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Anti-windup design for directionality compensation with application to quadrotor UAVs

Pietro Ghignoni[1], Nicolò Buratti[1], Davide Invernizzi[1], Marco Lovera[1]

***Abstract*—In this paper we propose an anti-windup strategy to counteract directionality effects arising in saturated MIMO systems in which independent dynamical subsystems are coupled through a static mixing of the inputs. Since such systems are affected by undesired input cross-couplings when saturation occurs, we propose an anti-windup augmentation scheme built on top of the baseline controller and tailored to achieve satisfactory time-domain performance for reference signals of interest. Motivated by the quadrotor application, in which position control has higher priority over yaw control for safety reasons, we embed in the anti-windup synthesis procedure the possibility to prioritize the level of performance degradation during saturation for the different system outputs.**

***Index Terms*—Anti-windup design, saturated MIMO systems, directionality, quadrotor, DLAW**

I. Introduction

THE design of control laws for quadrotor Unmanned Aerial Vehicles (UAVs) is typically carried out by referring to the six-dimensional dynamical model of a rigid body in which four inputs are available: a force acting along the positive direction of the axis orthogonal to the propeller rotors (thrust axis) and a torque in any direction. This setting allows one to deal with the quadrotor underactuation by implementing hierarchical control architectures in which the torque is used to track a desired yaw rotation while tilting the thrust axis in the direction of the force required for position tracking. Since the thrust and the torque are actually intermediate inputs obtained by a static mixing of the thrusts delivered by the propellers, onboard implementation requires an allocation step. Due to the peculiar structure of the quadrotor mixing map, applications which push the actuators to the saturation limit (*e.g.*, aggressive maneuvers, heavy payload) can give rise to motions in undesired directions, making quadrotors a good testbed for the design of compensation methods for directionality effects.

Directionality issues are commonly tackled in the literature about quadrotors by using iterative thrust-mixing schemes that prioritize roll-pitch control over thrust and yaw [1], [2]. While Anti-Windup (AW) augmentation designs with formal stability and performance guarantees have been developed in recent years for directionality compensation [3], [4], [5], the specific case of quadrotors has been dealt with in a systematic manner only in [6], which proposed full-order pseudo-decentralized and channel-by-channel AW compensators by extending the AW approach of [7].

While in a previous work [8] we considered the problem of saturated attitude control in quadrotors and we dealt with integral windup effects, in this paper directionality issues arising in the position and yaw control are addressed. Differently from [6], which addressed directionality issues for exponentially stable linear systems and sought a global result, this work addresses windup effects that come up when specific bounded reference signals are commanded, notably step-like references, which is one the most common operating scenario for quadrotors. To this aim, by referring to a generalized sector condition which includes the input mixing map, we appeal to the ideas of [9] and cast the AW design as a LMI-based optimization problem in which the objective is to penalize a weighted mismatch between the response of a suitably selected reference model and the response of the saturated system with AW compensation. Emphasis is placed on prioritization of control objectives and on time-domain performance in practical conditions by embedding in the AW synthesis model a filter which replicates the reference signals of interest.

After tuning a fixed-dynamics compensator having the structure of a static compensator cascaded with a unit delay (to avoid solving algebraic loops in real time implementations [10]), the performance of the augmented controller has been assessed in simulations by considering a combined position and yaw maneuver which induces cross-directional effects. Simulation results have shown that the proposed synthesis procedure can allow one to obtain a decentralized AW compensator (up to numerical errors), which is desirable in practice to reduce complexity as well as to avoid undesired cross-couplings that a full AW compensator would unavoidably induce during saturation. Following a consolidated approach in robust control for MIMO systems, by changing the weight of the performance outputs in the synthesis procedure we show that it is possible to achieve prioritization of the control objectives straightforwardly.

Notation. In this paper $\mathbb{Z}(\mathbb{Z}_{>0},\mathbb{Z}_{\geq 0})$ denotes the set of integers (positive, nonnegative integers), $\mathbb{R}\,(\mathbb{R}_{>0},\mathbb{R}_{\geq 0})$ denotes the set of real numbers (positive, nonnegative real numbers), $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ the set of $m\times n$ real matrices. The ith vector of the canonical basis of $\mathbb{R}^n$ is denoted as e_i and the identity matrix in $\mathbb{R}^{n\times n}$ is denoted as $I_n := [e_1 \cdots e_i \cdots e_n]$. Given $A \in \mathbb{R}^{n\times n}$, we use the compact notation $A \in \mathbb{R}^{n\times n}_{>0}(\mathbb{R}^{n\times n}_{<0})$ to represent a positive (negative) definite matrix. For a square matrix X, we denote $\mathrm{He}(X) := X + X^\top$. Given a sequence $x(t)$, $t \in \mathbb{Z}_{\geq 0}$, x^+ is a shorthand notation for $x(t+1)$. Func-

[1]P. Ghignoni, N. Buratti, D. Invernizzi and M. Lovera are with Dipartimento di Scienze e Tecnologie Aerospaziali, Politecnico di Milano, Via La Masa 34, 20156 Milano, Italy {pietro.ghignoni,nicolo1.buratti}@mail.polimi.it, {davide.invernizzi, marco.lovera}@polimi.it

tion $\text{sat}_{\underline{u}}^{\bar{u}}(\cdot)$ denotes the decentralized saturation function, *i.e.*, given $u \in \mathbb{R}^n$ and some bounds $\underline{u}, \bar{u} \in \mathbb{R}^n_{\geq 0}$, $\text{sat}_{\underline{u}}^{\bar{u}}(u) := (\max(\min(\bar{u}_1, u_1), -\underline{u}_1), \ldots, \max(\min(\bar{u}_n, u_n), -\underline{u}_n))$. Finally, $\overline{\text{co}}\{v_r \in \mathbb{R}^n,\ r = 1, \ldots, n_v\}$ is the closed convex hull, *i.e.*, the smallest closed convex set that contains the points identified by the vectors v_r. The S map $S(\cdot) : \mathbb{R}^3 \to \mathfrak{so}(3) := \{W \in \mathbb{R}^{3\times3} : W = -W^\top\}$ is defined such that given $a, b \in \mathbb{R}^3$ one has $S(a)b = a \times b$.

II. Modeling saturation effects in quadrotors

A quadrotor UAV is an aerial vehicle made by a central body and four arms, each of which carries a propeller. In this work we focus on operating conditions for which the model of quadrotors in near hovering flight, *i.e.*,

$$\dot{\alpha} = \omega, \qquad \dot{x} = v \tag{1}$$

$$J\dot{\omega} = \tau_c + \tau_e, \quad m\ddot{x} = mgS(\alpha)e_3 + (T_c - mg)e_3 + f_e, \tag{2}$$

can be considered sufficiently accurate [11]. In (1)-(2) $J = J^\top \in \mathbb{R}^{3\times3}_{>0}$ is the UAV inertia matrix with respect to the center of mass O_B, $m \in \mathbb{R}_{>0}$ is the mass, $g = 9.81\,\text{m/s}^2$ is the gravitational acceleration, $\alpha := (\phi, \theta, \psi) \in \mathbb{R}^3$ collects the roll (ϕ), pitch (θ) and yaw (ψ) angles, $x \in \mathbb{R}^3$ is the position vector from the origin of an inertial frame to O_B, $S(\alpha)e_3 = [\theta\ -\phi\ 0]^\top$, $\omega \in \mathbb{R}^3$ is the angular velocity, $v \in \mathbb{R}^3$ is the inertial translational velocity. Finally $(T_c e_3, \tau_c) \in \mathbb{R}^6$ is the control wrench delivered by the propellers while $(f_e, \tau_e) \in \mathbb{R}^6$ is the disturbance wrench including, *e.g.*, aerodynamic effects and gyroscopic torque of the rotors.

For a quadrotor UAV, the control force and torque are the resultant at O_B of the thrusts, delivered by the four propellers, which are applied along the unit vector b_3 representing the direction orthogonal to the propellers. The magnitudes of the thrusts, denoted henceforth with $T_i \in \mathbb{R}$ for $i \in \{1,2,3,4\}$, are the inputs for control design. Let us now introduce the relative (with respect to hovering) control torque $\Delta\tau_c := \tau_c$ and thrust $\Delta T_c := T_c - mg$ and the relative propeller thrusts $\Delta T_i := T_i - T_H$, where $T_H := \frac{mg}{4}$ is the hovering thrust for each propeller. Following a consolidated approach in the literature about small-scale quadrotors [2] these variables are related by the following static map:

$$\begin{bmatrix} \Delta T_c \\ \Delta\tau_{c_1} \\ \Delta\tau_{c_2} \\ \Delta\tau_{c_3} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ \ell\sin(\beta_1) & \ell\sin(\beta_2) & \ell\sin(\beta_3) & \ell\sin(\beta_4) \\ -\ell\cos(\beta_1) & -\ell\cos(\beta_2) & -\ell\cos(\beta_3) & -\ell\cos(\beta_4) \\ \sigma & -\sigma & \sigma & -\sigma \end{bmatrix}}_{X} \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \\ \Delta T_3 \\ \Delta T_4 \end{bmatrix}, \tag{3}$$

where $X \in \mathbb{R}^{4\times4}$ is the input map, $\ell \in \mathbb{R}_{>0}$ is the distance from the *i-th* rotor hub to O_B, σ is the ratio between the propeller thrust and torque coefficient and $\beta_i = \pi/4 + \pi/2(i-1)$ is the angle about the b_3 axis between each pair of axes of the arms.

A. Baseline control architecture

Given the desired relative thrust $\Delta T_c^d \in \mathbb{R}$ and control torque $\Delta\tau_c^d \in \mathbb{R}^3$, assuming no bound on the ΔT_is and relying on the invertibility of X, the selection $[\Delta T_1\ \Delta T_2\ \Delta T_3\ \Delta T_4]^\top = X^{-1}[\Delta T_c^d\ \Delta\tau_c^{d\top}]^\top$ allows one to write (1)-(2) directly in terms of ΔT_c^d and $\Delta\tau_c^d$, since $[\Delta T_c\ \Delta\tau_c^\top]^\top = X[\Delta T_1\ \Delta T_2\ \Delta T_3\ \Delta T_4]^\top = XX^{-1}[\Delta T_c^d\ \Delta\tau_c^{d\top}]^\top = [\Delta T_c^d\ \Delta\tau_c^{d\top}]^\top$. Then, the linear system $J\ddot{\alpha} = \Delta\tau_c^d + \tau_e$, $m\ddot{x} = mgS(\alpha)e_3 + \Delta T_c^d e_3 + f_e$ is underactuated but controllable, and a common control solution is based on a hierarchical design in which the roll and pitch angles are used as virtual inputs to stabilize the position dynamics in the x_1, x_2 plane. For instance, to track a desired position $x_d \in \mathbb{R}^3$ and yaw angle $\psi_d \in \mathbb{R}$, one of the most common architecture is based on P/PID loops (see, *e.g.*, [12]) with the following structure:

$$\Delta T_c^d := PI_{x_3}(z)\left(k^o_{p,x_3}(x_{d_3} - x_3) - v_3\right) - D_{x_3}(z)v_3 \tag{4}$$

$$\Delta\tau_c^d := PI_R(z)\left(K_{p,R}[\phi_v - \phi\ \ \theta_v - \theta\ \ \psi_d - \psi]^\top - \omega\right) - D_R(z)\omega, \tag{5}$$

where the virtual roll and pitch angles are

$$\phi_v := \tfrac{1}{mg}\left(PI_{x_2}(z)\left(k^o_{p,x_2}(x_{d_2} - x_2) - v_2\right) - D_{x_2}(z)v_2\right), \tag{6}$$

$$\theta_v := -\tfrac{1}{mg}\left(PI_{x_1}(z)\left(k^o_{p,x_1}(x_{d_1} - x_1) - v_1\right) - D_{x_1}(z)v_1\right) \tag{7}$$

in which $PI_{(\cdot)}(z) := k^i_{p,(\cdot)} + k^i_{i,(\cdot)} t_s \frac{1}{z-1}$, $D_{(\cdot)}(z) := k^i_{d,(\cdot)} N^i_{(\cdot)} \frac{z-1}{z-1+N^i_{(\cdot)} t_s}$ are discrete transfer functions defining, respectively, a proportional-integral and (filtered) derivative actions, $t_s \in \mathbb{R}_{>0}$ denotes the sampling time, $k^{(\cdot)}_{(\cdot,\cdot)} \in \mathbb{R}_{>0}$ are scalar gains while $K_{p,R} \in \mathbb{R}^{3\times3}_{>0}$ is a diagonal gain matrix and $N^{(i)}_{(\cdot)} \in \mathbb{R}_{>0}$ is the filter time constant.

B. Control allocation and directionality issues in quadrotors

Common propellers for quadrotors are unidirectional and have finite power, meaning that they can deliver only a positive and bounded thrust along their spinning axis, *i.e.*, $0 \leq T_i \leq T_M$ $\forall i \in \{1,2,3,4\}$. Equivalently, in terms of relative thrusts, $-T_H \leq \Delta T_i \leq T_M - T_H$ $\forall i \in \{1,2,3,4\}$. Since $T_M > T_H$ for the quadrotor to fly, the hovering thrust can be expressed as a fraction of the maximum one, *i.e.*, $T_H = \eta T_M$ $\eta := \frac{mg}{4T_M} \in (0,1]$. Therefore, when saturation occurs, one cannot transfer the commanded action $\Delta T_c^d, \Delta\tau_c^d$ to the quadrotor since the actual relative force and torque transferred are given by $[\Delta T_c\ \Delta\tau_c^\top]^\top = X\text{sat}_{\underline{u}}^{\bar{u}}(X^{-1}[\Delta T_c^d\ \Delta\tau_c^{d\top}]^\top)$, where $\underline{u}_i := -\eta T_M$, $\bar{u}_i := (1-\eta)T_M$ for $i \in \{1,2,3,4\}$. The associated *directional* change of the control vector induces cross coupling phenomena among the different inputs, giving rise to windup effects which are referred to as *directionality* issues in the literature on saturated multivariable plants. Note, in passing, that quadrotors are usually designed to hover at 50% or less of the available thrust, *i.e.*, $T_H \approx 0.5T_M$ ($\eta = 0.5$) so that there is more than enough margin to allocate control actions for standard maneuvers before reaching saturation. Nonetheless, when looking for high performance controllers, abrupt maneuvers (*e.g.*, step references or large initial errors with respect to the desired trajectory) or operations in off-design conditions (*e.g.*, heavy payload transportation, wind disturbances) can make the propellers saturate and therefore induce windup effects which are mainly associated with the induced cross-coupling among the inputs. In particular, due to the lower effectiveness of the yaw-torque generation mechanism with respect to the roll-pitch one (the coefficient σ in (3) has a small value compared to the arm length ℓ), combined position and yaw maneuvers can more easily push actuators to saturation bounds thereby inducing possibly dangerous effects due to directionality. One way to address this issue is to reduce the aggressiveness of yaw control or to scale down the commanded yaw torque τ_{c_3} until saturation is not reached any more when inverting (3) (at least until possible, see [1], [2]). In contrast, the approach that we propose herein is based on a systematic design of an AW augmentation scheme which allows one to adjust the relative level of position and yaw performance deterioration during saturation, while keeping the optimal performance of a baseline controller in an unsaturated regime.

III. Decentralized anti-windup augmentation strategy for input-coupled plants

In this work we focus on windup effects arising in quadrotors when step-like setpoints are commanded, which is one of the most typical operating conditions for this kind of UAVs. As discussed in Section II-B, the combination of yaw-position maneuvers and of an aggressive controller is more likely to induce directionality issues due to the peculiar allocation map of quadrotors, even when not so large steps are commanded. Accordingly, to tailor the anti-windup compensation design for reference signals which reflect practical operating conditions, a regional approach has been considered herein[1]. Specifically, the AW compensation problem will be addressed in the framework of the Direct Linear AW compensator (DLAW) design [13], extending the performance-oriented approach of [9] to discrete-time systems[2].

A. Performance-oriented AW design: preliminaries and problem formulation

The linearized quadrotor model in (1)-(2) belongs to a class of saturated systems in which the plant is made of several independent dynamical subsystems which are statically coupled through the inputs. Let us consider a linearized discrete time-invariant plant

$$(P) \quad \begin{cases} x_p^+ = A_p x_p + B_{p,u} u \\ \;\; y = C_{p,y} x_p + D_{p,yu} u \\ \;\; z = C_{p,z} x_p + D_{p,zu} u, \end{cases} \tag{8}$$

where the state space matrices are block-diagonal with n blocks, $x_p \in \mathbb{R}^{n_p}$, $y \in \mathbb{R}^{n_y}$, $z \in \mathbb{R}^{n_z}$ and $u \in \mathbb{R}^n$. The input u is a mixing of the actual plant inputs $u_a \in \mathbb{R}^n$ through a non-singular matrix $X \in \mathbb{R}^{n \times n}$, *i.e.*, $u = X u_a$. Consider a linear control law

$$(C) \quad \begin{cases} x_c^+ = A_c x_c + B_{c,y} y + B_{c,w} w + v_x \\ \;\; y_c = C_c x_c + D_{c,y} y + D_{c,w} w + v_y, \end{cases} \tag{9}$$

where again the state space matrices are made of n subsystems, $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $y_c \in \mathbb{R}^n$ is the corresponding output, $w \in \mathbb{R}^{n_w}$ is the set-point and $v_x \in \mathbb{R}^{n_c}$, $v_y \in \mathbb{R}^n$ are additional inputs to be used for the AW augmentation. When assuming no bound for u_a and exploiting the invertibility of X, the interconnection of (P) and (C) through $u_a = X^{-1} y_c$ yields n decoupled subsystems which are assumed to be well posed and internally stable. For quadrotors, the linearized closed-loop system (1)-(2) has a block-diagonal structure made by four subsystems: the altitude (x_3), the longitudinal-lateral (x_2, ϕ)-(x_1, θ) and the yaw (ψ) subsystems, respectively controlled by ΔT_c, $\Delta\tau_{c_1}$, $\Delta\tau_{c_2}$ and $\Delta\tau_{c_3}$; the performance output is $z = (x, \psi) \in \mathbb{R}^4$ while the reference input is $w = (x_d, \psi_d) \in \mathbb{R}^4$. In practice, due to actuator saturation, each u_{a_i} is bounded between $[-\underline{u}_i, \bar{u}_i]$, and the interconnection of (P) and (C) by means of

$$u_a = \mathrm{sat}_{\underline{u}}^{\bar{u}}(X^{-1} y_c), \tag{10}$$

forms the *constrained* closed-loop system. Whenever X is not diagonal, the closed loop loses its decentralized structure during saturation. To alleviate this effect, the compensation signals v_x, v_y are injected in (9); these signals are the outputs of the linear filter

$$(AW) \quad \begin{cases} x_{aw}^+ = B_{aw} q_X \\ \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ I_{n_u} \end{bmatrix} x_{aw} + \underbrace{\begin{bmatrix} \bar{D}_{aw} \\ 0_{n_u} \end{bmatrix}}_{D_{aw}} q_X, \end{cases} \tag{11}$$

$$q_X := y_c - X \mathrm{sat}_{\underline{u}}^{\bar{u}}(X^{-1} y_c) \tag{12}$$

where $x_{aw} \in \mathbb{R}^n$. As noted in [6], the use of $q := X^{-1} y_c - \mathrm{sat}_{\underline{u}}^{\bar{u}}(X^{-1} y_c)$ instead of (12) as the deadzone nonlinearity driving the AW compensator would hide the decentralized structure of the closed-loop system in unsaturated conditions (see Remark 2 for additional details). The AW compensator in equation (11) corresponds to a static full-authority AW in which the signal v_y is cascaded with a unit delay, which is included to avoid solving in real time implementations the algebraic loop arising in the controller (see, *i.e.*, [10])[3].

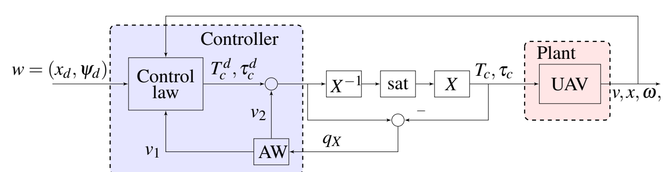

Figure 1. Anti-windup augmentation scheme.

The interconnection of (8), (9) and (11) through q_X can be written in compact form by introducing the *augmented* closed-loop (ACL):

$$(ACL) \quad \begin{cases} x_a^+ = A_a x_a + B_{a,q} q_X + B_{a,w} w \\ \;\; z = C_{a,z} x_a + D_{a,zq} q_X + D_{a,zw} w \\ \;\; y_c = C_{a,u} x_a + D_{a,uq} q_X + D_{a,uw} w, \end{cases} \tag{13}$$

where $x_a := (x_p, x_{cl}, x_{aw})$. In contrast to the majority of works in the framework of DLAW design [13], which are focused

[1] The quadrotor model in (1)-(2) is unstable (not exponentially), meaning that global results can be achieved only with a nonlinear compensation scheme [13, Section 2.3]. However, we are not interested in achieving global results here (which would also question the use of a linearized model for the quadrotor) but rather seek good performance in practical conditions.

[2] The emphasis on discrete-time plants is natural for the application considered in this work in view of onboard implementations. Moreover, identified flight models to be used in AW synthesis are typically obtained in the form of discrete-time systems (see, *e.g.*, [14]).

[3] The simple structure considered in (11) stands out for its computational efficiency among the ones that can be used in the proposed AW design. More complex solutions, possibly yielding better performance, can be included in our design following [9, Section 4].

on the minimization of the ℓ_2-gain from w to z, we propose a discrete-time version of the performance-oriented approach developed for continuous-time systems in [9]. Such an approach starts by defining a reference model (RM)

$$(RM) \quad \begin{cases} x_{rm}^{+} = A_{rm} x_{rm} + B_{rm,w} w \\ z_{rm} = C_{rm,z} x_{rm} + D_{rm,zw} w \end{cases} \tag{14}$$

which is used to describe the (desired) unconstrained closed-loop behavior. As conditions inducing propeller saturation can be assimilated with the use of step references, we include the filter

$$(F) \quad w^{+} = I_{n_w}(1-\varepsilon) w, \quad w(0) = w_0, \quad \mathbb{R}_{>0} \ni \varepsilon \ll 1, \tag{15}$$

into the closed-loop system used for the AW synthesis, as suggested in [9] to achieve good time-domain responses in practical conditions (the initial condition of the filter w_0 can be considered as the step amplitude). Thus, by defining the augmented state $\xi = (x_a, x_{rm}, w) \in \mathbb{R}^{n_\xi}$, the interconnection of (13), (14) and (15) through q_X is given compactly by

$$\begin{cases} \xi^{+} = A\xi + B_q q_X \\ z_e = C_z \xi + D_{zq} q_X \\ y_c = C_y \xi + D_{yq} q_X, \end{cases} \tag{16}$$

where all the involved matrices can be uniquely determined from (8), (9), (11), (14) and (15) and where $z_e := z - z_{rm}$ is a performance output introduced to evaluate the mismatch between the reference and the actual system response. Starting from the above representation, the problem that we will address can be formulated as follows.

Problem 1: Given the compact representation in (16), find the matrices of the AW compensator in (11) such that the ℓ_2-norm of the performance output z_e is as small as possible.

B. Fixed-dynamics AW compensator synthesis

In this section we provide a constructive solution to Problem 1 by exploiting an extension of the well-known generalized sector condition (see, *e.g.*, [13]), which is adapted from [5].

Lemma 1: Consider any diagonal matrix $M \in \mathbb{R}^{n\times n}_{>0}$, $H \in \mathbb{R}^{n\times n_\xi}$, a non singular matrix $X \in \mathbb{R}^{n\times n}$ and define $\bar{M} := X^{-T} M X^{-1}$. Then, the following condition holds:

$$-q_X^{\top} \bar{M} (q_X - y_c + XH\xi) \geq 0, \forall \xi \in \mathbb{R}^{n_\xi} : \mathrm{sat}(H\xi) = H\xi. \tag{17}$$

We now state our AW synthesis result, corresponding to the following theorem, whose proof is omitted due to space constraints, which leverages the generalized sector condition in Lemma 1 and can be considered as an extension of the results presented in [9] to discrete-time systems.

Theorem 1: Consider the system in (16), define n_r directions of interest $r_1, \ldots, r_{n_r} \in \mathbb{R}^{n_w}$ and select a diagonal matrix $W \in \mathbb{R}^{n_z \times n_z}_{\geq 0}$ to be used for control objectives prioritization. If there exist matrices $Q = Q^{\top} \in \mathbb{R}^{n_\xi \times n_\xi}_{>0}$, $Y \in \mathbb{R}^{n\times n_\xi}$, $U \in \mathbb{R}^{n\times n}_{>0}$ diagonal, $\hat{B}_{aw} \in \mathbb{R}^{n\times n}$, $\hat{D}_{aw} \in \mathbb{R}^{(n_c+n)\times n}$ and a scalar $\gamma \in \mathbb{R}_{>0}$ satisfying

$$\mathrm{He}\begin{bmatrix} -\frac{Q}{2} & QC_y^{\top} & 0 & 0 \\ -XY & -G + D_{cl,uq}G + D_{cl,uv}\hat{D}_{aw} & 0 & 0 \\ WC_zQ & W\left(D_{cl,zq}G + D_{cl,zv}\hat{D}_{aw}\right) & -\frac{\gamma}{2}I_{n_z} & 0 \\ AQ & \begin{bmatrix} B_{cl,q}G + B_{cl,v}\hat{D}_{aw} \\ \hat{B}_{aw} \\ 0 \end{bmatrix} & 0 & -\frac{Q}{2} \end{bmatrix} < 0, \tag{18}$$

$$\begin{bmatrix} \bar{\bar{u}}_i^2 & Y_i \\ Y_i^{\top} & Q \end{bmatrix} \geq 0, \quad \hat{D}_{aw_{jk}} = 0, \quad \begin{bmatrix} Q & \begin{bmatrix} 0 \\ r_h \end{bmatrix} \\ \begin{bmatrix} 0 & r_h^{\top} \end{bmatrix} & 1 \end{bmatrix} \geq 0, \tag{19}$$

where $G := XUX^{\top}$, Y_i denotes the *i-th* row of Y $(i = 1, \ldots, n)$, $\bar{\bar{u}}_i := \min(\underline{u}_i, \bar{u}_i)$ is the *i-th* input bound, $B_{cl,q}$, $B_{cl,v}$, $D_{cl,uq}$, $D_{cl,uv}$, $D_{cl,zq}$, $D_{cl,zv}$ are defined as in Appendix A, $\hat{D}_{aw_{jk}}$ is the element in the *j-th* row *k-th* column of $\hat{D}_{aw}$ $(j \in \{n_c+1, \ldots, n_c+n\}$, $k \in \{1, \ldots n_c\})$ and $h \in \{1, \ldots n_r\}$, then by selecting the anti-windup matrices in equation (11) as

$$B_{aw} = \hat{B}_{aw} G^{-1} \tag{20}$$

$$D_{aw} = \hat{D}_{aw} G^{-1}, \tag{21}$$

the ellipsoid $\mathscr{E}(Q^{-1}) := \left\{\xi \in \mathbb{R}^{n_\xi} : \xi^{\top} Q^{-1} \xi \leq 1\right\}$ is contained in the region of attraction of (16) and $\overline{\mathrm{co}}\{(0, r_h) \in \mathbb{R}^{n_\xi}, h = 1, \ldots, n_r\} \subset \mathscr{E}(Q^{-1})$. Moreover, the following condition on the ℓ_2 norm of z_e is satisfied:

$$\sum_{t=0}^{\infty} z_e^{\top} W^2 z_e \leq \gamma, \quad \forall \xi(0) \in \mathscr{E}(Q^{-1}). \tag{22}$$

The main idea behind the proposed design method consists in first selecting reasonable step amplitudes by defining the vectors r_i and then tuning the AW compensator in such a way that the unconstrained response is tracked at best for the given references. This objective, since the smaller γ, the lower the weighted mismatch z_e, can be achieved by solving the following semidefinite program:

$$\min_{Q,Y,U,\gamma,\hat{B}_{aw},\hat{D}_{aw}} \gamma, \quad \text{subject to (18)-(19).} \tag{23}$$

Then, the optimal anti-windup matrices are recovered using equations (20) and (21).

Remark 1: By suitably selecting the performance output weight W in (23), one can penalize as little as wanted the yaw performance output in the optimization step (23) (*i.e.*, reducing the corresponding diagonal entry in W) to guarantee that the position tracking performance is less deteriorated than the yaw one during saturation (see the simulation results in Section IV). This tuning possibility, which comes from well-known procedures in the robust control literature and has been exploited in an anti-windup design setting in [7], is particularly important for quadrotors since poor performance in position tracking is much more dangerous for safety reasons than poor yaw tracking, especially in cluttered environments. Whenever this issue is not so stringent, one can relatively increase the yaw weight in the AW synthesis according to a desired trade-off. ⌟

IV. SIMULATION RESULTS

This section is devoted to showing the capability of the proposed AW compensation method in addressing directionality issues affecting quadrotors. The AW synthesis has been carried out by using an identified linear discrete-time model of a lightweight quadrotor available from previous work in [8]. Instead, simulation results have been obtained by considering a nonlinear model in which the baseline controller is the one implemented in the PX4 autopilot [12] (which behaves for small errors as (4)-(7)). Taking into account the input map (3), the AW compensator proposed in (11) has been used to augment the baseline controller as shown in Figure 1, where the signal v_y is used to compensate the virtual inputs T_c^d, τ_c^d, while v_x is used to modify the states of the position and attitude controllers. Since the controller used in (9) is the equivalent representation of the cascade of the outer position and inner attitude controllers, v_x can be partitioned as $v_x = (v_{x,v}, v_{x,\omega})$, where $v_{x,v}$, $v_{x,\omega}$ are in charge of modifying, during saturation, the PID states of the position and attitude controller, respectively. The compensator has been tuned to face maneuvers characterized by combined longitudinal, vertical and directional motions giving priority to position control over yaw control ($W = \text{diag}(1,1,1,0)$). In particular, the convex set contained in the polyhedron made by eight vertices r_i corresponding to combinations of $(\pm 4\text{m}, 0\text{m}, 2\text{m}, \pm\pi/6\text{rad})$ and $(0\text{m}, \pm 4\text{m}, 2\text{m}, \pm\pi/6\text{rad})$ has been considered in the tuning procedure, *i.e.*, equation (23)), while according to the considerations made in Section II-B we assumed hovering at $T_H = 0.5T_M$, which implies that the relative propeller thrusts ΔT_i are constrained in the interval $[-0.5T_M, 0.5T_M]$. For the considered vertices r_i, the resulting AW compensator (11) has the following (almost) block-diagonal structure

$$\left[\frac{\bar{D}_{aw}}{B_{aw}}\right] = \left[\begin{array}{cccc} D^{(x_3)} & \bullet & \bullet & \bullet \\ \bullet & D^{(x_2,\phi)} & \bullet & \bullet \\ \bullet & \bullet & D^{(x_1,\theta)} & \bullet \\ \bullet & \bullet & \bullet & D^{(\psi)} \\ \hline B^{(T_c)} & \bullet & \bullet & \bullet \\ \bullet & B^{(\tau_{c_1})} & \bullet & \bullet \\ \bullet & \bullet & B^{(\tau_{c_2})} & \bullet \\ \bullet & \bullet & \bullet & B^{(\tau_{c_3})} \end{array}\right], \tag{24}$$

where, in virtue of the quadrotor control law (4)-(7), $\bar{D}_{aw} \in \mathbb{R}^{(1+4+4+2)\times(1+1+1+1)}$, $B_{aw} \in \mathbb{R}^{(1+1+1+1)\times(1+1+1+1)}$ and the terms $\bullet$ denote almost zero elements, while the superscripts are used to relate the gains with the corresponding partition of states/output of the controller.

Remark 2: In [6] an AW compensator with a decentralized structure has been developed for open-loop exponentially stable linear systems with the same form as the one considered in our work. Having a decentralized AW structure can be important in practice; for instance, in combined yaw-altitude maneuver for which propellers reach saturation bounds, the commanded roll and pitch torques are zero ($\Delta\tau_{c_1}^d = \Delta\tau_{c_2}^d = 0$) and, by construction, the saturation nonlinearity q_X would not introduce in-plane cross-couplings (the transferred roll and pitch torques are zero ($\Delta\tau_{c_1} = \Delta\tau_{c_2} = 0$)). Nonetheless, if the AW compensator is not decentralized, then such couplings would be induced by the corrective terms of the compensator. We cannot *a priori* impose a decentralized structure to the AW compensator through the synthesis (23); however, as shown by (24), thanks to the block diagonal structure of the plant

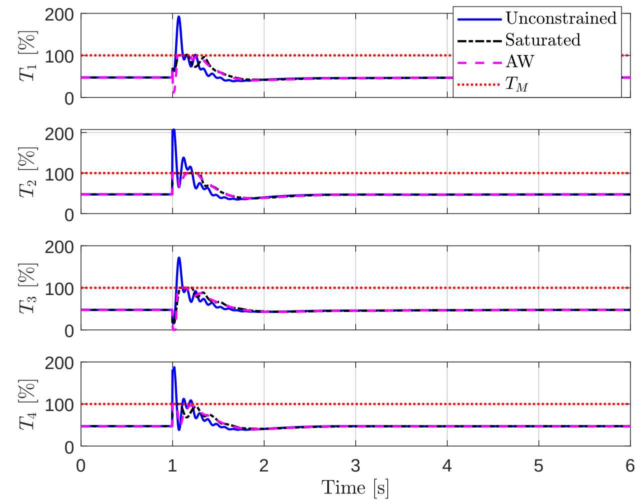

Figure 2. Quadrotor inputs time history.

(8) and the choice of the deadzone nonlinearity q_X over $q := X^{-1}y_c - \text{sat}_{\underline{u}}^{\bar{u}}(X^{-1}y_c)$, a decentralized structure (up to numerical errors) can be obtained. Numerical testing on the synthesis algorithm has shown that such a structure is obtained whenever the directions of interest r_i form a polyhedron with the eight vertices given by $(\pm a\,\text{m}, 0\,\text{m}, c\,\text{m}, \pm d\,\text{rad})$ and $(0\,\text{m}, \pm b\,\text{m}, c\,\text{m}, \pm d\,\text{rad})$ for $a,b,c,d \in \mathbb{R}_{>0}$. ⌟

In the following, the results of simulations considering a reference that combines a position step $x_d(t) := [4\ 0\ 2]^\top \text{step}(t-1)$m and a yaw step $\psi_d(t) := 30\,\text{step}(t-1)$ deg are presented. As can be seen from Figure 2, the requested set-points lead the unconstrained controller to require a throttle percentage exceeding the maximum one by a factor two, while the saturated and the AW solution satisfy the bound. Figure 3 shows instead a comparison of the set-point tracking performance for the different controllers. As can be seen from the figure, the presence of saturation in the propeller thrusts at the beginning of the maneuver induces visible directionality effects, which can be appreciated in the significant oscillations experienced by the UAV along the x_2 direction when using the baseline controller (Figure 3 - middle, black-dashed line). The slight overshoot visible in the unconstrained response of the baseline controller is related to the action commanded by the nonlinear version of (6)-(7) implemented in the simulator, which induces a small rolling motion even if only pitch and yaw set-points are commanded (see [15]). The AW compensator, on the other hand, allows reducing the amplitude of the x_2 oscillations and almost recovering the unconstrained behavior; in this sense the minimal AW architecture (11) shows very good position tracking properties, without the need of increasing the anti-windup order. At the same time, as desired by the control priority objective, the yaw tracking performance of the AW augmented controller exhibits a clear mismatch with respect to the response of the unconstrained controller (Figure 4 - bottom). Finally, by inspecting Figure 5, it is interesting to observe how changing the weight W allows one to trade control objectives off if needed. In particular, using the weight $W = \text{diag}(0,0,1,1)$, maximum priority can be given to the altitude, yaw response as shown in Figure 5. In this case, an almost unconstrained performance is obtained in the yaw

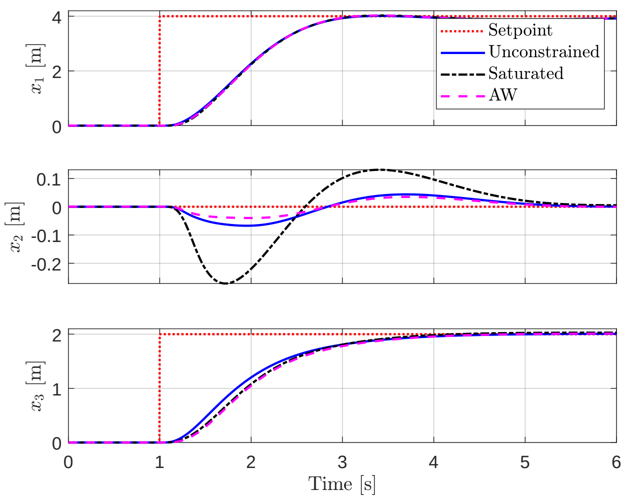

Figure 3. Quadrotor position time history.

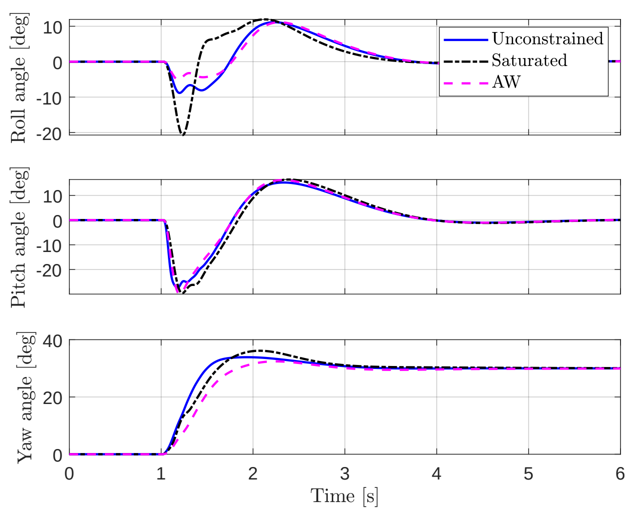

Figure 4. Quadrotor attitude time history.

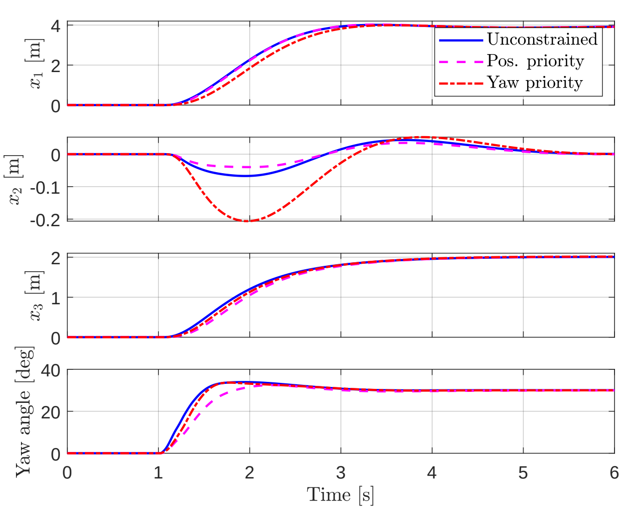

Figure 5. Tracking performance with weighted yaw.

altitude channels at the price of worse tracking performance on the longitudinal/lateral position.

V. Conclusions

In this paper we addressed the problem of compensation of directionality effects in quadrotor UAVs. By referring to a general class of input-coupled discrete-time linear plants, a LMI-based synthesis procedure for AW compensators has been proposed with a focus on time-domain performance and prioritization of control objectives. The synthesis procedure has been used to tune a fixed-dynamics AW controller: encouraging results achieved in simulations make the proposed augmentation scheme a viable remedy to some windup issues affecting quadrotors.

Appendix

Given $\Delta_u := (I - D_{c,y}D_{p,yu})^{-1}$, $\Delta_y := (I - D_{p,yu}D_{c,y})^{-1}$ the matrices to be defined in LMI (18) are:

$$\left[\begin{array}{c} B_{cl,q} \\ \hline D_{cl,zq} \\ \hline D_{cl,uq} \end{array}\right] := \left[\begin{array}{c} -B_{p,u}\Delta_u \\ -B_{c,y}\Delta_y D_{p,yu} \\ \hline -D_{p,zu}\Delta_u \\ \hline I - \Delta_u \end{array}\right], \quad \left[\begin{array}{c} B_{cl,v} \\ \hline D_{cl,uv} \\ \hline D_{cl,zv} \end{array}\right] := \left[\begin{array}{cc} 0 & B_{p,u}\Delta_u \\ I_{n_c} & B_{c,y}\Delta_y D_{p,yu} \\ \hline 0 & \Delta_u \\ \hline 0 & D_{p,zu}\Delta_u \end{array}\right]. \tag{25}$$

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