mathrm{gv}}$ (see (3.18)), $\Omega_{\mathrm{c}} > 0$.

Before stating the main result of this paper, we have to define more precisely the region we identify with the bulk of the condensate: we set for any $a > 0$

$$\mathcal{A}_{\mathrm{bulk}} := \left\{ \mathbf{x} \in \mathbb{R}^2 \mid g_{\mathrm{gv}}\left(\tfrac{x-1}{\varepsilon^2}\right) \geq |\log\varepsilon|^{-a} \right\}, \tag{2.8}$$

and observe that by the exponential decay proven in Proposition 3.3, $\left\|\psi^{\mathrm{GP}}\right\|_{L^2(\mathcal{A}_{\mathrm{bulk}})} = 1 + o(1)$, i.e., it certainly contains the bulk of the system.

Theorem 2.1 (Absence of vortices in $\mathcal{A}_{\mathrm{bulk}}$)**.**
If $\Omega = \Omega_0/\varepsilon^4$ with $\Omega_0 > \Omega_{\mathrm{c}}$ as $\varepsilon \to 0$, then no GP minimizer ψ^{GP} contains vortices in $\mathcal{A}_{\mathrm{bulk}}$. More preciselyfor any $\mathbf{x} \in \mathcal{A}_{\mathrm{bulk}}$

$$\boxed{\left|\psi^{\mathrm{GP}}(\mathbf{x})\right| = \frac{1}{\sqrt{2\pi\varepsilon}}\, g_{\mathrm{gv}}\left(\tfrac{x-1}{\varepsilon^2}\right) \Big(1 + \mathcal{O}(\varepsilon^{1/2}|\log\varepsilon|^\infty)\Big).} \tag{2.9}$$

Remark 2.1 (Giant vortex structure).
The pointwise estimate (2.9) *suggests that* $|\psi^{\mathrm{GP}}|$ *is approximately radial within* $\mathcal{A}_{\mathrm{bulk}}$. *As already mentioned, this does not imply that the rotational symmetry is restored, since one expects that* $|\psi^{\mathrm{GP}}|$ *is far from being radial in the inner region* $x \leq x_{\mathrm{in}}$, *where several vortices should presumably be distributed more or less uniformly. In any case no GP minimizer is invariant under rotations if* Ω *is large enough [CPRY3, Theorem 1.6], e.g., in the giant vortex regime.*

Remark 2.2 (Third critical velocity).
In Proposition 3.11 we will prove that the equation (2.7) *has a solution. Although we do not prove it, we strongly believe that such a solution is in fact unique and identifies the sharp constant in the value of the third critical speed.*
More precisely Theorem 2.1 indicates that above $\Omega_{\mathrm{c}}/\varepsilon^4$ *the system undergoes the phase transition to the giant vortex state and the bulk of the condensate becomes vortex free. Hence*

$$\Omega_{\mathrm{c}_3} \leq \frac{\Omega_{\mathrm{c}}}{\varepsilon^4}. \tag{2.10}$$

We actually expect that $\Omega_{\mathrm{c}_3} = \Omega_{\mathrm{c}}/\varepsilon^4$, *which obviously requires to prove that the solution to* (2.7) *is unique. In addition one should also prove that for slower rotations vortices are still present in the bulk of the system. We plan to attack such a problem in a future work, but here we want to stress that the negativity of the cost function (see next Section 2.1) for* $\Omega_0 < \Omega_{\mathrm{c}}$ *is a very strong indication that vortices are indeed convenient in this case and thus the sharp value of the critical speed is precisely* $\Omega_{\mathrm{c}}/\varepsilon^4$.

Remark 2.3 (Comparison with [CPRY3]).
We want here to discuss in more details the comparison between Theorem 2.1 and the analogous result proven in [CPRY3, Theorem 1.3]: in principle, one could indeed derive an estimate of the threshold $\bar{\Omega}_0$ *for the transition to the giant vortex state there and then it would be natural to compare it with the explicit value found here. However we provide here some heuristic arguments showing that such a comparison is actually not needed (see however next Remark 3.1 for futher details).*
First of all an explicit estimate of $\bar{\Omega}_0$ *is not an easy task to achieve, due to the proof structure in [CPRY3]: the result proven there is indeed obtained through an asyptotic analysis as* $\Omega_0 \to \infty$ *and one should then estimate all the coefficients of the error terms appearing in the formulae. Such quantities ultimately depends on the pointwise estimate of the difference between the giant vortex profile and the ground state of the harmonic oscillator given in [CPRY3, Proposition 3.5], which is not explicit at all.*
However, even assuming that one could obtain a sharp value $\bar{\Omega}_0$, *there are strong reasons to believe that, unlike* Ω_{c} *(see also the previous Remark 2.2), it can not be the coefficient of the critical speed. First of all the condition* $\Omega_0 > \bar{\Omega}_0$ *guarantees the positivity of the vortex energy cost in [CPRY3] (Remark 2.2) and therefore* $\bar{\Omega}_0 > \Omega_{\mathrm{c}}$. *Moreover, as explained in [CPRY3] (see also [CPRY2]), when* $\Omega_0 \to \infty$, *another transition takes place, i.e., the condensate density profile goes from a TF-like shape* (1.18) *to a gaussian function minimizing some suitable harmonic energy. The key fact is that such a transition is expected to take place* <u>*after*</u> *the giant vortex one. Indeed here we show that, for finite* Ω_0, *when the profile change has* <u>*not yet*</u> *occurred, the condensate is already in a giant vortex state. On the opposite, a quick inspection to the proof in [CPRY3] reveals that the transition to the giant vortex is proven there by imposing that the profile is already gaussian. Hence any so obtained threshold value can not be meaningful.*

Remark 2.4 (Giant vortex density).
We have formulated the pointwise estimate (2.9) *with* g_{gv}, *but an analogous statement holds true with* g_{gv} *replaced with* $g_\star$. *The error in* (2.9) *is indeed so large that one can not appreciate the difference between the two reference profiles (see Proposition 3.9). Let us stress however that the use of* $g_\star$ *as a reference profile in the proof is on the opposite crucial to obtain the result (compare, e.g., the asymptotics* (2.11) *and* (2.12)).

The absence of vortices proven in Theorem 2.1 and the pointwise estimate of ψ^{GP} follows from a refined result about the energy asymptotics in the same regime, that we state in the following

Theorem 2.2 (Energy asymptotics)**.**
If $\Omega = \Omega_0 \varepsilon^{-4}$ with $\Omega_0 > \Omega_{\mathrm{c}}$ as $\varepsilon \to 0$, then

$$\boxed{E^{\mathrm{GP}} = \frac{E^{\mathrm{gv}}_{\star}}{\varepsilon^4} + \mathcal{O}(1).} \tag{2.11}$$

Remark 2.5 (Energy expansion)**.**
The leading term $E^{\mathrm{gv}}_{\star}/\varepsilon^4$ contains the main energy contribution due to the inhomogeneity of the GP profile together with the subleading kinetic energy of $|\psi^{\mathrm{GP}}|$. The absence of vortices in $\mathcal{A}_{\mathrm{bulk}}$ can be read in the very small remainder term $\mathcal{O}(1)$. It is indeed interesting to compare (2.11) *with the analogous result [CPRY3, Theorem 1.4], where the error term is much larger, i.e., $\mathcal{O}(|\log\varepsilon|^{9/2})$, in addition to the fact that the result proven there holds true only for Ω_0 large enough.*

Notice however that the coefficient of the leading term $E^{\mathrm{gv}}_{\star}$ still depends on ε, through the boundaries of the integration domain as well as the optimal phase $\beta_\star$ and the potential $U_{\beta_\star}$. If one wanted to extract a proper asymptotic expansion then the natural statement would be

$$E^{\mathrm{GP}} = \frac{E^{\mathrm{gv}}}{\varepsilon^4} + \mathcal{O}(|\log\varepsilon|^7), \tag{2.12}$$

with a much worse error term.

Thanks to the pointwise statement (2.9), one can deduce that ψ^{GP} does not vanish on $\mathcal{A}_{\mathrm{bulk}}$. In particular for any $R = 1 + \mathcal{O}(\varepsilon^2)$, $|\psi^{\mathrm{GP}}| > 0$ on $\partial\mathcal{B}_R$. Hence it is possible to define the winding number of ψ^{GP} on $\partial\mathcal{B}_R$ for any such R. A consequence of the energy asymptotics and the estimate (2.9) is thus the following

Theorem 2.3 (Winding number)**.**
Let $\Omega = \Omega_0\varepsilon^{-4}$ with $\Omega_0 > \Omega_{\mathrm{c}}$ and R be any radius such that $R = 1 + \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$, then

$$\boxed{\deg\left(\psi^{\mathrm{GP}}, \partial\mathcal{B}_R\right) = \frac{\Omega_0}{\varepsilon^4} + \mathcal{O}(1).} \tag{2.13}$$

Note that the combination of the above result with the proof of the rotational symmetry breaking given in [CPRY3, Theorem 1.6] implies the presence of vortices in the inner hole region where ψ^{GP} is exponentially small.

2.1 Heuristics

Before discussing the proofs of the main results, we briefly expose the proof strategy from a heuristic point of view, i.e., not tracking down the error terms and neglecting most technical points. As usual the main result about the behavior of the condensate wave function is deduced from the energy asymptotics (2.11). We thus focus on such a proof.

Most of the relevant features of a fast rotating Bose-Einstein condensate were already discussed in details in [CPRY3] and recalled in the Introduction. Here we take as a starting point the effective functional (2.1) which is expected to provide the leading order term in the energy asymptotics in units ε^{-4}. Note that the ground state energy of $\mathcal{E}^{\mathrm{gv}}_{\beta}$ always provides an upper bound to E^{GP} for any *integer* phase, i.e., whenever $\Omega + \beta \in \mathbb{Z}$. Actually the same upper bound can be proven to hold true up to some small error term even if $\Omega + \beta$ is not an integer (see Section 4.1). Hence we can neglect the upper bound part of the proof and discuss only the lower estimate to E^{GP}.

A preliminary step which is already described in details in [CPRY3] is the restriction of the integration in $\mathcal{E}^{\mathrm{GP}}$ to the bulk of the condensate, i.e., to an annulus centered in the origin with radius $\simeq 1$ and width $\mathcal{O}(\varepsilon^2)$. This can be done by exploiting the exponential decay of ψ^{GP} outside. From now on we will then assume that the integration in $\mathbf{x}$ is restricted to the annulus $|1-x| \leq \mathcal{O}(\varepsilon^2 |\log \varepsilon|)$.

The main steps in the energy lower bound are then the following:

1. optimal giant vortex phase and profile: we minimize E^{gv}_{β} w.r.t. to $\beta \in \mathbb{R}$ and obtain a minimizing $\beta_\star$ and an associated density $g_\star$. It is crucial to observe that such a minimization yields an additional equation involving $g_\star$, which is in fact nothing but the vanishing of the first derivative of E^{gv}_{β} w.r.t. β. Such an equation will play a crucial role at point 4 below;

2. splitting of the energy: using a technique introduced in [LM], which is now rather standard, we decouple $\psi^{\mathrm{GP}} = \frac{1}{\sqrt{2\pi\varepsilon}} g_\star \left(\frac{x-1}{\varepsilon^2}\right) u(\mathbf{x})$ and, exploiting the variational equation satisfied by $g_\star$, we obtain
$$E^{\mathrm{GP}} = \frac{E^{\mathrm{gv}}_\star}{\varepsilon^4} + \frac{\mathcal{E}[u]}{2\pi\varepsilon^2}, \tag{2.14}$$
with u essentially minimizing the reduced energy functional
$$\mathcal{E}[u] = \int \mathrm{d}\mathbf{x}\, g_\star^2 \left\{ \tfrac{1}{2} |\nabla u|^2 + \mathbf{a}(\mathbf{x}) \cdot \mathbf{j}_u(\mathbf{x}) + \tfrac{1}{2\pi\varepsilon^4} g_\star^2 (1-|u|^2)^2 \right\}, \tag{2.15}$$
where the "magnetic potential" $\mathbf{a}$ depends on Ω and $\beta_\star$ and $\mathbf{j}_u$ is the *superconducting current*
$$\mathbf{j}_u(\mathbf{x}) = \tfrac{i}{2} \left(u \nabla u^* - u^* \nabla u \right). \tag{2.16}$$
Completing the lower bound means to show that $\mathcal{E}[u]$ is positive;

3. hydrodynamic estimate: we note that the "magnetic potential" is divergence free and therefore it exists a *potential function* $F(x)$ such that $2 g_\star^2(x) \mathbf{a}(\mathbf{x}) = -\nabla^\perp F(x)$. This trick was first used in [CRY] in the context of the GP theory for rotating condensates. For later applications to the GL function see also [CR2, CR3]. We can thus integrate by parts the second term in (2.15) obtaining
$$\int \mathrm{d}\mathbf{x}\, F(x) \, \mathrm{curl}\,(\mathbf{j}_u). \tag{2.17}$$
At this stage we observe that since $\beta_\star = \mathcal{O}(1)$ and it appears in (2.1) always multiplied by ε^2, a good approximation of the functional $\mathcal{E}^{\mathrm{gv}}_{\beta_\star}$ can be obtained by taking the limit $\varepsilon \to 0$, which yields the functional (2.3), with ground state energy E^{gv} and minimizer g_{gv}. We can also replace $F(x)$ with its limiting counterpart $F^{\mathrm{gv}}(x)$, which is in fact a negative function. The last step to estimate (2.17) is to use the trivial inequality $|\mathrm{curl}\,(\mathbf{j}_u)| \leq |\nabla u|^2$ and the negativity of F^{gv} to get the lower bound
$$\mathcal{E}[u] \geq \int \mathrm{d}\mathbf{x}\, \left(\tfrac{1}{2} g_{\mathrm{gv}}^2 + F^{\mathrm{gv}} \right) |\nabla u|^2, \tag{2.18}$$
where we have also dropped the last positive term in (2.15);

4. positivity of the cost function: the above lower bound suggests that any topological defect of u should carry an energy cost given by the *cost function*
$$K^{\mathrm{gv}} = \tfrac{1}{2} g_{\mathrm{gv}}^2 + F^{\mathrm{gv}}. \tag{2.19}$$
Positivity of such a function in the bulk would then imply that vortices are not energetically favorable anywhere in the condensate. This is turn can be proven by direct inspection of the function itself. First we observe that both g_{gv} and F^{gv} are radial functions and we therefore change coordinates $x = 1 + \varepsilon^2 y$, so that in the new variable y the bulk of the condensate is

basically the whole real line. In the new variable the explicit expression of F^{gv} (that we still denote by F^{gv}) is

$$F^{\mathrm{gv}}(y) = -2\Omega_0 \int_y^{\infty} \mathrm{d}t \, t \, g_{\mathrm{gv}}^2(t). \tag{2.20}$$

Notice that by symmetry[3] of g_{gv}, $F^{\mathrm{gv}}(-\infty) = 0$ and $F^{\mathrm{gv}}(y) \leq 0$ for any $y \in \mathbb{R}$. The cost function K^{gv} is therefore smooth and $K^{\mathrm{gv}}(\pm\infty) = 0$, so that, if it becomes negative, it must have a minimum. The derivative of K^{gv} can be easily computed

$$K^{\mathrm{gv}\prime}(y) = g_{\mathrm{gv}}(y) g_{\mathrm{gv}}^{\prime}(y) + 2\Omega_0 y g_{\mathrm{gv}}^2(y), \tag{2.21}$$

so that, by strict positivity of g_{gv}, at any critical point y_0 for K^{gv}, one has

$$g_{\mathrm{gv}}^{\prime}(y_0) = -2\Omega_0 y_0 g_{\mathrm{gv}}(y_0). \tag{2.22}$$

Now using the variational equation for g_{gv} and manipulating the expression (2.20) of the potential function, it is possible to show that the cost function can be equivalently rewritten as

$$K^{\mathrm{gv}} = \left[\frac{1}{2} + \Omega_0 y^2 + \frac{\Omega_0}{\pi\alpha^2} g_{\mathrm{gv}}^2 - \frac{2\Omega_0 \mu^{\mathrm{gv}}}{\alpha^2} \right] g_{\mathrm{gv}}^2 - \frac{\Omega_0}{\alpha^2} {g_{\mathrm{gv}}^{\prime}}^2 \tag{2.23}$$

and, inserting the condition (2.22) satisfied at any minimum point y_0 of K^{gv}, we get

$$K^{\mathrm{gv}}(y_0) = \left[\frac{1}{2} + \frac{\Omega_0(s+1)}{s+2} y_0^2 + \frac{\Omega_0}{\pi\alpha^2} g_{\mathrm{gv}}^2(y_0) - \frac{2\Omega_0 \mu^{\mathrm{gv}}}{\alpha^2} \right] g_{\mathrm{gv}}^2(y_0). \tag{2.24}$$

Using the parity of g_{gv} as well as the variational equation, one can prove that the quantity between brackets on the r.h.s. of the expression above is positive if and only if it is positive at the origin (see Proposition 3.10)

$$\frac{1}{2} + \frac{\Omega_0}{\pi\alpha^2} g_{\mathrm{gv}}^2(0) - \frac{2\Omega_0 \mu^{\mathrm{gv}}}{\alpha^2} = \frac{1}{2} + \frac{2}{\Omega_0(s+2)} \left[\frac{1}{2\pi} \left\| g_{\mathrm{gv}} \right\|_{\infty}^2 - \mu^{\mathrm{gv}} \right] \geq 0 \quad \Longleftarrow \quad \Omega_0 \geq \Omega_{\mathrm{c}}. \tag{2.25}$$

Once the energy asymptotics is proven, the pointwise estimate of $|\psi^{\mathrm{GP}}|$, which allows to exclude the presence of vortices in the bulk for $\Omega_0 > \Omega_{\mathrm{c}}$, is a simple consequence: putting back the positive term we have dropped in the lower bound, one first obtains an estimate of the region where $|u|$ can differ from 1. Then combining this with an L^{∞} estimate of the gradient of u, one gets the result.

It is worth mentioning at this stage a technical difference with previous approaches. Indeed in [CPRY3] two potential functions were actually used instead of one, in order to get rid of boundary terms coming from the integration by parts described at step 3 (see the discussion in [CPRY3, Sect. C]). Here on the opposite we are able to use only one potential function by estimating in a more refined way the boundary terms (compare, e.g., with next (4.16)). As in [CPRY3] we also exploit the symmetry properties of the profile $g_{\star}$, which is to a very good approximation invariant under reflections w.r.t. the origin.

3 Preliminary Estimates

Here we collect some useful technical results as well as the main properties of the effective functionals involved in the analysis. An important piece of information is contained in Section 3.4 where we prove the positivity of the cost function.

[3] Unlike g_{gv}, the profile $g_{\star}$ is not exactly symmetric, but F satisfies analogous properties thanks to the optimality condition of $\beta_{\star}$, i.e., the additional equation involving $g_{\star}$ and $\beta_{\star}$ which was mentioned at point 1.

3.1 Giant Vortex Functionals

We start by describing the derivation of the functional (2.1) from the GP energy. As anticipated in Section 2 the idea is to evaluate the energy of a trial state of the form $f(x)e^{in\vartheta}$ in polar coordinates $\mathbf{x} = (x, \vartheta)$ and with $n = \Omega + \beta$. In addition we assume that f is real as it will be for any giant vortex profile. The result of a rather simple computation is

$$\mathcal{E}^{\mathrm{GP}}[fe^{in\vartheta}] = 2\pi \int_0^\infty \mathrm{d}x\, x \left\{ \tfrac{1}{2} |\nabla f|^2 + \Omega^2 \left[\tfrac{1}{2} \left(x^2 - \tfrac{\Omega+\beta}{\Omega} \right)^2 \tfrac{1}{x^2} + W(x) \right] f^2 + \tfrac{1}{\varepsilon^2} f^4 \right\}. \tag{3.1}$$

Exploiting the exponential smallness of ψ^{GP} outside of the bulk of the condensate proven in [CPRY3] and recalled in next Proposition 3.3, we can restrict the integration domain to the annulus

$$\mathcal{A}_\eta := \left\{ \mathbf{x} \in \mathbb{R}^2 :\ |1 - \mathbf{x}| \leq \varepsilon^2 \eta \right\}, \qquad \eta := \tfrac{\eta_0}{2\sqrt{\Omega_0}} |\log \varepsilon|, \tag{3.2}$$

where $\eta_0 > 0$ is an arbitrary finite constant and the prefactor in the definition of η has been chosen of that form for further convenience. Thanks to (3.19)

$$\psi^{\mathrm{GP}}(\mathbf{x}) = \mathcal{O}(\varepsilon^\infty), \qquad \text{for any } \mathbf{x} \notin \mathcal{A}_\eta, \tag{3.3}$$

and the restriction is thus well motivated. In addition we will also see that a similar estimate holds true for any giant vortex profile. In terms of the one-dimensional functional (3.1) we are then integrating in the interval $[1 - \varepsilon^2\eta, 1 + \varepsilon^2\eta]$ and a change of variable is called for: setting

$$x = 1 + \varepsilon^2 y, \qquad g(y) = \sqrt{2\pi}\varepsilon\, f(1 + \varepsilon^2 y) \tag{3.4}$$

so that g is normalized in[4] $L^2_\eta := L^2([-\eta, \eta], (1 + \varepsilon^2 y)\mathrm{d}y)$, we obtain the energy

$$\tilde{\mathcal{E}}^{\mathrm{GP}}[g] = \frac{1}{\varepsilon^4} \int_{-\eta}^{\eta} \mathrm{d}y\, (1 + \varepsilon^2 y) \left\{ \tfrac{1}{2} (g')^2 + \right.$$
$$\left. + \varepsilon^4 \Omega^2 \left[\tfrac{1}{2} \left(1 + \varepsilon^2 y - \tfrac{\Omega+\beta}{\Omega(1+\varepsilon^2 y)} \right)^2 + W(1 + \varepsilon^2 y) \right] g^2 + \tfrac{1}{2\pi} g^4 \right\}. \tag{3.5}$$

Now we expand $W(1 + \varepsilon^2 y)$ in Taylor series around $y = 0$ to get

$$W(1 + \varepsilon^2 y) = \tfrac{s-2}{2} \varepsilon^4 y^2 + \tfrac{(s-2)(s-1)}{6} \varepsilon^6 y^3 + \varepsilon^8 \varphi(y) \tag{3.6}$$

where $\varphi(y) = \mathcal{O}(y^4)$. Using this fact we can rewrite the potential in (3.5) as (recall that $\alpha^2 = \Omega_0^2(s + 2)$)

$$\varepsilon^4 \Omega^2 \left[\tfrac{\left(2\varepsilon^2 y - \varepsilon^4 \beta/\Omega_0 + \varepsilon^4 y^2 \right)^2}{2(1+\varepsilon^2 y)^2} + W(1 + \varepsilon^2 y) \right] = U_\beta(y) + \varepsilon^2 y^3 v(y), \tag{3.7}$$

with v independent of β and of lower order w.r.t. to U_β. Explicitly

$$U_\beta(y) := \frac{1}{(1 + \varepsilon^2 y)^2} \left(\frac{\alpha^2}{2} y^2 - 2\Omega_0 \varepsilon^2 \beta y - \Omega_0 \varepsilon^4 \beta y^2 + \frac{1}{2} \varepsilon^4 \beta^2 \right), \tag{3.8}$$

$$v(y) := \frac{\Omega_0^2 (s + (s-1)\varepsilon^2 y)}{(1 + \varepsilon^2 y)^2} + \frac{(s-1)(s-2)\Omega_0^2}{6} + \frac{\varepsilon^2 \Omega_0^2}{y^3} \varphi(y). \tag{3.9}$$

Some trivial estimate using the Taylor expansion (3.6) implies that for $y \in [-\eta, \eta]$

$$U_\beta(y) = \tfrac{1}{2} \alpha^2 y^2 + \mathcal{O}\left(\varepsilon^2 (1 + |\beta|)\eta + \varepsilon^4 \beta^2 \right), \tag{3.10}$$

[4] We set in fact $L^p_\eta := L^p([-\eta, \eta], (1 + \varepsilon^2 y)\mathrm{d}y)$ for any $1 \leq p \leq \infty$.

which shows that, if, e.g., β is uniformly bounded in ε, the potential $U_\beta(y)$ is harmonic up to corrections of higher order in ε. Alternatively one can think of U_β as a shifted harmonic oscillator by writing

$$U_\beta(y) = \tfrac{1}{2}\alpha^2 \left(y - \tfrac{2\Omega_0 \varepsilon^2 \beta}{\alpha^2} \right)^2 + \mathcal{O}(\varepsilon^4 |\beta| \eta^2 + \varepsilon^4 \beta^2). \tag{3.11}$$

In fact, since the optimal value of β we are going to choose is $\mathcal{O}(1)$, both representations are equivalent since the shift will be $\mathcal{O}(\varepsilon^2)$. Concerning the rest $v(y)$ one trivially has the upper bound

$$|v(y)| \leq C_{\Omega_0} + \mathcal{O}(\varepsilon^2 \eta), \tag{3.12}$$

for $y \in [-\eta, \eta]$ and with a finite constant C_{Ω_0}. The rest in the above expression is a consequence of the bound $|\varphi(y)| \leq C|y|^4$, which follows from the Taylor expansion (3.6).

In conclusion we have recovered the expression (2.1), i.e.,

$$\mathcal{E}^{\mathrm{gv}}_\beta[g] = \int_{-\eta}^{\eta} \mathrm{d}y \, (1 + \varepsilon^2 y) \left\{ \tfrac{1}{2} (g')^2 + U_\beta(y) g^2 + \varepsilon^2 y^3 v(y) g^2 + \tfrac{1}{2\pi} g^4 \right\}.$$

We now discuss the minimization of such a function w.r.t. g and for that purpose we have to identify the proper minimization domain, i.e.,

$$\mathscr{D}^{\mathrm{gv}}_\beta := \left\{ g \in H^1(-\eta, \eta) \, \middle| \, g = g^*, \ \|g\|_{L^2_\eta} = 1 \right\}. \tag{3.13}$$

The ground state energy of $\mathcal{E}^{\mathrm{gv}}_\beta$ is defined as

$$E^{\mathrm{gv}}_\beta = \inf_{g \in \mathscr{D}^{\mathrm{gv}}_\beta} \mathcal{E}^{\mathrm{gv}}_\beta[g]. \tag{3.14}$$

Notice that the assumption $g = g^*$, i.e., reality of the argument, does not imply any loss of generality because the ground state can always be chosen real (see next Proposition).

Proposition 3.1 (Minimization of $\mathcal{E}^{\mathrm{gv}}_\beta$)**.**
There exists a minimizer $g_\beta \in \mathscr{D}^{\mathrm{gv}}_\beta$ of (2.1) *that is unique up to a sign, radial and can be chosen strictly positive. In addition $g_\beta \in C^\infty(-\eta, \eta)$ and it solves the variational equation*

$$-\tfrac{1}{2} g''_\beta - \tfrac{\varepsilon^2}{2(1+\varepsilon^2 y)} g'_\beta + U_\beta(y) g_\beta + \varepsilon^2 y^3 v(y) g_\beta + \tfrac{1}{\pi} g^3_\beta = \mu_\beta g_\beta \tag{3.15}$$

with Neumann boundary conditions $g'_\beta(\pm\eta) = 0$ and $\mu_\beta = E^{\mathrm{gv}}_\beta + \frac{1}{2\pi} \|g_\beta\|^4_{L^4_\eta}$.
Finally g_β has a unique maximum point at y_β and it decreases monotonically anywhere else.

Proof. Existence and uniqueness of the minimizer follow from strict convexity of the functional $\mathcal{E}^{\mathrm{gv}}_\beta[\sqrt{\rho}]$ with respect to the density $\rho = g^2$. The variational equation (3.15) is satisfied at least in weak sense. Then one deduces the strict positivity of g_β noticing that it is actually a ground state of a suitable one-dimensional Schrödinger operator. The equality for μ_β follows integrating the (3.15) and recalling the fact that g_β has L^2-norm equal to one. Finally a trivial bootstrap argument allows to deduce smoothness of g_β and therefore that (3.15) is solved in a classical sense.

The only non-trivial result is the one about the existence of the a single maximum point for g_β. However it follows from the property of the potential $U_\beta(y) + \varepsilon^2 y^3 v(y)$: going back to the expression of the potential in (3.1), one can easily compute, with $x = 1 + \varepsilon^2 y$,

$$\frac{\partial \left[U_\beta(y) + \varepsilon^2 y^3 v(y) \right]}{\partial x} = \frac{1}{x^3} \left[x^{s+2} - \left(1 + \tfrac{\varepsilon^4 \beta}{\Omega_0} \right)^2 \right],$$

which vanishes at a single point y_{pot}, i.e., where

$$1 + \varepsilon^2 y_{\mathrm{pot}} = \left(1 + \tfrac{\varepsilon^4 \beta}{\Omega_0} \right)^{\frac{2}{s+2}} = 1 + \tfrac{2\beta\varepsilon^4}{(s+2)\Omega_0} + \mathcal{O}(\varepsilon^8 \beta^2).$$

The Taylor expansion also shows that

$$y_{\mathrm{pot}} = \tfrac{2\beta\varepsilon^2}{(s+2)\Omega_0}(1 + \mathcal{O}(\varepsilon^4\beta)).$$

The monotonicity property of g_β can then be obtained by a simple rearrangement argument (see, e.g., [CPRY1, Proposition 2.2]): since the potential has a single maximum point, if g_β had more than one maximum besides y_β, one could move mass from the further maximum to the minimum in between and lower the energy. Since g_β is a minimizer one gets a contradiction. □

Another effective one-dimensional functional which is going to play an important role in the analysis is (2.3), i.e., the formal limit $\varepsilon \to 0$ of $\mathcal{E}^{\mathrm{gv}}_\beta$, assuming that $\beta = o(\varepsilon^{-2})$:

$$\mathcal{E}^{\mathrm{gv}}[g] = \int_{\mathbb{R}} \mathrm{d}y \left\{ \tfrac{1}{2}(g')^2 + \tfrac{\alpha^2}{2} y^2 g^2 + \tfrac{1}{2\pi} g^4 \right\}.$$

The minimization domain is in this case given by

$$\mathscr{D}^{\mathrm{gv}} := \left\{ g \in H^1(\mathbb{R}) \mid g = g^*, \|g\|_{L^2(\mathbb{R})} = 1 \right\}, \tag{3.16}$$

and the ground state energy will be denoted by $E^{\mathrm{gv}} = \inf_{g\in\mathscr{D}^{\mathrm{gv}}} \mathcal{E}^{\mathrm{gv}}[g]$.

Proposition 3.2 (Minimization of $\mathcal{E}^{\mathrm{gv}}$)**.**
There exists a minimizer $g_{\mathrm{gv}} \in \mathscr{D}^{\mathrm{gv}}$ of (2.3) that is unique up to a sign, radial and can be choose strictly positive. In addition $g_{\mathrm{gv}} \in C^\infty(\mathbb{R})$ and it solves the variational equation

$$-\tfrac{1}{2} g''_{\mathrm{gv}} + \tfrac{\alpha^2}{2} y^2 g_{\mathrm{gv}} + \tfrac{1}{\pi} g^3_{\mathrm{gv}} = \mu^{\mathrm{gv}} g_{\mathrm{gv}} \tag{3.17}$$

with $\mu^{\mathrm{gv}} = E^{\mathrm{gv}} + \frac{1}{2\pi}\|g_{\mathrm{gv}}\|_4^4$.
Finally g_{gv} is even w.r.t. the origin and has only one maximum at $y = 0$, which fulfills the inequality

$$g^2_{\mathrm{gv}}(0) = \|g_{\mathrm{gv}}\|^2_\infty \leq \pi\mu^{\mathrm{gv}}. \tag{3.18}$$

Proof. See the proof of Proposition 3.1. Parity of g_{gv} is a trivial consequence of the parity of the potential. The inequality (3.18) follows from direct inspection of the variational equation (3.17): at any maximum point $g''_{\mathrm{gv}} \leq 0$, which immediately implies the result. □

3.2 Estimates of the Gross-Pitaevskii and Giant Vortex Profiles

In this Section we collect several technical estimates of the profiles involved in the discussion. Such estimates will play a key role in the proofs but can be typically obtained by standard techniques in functional analysis.

We start by recalling a result which was in fact proven in [CPRY3, Propositions 3.1 and 3.2]: let η_0 be the parameter appearing in the definition (3.2) of $\mathcal{A}_\eta$, then

Proposition 3.3 (Exponential decay of ψ^{GP})**.**
If $\Omega = \Omega_0/\varepsilon^4$, there exists two finite constants $c, C > 0$ (independent of η_0) such that, for any $\mathbf{x} \notin \mathcal{A}_\eta$,

$$\left|\psi^{\mathrm{GP}}(\mathbf{x})\right|^2 \leq \frac{C}{\varepsilon^2} \max\left[\varepsilon^{\frac{c\eta_0^2}{4}}, \exp\left\{-\tfrac{\sqrt{\Omega_0}}{\varepsilon^2}|1-x|\right\}\right]. \tag{3.19}$$

In particular the above result implies that by taking η_0 large enough we can make ψ^{GP} arbitrarily small outside $\mathcal{A}_\eta$. This fact will be crucial in restricting the computation of the GP energy within $\mathcal{A}_\eta$. Notice also that as soon as $|1-x| \gg \varepsilon^2|\log\varepsilon|$, $\psi^{\mathrm{GP}} = \mathcal{O}(\varepsilon^\infty)$.

Let us now focus on the giant vortex profiles. Before stating the main technical estimates we first formulate a simple preliminary bound on the giant vortex energy E^{gv}_β:

Proposition 3.4 (Preliminary bound on $E^{\rm gv}_\beta$)**.**
If $\beta = \mathcal{O}\left(\varepsilon^{-2}\right)$ *as* $\varepsilon \to 0$, *then*

$$E^{\rm gv}_\beta = \mathcal{O}(1), \qquad \mu_\beta = \mathcal{O}(1). \tag{3.20}$$

Proof. Since $E^{\rm gv}_\beta$ is positive (compare with (3.5)), it suffices to prove a suitable upper bound: to that purpose one can simply evaluate the functional $\mathcal{E}^{\rm gv}_\beta$ on the ground state of the one-dimensional harmonic oscillator with frequency α. The result on μ_β follows from the trivial estimates $E^{\rm gv}_\beta \leq \mu_\beta \leq 2E^{\rm gv}_\beta$. □

The giant vortex profile g_β decays exponentially for large $|y|$ and one can actually show that this decay captures the correct asymptotics of g_β:

Proposition 3.5 (Pointwise estimates of g_β)**.**
If $\beta = \mathcal{O}\left(\varepsilon^{-2}\right)$ *as* $\varepsilon \to 0$, *then there exists a finite constant* C *such that*

$$g_\beta(y) \leq C e^{-2\sqrt{\Omega_0}|y|}, \qquad \textit{so that } g_\beta(\pm\eta) = \mathcal{O}\left(\varepsilon^{\eta_0}\right). \tag{3.21}$$

If $\beta = \mathcal{O}(1)$ *then there exist two finite constants* $C_1, C_2 > 0$ *such that the following inequalities hold true*

$$C_1 \|g_\beta\|^2_{L^4_\eta} \exp\left\{-\tfrac{\alpha}{2}y^2\right\} \leq g_\beta(y) \leq C_2 \exp\left\{-\tfrac{\alpha}{4}y^2\right\}. \tag{3.22}$$

Proof. The results are proven by means of standard super- and sub-solution techniques. We spell however the proofs in full details for the sake of clarity.

To prove (3.21) it is somehow more convenient to go back to the variational equation satisfied by $f_\beta(x) = (\sqrt{2\pi}\varepsilon)^{-1}\, g_\beta((x-1)/\varepsilon^2)$, i.e.,

$$-\tfrac{1}{2}f''_\beta - \tfrac{1}{2r}f'_\beta + \tfrac{\Omega^2}{2x^2}\left(x^2 - \tfrac{\Omega+\beta}{\Omega}\right)^2 f_\beta + \Omega^2 W(x) f_\beta + \tfrac{1}{\varepsilon^2}f^3_\beta = \tfrac{1}{\varepsilon^4}\mu_\beta f_\beta. \tag{3.23}$$

The first simple observation is that by positivity of f_β and $W(x)$, we get

$$-\tfrac{1}{2}f''_\beta - \tfrac{1}{2r}f'_\beta \leq \tfrac{1}{\varepsilon^4}\left(\mu_\beta - \varepsilon^2 f^2_\beta\right) f_\beta,$$

which, by negativity of the second derivative of f_β at any maximum point of f_β, immediately implies the upper bound

$$\|f_\beta\|^2_{L^\infty(\mathcal{A}_\eta)} \leq \tfrac{1}{\varepsilon^2}\mu_\beta,$$

which in terms of g_β becomes, via (3.20) (here we are assuming that $\beta = \mathcal{O}(\varepsilon^{-2})$),

$$\|g_\beta\|_{L^\infty_\eta} = \mathcal{O}(1). \tag{3.24}$$

In order to prove (3.21) we will provide an explicit supersolution to the equation (3.23). Notice that the first two terms of the equation form the two-dimensional Laplacian, i.e., for any radial function f, $-\Delta f = -\frac{1}{r}\partial_r(rf')$. We will use this fact to construct a supersolution in dimension two. Let then $a > 0$ be a parameter independent of ε that is going to be chosen later and consider the two-dimensional region

$$\mathcal{A} := \mathcal{B}_{1-a\varepsilon^2}(0) \cap \mathcal{A}_\eta = \left\{\mathbf{x} \in \mathbb{R}^2 \;\middle|\; 1 - \eta\varepsilon^2 \leq x \leq 1 - a\varepsilon^2\right\}. \tag{3.25}$$

Inside $\mathcal{A}$ one has the lower bound

$$W(x) \geq \tfrac{s-2}{2}(x-1)^2 + \mathcal{O}\left(|x-1|^3\right) \geq \tfrac{(s-2)}{2}a^2\varepsilon^4 + \mathcal{O}\left(\eta^3\varepsilon^6\right) \geq C_0 a^2 \varepsilon^4$$

with $C_0 > 0$, so that (3.23) and (3.20) yield

$$-\tfrac{1}{2}\Delta f_\beta \leq \tfrac{1}{\varepsilon^4}\mu_\beta f_\beta - \Omega^2 W(x) f_\beta \leq \tfrac{1}{\varepsilon^4}\left(C_1 - \Omega_0^2 C_0 a^2\right) f_\beta$$

where also $C_1 > 0$. If now we pick $a^2 \geq \frac{2C_1}{C_0\Omega_0^2}$, we get that f_β is a subsolution of the following differential problem

$$-\tfrac{1}{2}\Delta f + \tfrac{1}{2}C_0 a^2\Omega^2\varepsilon^4 f = 0. \tag{3.26}$$

To get rid of the inner boundary we now extend f_β to the whole ball $\mathcal{B}_{1-a\varepsilon^2}(0)$ in a smooth (in fact at least C^2) way. We denote by $\tilde{f}$ such a new function and we require that $\tilde{f}(x) = 0$ for $x \leq 1 - 2\varepsilon^2\eta$ and

$$-\tfrac{1}{2}\Delta f + \tfrac{1}{2}C_0 a^2\Omega^2\varepsilon^4 f \leq 0, \tag{3.27}$$

for any $\mathbf{x} \in \mathcal{B}_{1-a\varepsilon^2}(0)$. We omit the explicit details of such a construction for the sake of brevity. A supersolution to the same problem can be constructed by taking

$$f_{\mathrm{sup}}(x) := C_a \|f_\beta\|_\infty e^{-\sqrt{\Omega}\left(1-x^2\right)}$$

with C_a a constant to be suitably chosen:

$$-\tfrac{1}{2}\Delta f_{\mathrm{sup}} + \tfrac{1}{2}C_0 a^2\Omega^2\varepsilon^4 f_{\mathrm{sup}} = \tfrac{1}{2}\left(-2\sqrt{\Omega} - 2\Omega x^2 + C_0\Omega_0 a^2\Omega\right) f_{\mathrm{sup}} > 0,$$

if we choose $a^2 > \frac{4}{C_0\Omega_0}$. The constant C_a is then used to guarantees that f_{sup} satisfies the proper boundary conditions. In order to apply the maximum principle (see, e.g., [E, § 6.4.1, Theorem 2]), we need that $\tilde{f}(x) \leq f_{\mathrm{sup}}(x)$ on $\partial\mathcal{B}_{1-a\varepsilon^2}(0)$, which holds true if $C_a \geq e^{2\sqrt{\Omega_0}a}$:

$$f_{\mathrm{sup}}\big|_{\partial\mathcal{B}_{1-a\varepsilon^2}(0)} = C_a\|f_\beta\|_\infty e^{-\sqrt{\Omega_0}a\left(2-a\varepsilon^2\right)} \geq f_\beta\big|_{\partial\mathcal{B}_{1-a\varepsilon^2}(0)}.$$

Hence we conclude that $\tilde{f} \leq f_{\mathrm{sup}}$ in the whole $\mathcal{B}_{1-a\varepsilon^2}(0)$, and therefore, using the monotonicity of f_{sup}, $f_\beta \leq f_{\mathrm{sup}}$ in the whole region $\mathcal{B}_1(0) \cap \mathcal{A}_\eta$. Going back to g_β and using (3.24), we obtain (3.21) in $\mathcal{B}_1(0)\cap\mathcal{A}_\eta$. To extend the result to the complementary region, one can use a very similar argument with the trivial change $x^2 - 1 \to 1 - x^2$ in the supersolution.

For the refined estimates (3.22), we consider the variational equation (3.15) for $a \leq |y| \leq \eta$, with $a > 0$ such that $a^2 > \frac{8\mu_\beta}{3\alpha^2}$ and ε small enough, which imply

$$U_\beta(y) + \varepsilon^2 y^3 v(y) - \mu_\beta = \tfrac{\alpha^2}{2}y^2 - \mu_\beta + \mathcal{O}\left(\varepsilon^2\eta^3\right) \geq \tfrac{\alpha^2}{8}y^2$$

and therefore in that region g_β is a subsolution of the equation

$$-\tfrac{1}{2}g'' - \tfrac{\varepsilon^2}{2(1+\varepsilon^2 y)}g' + \tfrac{\alpha^2}{8}y^2 g = 0. \tag{3.28}$$

As before we extend g_β to the whole region $|y| \geq a$ in a C^2 way and preserving the differential inequality satisfied in $a \leq |y| \leq \eta$, i.e.,

$$-\tfrac{1}{2}g'' - \tfrac{\varepsilon^2}{2(1+\varepsilon^2 y)}g' + \tfrac{\alpha^2}{8}y^2 g \leq 0.$$

Again we skip the details for brevity.

Now for some $C > 0$ to be fixed later the following function

$$g_{\mathrm{sup}}(y) := Ce^{-\frac{\alpha}{4}y^2}$$

is a supersolution to (3.28): for ε small enough

$$-\tfrac{1}{2}g''_{\mathrm{sup}} - \tfrac{\varepsilon}{2(1+\varepsilon^2 y)}g'_{\mathrm{sup}} + \tfrac{\alpha^2}{8}y^2 g_{\mathrm{sup}} = \left(\tfrac{\alpha}{4} + \tfrac{\alpha y\varepsilon^2}{4(1+\varepsilon^2 y)}\right) g_{\mathrm{sup}} \geq 0.$$

Choosing the $C \geq \|g_\beta\|_\infty e^{\frac{\alpha a^2}{4}}$ to ensure that $g_\beta(\pm a) \leq g_{\mathrm{sup}}(a)$, we get the upper estimate.

Analogously we can choose $C > 0$ in such a way that

$$g_{\mathrm{sub}} := C e^{-\frac{\alpha}{2} y^2}$$

is a subsolution to (3.15): first one notes that

$$\tfrac{\alpha}{2} \leq \mu_\beta - \tfrac{1}{\pi} \|g_\beta\|^4_{L^4_\eta} + \mathcal{O}\left(\varepsilon^2 \eta\right),$$

which follows from the fact that the harmonic oscillator on the real line is bounded from below by $\alpha/2$; then using this inequality in (3.15), we obtain

$$\begin{aligned} - \tfrac{1}{2} g_{\mathrm{sub}}'' - \tfrac{\varepsilon^2}{2(1+\varepsilon^2 y)} g_{\mathrm{sub}}' + U_\beta(y) g_{\mathrm{sub}} + \varepsilon^2 y^3 v(y) g_{\mathrm{sub}} + \tfrac{1}{\pi} g_{\mathrm{sub}}^3 - \mu_\beta g_{\mathrm{sub}} \\ = \left[\tfrac{\alpha}{2} - \mu_\beta + \tfrac{1}{\pi} g_{\mathrm{sub}}^2 + U_\beta(y) - \tfrac{\alpha^2}{2} y^2 + \tfrac{\alpha y \varepsilon^2}{2(1+\varepsilon^2 y)} + \varepsilon^2 y^3 v(y) \right] g_{\mathrm{sub}} \\ \leq \left[\tfrac{1}{\pi} \left(g_{\mathrm{sub}}^2(y) - \|g_\beta\|^4_{L^4_\eta} \right) + \mathcal{O}\left(\varepsilon^2 \eta^3\right) \right] g_{\mathrm{sub}} < 0, \end{aligned}$$

if we pick $C < \|g_\beta\|^2_{L^4_\eta}$. To conclude we use the fact that g_{sub} goes to 0 as $|y|$ goes to infinity: indeed it is sufficient to observe that there certainly exists a point $\bar{y} > 0$ such that $g_{\mathrm{sub}}(\pm\bar{y}) = \min\left\{g_\beta(\eta), g_\beta(-\eta)\right\}$ and

$$\widetilde{g}(y) := \begin{cases} g_\beta(y) & |y| \leq \eta, \\ g_\beta(\eta) & \eta \leq y \leq \bar{y}, \\ g_\beta(-\eta) & -\bar{y} \leq y \leq -\eta, \end{cases}$$

is a supersolution to (3.15), satisfying $\tilde{g}(\pm\bar{y}) \geq g_{\mathrm{sub}}(\pm\bar{y})$. Hence $g_{\mathrm{sub}} \leq \tilde{g}$ for any $|y| \leq \bar{y}$, which implies the lower estimate (3.22) for $|y| \leq \eta$. □

We conclude this Section by stating analogous pointwise estimate for the limiting profile g_{gv}:

Proposition 3.6 (Pointwise estimates of g_{gv})**.**
There exists a finite constant $C > 0$ such that

$$\|g_{\mathrm{gv}}\|_4^2 \exp\left\{-\tfrac{\alpha}{2} y^2\right\} \leq g_{\mathrm{gv}}(y) \leq C \exp\left\{-\tfrac{\alpha}{4} y^2\right\}. \tag{3.29}$$

Proof. The estimate can be proven exactly as (3.22) in Proposition 3.5 and we skip the details. □

3.3 Optimal Giant Vortex Phase and Profile

In this Section we investigate the minimization of E^{gv}_β w.r.t. $\beta \in \mathbb{R}$. The main result is the following

Proposition 3.7 (Optimal phase)**.**
For ε small enough there exists a unique minimizer $\beta_\star \in \mathbb{R}$ such that

$$E^{\mathrm{gv}}_\star := \inf_{\beta \in \mathbb{R}} E^{\mathrm{gv}}_\beta = E^{\mathrm{gv}}_{\beta_\star}. \tag{3.30}$$

Such an optimal phase is explicitly given by

$$\beta_\star = - \frac{2}{\Omega_0 \left(s - 2\right)} \left[(s-2) V - Q + \mathcal{O}\left(\varepsilon^2\right) \right], \tag{3.31}$$

where we set $g_\star := g_{\beta_\star}$ and

$$V := \frac{\alpha^2}{2} \int_{-\eta}^{\eta} \mathrm{d}y \, y^2 g_\star^2, \qquad Q := \frac{1}{2\pi} \int_{-\eta}^{\eta} \mathrm{d}y \, g_\star^4. \tag{3.32}$$

Proof. The existence of a minimizer $\beta_\star$ is guaranteed from the fact that

$$U_\beta(y) \geq \tfrac{1}{(1+\varepsilon^2 y)^2} \left[\tfrac{s-2}{2(s+2)} \varepsilon^4 \beta^2 - \Omega_0 \varepsilon^4 \beta \eta \right]$$

which implies that $\lim_{|\beta| \to \infty} E^{\mathrm{gv}}_\beta = +\infty$ (recall that $s > 2$). By the same lower bound on the potential together with the trivial bound $E^{\mathrm{gv}}_\star \leq E^{\mathrm{gv}}_0 = \mathcal{O}(1)$, we also deduce that $\beta_\star = \mathcal{O}\left(\varepsilon^{-2}\right)$.

In order to find the explicit expression of $\beta_\star$, we first observe that by standard arguments E^{gv}_β is a smooth function of β and therefore by the Feynman-Hellmann principle[5]

$$\partial_\beta E^{\mathrm{gv}}_\beta = \left\langle g_\beta \left| \partial_\beta U_\beta \right| g_\beta \right\rangle_\eta = \left\langle g_\beta \left| \tfrac{\varepsilon^2}{(1+\varepsilon^2 y)^2} \left(-2\Omega_0 y - \Omega_0 \varepsilon^2 y^2 + \varepsilon^2 \beta \right) \right| g_\beta \right\rangle_\eta . \tag{3.33}$$

Since $\beta_\star$ is a minimizer, we must have $\partial_\beta E^{\mathrm{gv}}_\beta \Big|_{\beta_\star} = 0$, i.e.,

$$\varepsilon^2 \beta_\star \left\langle g_\star \left| \tfrac{1}{(1+\varepsilon^2 y)^2} \right| g_\star \right\rangle_\eta - 2\Omega_0 \left\langle g_\star \left| \tfrac{y}{(1+\varepsilon^2 y)^2} \right| g_\star \right\rangle_\eta - \Omega_0 \varepsilon^2 \left\langle g_\star \left| \tfrac{y^2}{(1+\varepsilon^2 y)^2} \right| g_\star \right\rangle_\eta = 0. \tag{3.34}$$

We compute the first and last terms of the expression above:

$$\left\langle g_\star \left| \tfrac{1}{(1+\varepsilon^2 y)^2} \right| g_\star \right\rangle_\eta = 1 + \mathcal{O}\left(\varepsilon^2\right), \qquad \left\langle g_\star \left| \tfrac{1}{(1+\varepsilon^2 y)^2} y^2 \right| g_\star \right\rangle_\eta = \frac{2V}{\alpha^2} + \mathcal{O}\left(\varepsilon^2\right). \tag{3.35}$$

Indeed thanks to the exponential decay proven in (3.21), one can easily realize that

$$\int_{-\eta}^{\eta} \mathrm{d}y \, |y|^k \, g_\star^2 = \mathcal{O}(1), \qquad \text{for any } k < \infty. \tag{3.36}$$

In fact an analogous estimate holds true if $g_\star$ is replaced with $g_\star'$, in particular

$$\int_{-\eta}^{\eta} \mathrm{d}y \, y \, \left(g_\star'\right)^2 = \mathcal{O}(1). \tag{3.37}$$

To see this it suffices to integrate by parts and use the variational equation (3.15) to go back to an expression involving only $g_\star$ and there one can use the above estimate. We omit the details for the sake of brevity. Notice that at this stage we are implicitly exploiting the bound $\beta_\star = \mathcal{O}(\varepsilon^{-2})$, which is among the hypothesis of Proposition 3.5. Next we integrate by parts the second term in (3.34) to get

$$\begin{aligned} \left\langle g_\star \left| \tfrac{1}{(1+\varepsilon^2 y)^2} y \right| g_\star \right\rangle_\eta &= \int_{-\eta}^{\eta} \mathrm{d}y \, \tfrac{1}{1+\varepsilon^2 y} \, y \, g_\star^2 \\ &= \left[\tfrac{1}{2(1+\varepsilon^2 y)} \, y^2 \, g_\star^2 \right]_{-\eta}^{\eta} - \int_{-\eta}^{\eta} \mathrm{d}y \, \tfrac{1}{1+\varepsilon^2 y} \, y^2 \, g_\star g_\star' + \frac{\varepsilon^2 V}{\alpha^2} \left(1 + \mathcal{O}\left(\varepsilon^2\right)\right) \end{aligned}$$

where the boundary terms (first term on the r.h.s. of the expression above) can be included in the

[5] The notation $\langle \cdot | \cdot \rangle_\eta$ stands for the scalar product in $L^2([-\eta, \eta], (1+\varepsilon^2 y)\mathrm{d}y)$.

remainder $\mathcal{O}(\varepsilon^2)$ if we choose $\eta_0 > 2$ (see again (3.21)). For the rest we can compute

$$
\begin{aligned}
- \int_{-\eta}^{\eta} \mathrm{d}y \, \tfrac{1}{1+\varepsilon^2 y} \, y^2 \, g_\star g_\star' = & - \frac{2}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, U_{\beta_\star}(y) g_\star g_\star' \\
& - \frac{2}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \frac{2\Omega_0 \varepsilon^2 \beta_\star y + \Omega_0 \varepsilon^4 \beta_\star y^2 - \frac{1}{2}\varepsilon^4 {\beta_\star}^2}{1+\varepsilon^2 y} \, g_\star g_\star' \\
= \frac{1}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \left(1+\varepsilon^2 y\right) \, \partial_y \left[-\tfrac{1}{2} \left(g_\star'\right)^2 + \tfrac{1}{2\pi} g_\star^4 - \mu_\star g_\star^2 \right] & + \frac{\varepsilon^2}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \left(1+\varepsilon^2 y\right) \, y^3 v(y) \partial_y g_\star^2 \\
- \frac{\varepsilon^2}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \left(g_\star'\right)^2 & - \frac{\varepsilon^2 \beta_\star}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \frac{2\Omega_0 y + \Omega_0 \varepsilon^2 y^2 - \frac{1}{2}\varepsilon^2 \beta_\star}{1+\varepsilon^2 y} \, \partial_y g_\star^2 = \\
= \frac{\varepsilon^2}{\alpha^2} \int_{-\eta}^{\eta} \mathrm{d}y \, \left\{ -\tfrac{1}{2} \left(g_\star'\right)^2 - \tfrac{1}{2\pi} g_\star^4 + \mu_\star g_\star^2 \right\} & - \frac{\varepsilon^2 (s+1) V}{\alpha^2} + \frac{2\varepsilon^2 \Omega_0 \beta_\star}{\alpha^2} + \mathcal{O}(\varepsilon^4 \beta_\star) + \mathcal{O}\left(\varepsilon^4\right) = \\
& = \frac{\varepsilon^2}{\alpha^2} \left[-K - Q + \mu_\star - (s+1) V + 2\Omega_0 \beta_\star + \mathcal{O}\left(\varepsilon^2 \beta_\star\right) + \mathcal{O}\left(\varepsilon^2\right) \right],
\end{aligned}
$$

where we have made use repeatedly of (3.36) and exploited the identity

$$
\left(y^3 v(y)\right)' = \tfrac{1}{2} \alpha^2 (s+1) y^2 + \mathcal{O}(\varepsilon^2 |y|^3).
$$

We have also set

$$
T := \frac{1}{2} \int_{-\eta}^{\eta} \mathrm{d}y \, \left(g_\star'\right)^2. \tag{3.38}
$$

Hence

$$
\left\langle g_\star \left| \tfrac{1}{(1+\varepsilon^2 y)^2} y \right| g_\star \right\rangle_\eta = \frac{\varepsilon^2}{\alpha^2} \left[-T - sV - Q + \mu_\star + 2\Omega_0 \beta_\star + \mathcal{O}\left(\varepsilon^2 \beta_\star\right) + \mathcal{O}\left(\varepsilon^2\right) \right] \tag{3.39}
$$

and plugging this together with (3.35) into (3.34), we obtain

$$
\frac{s-2}{s+2} \beta_\star (1 + \mathcal{O}(\varepsilon^2)) + \frac{2\Omega_0}{\alpha^2} \left[(s-2) V - Q \right] + \mathcal{O}(\varepsilon^2) = 0,
$$

since $\mu_\star = T + V + 2Q + \mathcal{O}(\varepsilon^2)$ (see (3.36) and (3.37)). The expression (3.31) is then recovered. □

Along the proof we have also proven in (3.34) that

$$
\int_{-\eta}^{\eta} \mathrm{d}y \, \tfrac{1}{1+\varepsilon^2 y} \left(y + \tfrac{1}{2} \varepsilon^2 y^2 - \tfrac{1}{2\Omega_0} \varepsilon^2 \beta_\star \right) g_\star^2 = 0, \tag{3.40}
$$

which, thanks to the result about $\beta_\star$, also implies that

$$
\left\langle g_\star \left| y \right| g_\star \right\rangle_\eta = \mathcal{O}(\varepsilon^2), \tag{3.41}
$$

i.e., the profile $g_\star$ is almost symmetric w.r.t. the origin.

In fact this latter information can be deduced also by looking at the relation between the functional $\mathcal{E}^{\mathrm{gv}}_{\beta_\star}$ and its minimization and the limiting model $\mathcal{E}^{\mathrm{gv}}$. From now on we fix β equal to the optimal value $\beta_\star$.

Before discussing this question further we have however to state an useful estimate on $g_\star$.

Lemma 3.1.
There exists a finite constant $C > 0$ such that for any $y \in [-\eta, \eta]$

$$
\left| g_\star'(y) \right| \leq C \eta^3 g_\star(y). \tag{3.42}
$$

Proof. It suffices to integrate the variational equation (3.15) between $y \geq y_{\beta_\star}$ and η or $-\eta$ and $y \leq y_{\beta_\star}$ (recall that $y_{\beta_\star}$ stands for the unique maximum point of $g_\star$): let us assume that $y \geq y_\beta$, then by positivity of $g_\star$

$$\begin{aligned}\tfrac{1}{2}\left|g_\star'(y)\right| = -\tfrac{1}{2}g_\star'(y) &= \int_y^\eta \mathrm{d}t \, \left\{ -\tfrac{\varepsilon^2}{2(1+\varepsilon^2 t)} g_\star' + \left(\tfrac{1}{\pi} g_\star^2 + U_{\beta_\star}(t) + \varepsilon^2 t^3 v(t) - \mu_{\beta_\star}^{\mathrm{gv}} \right) g_\star \right\} \\ &\leq \int_y^\eta \mathrm{d}t \, \left\{ -\tfrac{\varepsilon^4}{2(1+\varepsilon^2 t)^2} + \tfrac{1}{\pi} g_\star^2 + U_{\beta_\star}(t) + \varepsilon^2 t^3 v(t) \right\} g_\star + \tfrac{\varepsilon^2}{2(1+\varepsilon^2 y)} g_\star(y). \qquad (3.43)\end{aligned}$$

Now given that the quantity between brackets can be easily bounded from above by $C\eta^2$, it only remains to use the monotonicity of $g_\star$ to conclude the proof. □

We are now in position to prove the first result about the energy difference $E_\star^{\mathrm{gv}} - E^{\mathrm{gv}}$. As a matter of fact this will involve a corresponding statement about the closeness of $g_\star^2$ to g_{gv}^2 in L^2. We recall the expressions of the energy functionals

$$\mathcal{E}_\star^{\mathrm{gv}}[g] = \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \left\{ \tfrac{1}{2} (g')^2 + U_{\beta_\star}(y) g^2 + \varepsilon^2 y^3 v(y) g^2 + \tfrac{1}{2\pi} g^4 \right\},$$

$$\mathcal{E}^{\mathrm{gv}}[g] = \int_{\mathbb{R}} \mathrm{d}y \left\{ \tfrac{1}{2} (g')^2 + \tfrac{\alpha^2}{2} y^2 g^2 + \tfrac{1}{2\pi} g^4 \right\}.$$

Proposition 3.8 (Estimate of $E_\star^{\mathrm{gv}} - E^{\mathrm{gv}}$)**.**
As $\varepsilon \to 0$

$$E_\star^{\mathrm{gv}} = E^{\mathrm{gv}} + \mathcal{O}\left(\varepsilon^4 \eta^7\right), \qquad \left\| g_\star^2 - g_{\mathrm{gv}}^2 \right\|_{L_\eta^2}^2 = \mathcal{O}\left(\varepsilon^4 \eta^7\right). \qquad (3.44)$$

Proof. We test the two functionals $\mathcal{E}_{\beta_\star}^{\mathrm{gv}}$ and $\mathcal{E}^{\mathrm{gv}}$ on suitable test functions. Let us first regularize $g_\star$ outside $[-\eta, \eta]$ to make it an admissible test function for $\mathcal{E}^{\mathrm{gv}}$: we define

$$g_{\mathrm{trial}}(y) := c_\varepsilon \begin{cases} g_\beta(y), & |y| \leq \eta, \\ r_1(y), & \eta \leq y \leq 2\eta, \\ r_2(y), & -2\eta \leq y \leq \eta, \\ 0, & |y| \geq 2\eta, \end{cases}$$

with $r_{1,2}$ positive smooth functions chosen in such a way that g_{trial} is at least C^2. We also assume that both functions $r_{1,2}$ are also monotonically decreasing. The normalization constant c_ε, which ensures that $\|g_{\mathrm{trial}}\|_{L^2(\mathbb{R})} = 1$, can be easily estimated: assuming that $\eta_0 > 2$, we have

$$c_\varepsilon = 1 + \mathcal{O}(\varepsilon^4), \qquad (3.45)$$

since, e.g.,

$$\int_\eta^{2\eta} \mathrm{d}y \, r_1^2(y) \leq \eta g_\star^2(\eta) = \mathcal{O}(\eta \varepsilon^{2\eta_0}).$$

Notice also that we need to use (3.41) to reconstruct the norm of $g_\star$:

$$\int_{-\eta}^{\eta} \mathrm{d}y \, g_{\mathrm{trial}}^2 = c_\varepsilon^2 \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, g_\star^2 + c_\varepsilon^2 \varepsilon^2 \int_{-\eta}^{\eta} \mathrm{d}y \, y \, g_\star^2 = c_\varepsilon^2 + \mathcal{O}(\varepsilon^4).$$

Now we estimate

$$E^{\mathrm{gv}} \leq \mathcal{E}^{\mathrm{gv}}[g_{\mathrm{trial}}] = c_\varepsilon^2 \int_{-\eta}^{\eta} \mathrm{d}y \, \left\{ \tfrac{1}{2} (g_\beta')^2 + \tfrac{\alpha^2}{2} y^2 g_\beta^2 + \tfrac{1}{2\pi} g_\beta^4 \right\} + \mathcal{O}(\varepsilon^4).$$

Thanks to (3.41) we can easily estimate the error we make by replacing $\frac{\alpha^2}{2}y^2$ with $U_\beta + \varepsilon^2 y^3 v(y)$: denoting for short $\langle f \rangle := \langle g_\star \, |f| \, g_\star \rangle$, we have

$$\left\langle g_\star \left| \tfrac{\alpha^2}{2} y^2 - U_\beta(y) - \varepsilon^2 y^3 v(y) \right| g_\star \right\rangle_{L^2(-\eta,\eta)}$$
$$= \left\langle g_\star \left| \tfrac{\alpha^2 \varepsilon^2 y^3 \left(2+\varepsilon^2 y\right)}{2(1+\varepsilon^2 y)^2} + \tfrac{\varepsilon^2 \beta_\star}{(1+\varepsilon^2 y)^2} \left(2\Omega_0 y + \Omega_0 \varepsilon^2 y - \tfrac{1}{2}\varepsilon^2 \beta_\star\right) - \varepsilon^2 y^3 v(y) \right| g_\star \right\rangle = \mathcal{O}\left(\varepsilon^4 + \varepsilon^2 \left\langle y^3 \right\rangle\right),$$

so that

$$E^{\mathrm{gv}} \leq E^{\mathrm{gv}}_\star + \mathcal{O}\left(\varepsilon^4 + \varepsilon^2 \left\langle y^3 \right\rangle\right), \tag{3.46}$$

where we have also used (3.20), which in turn requires $\beta_\star = \mathcal{O}(\varepsilon^{-2})$.

The trial state for the functional $\mathcal{E}^{\mathrm{gv}}_{\beta_\star}$ is simply the truncation of g_{gv}, i.e., $c_\varepsilon g_{\mathrm{gv}}$, where now the normalization factor can be estimated in this case as

$$c_\varepsilon = 1 + \mathcal{O}(\varepsilon^\infty), \tag{3.47}$$

since by symmetry of g_{gv}

$$\int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, g_{\mathrm{gv}}^2 = 1 - 2 \int_{\eta}^{\infty} \mathrm{d}y \, g_{\mathrm{gv}}^2 = 1 + \mathcal{O}(\varepsilon^\infty),$$

where we have used the pointwise estimate (3.29) on g_{gv} and the fact that the integral of a gaussian, i.e., the error function, is bounded by the value of the gaussian at the boundary, i.e., $\exp\{-c|\log\varepsilon|^2\} = \mathcal{O}(\varepsilon^\infty)$ (see, e.g., [AS, Eq. (5.1.19)]). Then we have

$$E^{\mathrm{gv}}_\star \leq \mathcal{E}^{\mathrm{gv}}_{\beta_\star}[c_\varepsilon g_{\mathrm{gv}}] = (1+\mathcal{O}(\varepsilon^\infty)) \int_{-\eta}^{\eta} \mathrm{d}y \, \left(1+\varepsilon^2 y\right) \left\{ \tfrac{1}{2}(g_{\mathrm{gv}}')^2 + U_{\beta_\star}(y) g_{\mathrm{gv}}^2 + \varepsilon^2 y^3 v(y) g_{\mathrm{gv}}^2 + \tfrac{1}{2\pi} g_{\mathrm{gv}}^4 \right\}$$
$$= \int_{-\eta}^{\eta} \mathrm{d}y \left\{ \tfrac{1}{2}(g_{\mathrm{gv}}')^2 + \tfrac{\alpha^2}{2} g_{\mathrm{gv}}^2 + \tfrac{1}{2\pi} g_{\mathrm{gv}}^4 \right\} + \mathcal{O}(\varepsilon^4) = E^{\mathrm{gv}} + \mathcal{O}(\varepsilon^4). \tag{3.48}$$

Putting together (3.46) with (3.48), we obtain

$$E^{\mathrm{gv}}_\star = E^{\mathrm{gv}} + \mathcal{O}\left(\varepsilon^4 + \varepsilon^2 \left\langle y^3 \right\rangle\right). \tag{3.49}$$

Now we decouple the energy E^{gv}: first we bound from below E^{gv} as

$$E^{\mathrm{gv}} \geq \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \left\{ \tfrac{1}{2}(g_{\mathrm{gv}}')^2 + \tfrac{\alpha^2}{2} y^2 g_{\mathrm{gv}}^2 + \tfrac{1}{2\pi} g_{\mathrm{gv}}^4 \right\},$$

where we have just dropped some positive quantities and used the symmetry of g_{gv}. Then we set $g_{\mathrm{gv}} = u g_\star$ for some unknown smooth function u (recall that $g_\star$ never vanishes in $[-\eta,\eta]$) and using the variational equation for $g_\star$ as well as Neumann boundary conditions, we obtain

$$E^{\mathrm{gv}} \geq E^{\mathrm{gv}}_\star + \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, g_\star^2 \left\{ \tfrac{1}{2}(u')^2 + \left(\tfrac{\alpha^2}{2} y^2 - U_{\beta_\star}(y) - \varepsilon^2 y^3 v(y)\right) u^2 + \tfrac{1}{2\pi} g_\star^2 \left(1-u^2\right)^2 \right\}$$
$$- \tfrac{1}{2}\varepsilon^2 \int_{-\eta}^{\eta} \mathrm{d}y \, u^2 g_\star g_\star' + \mathcal{O}(\varepsilon^\infty).$$

Then we estimate

$$\left| \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, \left(\tfrac{\alpha^2}{2} y^2 - U_{\beta_\star}(y) - \varepsilon^2 y^3 v(y)\right) g_\star^2 u^2 \right| \leq C\varepsilon^2 \int_{-\eta}^{\eta} \mathrm{d}y \, |y| g_\star^2 \left|1-u^2\right|$$
$$+ \left| \int_{-\eta}^{\eta} \mathrm{d}y \, (1+\varepsilon^2 y) \, \left(\tfrac{\alpha^2}{2} y^2 - U_{\beta_\star}(y) - \varepsilon^2 y^3 v(y)\right) g_\star^2 \right|$$
$$\leq C\varepsilon^2 \eta^{3/2} \left\| g_\star^2 (1-u^2) \right\|_{L^2_\eta} + \mathcal{O}\left(\varepsilon^2 \left\langle y^3 \right\rangle + \varepsilon^4\right),$$

and by (3.42) (notice that the factor $1+\varepsilon^2 y$ is uniformly bounded from above and below by a constant)

$$\left| \int_{-\eta}^{\eta} \mathrm{d}y \, u^2 g_\star g_\star' \right| = \int_{-\eta}^{\eta} \mathrm{d}y \, \left|1-u^2\right| |g_\star||g_\star'| + \mathcal{O}(\varepsilon^4) \le C\eta^3 \int_{-\eta}^{\eta} \mathrm{d}y \, \left|1-u^2\right| g_\star^2 + \mathcal{O}(\varepsilon^4)$$
$$\le C\eta^{7/2} \left\| g_\star^2 (1-u^2) \right\|_{L^2_\eta} + \mathcal{O}(\varepsilon^4),$$

which imply by dropping the kinetic term

$$E^{\mathrm{gv}} \ge E^{\mathrm{gv}}_\star + \tfrac{1}{2} \left\| g_\star u' \right\|^2_{L^2_\eta} + \tfrac{1}{\pi} \left\| g_\star^2 (1-u^2) \right\|^2_{L^2_\eta} - C\varepsilon^2 \eta^{7/2} \left\| g_\star^2 (1-u^2) \right\|_{L^2_\eta} + \mathcal{O}\left(\varepsilon^2 \left\langle y^3 \right\rangle + \varepsilon^4 \right)$$
$$\ge E^{\mathrm{gv}}_\star + \tfrac{1}{\pi} \left(\left\| g_\star^2 (1-u^2) \right\|_{L^2_\eta} - C\varepsilon^2 \eta^{7/2} \right)^2 + \mathcal{O}\left(\varepsilon^2 \left\langle y^3 \right\rangle + \varepsilon^4 \eta^7 \right). \tag{3.50}$$

If we compare what we have obtained with (3.49), we conclude that

$$\left\| g_\star^2 - g_{\mathrm{gv}}^2 \right\|^2_{L^2_\eta} = \left\| g_\star^2 (1-u^2) \right\|^2_{L^2_\eta} = \mathcal{O}\left(\varepsilon^2 \left\langle y^3 \right\rangle + \varepsilon^4 \eta^7 \right), \tag{3.51}$$

but on the other hand

$$\left| \int_{-\eta}^{\eta} \mathrm{d}y \, y^3 \, g_\star^2 \right| = \left| \int_{-\eta}^{\eta} \mathrm{d}y \, y^3 \, \left(g_\star^2 - g_{\mathrm{gv}}^2 \right) \right| \le C\eta^{7/2} \left\| g_\star^2 - g_{\mathrm{gv}}^2 \right\|_{L^2_\eta}, \tag{3.52}$$

so that finally

$$\left\| g_\star^2 - g_{\mathrm{gv}}^2 \right\|^2_{L^2_\eta} = \mathcal{O}\left(\varepsilon^4 \eta^7 \right).$$

This proves the second inequality in (3.44) but the first is obtained by replacing the above estimate into (3.49). □

It is interesting to remark that a by-product of the proof of Proposition 3.8 is that (see (3.52))

$$\left\langle g_\star \left| y^3 \right| g_\star \right\rangle = \mathcal{O}(\varepsilon^2 \eta^7), \tag{3.53}$$

which in combination with (3.41) is a strong indication of $g_\star$ being symmetric w.r.t. the origin with a very high precision. In fact this is also made apparent by the estimate of the difference $g_\star - g_{\mathrm{gv}}$.

The L^2-statement in (3.44) can indeed be improved to an L^∞-one, showing that $g_\star$ and g_{gv} are pointwise close. The price to pay to have a result in a stronger norm is the restriction of the region under consideration to the annulus $\widetilde{\mathcal{A}}_\eta \subset \mathcal{A}_\eta$ defined as

$$\widetilde{\mathcal{A}}_\eta := \left\{ y \in \mathbb{R} \; \middle| \; g_{\mathrm{gv}}(y) \ge \tfrac{1}{\eta^\nu} \right\}, \tag{3.54}$$

for some $\nu > 0$ independent of ε. Note that thanks to the pointwise estimates (3.29) and the monotonicity of g_{gv},

$$\widetilde{\mathcal{A}}_\eta = [-y_\eta, y_\eta], \qquad \text{with } y_\eta \gg 1. \tag{3.55}$$

Proposition 3.9 (Pointwise estimate of $g_\star - g_{\mathrm{gv}}$).
As $\varepsilon \to 0$ and for any $\nu > 0$

$$\left\| g_\star - g_{\mathrm{gv}} \right\|_{L^\infty(\widetilde{\mathcal{A}}_\eta)} = \mathcal{O}\left(\varepsilon^2 \eta^{\frac{7+4\nu}{2}} \right). \tag{3.56}$$

Proof. Going back to (3.50) and retaining the kinetic term, we see that we obtain via (3.44) the upper bound

$$\|g_\star u'\|^2_{L^2_\eta} = \mathcal{O}(\varepsilon^4\eta^7), \tag{3.57}$$

where we recall that $u = g_{\mathrm{gv}}/g_\star$. Now let us introduce the set

$$\widehat{\mathcal{A}}_\eta := \left\{ y \in \mathbb{R} \;\middle|\; g_\star(y) \geq \tfrac{1}{2\eta^\nu} \right\},$$

so that above inequality together with (3.44) and the bound $|1-u| \leq |1-u^2|$ imply

$$\|u'\|^2_{L^2(\widehat{\mathcal{A}}_\eta)} = \mathcal{O}(\varepsilon^4\eta^{7+2\nu}), \qquad \|1-u\|^2_{L^2(\widehat{\mathcal{A}}_\eta)} = \mathcal{O}(\varepsilon^4\eta^{7+4\nu}).$$

Then it suffices to use Sobolev inequality in one-dimension:

$$\|1-u\|^2_{L^\infty(\widehat{\mathcal{A}}_\eta)} \leq C\left(\|u'\|^2_{L^2(\widehat{\mathcal{A}}_\eta)} + \|1-u\|^2_{L^2(\widehat{\mathcal{A}}_\eta)}\right) = \mathcal{O}\left(\varepsilon^4\eta^{7+4\nu}\right). \tag{3.58}$$

Finally to obtain the result it remains to observe that $\widetilde{\mathcal{A}}_\eta \subset \widehat{\mathcal{A}}_\eta$, because in the region where $g_\star \geq 1/(2\eta^\nu)$, by the pointwise estimate, g_{gv} is larger than $(1+o(1))/(2\eta^\nu)$, which is obviously satisfied if $g_{\mathrm{gv}} \geq 1/\eta^\nu$. □

The above bound shows that inside $\widetilde{\mathcal{A}}_\eta$ one can estimate the distance of $g_\star$ from a perfectly even function: for any $y \in \widetilde{\mathcal{A}}_\eta$

$$g_\star(-y) = g_\star(y) + \mathcal{O}\left(\varepsilon^2\eta^{\frac{7+4\nu}{2}}\right),$$

which is perfectly compatible with the estimates (3.41) and (3.53).

Another useful consequence of the above pointwise statement is the following

Corollary 3.1 (Maximum point of $g_\star$)**.**
Let $y_{\beta_\star}$ be the unique maximum point of $g_\star$, then as $\varepsilon \to 0$

$$y_{\beta_\star} = \mathcal{O}\left(\varepsilon^2\eta^{\frac{7+4\nu}{2}}\right). \tag{3.59}$$

Proof. The result is a straightforward consequence of the pointwise estimate (3.56) and the properties of g_{gv} (see Proposition 3.2). □

3.4 Critical Velocity and Positivity of the Cost Function

From now we fix the phase to be optimal one, i.e., $\beta = \beta_\star$. The *potential function* is defined as

$$F(y) := -\frac{1}{\varepsilon^2}\int_{-\eta}^{y} \mathrm{d}t\,(1+\varepsilon^2 t)\,\partial_\beta U_\beta(t)\big|_{\beta=\beta_\star} g_\star^2 = 2\Omega_0 \int_{-\eta}^{y} \mathrm{d}t\,\frac{1}{1+\varepsilon^2 t}\left(t + \frac{1}{2}\varepsilon^2 t^2 - \frac{\varepsilon^2\beta_\star}{2\Omega_0}\right) g_\star^2. \tag{3.60}$$

The main object under investigation is the *cost function*

$$K(y) := \tfrac{1}{2} g_\star^2(y) + F(y), \tag{3.61}$$

and our main goal in this Section is to prove that it is positive in the bulk of the condensate when $\Omega_0 \geq \Omega_{\mathrm{c}}$. To this purpose we will clearly have to investigate the equation (2.7) and prove at least that there exists a positive solution to it. Notice the equation (2.7) involves only quantities relative to the limiting functional $\mathcal{E}^{\mathrm{gv}}$ and is independent of ε. We thus introduce the analogue of (3.61) for the limiting case, i.e., the function (2.19)

$$K^{\mathrm{gv}} = \tfrac{1}{2} g_{\mathrm{gv}}^2 + F^{\mathrm{gv}},$$

where F^{gv} is defined in (2.20):

$$F^{\mathrm{gv}}(y) = -2\Omega_0 \int_y^{\infty} \mathrm{d}t\, t\, g_{\mathrm{gv}}^2(t).$$

We will start by studying the positivity of (2.19) and show that the condition $\Omega_0 > \Omega_{\mathrm{c}}$, where the latter is defined as the biggest solution to (2.7), is sufficient to deduce that $K^{\mathrm{gv}}(y) \geq 0$ for any $y \in \mathbb{R}$. In the second part of the Section we will turn our attention to the cost function (3.61) and prove that the same condition on Ω_0 guarantees positivity of K as well.

We first observe that F^{gv} is a negative function vanishing at $\pm\infty$: at $y = +\infty$ it is obvious, at $-\infty$ it is a consequence of parity of g_{gv}. By this property one can rewrite

$$F^{\mathrm{gv}}(y) = 2\Omega_0 \int_{-\infty}^{y} \mathrm{d}t\, t\, g_{\mathrm{gv}}^2(t).$$

In fact there is another explicit expression of F^{gv}, which can be obtained by using the variational equation (3.17):

$$\begin{aligned} F^{\mathrm{gv}}(y) = -\Omega_0 \int_y^{\infty} \mathrm{d}t\, \partial_t(t^2)\, g_{\mathrm{gv}}^2 &= \Omega_0 y^2 g_{\mathrm{gv}}^2(y) + 2\Omega_0 \int_y^{\infty} \mathrm{d}t\, t^2\, g_{\mathrm{gv}} g_{\mathrm{gv}}' \\ &= \Omega_0 y^2 g_{\mathrm{gv}}^2(y) + \frac{4\Omega_0}{\alpha^2} \int_y^{\infty} \mathrm{d}t\, g_{\mathrm{gv}}' \left[\tfrac{1}{2} g_{\mathrm{gv}}'' - \tfrac{1}{\pi} g_{\mathrm{gv}}^3 + \mu^{\mathrm{gv}} g_{\mathrm{gv}} \right] \\ &= -\tfrac{1}{\Omega_0(s+2)} \left(g_{\mathrm{gv}}'(y) \right)^2 + \left[\Omega_0 y^2 + \tfrac{1}{\pi \Omega_0 (s+2)} g_{\mathrm{gv}}^2(y) - \tfrac{2\mu^{\mathrm{gv}}}{\Omega_0(s+2)} \right] g_{\mathrm{gv}}^2(y), \end{aligned} \tag{3.62}$$

where we have used the exponential decay at ∞ of g_{gv} to cancel the missing boundary terms. Consequently we can rewrite K^{gv} as

$$K^{\mathrm{gv}}(y) = -\tfrac{1}{\Omega_0(s+2)} \left(g_{\mathrm{gv}}'(y) \right)^2 + \left[\tfrac{1}{2} + \Omega_0 y^2 + \tfrac{1}{\pi \Omega_0(s+2)} g_{\mathrm{gv}}^2(y) - \tfrac{2\mu^{\mathrm{gv}}}{\Omega_0(s+2)} \right] g_{\mathrm{gv}}^2(y). \tag{3.63}$$

The main result about K^{gv} is the following

Proposition 3.10 (Positivity of K^{gv})**.**
Let $\Omega_0 > 0$, then

$$K^{\mathrm{gv}}(y) \geq 0 \textit{ for any } y \in \mathbb{R} \Longleftrightarrow \Omega_0 \geq \tfrac{4}{s+2} \left[\mu^{\mathrm{gv}} - \tfrac{1}{2\pi} g_{\mathrm{gv}}^2(0) \right]. \tag{3.64}$$

Moreover if the strict inequality is verified on the r.h.s., $K^{\mathrm{gv}}(y) > 0$ for any y finite.

Remark 3.1 (Comparison with [CPRY3])**.**
Despite the use of two different potential functions F_1 and F_2 in [CPRY3], one should realize that [CPRY3, Lemma 3.3] yields the poitwise positivity of a cost function, which is the analogue of K^{gv} in the asymptotic regime $\Omega_0 \gg 1$. In fact it can be easily seen that the cost function in [CPRY3] is bounded from below by K^{gv} and therefore the positivity of the latter implies the positivity of the first. Hence any threshold $\bar{\Omega}_0$ one might deduce there must be larger than Ω_{c} by definition.

Proof. One side of the statement, i.e., the fact that the condition $K^{\mathrm{gv}}(0) \geq 0$ is necessary for the positivity of K^{gv} everywhere, is obviously trivial, so we focus on the other side of the implication, namely that $K^{\mathrm{gv}}(0) \geq 0$ is also sufficient.

The core of the proof is to show that

$$K^{\mathrm{gv}}(y) \geq 0 \text{ for any } y \in \mathbb{R} \Longleftrightarrow K^{\mathrm{gv}}(0) \geq 0. \tag{3.65}$$

Indeed if we assume that this double implication is true, a straightforward computation yields

$$K^{\mathrm{gv}}(0) = \left[\frac{1}{2} + \frac{1}{\pi\Omega_0(s+2)} g_{\mathrm{gv}}^2(0) - \frac{2\mu^{\mathrm{gv}}}{\Omega_0(s+2)}\right] g_{\mathrm{gv}}^2(0), \tag{3.66}$$

since g_{gv} is symmetric w.r.t. the origin and has a maximum at $y = 0$ (see Proposition 3.2). The result is then a trivial consequence of strict positivity of $g_{\mathrm{gv}}(0)$.

In order to prove (3.65), we first observe that $K^{\mathrm{gv}}(\pm\infty) = 0$ and K^{gv} is smooth, so, if there was a point y_0 where K^{gv} becomes negative, it must be $|y_0| < +\infty$. Moreover as g_{gv}, K^{gv} is symmetric w.r.t. to the origin, so it suffices to consider $y \in \mathbb{R}^+$. The derivative of K^{gv} is easily computed from the expression (2.19):

$$K^{\mathrm{gv}\prime}(y) = g_{\mathrm{gv}} g_{\mathrm{gv}}' + 2\Omega_0 y g_{\mathrm{gv}}^2, \tag{3.67}$$

and one immediately has that $K^{\mathrm{gv}\prime}(0) = 0$, i.e., K^{gv} has a critical point there. Whether it is a minimum or a maximum depends on s and Ω_0, but as we are going to see this does not matter. We can in any case compute easily the second derivative of K^{gv} exploiting once more (3.17):

$$K^{\mathrm{gv}\prime\prime}(y) = \left(g_{\mathrm{gv}}'\right)^2 + 4\Omega_0 y g_{\mathrm{gv}} g_{\mathrm{gv}}' + 2\left[\tfrac{1}{2}\alpha^2 y^2 + \Omega_0 - \mu^{\mathrm{gv}} + \tfrac{1}{\pi} g_{\mathrm{gv}}^2\right] g_{\mathrm{gv}}^2. \tag{3.68}$$

Then we prove the crucial property of K^{gv}: suppose that K^{gv} has a maximum at $y_1 \geq 0$ and then a minimum at $y_2 > y_1$, then

$$\frac{K^{\mathrm{gv}}(y_2)}{g_{\mathrm{gv}}^2(y_2)} \geq \frac{K^{\mathrm{gv}}(y_1)}{g_{\mathrm{gv}}^2(y_1)}, \tag{3.69}$$

and in particular $K^{\mathrm{gv}}(y_2) \geq 0$ if $K^{\mathrm{gv}}(y_1) \geq 0$.

To conclude the argument once (3.69) is proven, it is sufficient to observe that K^{gv} has a critical point in $y = 0$, which by parity must be either a maximum or a minimum: if it is a maximum, then (3.69) shows that at any minimum point $y_2 > 0$, $K^{\mathrm{gv}}(y_2) \geq 0$. Notice that it does not matter whether K^{gv} has a single or multiple minima, because any minimum after the first requires the presence of a preceding maximum point, where K^{gv} is larger than its first minimum and therefore positive. If on the opposite K^{gv} has a minimum at the origin, then it means that there must be a maximum at some $y_1 > 0$, where obviously $K^{\mathrm{gv}}(y_1) \geq K^{\mathrm{gv}}(0) \geq 0$ and we can repeat the argument for any minimum after y_1.

Let us now prove (3.69): we assume again that K^{gv} has a maximum in $y_1 \geq 0$ and a minimum in $y_2 > y_1$. Then it must be $K^{\mathrm{gv}\prime}(y_{1,2}) = 0$, i.e.,

$$g_{\mathrm{gv}}'(y_{1,2}) = -2\Omega_0 y_{1,2} g_{\mathrm{gv}}^2(y_{1,2}). \tag{3.70}$$

Moreover replacing this condition in (2.19) and (3.68), we get

$$K^{\mathrm{gv}}(y_{1,2}) = \left[\Omega_0 \tfrac{s-2}{s+2} y_{1,2}^2 + \tfrac{1}{2} - \tfrac{2}{\Omega_0(s+2)}\left(\mu^{\mathrm{gv}} - \tfrac{1}{2\pi} g_{\mathrm{gv}}^2(y_{1,2})\right)\right] g_{\mathrm{gv}}^2(y_{1,2}), \tag{3.71}$$

$$K^{\mathrm{gv}\prime\prime}(y_{1,2}) = \left[\Omega_0^2(s-2) y_{1,2}^2 + 2\Omega_0 - 2\mu^{\mathrm{gv}} + \tfrac{2}{\pi} g_{\mathrm{gv}}^2(y_{1,2})\right] g_{\mathrm{gv}}^2(y_{1,2}). \tag{3.72}$$

Moreover

$$K^{\mathrm{gv}\prime\prime}(y_1) \leq 0 \leq K^{\mathrm{gv}\prime\prime}(y_2), \tag{3.73}$$

which implies

$$\Omega_0^2(s-2)y_1^2 + 2\Omega_0 - 2\mu + \tfrac{2}{\pi} g_{\mathrm{gv}}^2(y_1) \leq \Omega_0^2(s-2)y_2^2 + 2\Omega_0 - 2\mu + \tfrac{2}{\pi} g_{\mathrm{gv}}^2(y_2), \tag{3.74}$$

and using this inequality in the expression of $K^{\mathrm{gv}}(y_2)$, we obtain

$$\frac{K^{\mathrm{gv}}(y_2)}{g_{\mathrm{gv}}^2(y_2)} \geq \Omega_0 \frac{s-2}{s+2} y_1^2 - \frac{2}{\pi\Omega_0(s+2)} \left(g_{\mathrm{gv}}^2(y_2) - g_{\mathrm{gv}}^2(y_1)\right) + \frac{1}{2} - \frac{2}{\Omega_0(s+2)} \left(\mu^{\mathrm{gv}} - \frac{1}{2\pi} g_{\mathrm{gv}}^2(y_2)\right)$$
$$= \frac{K^{\mathrm{gv}}(y_1)}{g_{\mathrm{gv}}^2(y_1)} + \frac{1}{\pi\Omega_0(s+2)} \left(g_{\mathrm{gv}}^2(y_1) - g_{\mathrm{gv}}^2(y_2)\right) \geq \frac{K^{\mathrm{gv}}(y_1)}{g_{\mathrm{gv}}^2(y_1)}, \quad (3.75)$$

because by hypothesis $y_2 > y_1$ and g_{gv} is decreasing.

Notice that as a by-product of our analysis we found out that can have no global minima, since $\inf_{y\in\mathbb{R}} K^{\mathrm{gv}}(y) = 0$ and therefore at any such minimum point y_0, we would have $K^{\mathrm{gv}}(y_0) = 0$, but this clearly contradicts (3.69). Hence if the inequality on r.h.s. of (3.64) is strict then $K^{\mathrm{gv}}(y) > 0$ for any finite y. □

Proposition 3.10 introduces the equation (2.7). The next step is obviously to prove that such an equation as at least one solution:

Proposition 3.11 (Equation (2.7))**.**
The equation (2.7)

$$\Omega_0 = \frac{4}{s+2}\left[\mu^{\mathrm{gv}} - \frac{1}{2\pi} g_{\mathrm{gv}}^2(0)\right].$$

has at least one solution $\Omega_0 > 0$.

Proof. Let us first set

$$G(\Omega_0) := \frac{4}{s+2}\left[\mu^{\mathrm{gv}} - \frac{1}{2\pi} g_{\mathrm{gv}}^2(0)\right],$$

so that (2.7) reads $\Omega_0 = G(\Omega_0)$. We will show that $G(\Omega_0)$ is asymptotically smaller than Ω_0 (resp. larger) Ω_0 for large (resp. small) Ω_0.

Let us first consider $\Omega_0 \gg 1$: using the trivial bound $\|g_{\mathrm{gv}}\|_4^4 \leq \|g_{\mathrm{gv}}\|_\infty^2$ and the definition of μ^{gv}, we get

$$G(\Omega_0) \leq \tfrac{4}{s+2} E^{\mathrm{gv}}.$$

If now we plug into $\mathcal{E}^{\mathrm{gv}}$ as a trial state the ground state of the harmonic oscillator $h_{\mathrm{osc}} = -\frac{1}{2}\Delta + \frac{1}{2}\alpha^2 y^2$, we easily obtain

$$E^{\mathrm{gv}} \leq \tfrac{1}{2}\Omega_0\sqrt{s+2}\left(1 + \mathcal{O}(\Omega_0^{-1/2})\right),$$

so that

$$G(\Omega_0) \leq \tfrac{2}{\sqrt{s+2}}\Omega_0\left(1 + \mathcal{O}(\Omega_0^{-1/2})\right) < \Omega_0,$$

if $\Omega_0 \gg 1$, because $s > 2$.

On the other hand for small Ω_0, thanks to the estimate (3.18) and again the definition of μ^{gv}, we have

$$G(\Omega_0) \geq \tfrac{2}{s+2} E^{\mathrm{gv}}.$$

To bound from below E^{gv} for small Ω_0, we can simply drop the kinetic term to get

$$E^{\mathrm{gv}} \geq \inf_{\|\rho\|_1=1} \int_{\mathbb{R}} \mathrm{d}y \,\left\{\tfrac{1}{2}\alpha^2 y^2 \rho + \tfrac{1}{2\pi}\rho^2\right\},$$

i.e., a TF-like functional. By scaling we immediately obtain

$$\inf_{\|\rho\|_1=1} \int_{\mathbb{R}} \mathrm{d}y \,\left\{\tfrac{1}{2}\alpha^2 y^2 \rho + \tfrac{1}{2\pi}\rho^2\right\} = C\Omega_0^{2/3},$$

as $\Omega_0 \to 0$, so that

$$G(\Omega_0) \geq C\Omega_0^{2/3} > \Omega_0,$$

for Ω_0 small enough. □

In the above Proposition we have not investigated the uniqueness of the solution. We indeed expect that such a solution is in fact unique, but without a proof of this fact, we have to choose Ω_{c} equal to the *largest* possible solution.

Now we turn our attention back to the cost function K: as K^{gv} it is given by the sum of a the positive density $\frac{1}{2}g_\star^2$ and the negative potential function F (see next Proposition 3.12). In addition $g_\star$ is monotonically decreasing for $y \geq y_{\beta_\star} = o(1)$ and, like F, almost symmetric. Close the boundary of the interval $[-\eta, \eta]$, $g_\star$ gets extremely small (in fact exponentially small in ε) but F vanishes identically at $\pm\eta$. In conclusion it is clear that the overall positivity of K should then emerge from a very delicate balance between the two opposite contributions.

We first state some simple properties of F collected in the following

Proposition 3.12 (Properties of F).
The potential function defined in (3.60) *is such that*

$$F(y) \leq 0, \qquad \text{for any } y \in [-\eta, \eta], \tag{3.76}$$

$$F(\pm\eta) = 0. \tag{3.77}$$

Proof. One of the identities (3.77) is trivial, the other is a direct consequence of (3.40). In order to show that F is negative everywhere we compute the derivative

$$F'(y) = \tfrac{1}{1+\varepsilon^2 y} \left(2\Omega_0 y + \Omega_0 \varepsilon^2 y^2 - \beta_\star \varepsilon^2\right) g_\star^2(y), \tag{3.78}$$

and it is easy to verify that because of the first term $F'(-\eta) < 0$ while $F'(\eta) > 0$. Moreover F' vanishes at a single point $y_F = \mathcal{O}(\varepsilon^2)$, where F has a global minimum. Hence it is negative everywhere in $[-\eta, \eta]$. □

A very crucial piece of information about the potential function formulated in the next Proposition is an alternative expression of it, which relies on the variational equation (3.15) and is the analogue of (3.62) for F^{gv}.

Proposition 3.13 (Alternative expressions of F).
For any $y \in [-\eta, \eta]$ the potential function F admits the following alternative expressions

$$F(y) = -\frac{\Omega_0}{\alpha^2}\left(g_\star'\right)^2 + \frac{2\Omega_0}{\alpha^2}\left[\frac{1}{2}\alpha^2 y^2 + \frac{1}{2\pi} g_\star^2 - \mu_\star\right] g_\star^2 + \begin{cases} R_+(y) + R_+, & \text{if } y \geq y_{\beta_\star}, \\ R_-(y) + R_-, & \text{if } y \leq y_{\beta_\star}, \end{cases} \tag{3.79}$$

where

$$R_\pm(y) = \mathcal{O}(\varepsilon^2 \eta^7) g_\star^2(y), \qquad R_\pm = -\Omega_0 \eta^2 g_\star^2(\pm\eta)\left(1 + o(1)\right). \tag{3.80}$$

Proof. We consider only the case $y \geq y_\beta$, since the other one is analogous. The key ingredient of the proof is an integration by parts, exactly as for (3.62). We spell all the details nevertheless for the sake of clarity. Thanks to the vanishing of F at η, we have

$$\begin{aligned} F(y) &= \int_y^\eta \mathrm{d}t\; \tfrac{1}{1+\varepsilon^2 t}\left(-2\Omega_0 t - \Omega_0 \varepsilon^2 t^2 + \beta_\star \varepsilon^2\right) g_\star^2 = \int_y^\eta \mathrm{d}t\; \tfrac{1}{1+\varepsilon^2 t}\, g_\star^2\, \partial_t\left(-\Omega_0 t^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 t^3 + \beta_\star \varepsilon^2 t\right) \\ &= \tfrac{1}{1+\varepsilon^2 \eta}\left(-\Omega_0 \eta^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 \eta^3 + \beta_\star \varepsilon^2 \eta\right) g_\star^2(\eta) - \tfrac{1}{1+\varepsilon^2 y}\left(-\Omega_0 y^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 y^3 + \beta_\star \varepsilon^2 y\right) g_\star^2(y) + \\ &\quad + \int_y^\eta \mathrm{d}t\; \left(-\Omega_0 t^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 t^3 + \beta_\star \varepsilon^2 t\right)\left[\tfrac{\varepsilon^2}{(1+\varepsilon^2 y)^2} g_\star^2 - \tfrac{2}{(1+\varepsilon^2 y)} g_\star g_\star'\right] \\ &= -\Omega_0 \eta^2 g_\star^2(\eta)(1 + o(1)) + \left(\Omega_0 y^2 + \mathcal{O}(\varepsilon^2 \eta^4)\right) g_\star^2(y) \\ &\quad - 2\int_y^\eta \mathrm{d}t\; \tfrac{1}{(1+\varepsilon^2 y)}\left(-\Omega_0 t^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 t^3 + \beta_\star \varepsilon^2 t\right) g_\star g_\star'. \end{aligned} \tag{3.81}$$

We now rewrite the last term by reconstructing the potential $U_{\beta_\star}$ and using the variational equation (3.15): since

$$
\begin{aligned}
- \tfrac{1}{1+\varepsilon^2 y} \left(-\Omega_0 t^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 t^3 + \beta_\star \varepsilon^2 t \right) = \tfrac{2\Omega_0}{\alpha^2} (1+\varepsilon^2 t) U_{\beta_\star}(t) \\
+ \tfrac{\varepsilon^2}{(1+\varepsilon^2 t)} \left(\tfrac{1}{6}\alpha^2 t^3 - \tfrac{1}{2}\Omega_0 (s-2)\beta_\star t + \Omega_0 \varepsilon^2 \beta_\star t^2 - \tfrac{1}{2}\varepsilon^2 {\beta_\star}^2 \right),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
- 2 \int_y^\eta \mathrm{d}t \; \frac{1}{(1+\varepsilon^2 y)} \left(-\Omega_0 t^2 - \tfrac{1}{3}\Omega_0 \varepsilon^2 t^3 + \beta_\star \varepsilon^2 t \right) g_\star g_\star' = \frac{4\Omega_0}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, U_{\beta_\star}(t) g_\star g_\star' \\
+ 2\varepsilon^2 \int_y^\eta \mathrm{d}t \; \tfrac{1}{(1+\varepsilon^2 t)} \left(\tfrac{1}{6}\alpha^2 t^3 - \tfrac{1}{2}\Omega_0 (s-2)\beta_\star t + \Omega_0 \varepsilon^2 \beta_\star t^2 - \tfrac{1}{2}\varepsilon^2 {\beta_\star}^2 \right) g_\star g_\star' \\
= \frac{4\Omega_0}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, U_{\beta_\star}(t) g_\star g_\star' + \mathcal{O}(\varepsilon^2 \eta^7) g_\beta^2(y), \quad (3.82)
\end{aligned}
$$

where we have used the bound (3.42) and the monotonicity of $g_\star$ for $y \geq y_{\beta_\star}$. The first term on the r.h.s. can be rewritten by means of (3.15):

$$
\begin{aligned}
\frac{4\Omega_0}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, U_{\beta_\star}(t) g_\star g_\star' = \frac{2\Omega_0}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, \left[\tfrac{1}{2} (g_\star')^2 - \tfrac{1}{2\pi} g_\star^4 + \mu_\star g_\star^2 \right]' \\
+ \frac{2\Omega_0 \varepsilon^2}{\alpha^2} \int_y^\eta \mathrm{d}t \; (g_\star')^2 - \frac{4\Omega_0 \varepsilon^2}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, t^3 v(t) g_\star g_\star' = \\
= \frac{2\Omega_0}{\alpha^2} \left(1+\varepsilon^2 \eta\right) \left[-\tfrac{1}{2\pi} g_\star^4(\eta) + \mu_\star g_\star^2(\eta) \right] - \frac{2\Omega_0}{\alpha^2} \left(1+\varepsilon^2 y\right) \left[\tfrac{1}{2} (g_\star'(y))^2 - \tfrac{1}{2\pi} g_\star^4(y) + \mu_\star g_\star^2(y) \right] - \\
+ \frac{2\Omega_0 \varepsilon^2}{\alpha^2} \int_y^\eta \mathrm{d}t \; \left[\tfrac{1}{2} (g_\star')^2 + \tfrac{1}{2\pi} g_\star^4 - \mu_\star g_\star^2 \right] - \frac{4\Omega_0 \varepsilon^2}{\alpha^2} \int_y^\eta \mathrm{d}t \, (1+\varepsilon^2 t) \, t^3 v(t) g_\star g_\star' \\
= \frac{2\Omega_0}{\alpha^2} \mu_\star g_\star^2(\eta) - \frac{2\Omega_0}{\alpha^2} \left[\tfrac{1}{2} (g_\star'(y))^2 - \tfrac{1}{2\pi} g_\star^4(y) + \mu_\star g_\star^2(y) \right] + \mathcal{O}(\varepsilon^2 \eta^7) g_\star^2(y).
\end{aligned}
$$

Putting together the above estimate with (3.81) and (3.82), we obtain the result. □

Thanks to Proposition 3.13, the cost function K can also be expressed as

$$
K(y) = -\frac{\Omega_0}{\alpha^2} (g_\star')^2 + \frac{2\Omega_0}{\alpha^2} \left[\frac{\alpha^2}{4\Omega_0} + \frac{1}{2}\alpha^2 y^2 - \frac{1}{2\pi} g_\star^4 + \mu_\star \right] g_\star^2 + \begin{cases} R_+(y) + R_+, & \text{if } y \geq y_{\beta_\star}, \\ R_-(y) + R_-, & \text{if } y \leq y_{\beta_\star}. \end{cases} \quad (3.83)
$$

This alternative expression will play an important role in the proof of its positivity, exactly as for K^{gv}. Another important ingredient of the proof is also the closeness of K to K^{gv} as $\varepsilon \to 0$:

Lemma 3.2.
For any $\Omega_0 > 0$ and $y \in \widetilde{\mathcal{A}}_\eta$

$$
K(y) - K^{\mathrm{gv}}(y) = \mathcal{O}(\varepsilon^2 |\log \varepsilon|^\infty). \quad (3.84)
$$

Proof. The result is a direct consequence of the pointwise estimate (3.56). □

The above Lemma in combination with Proposition 3.10 might seem to give also the positivity of K inside $\widetilde{\mathcal{A}}_\eta$. However this is not the case because, although we proved that K^{gv} is positive on the whole real line, we did not provide any lower bound to it. In fact even by just looking at its minima, one could conclude from (3.69) that $K^{\mathrm{gv}}(y_2) \geq K^{\mathrm{gv}}(y_1) g_{\mathrm{gv}}^2(y_2)/g_{\mathrm{gv}}^2(y_1)$, where y_2, y_1 are the positions of the minimum point and the preceding maximum point (consider for simplicity the half-line $\mathbb{R}^+$). Now even if $K^{\mathrm{gv}}(y_1) > C > 0$ as it occurs for instance at the origin, the ratio between the densities can become extremely small in $\mathcal{A}_\eta$. In addition to that the inequality holds true only for the minima of K^{gv} and it might be that it has no minimum inside $\widetilde{\mathcal{A}}_\eta$ or $\mathcal{A}_\eta$, in which case we only know that it is positive there, but without any meaningful lower bound.

In fact we will be able to prove positivity of K only in domain strictly smaller than $\mathcal{A}_\eta$, because of the additional constant terms $R_\pm$ in (3.83), irrespective of their smallness. We thus set

$$\mathcal{A}_> := \left\{ y \in [-\eta, \eta] \,\middle|\, g_\star^2(y) \geq \eta^6 \max\left\{ g_\star^2(\eta), g_\star^2(-\eta) \right\} \right\}. \tag{3.85}$$

By monotonicity of $g_\star$ for large y is easy to see that $\mathcal{A}_> = [-y_-, y_+]$ with $y_\pm \to \infty$ as $\varepsilon \to 0$. Notice also that $g_\star$ is very small at the boundary of $\mathcal{A}_>$, although not as small as $g_\star(\eta)$.

We can now state the main result of this Section:

Proposition 3.14 (Positivity of K)**.**
If $\Omega_0 > \Omega_{\mathrm{c}}$ as $\varepsilon \to 0$,

$$K(y) > 0, \qquad \textit{for any } y \in \mathcal{A}_>. \tag{3.86}$$

Proof. As in the proof of positivity of K^{gv} in Proposition 3.10 the key idea is to show that positivity at the origin is indeed sufficient to get the result. This in turn is easily inherited from positivity of K^{gv} whenever $\Omega_0 > \Omega_{\mathrm{c}}$, via (3.84): by (3.66)

$$K^{\mathrm{gv}}(0) > C > 0,$$

but

$$K(0) - K^{\mathrm{gv}}(0) = \mathcal{O}(\varepsilon^2 |\log \varepsilon|^\infty),$$

and thus $K(0) > C > 0$ for a possibly different constant C.

The rest of the proof follows the same line of reasoning of the proof of Proposition 3.10. There are however two complications: first we have two alternative expressions of K in $[y_{\beta_\star}, \eta]$ and $[-\eta, y_{\beta_\star}]$ respectively. Recall that $y_{\beta_\star} = o(1)$ denotes the unique maximum point of $g_\star$. Second the presence of the constant terms $R_\pm$ in (3.83) is very annoying and in fact it is responsible of the restriction to $\mathcal{A}_>$.

In order to handle the first issue it is sufficient to take into account the two intervals $[y_{\beta_\star}, \eta]$ and $[-\eta, y_{\beta_\star}]$ separately and use a different expressions for K (see (3.83)).

The second issue on the other hand leads to the introduction of the modified cost function

$$\widetilde{K}(y) := K(y) - \delta_\varepsilon g_\star^2(y) - R_\pm \tag{3.87}$$

for some

$$0 < \delta_\varepsilon \ll \eta^{-2} \ll 1 \tag{3.88}$$

to be chosen later. Here we have used a compact notation to mean that we subtract R_+ (resp. R_-) in $[y_{\beta_\star}, \eta]$ (resp. $[-\eta, y_{\beta_\star}]$).

Now we observe that if $\Omega_0 > \Omega_{\mathrm{c}}$

$$\widetilde{K}(y_{\beta_\star}) = \widetilde{K}(0) + o(1) = K(0) + o(1) > 0, \tag{3.89}$$

thanks to the pointwise estimate (3.84) and since $\|g_\star\|_\infty \leq C$. It is interesting to remark that this is the only point in the proof where we use the condition $\Omega_0 > \Omega_{\mathrm{c}}$, although several later estimates are affected by this one. Moreover at the boundary of the domain we have

$$\widetilde{K}(\pm\eta) = \left(\tfrac{1}{2} - \delta_\varepsilon + \mathcal{O}(\varepsilon^3 \eta^7) \right) g_\star^2(\pm\eta) > 0. \tag{3.90}$$

Therefore in order to exclude that $\widetilde{K}$ becomes negative, it suffices to prove that it is positive at any possible *global* minimum point $-\eta < y_m < \eta$.

We claim that, for $\Omega_0 > \Omega_{\mathrm{c}}$, any global minimum point y_m of $\widetilde{K}$ must satisfy the condition

$$|y_m| \gg 1. \tag{3.91}$$

The reason is the pointwise estimate (3.84) and the observation contained in Proposition 3.10: $\widetilde{K}$ and K^{gv} are pointwise close and therefore we would have

$$K^{\mathrm{gv}}(y_m) \leq \min[K(-\eta), K(\eta)] + o(1) \leq \tfrac{1}{2}\min[g_\star^2(-\eta), g_\star^2(\eta)] + o(1) = o(1),$$

which in turn implies $K^{\mathrm{gv}}(y_m) \leq 0$ since K^{gv} is independent of ε. For $\Omega_0 > \Omega_{\mathrm{c}}$ this contradicts the statement of Proposition 3.10.

The key point in the proof is the following property: let y_m be a point where $\widetilde{K}$ reaches its global minimum $\widetilde{K}(y_m) < \widetilde{K}(y_{\beta_\star})$ (otherwise there would be nothing to prove) and y_M any maximum point of $\widetilde{K}$ such that $y_M < y_m$, if $y_m > y_{\beta_\star}$, or $y_M > y_m$ in the opposite case $y_m < y_{\beta_\star}$. Notice that such a maximum needs not to be the global maximum but its existence is a consequence of smoothness of $\widetilde{K}$ and the inequalities

$$\widetilde{K}(\pm\eta) < \widetilde{K}(y_{\beta_\star}), \qquad \widetilde{K}(y_m) < \widetilde{K}(y_{\beta_\star}).$$

Then we are going to prove that

$$\frac{\widetilde{K}(y_m)}{g_\star^2(y_m)} \geq \frac{\widetilde{K}(y_M)}{g_\star^2(y_M)} + o(1). \tag{3.92}$$

Now suppose that this is true, then we can pick a maximum point y_M of $\widetilde{K}$ where $\widetilde{K}(y_M) \geq \widetilde{K}(y_{\beta_\star}) > C > 0$. In addition it must be

$$y_M = \mathcal{O}(1), \tag{3.93}$$

because $\widetilde{K}(y) \leq Cg^2(y)$ and the decay estimate (3.22) or the pointwise estimate (3.56) imply that $\widetilde{K}(y) = o(1)$, if $|y| \gg 1$. Hence by the lower bound (3.22) $g_\star(y_M) \geq C > 0$, (3.92) yields

$$\widetilde{K}(y_m) \geq g_\star^2(y_m)\left(C^{-2}\widetilde{K}(y_{\beta_\star}) + o(1)\right) \geq C_0 g_\star^2(y_m) > 0 \tag{3.94}$$

for some $C_0 > 0$. In fact we have obtained something more: for any $y \in \mathcal{A}_\eta$ either $\widetilde{K}(y) \geq \min\{\widetilde{K}(-\eta), \widetilde{K}(\eta)\} > 0$ or

$$\frac{\widetilde{K}(y)}{g_\star^2(y)} \geq \frac{\widetilde{K}(y_m)}{g_\star^2(y)} \geq C\frac{g_\star^2(y_m)}{g_\star^2(y)} > 0.$$

Either way $\widetilde{K}$ is positive everywhere in $[-\eta, \eta]$. Moreover the positivity of $\widetilde{K}$ implies that

$$K(y) > R_\pm + \delta_\varepsilon g_\star^2(y) \geq 0, \qquad \text{if } g_\star^2(y) \geq \delta_\varepsilon^{-1}|R_+|,$$

for any $y \in \mathcal{A}_\eta$. If now we restrict the inequality to $\mathcal{A}_>$ and we choose, e.g., $\delta_\varepsilon = \eta^{-3}$, the estimates (3.80) imply that inside $\mathcal{A}_>$

$$g_\star^2(y) \geq \eta^6 \max\left\{g_\star^2(-\eta), g_\star^2(\eta)\right\} \gg C\eta^5 \max\left\{g_\star^2(-\eta), g_\star^2(\eta)\right\} \geq \delta_\varepsilon^{-1}|R_\pm|,$$

so that $K(y)$ is strictly positive for any $y \in \mathcal{A}_>$. In fact a closer look to the chain of inequalities reveals that we have proven something more, i.e., for ε small enough

$$K(y) \geq \eta^{-3} g_\star^2(y), \qquad \text{for any } y \in \mathcal{A}_>. \tag{3.95}$$

We have now to prove (3.92). Recall the assumption: we have a global minimum of $\widetilde{K}$ at y_m and a maximum at y_M, which is on the left (resp. right) of y_m, if $y_m > y_{\beta_\star}$ (resp. $y_m < y_{\beta_\star}$). The idea is the same used in the proof of Proposition 3.10: the derivative of $\widetilde{K}$, i.e.,

$$\widetilde{K}'(y) = \left[(1-\delta_\varepsilon) g_\star' + \tfrac{1}{1+\varepsilon^2 y} \left(2\Omega_0 y + \Omega_0 \varepsilon^2 y^2 - \beta_\star \varepsilon^2 \right) g_\star \right] g_\star, \tag{3.96}$$

must vanish both at y_m and y_M and therefore

$$g_\star'(y_{m,M}) = -\frac{1}{(1-\delta_\varepsilon)(1+\varepsilon^2 y_{m,M})} \left(2\Omega_0 y_{m,M} + \Omega_0 \varepsilon^2 y_{m,M}^2 - \beta_\star \varepsilon^2 \right) g_\star(y_{m,M}). \tag{3.97}$$

The second derivative of K can be computed as well:

$$\widetilde{K}''(y) = (1-\delta_\varepsilon) \left(g_\star' \right)^2 + 4\Omega_0 y g_\star g_\star' + \left[(1-\delta_\varepsilon) \left(\alpha^2 y^2 + \tfrac{2}{\pi} g_\star^2 - 2\mu_\star \right) + 2\Omega_0 + \mathcal{O}(\varepsilon^2 \eta^5) \right] g_\star^2, \tag{3.98}$$

so that at any extreme point of $\widetilde{K}$, one has

$$\widetilde{K}''(y_{m,M}) = \left[\Omega_0^2 (s-2) y_{m,M}^2 + 2\Omega_0 + \tfrac{2}{\pi} g_\star^2(y_{m,M}) - 2\mu_\star + o(1) \right] g_\star^2(y_{m,M}), \tag{3.99}$$

where we have exploited the condition (3.88). Similarly by (3.83) we get

$$\widetilde{K}(y_{m,M}) = \left[\Omega_0 \tfrac{s-2}{s+2} y_{m,M}^2 + \tfrac{1}{2} - \tfrac{1}{\pi\Omega_0(s+2)} g_\star^2(y_{m,M}) - \tfrac{2}{\Omega_0(s+2)} \mu_\star + o(1) \right] g_\star^2(y_{m,M}), \tag{3.100}$$

and a direct comparison between (3.99) and (3.100) yields

$$\frac{\widetilde{K}(y_{m,M})}{g_\star^2(y_{m,M})} = \frac{s-2}{s+2} - \frac{g_\star^2(y_{m,M})}{\pi\Omega_0(s+2)} + \frac{\widetilde{K}''(y_{m,M})}{\Omega_0(s+2) g_\star^2(y_{m,M})} + o(1). \tag{3.101}$$

Now this is the key identity because by assumption (recall also (3.91))

$$\widetilde{K}''(y_M) \leq 0 \leq \widetilde{K}''(y_m), \qquad g_\star(y_m) < g_\star(y_M),$$

so that

$$\frac{\widetilde{K}(y_m)}{g_\star^2(y_m)} \geq \frac{\widetilde{K}(y_M)}{g_\star^2(y_M)} + o(1),$$

i.e., (3.92) is proven. Note that the fact that we have two different explicit expressions of $\widetilde{K}$ for $y > y_{\beta_\star}$ and $y < y_{\beta_\star}$ did not affect the proof, because the difference between the two expressions is $o(1)$ and therefore can be included in the error term. □

4 Energy Asymptotics

We attack in this Section the proof of Theorem 2.2, which will imply the main result of the paper. The result is obtained by combining upper (Proposition 4.1) and lower (Proposition 4.2) bounds on E^{GP}.

4.1 Upper Bound

The upper bound on E^{GP} is stated in next

Proposition 4.1 (GP energy upper bound)**.**
As $\varepsilon \to 0$,

$$E^{\mathrm{GP}} \leq \frac{E_\star^{\mathrm{gv}}}{\varepsilon^4} + \mathcal{O}(1). \tag{4.1}$$

Proof. The proof is rather simple because it is sufficient to test $\mathcal{E}^{\mathrm{GP}}$ on suitable trial function of the form

$$\Psi_{\mathrm{trial}}(\mathbf{x}) := \frac{1}{\sqrt{2\pi\varepsilon}} g_{\mathrm{trial}}\left(\tfrac{|\mathbf{x}|-1}{\varepsilon^2}\right) \exp\left\{i \left\lfloor \Omega + \beta_\star \right\rfloor \theta\right\}, \tag{4.2}$$

where g_{trial} coincides up to a normalization constant with $g_\star$ within $\mathcal{A}_\eta$ and is suitably regularized outside. The calculation is rather straightforward and we omit it for the sake of brevity. Note that the remainder $\mathcal{O}(1)$ is entirely due to the fact that the phase $\Omega + \beta_\star$ might not be an integer number. Otherwise one would obtain a much better error term $\mathcal{O}(\varepsilon^4)$. □

4.2 Lower Bound

A lower bound for E^{GP} matching the upper bound of Proposition 4.1 is formulated in next

Proposition 4.2 (GP energy lower bound)**.**
If $\Omega_0 > \Omega_{\mathrm{c}}$, *as* $\varepsilon \to 0$,

$$E^{\mathrm{GP}} \geq \frac{E^{\mathrm{gv}}_\star}{\varepsilon^4} + \mathcal{O}(\varepsilon^\infty). \tag{4.3}$$

Proof. We first restrict the integration in the GP energy functional to the domain $\mathcal{A}_\eta$ (recall its definition in (3.2): to this purpose we just have to observe that all the three terms in the GP energy functional are pointwise positive and thus we can simply drop their integrals outside $\mathcal{A}_\eta$. Of course ψ^{GP} is not normalized in $L^2(\mathcal{A}_\eta)$ but the exponential decay proven in Proposition 3.3 guarantees that

$$\left\| \psi^{\mathrm{GP}} \right\|_{L^2(\mathcal{A}_\eta)} = 1 + \mathcal{O}(\varepsilon^\infty), \tag{4.4}$$

by taking η_0 large enough.

The first step in the proof is a splitting of the energy, in order to extract the leading order term $E^{\mathrm{gv}}_\star/\varepsilon^4$. This is now rather standard and we do not spell all the details of the computation. We just note that one sets

$$\psi^{\mathrm{GP}}(\mathbf{x}) =: \frac{1}{\sqrt{2\pi\varepsilon}} u\left(x, \vartheta\right) g_\star\left(\tfrac{x-1}{\varepsilon^2}\right) e^{i(\Omega+\beta_\star)\theta}. \tag{4.5}$$

Since $\Omega + \beta_\star$ needs not to be an integer, u is not single-valued in general, but

$$u(x, \vartheta + 2k\pi) = e^{-i2\pi k(\Omega+\beta_\star)} u(x, \vartheta), \tag{4.6}$$

for any $k \in \mathbb{Z}$. A part from that u is finite for any $\mathbf{x} \in \mathcal{A}_\eta$ thanks to the strict positivity of $g_\star$. A long but simple computation using the variational equation for $g_\star$ gives

$$E^{\mathrm{GP}} \geq \frac{E^{\mathrm{gv}}_\star}{\varepsilon^4} + \frac{\mathcal{E}[u]}{2\pi\varepsilon^2} + \mathcal{O}(\varepsilon^\infty), \tag{4.7}$$

where the inequality is mainly due to the restriction of the integration domain and setting $y = 1 + \varepsilon^2 x$ for short

$$\mathcal{E}[u] = \int_{\mathcal{A}_\eta} \mathrm{d}\mathbf{x}\; g_\star^2(y) \left\{ \tfrac{1}{2} \left|\nabla u\right|^2 + \mathbf{a} \cdot \mathbf{j}_u + \tfrac{1}{2\pi\varepsilon^4} g_\star^2(y)(1 - |u|^2)^2 \right\}, \tag{4.8}$$

$$\mathbf{a}(x) := \left(\frac{\Omega + \beta_\star}{x} - \Omega x \right) \mathbf{e}_\vartheta, \tag{4.9}$$

and the superfluid current is defined in (2.16). The rest of the proof is devoted to prove that

$$\mathcal{E}[u] \geq \mathcal{O}(\varepsilon^\infty). \tag{4.10}$$

In order to exploit the cost function trick mentioned in Section 2.1 and the positivity of K proven in Proposition 3.14, we need to restrict again the integration domain in $\mathcal{E}[u]$ to $\mathcal{A}_{>}^{2\mathrm{D}} \subset \mathcal{A}_{\eta}$, where

$$\mathcal{A}_{>}^{2\mathrm{D}} = \left\{ \mathbf{x} \in \mathbb{R}^2 \;\middle|\; |1 - |\mathbf{x}|| / \varepsilon^2 \in \mathcal{A}_{>} \right\}.$$

The only annoying term is the only one which is not positive, i.e., the second one in (4.8):

$$\left| \int_{\mathcal{A}_\eta \setminus \mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^2(y)\, \mathbf{a} \cdot \mathbf{j}_u \right| \leq \|\mathbf{a}\|_{L^\infty(\mathcal{A}_\eta)} \int_{\mathcal{A}_\eta \setminus \mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^2(y)\, |u|\, |\nabla_\vartheta u|$$
$$\leq C\eta^2 \int_{\mathcal{A}_\eta \setminus \mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, \left|\psi^{\mathrm{GP}}\right| \left|\nabla_\vartheta \psi^{\mathrm{GP}}\right| \leq C\varepsilon^2 \eta^4 \left\|\psi^{\mathrm{GP}}\right\|_{L^\infty(\mathcal{A}_\eta \setminus \mathcal{A}_{>}^{2\mathrm{D}})} \left\|\nabla \psi^{\mathrm{GP}}\right\|_{L^\infty(\mathcal{A}_\eta)}$$
$$\leq C\varepsilon^{-4} \eta^4 \left\|\psi^{\mathrm{GP}}\right\|_{L^\infty(\mathcal{A}_\eta \setminus \mathcal{A}_{>}^{2\mathrm{D}})}, \tag{4.11}$$

where we have used the bound $\left\|\nabla \psi^{\mathrm{GP}}\right\|_{L^\infty(\mathcal{A}_\eta)} \leq C\varepsilon^{-6}$, following from

$$\|\nabla \psi\|_\infty \leq C \left(\|\Delta \psi\|_\infty^{1/2} \|\psi\|_\infty^{1/2} + \|\psi\|_\infty \right), \tag{4.12}$$

which can be proven from Gagliardo-Nirenberg inequalities exactly as in [CRY, Lemma 5.1]. However the lower bound (3.22) easily implies that if we set $\mathcal{A}_{>} =: [-y_-, y_+]$, then

$$y_\pm = \eta(1 + o(1)),$$

so that

$$\left|\psi^{\mathrm{GP}}\right|_{\partial \mathcal{A}_{>}^{2\mathrm{D}}} \leq \mathcal{O}(\varepsilon^\infty), \tag{4.13}$$

again by (3.19) and the arbitrariness in the choice of η_0. Hence (4.11) yields an error which can be made smaller than any power of ε and we get the lower bound

$$\mathcal{E}[u] \geq \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\; g_\star^2(y) \left\{ \tfrac{1}{2} |\nabla u|^2 + \mathbf{a} \cdot \mathbf{j}_u + \tfrac{1}{2\pi\varepsilon^4} g_\star^2(y)(1 - |u|^2)^2 \right\} + \mathcal{O}(\varepsilon^\infty). \tag{4.14}$$

We can now finally integrate by the angular momentum term by using the potential function F defined in (3.60): it is trivial to verify that

$$2 g_\star^2 \left(\tfrac{x-1}{\varepsilon^2} \right) \mathbf{a}(x) = -\partial_x F \left(\tfrac{x-1}{\varepsilon^2} \right) \mathbf{e}_\vartheta, \tag{4.15}$$

so that

$$\int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^2(y)\, \mathbf{a} \cdot \mathbf{j}_u = -\frac{1}{2} \int_0^{2\pi} \mathrm{d}\vartheta \int_{1-\varepsilon^2 y_-}^{1+\varepsilon^2 y_+} \mathrm{d}x\, \partial_x F \left(\tfrac{x-1}{\varepsilon^2} \right) \Re \left[i u(x,\vartheta) \partial_\vartheta u^*(x,\vartheta) \right]$$
$$= \frac{1}{2} \int_0^{2\pi} \mathrm{d}\vartheta \int_{1-\varepsilon^2 y_-}^{1+\varepsilon^2 y_+} \mathrm{d}x\, F \left(\tfrac{x-1}{\varepsilon^2} \right) \Re \left[i \partial_x u(x,\vartheta) \partial_\vartheta u^*(x,\vartheta) + i u(x,\vartheta) \partial_{x,\vartheta}^2 u^*(x,\vartheta) \right]$$
$$- \frac{1}{2} \int_0^{2\pi} \mathrm{d}\vartheta \left| F \left(\tfrac{x-1}{\varepsilon^2} \right) \Re \left[i u(x,\vartheta) \partial_\vartheta u^*(x,\vartheta) \right] \right|_{1-\varepsilon^2 y_-}^{1+\varepsilon^2 y_+}. \tag{4.16}$$

The boundary term can be easily proven to provide an exponentially small correction: consider, e.g., the term at $1 + \varepsilon^2 y_+$, since $|F(y_\pm)| \leq C\eta^8 g_\star^2(\pm\eta)$, one can reconstruct a term, which can be bounded exactly as (4.11). The result is an error $\mathcal{O}(\varepsilon^\infty)$. The rest is integrated by parts once more but this time w.r.t. ϑ:

$$\frac{1}{2} \int_0^{2\pi} \mathrm{d}\vartheta \int_{1-\varepsilon^2 y_-}^{1+\varepsilon^2 y_+} \mathrm{d}x\, F \left(\tfrac{x-1}{\varepsilon^2} \right) \Re \left[i u(x,\vartheta) \partial_{x,\vartheta}^2 u^*(x,\vartheta) \right]$$
$$= -\frac{1}{2} \int_0^{2\pi} \mathrm{d}\vartheta \int_{1-\varepsilon^2 y_-}^{1+\varepsilon^2 y_+} \mathrm{d}x\, F \left(\tfrac{x-1}{\varepsilon^2} \right) \Re \left[i \partial_\vartheta u(x,\vartheta) \partial x u^*(x,\vartheta) \right],$$

where the vanishing of boundary terms is due to the periodicity of $u^* \partial_x u$ and its complex conjugate (compare with (4.6)). Altogether we have thus obtained that

$$\int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^2(y)\, \mathbf{a} \cdot \mathbf{j}_u = \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, F\left(\tfrac{x-1}{\varepsilon^2}\right) \Re\left[i \nabla_x u(x,\vartheta) \nabla_\vartheta u^*(x,\vartheta)\right] + \mathcal{O}(\varepsilon^\infty)$$
$$\geq - \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, \left|F\left(\tfrac{x-1}{\varepsilon^2}\right)\right| |\nabla u|^2 + \mathcal{O}(\varepsilon^\infty) = - \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, F\left(\tfrac{x-1}{\varepsilon^2}\right) |\nabla u|^2 + \mathcal{O}(\varepsilon^\infty), \quad (4.17)$$

and therefore

$$\mathcal{E}[u] \geq \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, \left\{ K\left(\tfrac{x-1}{\varepsilon^2}\right) |\nabla u|^2 + \frac{1}{2\pi\varepsilon^4} g_\star^2(y)(1-|u|^2)^2 \right\} + \mathcal{O}(\varepsilon^\infty)$$
$$\geq \eta^{-3} \int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^2(y)\, |\nabla u|^2 + \mathcal{O}(\varepsilon^\infty) \geq \mathcal{O}(\varepsilon^\infty), \quad (4.18)$$

thanks to Proposition 3.14 and in particular (3.95). □

5 Giant Vortex Transition

In this Section we prove the results regarding absence of vortices and total vorticity of the condensate.

Proof of Theorem 2.1. Combining (4.18) with (4.7) and the upper bound proven in Proposition 4.1, we get

$$\int_{\mathcal{A}_{>}^{2\mathrm{D}}} \mathrm{d}\mathbf{x}\, g_\star^4(y)(1-|u|^2)^2 = \mathcal{O}(\varepsilon^6), \quad (5.1)$$

which already means that $|u|$ can not differ too much from 1. To deduce the pointwise estimate of Theorem 2.1, we need to combine this with an estimate of $\|\nabla u\|_\infty$.

As in [CRY, Lemma 4.3] we obtain from (1.16) and (3.15) the following variational equation for u:

$$-\tfrac{1}{2} g_\beta \Delta u - \tfrac{1}{\varepsilon^2} g_\star' \partial_x u - i g_\star \mathbf{a} \cdot \nabla u + \tfrac{1}{\pi\varepsilon^4} g_\star^3 \left(|u|^2 - 1\right) u = \left(\mu^{\mathrm{GP}} - \tfrac{1}{\varepsilon^4}\mu_\star\right) g_\star u, \quad (5.2)$$

which yields (recall the definition of $\mathcal{A}_{\mathrm{bulk}} \subset \mathcal{A}_{>}^{2\mathrm{D}}$ in (2.8))

$$\|\Delta u\|_{L^\infty(\mathcal{A}_{\mathrm{bulk}})} \leq C \left[\varepsilon^{-2}\eta \|\nabla u\|_{L^\infty(\mathcal{A}_{\mathrm{bulk}})} + \varepsilon^{-4}\eta^{3a}\right].$$

Now using the elliptic estimate (4.12) we conclude that

$$\|\nabla u\|_{L^\infty(\mathcal{A}_{\mathrm{bulk}})} = \mathcal{O}\left(\varepsilon^{-2}\eta^{1+\frac{3a}{2}}\right). \quad (5.3)$$

Suppose now that it exists $\mathbf{x}_0 \in \mathcal{A}_{\mathrm{bulk}}$ such that $|u(\mathbf{x}_0) - 1| \geq \varepsilon^{1/2}|\log\varepsilon|^b$, for some $b > 0$ to be chosen later. Then from (5.3) we get that

$$||u| - 1| \geq \tfrac{1}{2}\varepsilon^{1/2}|\log\varepsilon|^b, \qquad \text{for } \mathbf{x} \in \mathcal{B}_\varrho(\mathbf{x}_0) \cap \mathcal{A}_a,$$

with $\varrho = \varepsilon^{5/2}|\log\varepsilon|^{b-1-\frac{3a}{2}}$, and

$$\mathcal{O}(\varepsilon^6) = \int_{\mathcal{A}_{\mathrm{bulk}} \cap \mathcal{B}_\varrho(\mathbf{x}_0)} \mathrm{d}\mathbf{x}\, g_\star^4(y)\left(1-|u|^2\right)^2 \geq C\varepsilon^6|\log\varepsilon|^{2b-7a-2},$$

which is a contradiction for all $b \geq 4a - 1$. □

We now focus on the proof of Theorem 2.3 and for later purposes we state a useful Lemma, which is the analogue of [CPRY3, Lemma 3.5]:

Lemma 5.1.
Let $\Omega_0 > \Omega_{\mathrm{c}}$ and R be a radius satisfying $R = 1 + \mathcal{O}\left(\varepsilon^2\right)$, then

$$\left|\deg\left(u, \partial\mathcal{B}_R\right)\right| = \mathcal{O}(1) \tag{5.4}$$

Proof. We use a smooth radial cut-off function χ with support in $[\widetilde{R}, R]$ such that $\chi(\widetilde{R}) = 0$ and $\chi(R) = 1$, for some radius

$$\widetilde{R} = R - c\varepsilon^2$$

with $c > 0$. We also require that $|\chi| \leq 1$ and $|\nabla\chi| = \mathcal{O}(\varepsilon^{-2})$. Then by Stokes formula

$$\deg\left(u, \partial\mathcal{B}_R\right) = \frac{1}{\pi}\int_{\partial\mathcal{B}_R} \mathrm{d}\sigma\, \Im\left(\frac{\nabla_\vartheta u}{u}\right) = \frac{1}{\pi}\int_{\partial\mathcal{B}_R} \mathrm{d}\sigma\, \chi(R)\, \Im\left(\frac{\nabla_\vartheta u}{u}\right)$$
$$= \frac{1}{\pi}\int_{\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}} \mathrm{d}\mathbf{x}\, \nabla^\perp\chi \cdot \Im\left(\frac{\nabla u}{u}\right). \tag{5.5}$$

Therefore

$$\left|\deg\left(u, \partial\mathcal{B}_R\right)\right| \leq \frac{C}{\varepsilon^2}\int_{\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}} \mathrm{d}\mathbf{x}\, \frac{|\nabla u|}{|u|} \leq \frac{C}{\varepsilon^2}\left|\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}\right|^{1/2} \left\|\nabla u\right\|_{L^2(\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}})}, \tag{5.6}$$

where we used that $\left\|1 - |u|\right\|_{L^\infty(\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}})} = o(1)$. Now the result proven in Proposition 3.10 in fact says that for any $\mathbf{x} \in \mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}$ and for $\Omega_0 > \Omega_{\mathrm{c}}$, there exists a constant $C > 0$ such that

$$K^{\mathrm{gv}}\left(\tfrac{x-1}{\varepsilon^2}\right) \geq C,$$

and thanks to (3.84) the same inequality holds true for K, i.e.,

$$K\left(\tfrac{x-1}{\varepsilon^2}\right) \geq C > 0, \qquad \text{for any } \mathbf{x} \in \mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}. \tag{5.7}$$

Going back to (4.18) this yields

$$\mathcal{O}(\varepsilon^2) \geq \int_{\mathcal{A}^{\mathrm{2D}}_{>}} \mathrm{d}\mathbf{x}\, K\left(\tfrac{x-1}{\varepsilon^2}\right)|\nabla u|^2 \geq \int_{\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}}} \mathrm{d}\mathbf{x}\, K\left(\tfrac{x-1}{\varepsilon^2}\right)|\nabla u|^2 \geq C\left\|\nabla u\right\|_{L^2(\mathcal{B}_R \setminus \mathcal{B}_{\widetilde{R}})},$$

which gives the result once plugged into (5.6). □

We are now in position to complete the estimate of the winding number of ψ^{GP}:

Proof of Theorem 2.3. We follow [CPRY3, proof of Theorem 1.5]. The positivity of $|\psi^{\mathrm{GP}}|$ on $\partial\mathcal{B}_R$ is guaranteed for any radius $R = 1 + \mathcal{O}\left(\varepsilon^2\right)$ thanks to (2.9). A simple computation shows that

$$\deg\left(\psi^{\mathrm{GP}}, \partial\mathcal{B}_R\right) = \Omega + \beta_\star + \deg\left(u, \partial\mathcal{B}_R\right),$$

which yields the result in combination with (5.4). □

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