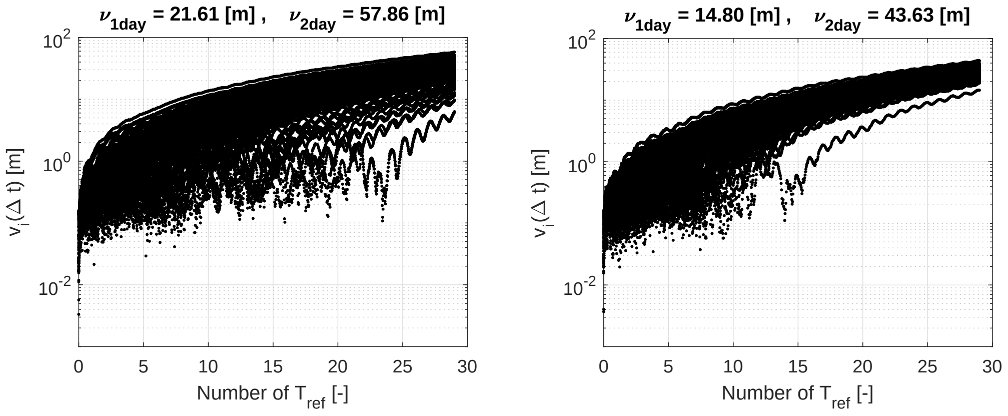


a) T_2 with KA-6×6.

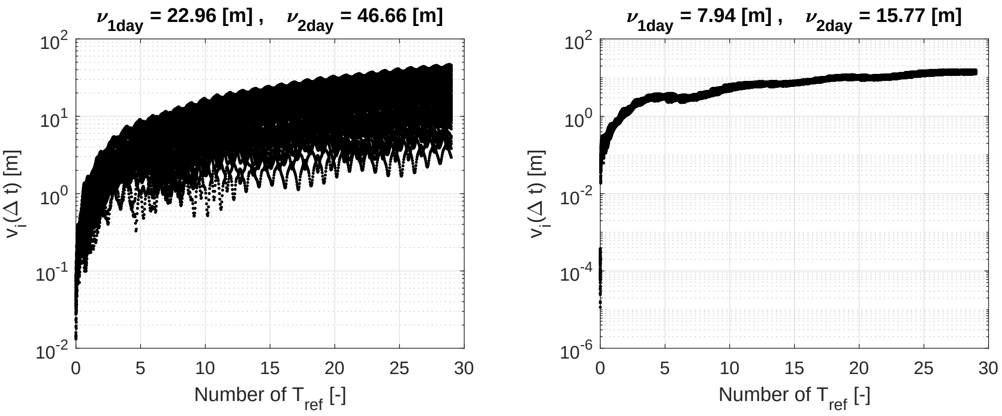
b) T_2 with KA-10×10.

Fig. 9 Accuracy of the navigation set-up: reference orbits from numerical integration in 30×30 gravity field and remaining orbital perturbations.

the orbital dynamics is perturbed only by a degree-6 order-6 gravity field. The 99 cases are plotted altogether to show the error dispersion due to the different initial conditions, and the y-axis shows the trend over time of each $\nu_i(\Delta t)$ of the population as defined in Eq. (12). As discussed in [25], a remarkable improvement is already achievable by including J_2 to the second order. In this case the transformations employed in Fig. 7.b) account for the same geopotential terms of the environment (i.e., 6×6). In such limit case, the fully analytical propagation of the relative motion would remain accurate at meter level even after 2 days.

Once verified the effectiveness of the proposed approach, the remaining simulations of Fig. 8 and 9 address the trade-off between transformation complexity (i.e., in the sense on included terms KA-6×6 or KA-10×10) and achievable accuracy performance when the dynamics resembles a more realistic environment. One can see that the propagation accuracy improves considering more terms in the transformations. In Fig. 9, the obtained error is also due to the lack of inclusion of the additional perturbations in the relative motion model, as well as in the transformations.

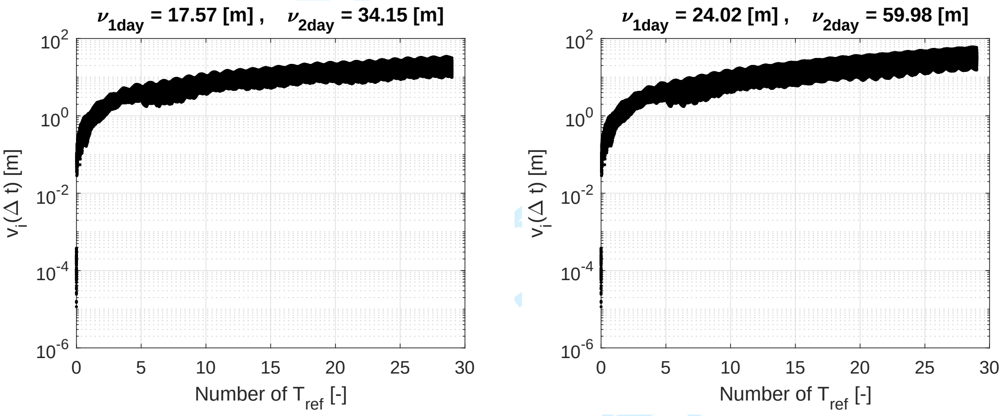
Figures 10 and 11 address the achievable accuracy for the guidance set-up. Again, the benefits of using the proposed orbital elements' conversion algorithm is remarkable, as one can see from the comparison in the order-6 degree-6 gravity field. Nevertheless, the overall accuracy is worsened compared to the navigation set-up, due to the error introduced in the propagation of the absolute



a) T_2 with SJ-1st.

b) T_2 with KA-6×6.

Fig. 10 Accuracy of the guidance set-up: reference orbits from numerical integration in 6×6 gravity field.



a) Geopot. 30×30.

b) Geopot. 30×30 & other perturbations.

Fig. 11 Accuracy of the guidance set-up: in both simulation environments the KA-6×6 transformation is used.

orbital elements of the chief. The considerations concerning the trade-off between model complexity and realistic achievable performance, as well as the impact of non-modeled phenomena apply as before.

B. Performance of the ROE-based model

This section focuses on the propagation performance of the developed ROE-based second-order model accounting for zonal harmonics up to J_6. To this end, a relative orbit of the family of Fig. 6, with $|a\delta a| = 200$ m is considered, which is also the case presented in [20]. Figure 12 shows the evolution over time of the propagation error defined as $\mathbf{e} = a(\delta\boldsymbol{\alpha} - \delta\boldsymbol{\alpha}_{\text{true}})$, being the $\delta\boldsymbol{\alpha}_{\text{true}}$ term computed from the numerically integrated reference orbit. The environment accounts only for an order-6 degree-6 gravity field and the KA-6×6 algorithm is used for the transformations T_2. This set-up is used to focus only on the performance of the core block of the framework. The error is considered for the ROE-based developed model (in black) as well as for the model of Yang et al. [15] (in light gray). Thus, both models include the second-order expansion of the unperturbed orbit and of the J_2 first-order term. The remaining, tiny, discrepancy - for this scenario appreciable only for the relative mean longitude component - is due to the relative contribution of the considered first-order higher zonal terms. Indeed, at practical level, in the application case of a far-range rendezvous to a noncooperative target (as the scenario considered, inspired from the AVANTI demonstration), the relative effects of higher zonal terms can be neglected to minimize the complexity of the model. Thus, the proposed model behaves as Yang et al. [15], though with the simpler structure of Φ and Ψ discussed in the previous sections. This last aspect is beneficial for designing the GNC algorithms for onboard applications.

VI. Conclusion

This paper presented a framework to model analytically and precisely the relative motion in low Earth orbits taking into account the perturbations due to the non-spherically symmetric mass of the Earth. With focus on formation-flying applications, where the relative GNC algorithms are conveniently developed in the orbital elements' space, the main functionalities of the framework include the extraction of mean orbital elements, out of the orbit of the chief satellite known in a Cartesian inertial frame, and the propagation of the relative motion in the mean state variables. In the paper, an accurate description of the interfaces between those functions is provided, since consistency in reference systems and transformation errors is crucial to assess the true achievable

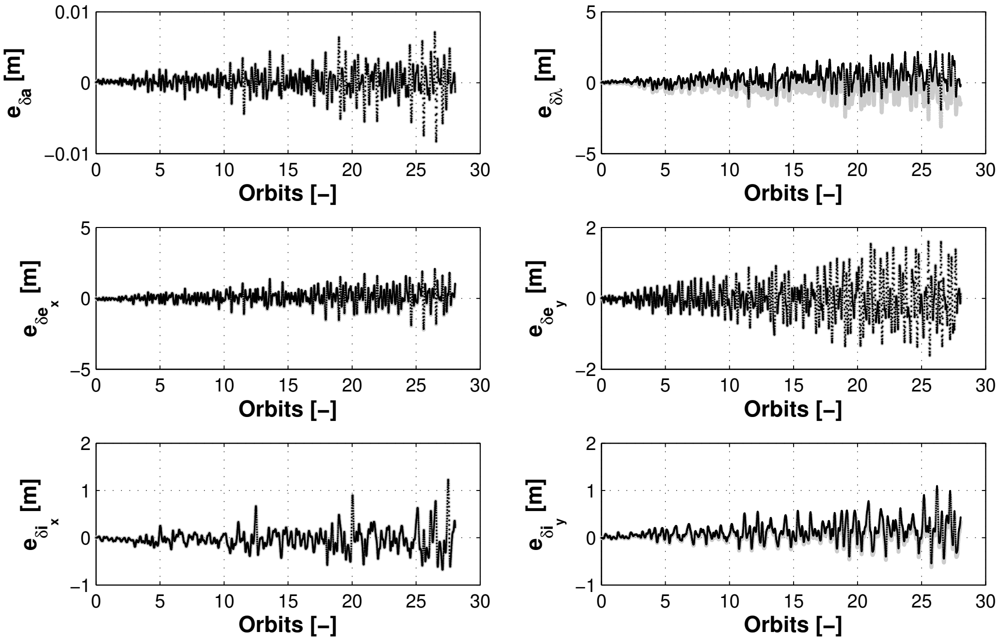


Fig. 12 ROEs propagation error: in black for the model of Eq. (10)**, in gray for the model of Yang et al. [15].**

modeling precision.

In the framework one can choose order and degree of the geopotential field considered in the mean/osculating elements' transformations, order of zonal terms considered in the propagation of the mean relative motion, as well as order of the Taylor expansion employed to model the relative dynamics (e.g., first-order state transition matrix or second-order state transition tensor). These design parameters have to be chosen to meet the propagation requirements posed by the specific application under consideration (e.g., sensitivity of relative navigation sensors, large reconfigurations or station keeping, size of the relative orbit, average time between successive orbit corrections, impact of additional orbit perturbations).

About the trade-off between complexity of the algorithms involved in the relative motion model and achievable modeling accuracy, the paper showed that for almost bounded, centered, relative orbits the effect of the second-order expansion of the terms due to J_2 is larger than the effect of first-order higher zonal terms (i.e., J_2^2, J_4, and J_6) for relative orbits of size larger than 5000-9000 m. For far-range rendezvous applications exploiting passively safe relative orbits of small size (i.e., 300 m), the second-order expansion of the Keplerian term is advised for relative semi-major axis larger than 130 m.

For what concerns the mean/osculating elements' transformations, the consideration of J_2 to the second order is strongly advised. This, in fact, allows computing more precisely the value of the initial relative semi-major axis, and therefore allows improving remarkably the propagation accuracy over time. The approach proposed in the paper combines a Hamiltonian technique applied to the J_2 problem with Kaula's linear perturbation method for the remaining terms of the geopotential. The resulting algorithm is compact, fully analytical in both transformation directions, and free from singularities (but not applicable in the vicinity of the critical inclination). The improvements proposed in this work are not strictly required in the case of station keeping of almost bounded relative orbits, since in this case the value of relative semi-major axis is close to zero and orbit corrections are anyway performed frequently (i.e., in less than an orbital period of time). The inclusion of additional terms to J_2 up to given order and degree has to be traded-off considering the orbit scenario and the time elapse of the propagation. If only the geopotential effect is meaningful (e.g., up to 30 × 30), then transformations with 6 × 6 terms guarantee a worst-case propagation error within 15 m after one day and 10 × 10 transformations reduce it to the half. Whenever meter level precision is required for 1-day or 2-day propagation legs, the modeling of the effects on the relative motion produced by the remaining orbital perturbations becomes necessary. In this case, the framework can be straightway complemented including available ROE-based formulations from the literature, exploiting its modular structure. Note that in this case additional terms/parameters have to be included in the state variable.

Several recent in-flight demonstrations of formation-flying and noncooperative rendezvous exploited the relative orbital elements' parametrization for their spaceborne relative GNC algorithms. The relative motion model developed in this paper improves ROE-based available methods in terms of: validity (applicable also to eccentric reference orbits with no inclusion of approximations), included zonal terms (up to p-order), and order of the expansion (up to second-order state transition tensor). Therefore, the developed framework can be readily used to enhance the performances of available GNC algorithms, with little impact on the complexity of their code.

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References

[1] Liou, J.-C., "An active debris removal parametric study for LEO environment remediation," *Advances in Space Research*, Vol. 47, 2011, pp. 1865–1876. doi: 10.1016/j.asr.2011.02.003.

[2] Braun, V., Lüpken, A., Flegel, S., Gelhaus, J., Möckel, M., Kebschull, C., Wiedemann, C., and Vörsmann, P., "Active debris removal of multiple priority targets," *Advances in Space Research*, Vol. 51, 2013, pp. 1638–1648. doi: 10.1016/j.asr.2012.12.003.

[3] Gaias, G. and Ardaens, J.-S., "Flight Demonstration of Autonomous Noncooperative Rendezvous in Low Earth Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 6, 2018, pp. 1137–1354. doi: 10.2514/1.G003239.

[4] Sullivan, J., Lovell, T. A., and D'Amico, S., "Angles-Only Navigation for Autonomous On-Orbit Space Situational Awareness Applications," in "Proceedings of the AAS/AIAA Astrodynamics Specialist Conference," Univelt, Inc., San Diego, CA, Vol. 67, 2019. Paper AAS 18-468, August 19-23, 2018, Snowbird, UT.

[5] Ardaens, J.-S. and Gaias, G., "Flight Demonstration of Spaceborne Real-Time Angles-Only Navigation to a Noncooperative Target in Low-Earth Orbit," *Acta Astronautica*, Vol. 153, 2018, pp. 367–382. doi: 10.1016/j.actaastro.2018.01.044.

[6] Ardaens, J.-S. and Gaias, G., "Angles-Only Relative Orbit Determination in Low Earth Orbit," *Advances in Space Research*, Vol. 31, No. 11, 2018, pp. 2740–2760. doi: 10.1016/j.asr.2018.03.016.

[7] Gaias, G., Ardaens, J.-S., and Colombo, C., "Precise Line-Of-Sight Modelling for Angles-Only Relative Navigation," 10th International Workshop on Satellite Constellations and Formation Flying, University of Strathclyde, Glasgow, UK, 19-48, 2019.

[8] Sullivan, J., Grimberg, S., and D'Amico, S., "Comprehensive Survey and Assessment of Spacecraft Relative Motion Dynamics Models," *Journal of Guidance, Control, and Dynamics*, Vol. 40, No. 8, 2017, pp. 1837–1859. doi: 10.2514/1.G002309.

[9] Schaub, H., Vadali, S. R., and Alfriend, K. T., "Spacecraft Formation Flying Control Using Mean Orbit

Elements," *Journal of the Astronautical Sciences*, Vol. 48, No. 1, 2000, pp. 69–87.

[10] Gim, D.-W. and Alfriend, K. T., "State Transition Matrix of Relative Motion for the Perturbed Noncircular Reference Orbit," *Journal of Guidance, Control and Dynamics*, Vol. 26, No. 6, 2003, pp. 956–971. doi: 10.2514/2.6924.

[11] Hoots, F. R., "Reformulation of the Brouwer geopotential theory for improved computational efficiency," *Celestial Mechanics*, Vol. 24, No. 4, 1981, pp. 367–375. doi: 10.1007/BF01230396.

[12] Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, New York: Dover Publications, 4th ed., 1944.

[13] Gim, D.-W. and Alfriend, K. T., "Satellite Relative Motion using Differential Equinoctial Elements," *Celestial Mechanics and Dynamical Astronomy*, Vol. 92, 2005, pp. 295–336. doi: 10.1007/s10569-004-1799-0.

[14] Sengupta, P., Vadali, S. R., and Alfriend, K. T., "Second-order state transition for relative motion near perturbed, elliptic orbits," *Celestial Mechanics and Dynamical Astronomy*, Vol. 97, 2007, pp. 101–129. doi: 10.1007/s10569-006-9054-5.

[15] Yang, Z., Luo, Y.-Z., and Zhang, J., "Second-Order Analytical Solution of Relative Motion in J2-Perturbed Elliptic Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 41, No. 10, 2018, pp. 2258–2270. doi: 10.2514/1.G003573.

[16] Härting, A., Rajasingh, C. K., Eckstein, M. C., Leibold, A. F., and Srinivasamurthy, K. N., "On the collision hazard of colocated geostationary satellites," AIAA/AAS Astrodynamics conference, Minneapolis, USA, 88-4239, 1988.

[17] D'Amico, S., *Autonomous Formation Flying in Low Earth Orbit*, Ph.D. thesis, Technical University of Delft, The Netherlands, 2010. https://elib.dlr.de/63481/.

[18] Gaias, G., Ardaens, J.-S., and Montenbruck, O., "Model of J2 Perturbed Satellite Relative Motion with Time-Varying Differential Drag," *Celestial Mechanics and Dynamical Astronomy*, Vol. 123, No. 4, 2015, pp. 411–433. doi: 10.1007/s10569-015-9643-2.

[19] Koenig, A. W., Guffanti, T., and D'Amico, S., "New State Transition Matrices for Spacecraft Relative Motion in Perturbed Orbits," *Journal of Guidance, Control, and Dynamics*, Vol. 40, No. 7, 2017, pp. 1749–1768. doi: 10.2514/1.G002409.

[20] Gaias, G. and Colombo, C., "Semi-Analytical Framework for Precise Relative Motion in Low Earth Orbits," in "7th International Conference on Astrodynamics Tools and Techniques (ICATT)," European Space Agency, ESTEC, Noordwijk, The Netherlands, 2018.

[21] Blitzer, L., *Handbook of Orbital Perturbations*, Univ. of Arizona Press, Tucson, AZ, 1970. p. 32.

[22] Gaias, G., Lavagna, M., Golikov, A., and Ovchinnikov, M. Y., "Formation Flying: Relative Orbits' Modelling and Control through Eulerian Orbital Elements," 19$^{\text{th}}$ AAS/AIAA Space Flight Mechanics Meeting, Savannah, USA, 09-186, 2009.

[23] Biria, A. and Russel, R., "A Satellite Relative Motion Model Using J_2 and J_3 via Vinti's Intermediary," 26$^{\text{th}}$ AAS/AIAA Space Flight Mechanics Conference, Napa, California, 16-537, 2016.

[24] Yan, H., Vadali, S., and Alfriend, K., "State transition matrix for relative motion including higher-order gravity perturbations," *Advances in the Astronautical Sciences*, Vol. 150, 2014, pp. 1317–1336.

[25] Gaias, G., Lara, M., and Colombo, C., "Accurate Osculating/Mean Orbital Elements Conversions for Spaceborne Formation Flying," in "27th International Symposium on Space Flight Dynamics (ISSFD)," Engineers Australia, Melbourne, Australia, 2019.

[26] Johnson, K. W., *Approaches for Modeling Satellite Relative Motion*, Ph.D. thesis, Texas A&M Univ., U.S.A, 2016. https://oaktrust.library.tamu.edu/handle/1969.1/158945.

[27] Mahajan, B., Vadali, S., and Alfriend, K., "Analytic Solution for Satellite Relative Motion: The Complete Zonal Gravitational Problem," in "Proceedings of the 26th AAS/AIAA Space Flight Mechanics Meeting," Univelt, Inc., San Diego, CA, Vol. 158, 2016, pp. 3325–3348.

[28] Mahajan, B., Vadali, S., and Alfriend, K., "Analytic Solution for Satellite Relative Motion: The Complete Zonal Gravitational Problem," *Journal of Guidance, Control, and Dynamics*. doi: 10.2514/1.G004133, in press.

[29] D'Amico, S., Ardaens, J.-S., and Larsson, R., "Spaceborne Autonomous Formation-Flying Experiment on the PRISMA Mission," *Journal of Guidance, Control, and Dynamics*, Vol. 35, No. 3, 2012, pp. 834–850. doi: 10.2514/1.55638.

[30] Ardaens, J.-S., D'Amico, S., and Fischer, D., "Early Flight Results from the TanDEM-X Autonomous Formation Flying System," in "Proceedings of the 4$^{\text{th}}$ International Conference on Spacecraft Formation Flying Missions & Technologies (SFFMT)," Canadian Space Agency, St-Hubert, Quebec, 2011.

[31] D'Amico, S., Ardaens, J.-S., Gaias, G., Benninghoff, H., Schlepp, B., and Jørgensen, J. L., "Noncooperative Rendezvous Using Angles-Only Optical Navigation: System Design and Flight Results," *Journal of Guidance, Control, and Dynamics*, Vol. 36, No. 6, 2013, pp. 1576–1595. doi: 10.2514/1.59236.

[32] Gill, E. and Montenbruck, O., "Comparison of GPS-based Orbit Determination Strategies," in "Proceedings of the 18$^{\text{th}}$ International Symposium on Space Flight Dynamics, Missions and Technologies," ESA Communication Production Office (ESA SP-548), ESTEC, Noordwijk, The Netherland, 2004, p. 169âĂŞ174.

[33] Wnuk, E., "Recent Progress in Analytical Orbit Theories," *Advances in Space Research*, Vol. 23, No. 4,

1999, pp. 677–687. doi: 10.1016/S0273-1177(99)00148-9.

[34] Guinn, J. R., “Periodic Gravitational Perturbations for Conversion Between Osculating and Mean Orbit Elements,” in “Proceedings of the AAS/AIAA Astrodynamics Conference,” Univelt, Inc., San Diego, CA, Vol. 63, 1991.

[35] Kaula, W., *Theory of Satellite Geodesy*, Blaisdell publ. Co., Walthman, Massachusetts, 1966. ch. 3.

[36] Spiridonova, S., Kirschner, M., and Hugentobler, U., “Precise Mean Orbital Elements Determination for LEO Monitoring and Maintenance,” 24th International Symposium on Space Flight Dynamics, Laurel, Maryland, USA, 2014.

[37] Eckstein, M. and Hechler, H., “A reliable derivation of the perturbations due to any zonal and tesseral harmonics of the geopotential for nearly-circular satellite orbits,” ESRO sr-13, ESOC, Darmstadt, Germany, 1970.

[38] Montenbruck, O. and Gill, E., *Satellite Orbits - Models, Methods, and Applications*, Springer Verlag, 2001. ch. 5.

[39] Montenbruck, O., Kirschner, M., D’Amico, S., and Bettadpur, S., “E/I-Vector Separation for Safe Switching of the GRACE Formation,” *Aerospace Science and Technology*, Vol. 10, No. 7, 2006, pp. 628–635. doi: 10.1016/j.ast.2006.04.001.

[40] D’Amico, S. and Montenbruck, O., “Proximity Operations of Formation Flying Spacecraft using an Eccentricity/Inclination Vector Separation,” *Journal of Guidance, Control and Dynamics*, Vol. 29, No. 3, 2006, pp. 554–563. doi: 10.2514/1.15114.

[41] Han, C. and Yin, J., “Formation design in elliptical orbit using relative orbit elements,” *Acta Astronautica*, Vol. 77, 2012, pp. 34–47. doi:10.1016/j.actaastro.2012.02.026.

[42] Gaias, G. and Ardaens, J.-S., “Design challenges and safety concept for the AVANTI experiment,” *Acta Astronautica*, Vol. 123, 2016, pp. 409–419. doi: 10.1016/j.actaastro.2015.12.034.

[43] Gaias, G. and D’Amico, S., “Impulsive Maneuvers for Formation Reconfiguration using Relative Orbital Elements,” *Journal of Guidance, Control, and Dynamics*, Vol. 38, No. 6, 2015, pp. 1036–1049. doi: 10.2514/1.G000191.

[44] Gaias, G., D’Amico, S., and Ardaens, J.-S., “Generalized Multi-Impulsive Maneuvers for Optimum Spacecraft Rendezvous in Near-Circular Orbit,” *Int. J. Space Science and Engineering*, Vol. 3, No. 1, 2015, pp. 68–88. doi: 10.1504/IJSPACESE.2015.069361.

[45] Ardaens, J.-S., Gaias, G., and Kahle, R., “From GRACE to AVANTI: 15 years of Formation-Flying Experience at DLR,” 69th International Astronautical Congress, Bremen, Germany, IAC-18,D1,2,3, 2018.

[46] Deprit, A., "Canonical transformations depending on a small parameter," *Celestial Mechanics*, Vol. 1, No. 1, 1969, pp. 12–30. doi: 10.1007/BF01230629.

[47] Deprit, A., "The elimination of the parallax in satellite theory," *Celestial Mechanics*, Vol. 24, No. 2, 1981, pp. 111–153. doi: 10.1007/BF01229192.

[48] Lara, M., San-Juan, J., and López-Ochoa, L., "Proper Averaging Via Parallax Elimination," *Advances in the Astronautical Sciences*, Vol. 150, 2014, pp. 315–331. AAS 13-722 paper.

[49] Lara, M., San-Juan, J., and López-Ochoa, L., "Delaunay variables approach to the elimination of the perigee in Artificial Satellite Theory," *Celestial Mechanics and Dynamical Astronomy*, Vol. 120, No. 1, 2014, pp. 39–56. doi: 10.1007/s10569-014-9559-2.

[50] Deprit, A., "Delaunay normalisations," *Celestial Mechanics*, Vol. 26, No. 1, 1982, pp. 9–21. doi: 10.1007/BF01233178.

[51] Schaub, H. and Junkins, J. L., *Analytical Mechanics of Space Systems*, AIAA Education Series, Reston, Virginia (USA), 2014.

[52] Brouwer, D., "Solution of the problem of artificial satellite theory without drag," *Astronomical Journal*, Vol. 64, No. 1274, 1959, pp. 378–397. doi: 10.1086/107958.

[53] Lyddane, R., "Small Eccentricities or Inclination in the Brouwer Theory of the Artificial Satellite," *Astronomical Journal*, Vol. 68, No. 8, 1963, pp. 555–558.

[54] Ustinov, B., "Motion of Satellites along low-eccentricity Orbits in a noncentral terrestrial gravitational Field," *Cosmic Research*, Vol. 2, 1967, pp. 184–193. (in Russian).

[55] Cain, B., "Determination of Mean Elements for Brouwer's Satellite Theory," *Astronomical Journal*, Vol. 67, No. 6, 1962, pp. 391–392.

[56] Walter, H. G., "Conversion of osculating orbital elements into mean elements," *Astronomical Journal*, Vol. 72, No. 8, 1967, pp. 994–997.

[57] Szeto, A. and Lambeck, K., "On eccentricity functions for eccentric orbits," *Celestial Mechanics*, Vol. 27, 1982, pp. 325–337. doi: 10.1007/BF01228558.

[58] Persson, S., Jakobsson, B., and Gill, E., "PRISMA - Demonstration Mission for Advanced Rendezvous and Formation Flying Technologies and Sensors," 56th International Astronautical Congress, Fukuoka, Japan, 05-B56B07, 2005.

[59] Alfriend, K. T. and Yan, H., "Evaluation and Comparison of Relative Motion Theories," *Journal of Guidance, Control and Dynamics*, Vol. 28, No. 2, 2005, pp. 254–261. doi: 10.2514/1.6691.