

When the associated graded ring of a semigroup ring is Complete Intersection

M. D'Anna* V. Micale A. Sammartano

Abstract

Let (R, \mathfrak{m}) be the semigroup ring associated to a numerical semigroup S . In this paper we study the property of its associated graded ring $\text{gr}_{\mathfrak{m}}(R)$ to be Complete Intersection. In particular, we introduce and characterize β -rectangular and γ -rectangular Apéry sets, which will be the fundamental concepts of the paper and will provide, respectively, a sufficient condition and a characterization for $\text{gr}_{\mathfrak{m}}(R)$ to be Complete Intersection. Then we use these notions to give four equivalent conditions for $\text{gr}_{\mathfrak{m}}(R)$ in order to be Complete Intersection.

MSC: 13A30; 13H10.

Introduction

Let (R, \mathfrak{m}) be a Noetherian local ring with $|R/\mathfrak{m}| = \infty$ and let $\text{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$ be the associated graded ring of R with respect to \mathfrak{m} . The study of the properties of $\text{gr}_{\mathfrak{m}}(R)$ is a classical subject in local algebra, not only in the general d -dimensional case, but also under particular hypotheses (that allow to obtain more precise results). One main problem in this context is to estimate the depth of $\text{gr}_{\mathfrak{m}}(R)$ and to understand when this ring is a Cohen-Macaulay ring (see, e.g., [17], [18] and [21]). In connection to this problem, it is natural to investigate if $\text{gr}_{\mathfrak{m}}(R)$ is a Buchsbaum ring (see [11], [12]), a Gorenstein ring or if it is Complete Intersection (see [13]).

In this paper we are interested in the properties of $\text{gr}_{\mathfrak{m}}(R)$, when R is a numerical semigroup ring. The study of numerical semigroup rings is motivated

* *email* mdanna@dmi.unict.it

by their connection to singularities of monomial curves and by the possibility of translating algebraic properties into numerical properties. However, even in this particular case, many pathologies occur, hence these rings are also a great source of interesting examples.

In the numerical semigroup case, the Cohen-Macaulayness of $\text{gr}_{\mathfrak{m}}(R)$ has been extensively studied (see, e.g., [10], [15] and [20]); recently, different authors studied the Buchsbaumness and the Gorensteinness of $\text{gr}_{\mathfrak{m}}(R)$ (see [3], [5], [6] and [19]).

In this paper we investigate when $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection. About this problem not much is known (see [1]). When the embedding dimension of R is small, it is possible to list the generators of the defining ideals of R and of $\text{gr}_{\mathfrak{m}}(R)$ (as it is done in [14] and [20] when the embedding dimension is 2 or 3), but, as soon as the number of generators of \mathfrak{m} increases, the computations become too huge. On the other hand, a useful tool to study the general case (when \mathfrak{m} is n -generated) is the so called Apéry set of the semigroup. The properties of the Apéry set reveal much information on the Cohen-Macaulayness, the Buchsbaumness and the Gorensteinness of $\text{gr}_{\mathfrak{m}}(R)$. In this paper, using the Apéry set, we are able to characterize when $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection.

The structure of the paper is the following. In Section 1 we fix the notation and give some preliminaries about numerical semigroups, semigroup rings associated to a numerical semigroup and their associated graded ring. In particular, we qualitatively discuss the form of the elements in the defining ideals of R , of its quotient \overline{R} modulo an element of minimal value and of their associated graded rings (cf. Discussion 1.5).

In Section 2 we define two sets of integers β_i and γ_i , that yield to the definition and the characterization of two classes of numerical semigroups: semigroups with β -rectangular and γ -rectangular Apéry set (cf. Definition 2.9, Theorem 2.16 and Theorem 2.22). These notions will provide, respectively, a sufficient condition and a characterization for $\text{gr}_{\mathfrak{m}}(R)$ to be Complete Intersection. These classes are strictly connected and it is useful to study both of them together. The idea in these definitions is to use the “shape” of the Apéry set of the semigroup in order to find integers that give information on the degree and on the nature of the generators of the defining ideals of R , of its quotient \overline{R} modulo an element of minimal value and of their associated graded rings.

In Section 3 we prove the main theorem of the paper (cf. Theorem 3.6),

that is the characterization of those numerical semigroup rings whose associated graded ring is Complete Intersection. More precisely, we give four equivalent conditions for $\text{gr}_{\mathfrak{m}}(R)$ in order to be Complete Intersection. To obtain this result we need to deepen carefully the nature of the elements of the defining ideals of R , of its quotient \overline{R} modulo an element of minimal value and of their associated graded rings, using the integers β_i and γ_i introduced in the previous section (cf. Discussion 3.2 and Lemma 3.4). Finally, we give an alternative proof of a sufficient condition for $\text{gr}_{\mathfrak{m}}(R)$ being Complete Intersection presented in [2] (cf. Corollary 3.13) and we briefly study the case of embedding dimension 3 (cf. Theorem 3.14).

The computations made for this paper are performed by using the GAP system [9] and, in particular, the NumericalSgps package [7].

1 Preliminaries

Let \mathbb{N} denote the set of natural numbers, including 0. A *numerical semigroup* is a submonoid S of the monoid $(\mathbb{N}, +)$ with finite complement in it. Each numerical semigroup S has a natural partial ordering \preceq where, for every s and t in S , $s \preceq t$ if there is an element $u \in S$ such that $t = s + u$. The set $\{g_i\}$ of the minimal elements in the poset $(S \setminus \{0\}, \preceq)$ is called *minimal set of generators* for S ; indeed all the elements in S are linear combinations, with coefficients in \mathbb{N} , of minimal elements. Note that the set $\{g_i\}$ is finite since for any $s \in S$, $s \neq 0$, we have $g_i \not\equiv g_j \pmod{s}$ if $i \neq j$. A numerical semigroup minimally generated by $g_1 < g_2 < \dots < g_\nu$ is denoted by $\langle g_1, g_2, \dots, g_\nu \rangle$; the condition $|\mathbb{N} \setminus S| < \infty$ is equivalent to $\text{gcd}(g_1, \dots, g_\nu) = 1$.

There are several invariants associated to a numerical semigroup S . The integer $m = g_1 = \min\{s \in S, s > 0\}$ is called *multiplicity*, while the minimal number of generators ν is called *embedding dimension*; it is well known that $\nu \leq m$. Finally, the integer $f = \max\{z \in \mathbb{Z}, z \notin S\}$ is called *Frobenius number* of S .

Following the notation in [2], we denote by $\text{Ap}(S) = \{\omega_0, \dots, \omega_{m-1}\}$ the *Apéry set* of S with respect to m , that is, the set of the smallest elements in S in each congruence class modulo m . More precisely, $\omega_0 = 0$ and $\omega_i = \min\{s \in S \mid s \equiv i \pmod{m}\}$. The largest element in the Apéry set is always $f + m$.

Very often we will use the following result.

Proposition 1.1 ([8], Lemma 6). *Let $t \in \text{Ap}(S)$ and $u \preceq t$, then $u \in \text{Ap}(S)$.*

A numerical semigroup S is called *symmetric* if $f - x \notin S$ implies that $x \in S$ for every integer x (notice that the converse is true for every numerical semigroup S).

Proposition 1.2 ([16], Corollary 4.12). *S is symmetric if and only if $f + m$ is the unique maximal element in $(\text{Ap}(S), \preceq)$.*

An *ideal* of a semigroup S is a nonempty subset H of S such that $H + S \subseteq H$. The ideal $M = \{s \in S \mid s \neq 0\}$ is called *maximal ideal* of S . It is straightforward to see that, if H and L are ideals of S , then $H + L = \{h + l \mid h \in H, l \in L\}$ and $kH (= H + \dots + H, k \text{ summands, for } k \geq 1)$ are also ideals of S .

Let \mathbb{k} be an infinite field; the rings $R = \mathbb{k}[[t^S]] = \mathbb{k}[[t^{g_1}, \dots, t^{g_\nu}]]$ and $R = \mathbb{k}[t^S]_{\mathfrak{m}}$ are called the *numerical semigroup rings* associated to S . The ring R is a one-dimensional local domain, with maximal ideal $\mathfrak{m} = (t^{g_1}, \dots, t^{g_\nu})$ and quotient field $\mathbb{k}((t))$ and $\mathbb{k}(t)$, respectively. In both cases the associated graded ring of R with respect to \mathfrak{m} , $\text{gr}_{\mathfrak{m}}(R) = \bigoplus_{i \geq 0} \mathfrak{m}^i / \mathfrak{m}^{i+1}$, is the same. From now on, we will assume that $R = \mathbb{k}[[t^S]]$, but the other case is perfectly analogous.

Let (A, \mathfrak{n}) be the local ring of formal power series $\mathbb{k}[[x_1, \dots, x_\nu]]$ and let $\varphi : A \rightarrow R$ be the map defined by $\varphi(x_i) = t^{g_i}$. Clearly $R = A/I$ and $\mathfrak{m} = \mathfrak{n}/I$, with $I = \ker \varphi$. Notice that I is a binomial ideal generated by all the elements of the form

$$(*) \quad x_1^{j_1} \cdots x_\nu^{j_\nu} - x_1^{h_1} \cdots x_\nu^{h_\nu},$$

with $j_1 g_1 + \dots + j_\nu g_\nu = h_1 g_1 + \dots + h_\nu g_\nu$.

It is well known that this presentation induces a presentation of the corresponding associated graded rings:

$$\psi : \text{gr}_{\mathfrak{n}}(A) \rightarrow \text{gr}_{\mathfrak{m}}(R),$$

where the kernel is the initial ideal of I , i.e. the ideal I^* generated by the initial forms of the elements of I ; hence $\text{gr}_{\mathfrak{m}}(R) \cong \text{gr}_{\mathfrak{n}}(A)/I^* \cong \mathbb{k}[x_1, \dots, x_\nu]/I^*$ canonically.

Notice that I^* is an homogenous ideal generated by all the monomials of the form $x_1^{j_1} \cdots x_\nu^{j_\nu}$ coming from a binomial $(*)$ for which $j_1 + \dots + j_\nu < h_1 + \dots + h_\nu$ and by all the binomials $(*)$ such that $j_1 + \dots + j_\nu = h_1 + \dots + h_\nu$.

We denote by $\mu(\cdot)$ the minimal number of generators of an ideal. The ring R is Complete Intersection if $\mu(I) = \nu - 1$ and the associated graded ring $\text{gr}_{\mathfrak{m}}(R)$ is *Complete Intersection* if $\mu(I^*) = \nu - 1$. It is well known that, if $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection, then also R is Complete Intersection.

Numerical semigroups for which R is Complete Intersection are well known (and they are called Complete Intersection numerical semigroups; for the definition see, e.g., [16]). We are interested in studying when $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection. Let $\overline{R} = R/(t^m)$ and $\overline{G} = \text{gr}_{\overline{\mathfrak{m}}}(\overline{R})$, where $\overline{\mathfrak{m}}$ is the maximal ideal of \overline{R} .

Remark 1.3. In our hypotheses, it is clear that $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection if and only if \overline{G} is Complete Intersection and $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay (i.e., as it is proved in [10], $(t^m)^* \in \mathfrak{m}/\mathfrak{m}^2$ is not a zero-divisor in $\text{gr}_{\mathfrak{m}}(R)$). In fact, in case $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, from the isomorphism $R/(t^m) \cong \overline{R}$ we get the isomorphism $\text{gr}_{\mathfrak{m}}(R)/((t^m)^*) \cong \overline{G}$. We notice also that, in general, there is a surjective homomorphism of graded rings $\text{gr}_{\mathfrak{m}}(R) \rightarrow \overline{G}$, whose kernel is the initial ideal of (t^m) in $\text{gr}_{\mathfrak{m}}(R)$.

Remark 1.4. We note that $\overline{R} = \langle \overline{t^{\omega_i}} \mid \omega_i \in \text{Ap}(S) \rangle_{\mathbb{k}}$, since

$$\overline{t^s} = \overline{0} \text{ in } \overline{R} \iff t^s \in (t^m) \iff s - m \in S.$$

We also have $\overline{G} = \overline{R}$ as \mathbb{k} -vector spaces (but not as rings) since a nonzero monomial in \overline{R} is still nonzero in \overline{G} .

Discussion 1.5. Recalling the isomorphism $R \cong \mathbb{k}[[x_1, x_2, \dots, x_{\nu}]]/I$, where $\overline{x_i}$ corresponds to t^{g_i} (hence $\overline{x_1}$ corresponds to t^m), we have the isomorphism $\overline{R} \cong \mathbb{k}[[x_2, x_3, \dots, x_{\nu}]]/H$. More precisely H is the kernel of the homomorphism defined by $x_i \mapsto \overline{t^{g_i}}$ and it is generated by all the binomials of the form

$$(**) \quad x_2^{j_2} \cdots x_{\nu}^{j_{\nu}} - x_2^{h_2} \cdots x_{\nu}^{h_{\nu}},$$

with $j_2 g_2 + \cdots + j_{\nu} g_{\nu} = h_2 g_2 + \cdots + h_{\nu} g_{\nu} \in \text{Ap}(S)$, and by all the monomials of the form

$$(***) \quad x_2^{j_2} \cdots x_{\nu}^{j_{\nu}},$$

where $j_2 g_2 + \cdots + j_{\nu} g_{\nu} \notin \text{Ap}(S)$.

It follows that $\overline{G} = \text{gr}_{\overline{\mathfrak{m}}}(\overline{R}) \cong \mathbb{k}[x_2, x_3, \dots, x_{\nu}]/J$, where J is the kernel of the homomorphism defined by $x_i \mapsto \overline{t^{g_i}}$ (where now $\overline{t^{g_i}}$ is viewed as an

element of \overline{G}) and it is the initial ideal of H . Hence J is a binomial ideal generated by all the the binomials of the form

$$(+) \quad x_2^{j_2} \cdots x_\nu^{j_\nu} - x_2^{h_2} \cdots x_\nu^{h_\nu},$$

with $j_2 g_2 + \cdots + j_\nu g_\nu = h_2 g_2 + \cdots + h_\nu g_\nu \in \text{Ap}(S)$ and $j_1 + \cdots + j_\nu = h_1 + \cdots + h_\nu$, and by all the monomials of the form

$$(++) \quad x_2^{j_2} \cdots x_\nu^{j_\nu},$$

where either $j_2 g_2 + \cdots + j_\nu g_\nu \notin \text{Ap}(S)$ or $j_2 g_2 + \cdots + j_\nu g_\nu \in \text{Ap}(S)$ and there exist $h_2, \dots, h_\nu \in \mathbb{N}$, such that $j_2 g_2 + \cdots + j_\nu g_\nu = h_2 g_2 + \cdots + h_\nu g_\nu$ and $j_2 + \cdots + j_\nu < h_2 + \cdots + h_\nu$.

In particular, let $j = j_2 + \cdots + j_\nu$; then a binomial of the form (+) is not necessary as generator of J , if $x_2^{j_2} \cdots x_\nu^{j_\nu} \in (x_2, \dots, x_\nu)^{j+1}$. Furthermore, the monomial $x_2^{j_2} \cdots x_\nu^{j_\nu} \in (x_2, \dots, x_\nu)^j \setminus (x_2, \dots, x_\nu)^{j+1}$, such that $j_2 g_2 + \cdots + j_\nu g_\nu \in \text{Ap}(S)$, does not belong to J .

Finally, since the Krull dimension of \overline{G} is 0, we must have $\mu(J) \geq \nu - 1$. Hence \overline{G} is Complete Intersection if and only if $\mu(J) = \nu - 1$.

From the previous remarks and discussion, it is clear why, to study the Complete Intersection property for $\text{gr}_{\mathfrak{m}}(R)$ it is necessary to study the Apéry set of S . When \overline{G} is Complete Intersection, its Hilbert function is completely determined by the degree of the generators of J and its dimension as \mathbb{k} -vector space is the product of these degrees. Hence we will have to determine these degrees using numerical conditions; moreover, since monomials in \overline{G} correspond bijectively to elements of the Apéry set, we will have to determine the “shape” of $\text{Ap}(S)$ corresponding to \overline{G} Complete Intersection.

2 β - and γ -rectangular Apéry Sets

Within this section we introduce two sets of integers and two corresponding classes of numerical semigroups, defined via the shape of the Apéry set. The first class will provide a sufficient condition for \overline{G} to be complete intersection, while the second one will give a characterization.

Given a numerical semigroup $S = \langle g_1, g_2, \dots, g_\nu \rangle$ and $s, t \in S$, we recall that $s \preceq t$ if there exists $u \in S$ such that $s + u = t$. Now we want to define another partial ordering on S as in [3]. If $s \in S$ and $M = S \setminus \{0\}$ then

there exists a unique $h \in \mathbb{N}$ such that $s \in hM \setminus (h+1)M$; this integer is defined as the *order* of s and we will write $\text{ord}(s) = h$. Given $s, t \in S$, we say that $s \preceq_M t$ if there exists $u \in S$ such that $s + u = t$ (hence $s \preceq t$) and $\text{ord}(s) + \text{ord}(u) = \text{ord}(t)$. The partial order \preceq_M is particularly helpful in the study of the associated graded ring.

The sets of maximal elements of $\text{Ap}(S)$ with respect to \preceq and \preceq_M are denoted with $\text{maxAp}(S)$ and $\text{maxAp}_M(S)$, respectively.

Remark 2.1. We note that $\text{maxAp}(S) \subseteq \text{maxAp}_M(S)$ and the inclusion can be strict. For example, let $S = \langle 8, 9, 15 \rangle$. The only maximal element in $\text{Ap}(S)$ with respect to \preceq is 45. Anyway $\text{maxAp}_M(S) = \{30, 45\}$. Note that $\text{ord}(45) = 5 > 3 = \text{ord}(30) + \text{ord}(15)$.

A numerical semigroup S is called *M-pure* if every element in $\text{maxAp}_M(S)$ has the same order. *M-pure symmetric semigroups* are characterized in a similar way to symmetric semigroups:

Proposition 2.2 ([3], Proposition 3.7). *A semigroup S is M-pure symmetric if and only if $\omega \preceq_M f + m$, for every $\omega \in \text{Ap}(S)$.*

Every element $s \in S$ can be written, not necessarily in a unique way, as $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$; we call this combination of the generators a *representation* of s . Throughout the paper, we call “representation” both the expression $s = \lambda_1 g_1 + \cdots + \lambda_\nu g_\nu$ and the tuple $(\lambda_1, \lambda_2, \dots, \lambda_\nu)$. We say that an element $s \in S$ has a *unique representation* if it can be written in a unique way as a linear combination of g_1, g_2, \dots, g_ν . Notice that, by definition of Apéry set, an element $\omega \in \text{Ap}(S)$ can have only representations where g_1 does not appear.

A representation of an element $s \in S$ as $s = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_\nu g_\nu$ is called *maximal* if $\lambda_1 + \lambda_2 + \cdots + \lambda_\nu = \text{ord}(s)$. This kind of representations and, in particular, the number of maximal representations of elements of S have been studied in [4]. We say that $\text{Ap}(S)$ is of *unique maximal expression* if every $\omega \in \text{Ap}(S)$ has a unique maximal representation.

Let us define now the following integers, for every $i = 2 \dots, \nu$:

$$\begin{aligned} \beta_i &= \max\{h \in \mathbb{N} \mid hg_i \in \text{Ap}(S) \text{ and } \text{ord}(hg_i) = h\}; \\ \gamma_i &= \max\{h \in \mathbb{N} \mid hg_i \in \text{Ap}(S), \text{ord}(hg_i) = h \text{ and} \\ &\quad hg_i \text{ has a unique maximal representation}\}. \end{aligned}$$

Notice that, in the second definition, from $\text{ord}(hg_i) = h$ it follows that hg_i must be the unique maximal representation. The following proposition is straightforward:

Proposition 2.3. *For each index $i = 2, \dots, \nu$, we have $\gamma_i \leq \beta_i$.*

Examples 2.4. Let $S = \langle 8, 10, 15 \rangle$; so $\text{Ap}(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$. Since $30, 45 \in \text{Ap}(S)$ and $40, 60 \notin \text{Ap}(S)$ then $\beta_2, \beta_3 \leq 3$. We have the double representation $30 = 3 \cdot 10 = 2 \cdot 15$, implying that $\text{ord}(2 \cdot 15) = 3 > 2$; hence $\beta_2 = 3$ and $\beta_3 = 1$. It is easy to check that every element in $\text{Ap}(S)$ has a unique maximal representation, thus $\gamma_2 = \beta_2 = 3$ and $\gamma_3 = \beta_3 = 1$.

Let $S = \langle 7, 9, 10, 11, 12 \rangle$; we have $\text{Ap}(S) = \{0, 9, 10, 11, 12, 20, 22\}$. Analogously to the previous example, we have $\beta_2 = 1, \beta_3 = 2, \beta_4 = 2, \beta_5 = 1$. The only double representations in $\text{Ap}(S)$ are $20 = 9 + 11 = 10 + 10$ and $22 = 10 + 12 = 11 + 11$, and they are all maximal. In particular, it follows $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 1$.

In correspondence to the two families of numbers introduced above, we can define the following two sets:

$$B = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \leq \lambda_i \leq \beta_i \right\};$$

$$\Gamma = \left\{ \sum_{i=2}^{\nu} \lambda_i g_i \mid 0 \leq \lambda_i \leq \gamma_i \right\}.$$

The sets B, Γ , consist of the elements of S representable via a tuple $(\lambda_2, \dots, \lambda_{\nu})$ belonging to the hyper-rectangle of $\mathbb{N}^{\nu-1}$ whose vertices are respectively given by β_i and γ_i . By Proposition 2.3, it follows that $\Gamma \subseteq B$. Notice also that, as can be easily seen by the previous examples, since elements in B and Γ can have more than one representation, $|B| \leq \prod_{i=2}^{\nu} (\beta_i + 1)$ and $|\Gamma| \leq \prod_{i=2}^{\nu} (\gamma_i + 1)$.

It is natural to ask how the sets B and Γ , are related to the Apéry set. The inclusion $\text{Ap}(S) \subseteq B$ is always true, as we can deduce from the next lemma:

Lemma 2.5. *Let $\omega \in \text{Ap}(S)$ and let $\omega = \sum_{i=2}^{\nu} \lambda_i g_i$ be a maximal representation. Then $\lambda_i \leq \beta_i$ for each i .*

Proof. By Lemma 1.1 we have $\lambda_i g_i \in \text{Ap}(S)$. If there is a index i such that $\beta_i < \lambda_i$, then $\text{ord}(\lambda_i g_i) > \lambda_i$, by definitions of β_i . It follows that $\text{ord}(\sum_{i=2}^{\nu} \lambda_i g_i) > \sum_{i=2}^{\nu} \lambda_i$ and thus it is not a maximal representation of ω ; contradiction. \square

Notice also that an integer in B could have more different maximal representations, as can be seen in the same example as above $S = \langle 7, 9, 10, 11, 12 \rangle$.

We want to show that actually the stronger inclusion $\text{Ap}(S) \subseteq \Gamma$ holds; this is a consequence of the following useful result, which is somehow analogous to Lemma 2.5. In what follows, we denote with **lex** and **grlex** respectively the usual lexicographic order and graded lexicographic order in $\mathbb{N}^{\nu-1}$.

Lemma 2.6. *Let $\omega \in \text{Ap}(S)$ and set*

$$\mathcal{R} = \left\{ (\lambda_2, \dots, \lambda_{\nu}) \in \mathbb{N}^{\nu-1} \mid \sum_{i=2}^{\nu} \lambda_i g_i = \omega \text{ and } \sum_{i=2}^{\nu} \lambda_i = \text{ord}(\omega) \right\}$$

*i.e. the set of maximal representations of w . Let $(\mu_2, \dots, \mu_{\nu})$ be the maximum in \mathcal{R} with respect to **lex**, then we have $\mu_i \leq \gamma_i$ for each i .*

Proof. By Lemma 2.5 we have $\mu_i \leq \beta_i$. Let us suppose that there exists an index i such that $\gamma_i < \mu_i \leq \beta_i$, and take the minimum i with this property. By definitions of β_i, γ_i , the fact that $\gamma_i < \beta_i$ implies a double maximal representation of $(\gamma_i + 1)g_i$; moreover, since $\gamma_i + 1$ is the least integer with this property, the other maximal representation does not involve g_i . Explicitly, we have the relation

$$(2.1) \quad (\gamma_i + 1)g_i = \sum_{j \neq i} \eta_j g_j, \quad \gamma_i + 1 = \sum_{j \neq i} \eta_j.$$

Let us substitute the relation just found in $(\mu_2, \dots, \mu_{\nu})$, i.e. consider the tuple $(\mu'_2, \dots, \mu'_{\nu})$ where

$$\mu'_i = \mu_i - \gamma_i - 1, \quad \mu'_j = \mu_j + \eta_j \quad \text{for } j \neq i$$

in particular by (2.1) $(\mu'_2, \dots, \mu'_{\nu}) \in \mathcal{R}$. Now, if there is a index $j < i$ such that $\eta_j \neq 0$, then $(\mu_2, \dots, \mu_{\nu}) < (\mu'_2, \dots, \mu'_{\nu})$ with respect to **lex**, yielding a contradiction to the choice of $(\mu_2, \dots, \mu_{\nu})$. Hence $\eta_j = 0$ for each $j < i$. But this is a contradiction to (2.1), since $g_i < g_j$ for $i < j$ and $\gamma_i + 1 = \sum_{j > i} \eta_j$. The lemma is proved. \square

Corollary 2.7. *Let S be a numerical semigroup, then $\text{Ap}(S) \subseteq \Gamma$.*

Remark 2.8. It is straightforward to check that, under the notation of Lemma 2.6, (μ_2, \dots, μ_ν) is also the maximum with respect to grlex in the set of (not necessarily maximal) representations of ω .

We are interested in numerical semigroups for which the inclusions shown so far turn out to be equalities.

Definition 2.9. Let S be a numerical semigroup:

1. the semigroup S has β -rectangular Apéry set if $\text{Ap}(S) = B$;
2. the semigroup S has γ -rectangular Apéry set if $\text{Ap}(S) = \Gamma$.

Corollary 2.10. *We have the following implication:*

$$\text{Ap}(S) \text{ is } \beta\text{-rectangular} \Rightarrow \text{Ap}(S) \text{ is } \gamma\text{-rectangular}.$$

Proof. It follows immediately by the inclusions $\text{Ap}(S) \subseteq \Gamma \subseteq B$ □

Remark 2.11. It would be natural to introduce also another set of integers: $\alpha_i = \max\{h \in \mathbb{N} \mid hg_i \in \text{Ap}(S)\}$; it is clear that $\alpha_i \geq \beta_i$. These integers, as for the β_i 's and the γ_i 's, yield to another class of semigroups (that we could call semigroups with α -rectangular Apéry set) interesting to discuss and somehow similar to the two classes just defined. However, this class would provide a too strong condition with respect to the property of \overline{G} to be Complete Intersection, hence, for brevity, we will omit its study.

Examples 2.12. (1) Let $S = \langle 8, 10, 15 \rangle$; we have seen in Examples 2.4 that $\text{Ap}(S) = \{0, 10, 15, 20, 25, 30, 35, 45\}$ and $\beta_2 = 3, \beta_3 = 1$. Hence we obtain that $\text{Ap}(S)$ is β -rectangular.

(2) Let $S = \langle 8, 10, 11, 12 \rangle$; we have $\text{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$. Since $20, 44, 24 \notin \text{Ap}(S)$ and $\text{ord}(10) = 1, \text{ord}(33) = 3, \text{ord}(12) = 1$ we have $\beta_2 = 1, \beta_4 = 3, \beta_4 = 1$. It is clear that $\gamma_2 = \gamma_4 = 1$; since $22 = 11 + 11 = 10 + 12$ has two maximal representation, we find $\gamma_3 = 1$. Thus $\text{Ap}(S)$ is γ -rectangular but $\text{Ap}(S)$ is not β -rectangular.

(3) Let $S = \langle 5, 6, 9 \rangle$; we have $\text{Ap}(S) = \{0, 6, 9, 12, 18\}$ and the only double representation is $18 = 3 \cdot 6 = 2 \cdot 9$. In this case $\gamma_2 = 3$, as $18 = 3 \cdot 6$ is the unique maximal representation, while $\gamma_3 = 1$ because $\text{ord}(2 \cdot 18) = 3 > 2$. It follows that $\text{Ap}(S)$ is not γ -rectangular, since $\lambda_2 \cdot 6 + \lambda_3 \cdot 9 \notin \text{Ap}(S)$ as soon as both $\lambda_2 > 0$ and $\lambda_3 > 0$.

Our main purpose is to characterize these kinds of numerical semigroups in terms of some of their invariants, namely the Frobenius number and the multiplicity. This will allow simpler tests for those properties. We also want to study the relations between these classes of semigroups and the properties of unique maximal expression of $\text{Ap}(S)$ and of the partial orders \preceq and \preceq_M .

The next lemmas concern maximal representations and are necessary to establish a characterization of semigroups with β -rectangular $\text{Ap}(S)$.

Lemma 2.13. *If $s \preceq_M t$ and t has a unique maximal representation, then s has a unique maximal representation.*

Proof. We have that $s + u = t$ for some $u \in S$ and $\text{ord}(s) + \text{ord}(u) = \text{ord}(t)$. If s had two maximal representations, the same should be true for t ; contradiction. \square

Lemma 2.14. *Let S be a numerical semigroup with β -rectangular Apéry set and let $\omega \in \text{Ap}(S)$. Then any representation $\omega = \lambda_2 g_2 + \cdots + \lambda_\nu g_\nu$, with $\lambda_i \leq \beta_i$ for each $i = 2, \dots, \nu$, is maximal.*

Proof. Assume that the representation $\omega = \lambda_2 g_2 + \cdots + \lambda_\nu g_\nu$ is not maximal. Let $\omega = \mu_2 g_2 + \cdots + \mu_\nu g_\nu$ be a maximal representation of ω ; since $\omega \in \text{Ap}(S)$, by Lemma 2.5, we have $\mu_i \leq \beta_i$ for every $i = 2, \dots, \nu$. Moreover, by maximality, we have $\sum_{i=2}^\nu \lambda_i < \sum_{i=2}^\nu \mu_i$, hence there exists an index i such that $\lambda_i < \mu_i$. On the other hand, since $\lambda_2 g_2 + \cdots + \lambda_\nu g_\nu = \mu_2 g_2 + \cdots + \mu_\nu g_\nu$, there exists another index j such that $\lambda_j > \mu_j$. Subtracting from both sides of the previous equality the common summands, we get the equality

$$(2.2) \quad \sum_{i \in T_1} \eta_i g_i = \sum_{i \in T_2} \eta_i g_i$$

where T_1, T_2 are two non-empty disjoint subsets of $\{2, \dots, \nu\}$, $0 \leq \eta_i \leq \beta_i$, for each $i = 2, \dots, \nu$, and $\sum_{i \in T_1} \eta_i < \sum_{i \in T_2} \eta_i$.

Since S has β -rectangular Apéry set, the element $t = \sum_{i=2}^\nu \beta_i g_i$ belongs to $\text{Ap}(S)$. Let us substitute the relation (2.2) in this representation of t , that is to say consider the tuple (ξ_2, \dots, ξ_ν) where

$$\xi_i = \beta_i - \eta_i \text{ if } i \in T_1, \quad \xi_i = \beta_i + \eta_i \text{ if } i \in T_2, \quad \xi_i = \beta_i \text{ otherwise.}$$

This is another representation of t and $\sum_{i=2}^\nu \beta_i < \sum_{i=2}^\nu \xi_i$ by (2.2). It follows that $(\beta_2, \dots, \beta_\nu)$ is not a maximal representation and, therefore, t has a maximal representation with some coefficient bigger than β_i ; contradiction to Lemma 2.5. \square

Remark 2.15. Combining the last lemma and Lemma 2.5, we have that the maximality of a representation $(\lambda_2, \dots, \lambda_\nu)$ of an element of $\text{Ap}(S)$ is equivalent to have $\lambda_i \leq \beta_i$ for each i , under the assumption of β -rectangular Apéry set.

We are ready to give some characterizations of semigroups with β -rectangular Apéry set.

Theorem 2.16. *The following conditions are equivalent:*

- (i) $\text{Ap}(S)$ is β -rectangular;
- (ii) $\text{Ap}(S)$ has a unique maximal element with respect to \preceq_M and this element has unique maximal representation;
- (iii) S is M -pure, symmetric and $\text{Ap}(S)$ is of unique maximal expression;
- (iv) $f + m = \sum_{i=2}^\nu \beta_i g_i$;
- (v) $m = \prod_{i=2}^\nu (\beta_i + 1)$.

Proof. (i) \Rightarrow (ii) Since $\text{Ap}(S)$ is β -rectangular, we immediately get that $\sum_{i=2}^\nu \beta_i g_i$ is the unique maximal element in $(\text{Ap}(S), \preceq)$ and hence it is $f + m$. Moreover, by Lemma 2.14, any element $\omega \in \text{Ap}(S)$ is maximally represented by a tuple $(\lambda_2, \dots, \lambda_\nu)$ with $\lambda_i \leq \beta_i$; in particular $\text{ord}(\omega) = \sum_{i=2}^\nu \lambda_i$ and we can deduce that $\omega \preceq_M f + m$, for each $\omega \in \text{Ap}(S)$.

Finally, by Remark 2.15, $\sum_{i=2}^\nu \beta_i g_i$ is the unique maximal representation of $f + m$.

(ii) \Rightarrow (iii) By Proposition 2.2, S is M -pure symmetric; by Lemma 2.13, every $\omega \in \text{Ap}(S)$ has a unique maximal representation.

(iii) \Rightarrow (iv) The unique maximal element in $(\text{Ap}(S), \preceq_M)$ is necessarily $f + m$. Its unique maximal representation is of the form $f + m = \sum_{i=2}^\nu \lambda_i g_i$ with $\lambda_i \leq \beta_i$ by Lemma 2.5.

Since S is M -pure symmetric $\beta_i g_i \preceq_M f + m$ for each $i = 2, \dots, \nu$. Hence $f + m - \beta_2 g_2 \in \text{Ap}(S)$ and, by Lemma 2.13, it has unique maximal representation $f + m - \beta_2 g_2 = \sum_{i=2}^\nu \varepsilon_i g_i$; moreover $\varepsilon_i \leq \beta_i$, by maximality of the representation. It follows that $f + m = \beta_2 g_2 + \sum_{i=2}^\nu \varepsilon_i g_i$ is a maximal representation, hence $\varepsilon_2 = 0$ (again by Lemma 2.5). Moreover, since $f + m$ has a unique maximal representation, $\lambda_2 = \beta_2$; arguing recursively, the thesis follows.

(iv) \Rightarrow (i) It is a consequence of Proposition 1.1.

(i) \Rightarrow (v) It follows by $m = |\text{Ap}(S)|$, the fact that $\text{Ap}(S)$ is of unique maximal expression and Remark 2.15.

(v) \Rightarrow (i) We already noticed that $\text{Ap}(S) \subseteq B$ and since $m = \prod_{i=2}^{\nu}(\beta_i + 1) = |\text{Ap}(S)|$, we must have an equality. \square

Example 2.17. We may apply the last theorem to show that the Apéry set of $S = \langle 12, 14, 16, 23 \rangle$ is β -rectangular, without even computing the whole set $\text{Ap}(S)$. We need to determine the β_i 's:

$$\begin{aligned} 2 \cdot 14 &= 12 + 16 \in 12 + S \\ 2 \cdot 16 &= 32 \in 2M \setminus 3M \text{ and } 32 - 12 \notin S \\ 3 \cdot 16 &= 4 \cdot 12 \in 12 + S \\ 2 \cdot 23 &= 2 \cdot 16 + 14 \in 3M \end{aligned}$$

and so $\beta_2 = 1$, $\beta_3 = 2$, $\beta_4 = 1$ and $m = 12 = 2 \cdot 3 \cdot 2 = \prod_{i=2}^{\nu}(\beta_i + 1)$.

We have seen (cf. Corollary 2.10) that the following implication holds:

$$\text{Ap}(S) \text{ is } \beta\text{-rectangular} \Rightarrow \text{Ap}(S) \text{ is } \gamma\text{-rectangular.}$$

But a priori we still might have $\beta_i > \gamma_i$ when $\text{Ap}(S)$ is β -rectangular, for some i . We show that this is not the case, using the theorem just proved.

Corollary 2.18. *Let S be a numerical semigroup. If $\text{Ap}(S)$ is β -rectangular, then $\beta_i = \gamma_i$, for every $i = 2, \dots, \nu$.*

Proof. If there is an index i such that $\gamma_i < \beta_i$, then, by definition of β_i and of γ_i , $\beta_i g_i$ is in the Apéry set and it has more maximal representations. Contradiction to Theorem 2.16, (ii). \square

We now turn to the study of semigroups with γ -rectangular Apéry set, starting with a result that is analogous to Lemma 2.14.

Lemma 2.19. *Let S be a semigroup with γ -rectangular Apéry set and let $\omega \in \text{Ap}(S)$. Then any representation $\omega = \lambda_2 g_2 + \dots + \lambda_{\nu} g_{\nu}$, with $\lambda_i \leq \gamma_i$ (for every $i = 2 \dots, \nu$), is maximal.*

Proof. Assume, by absurd, that there exists $\omega \in \text{Ap}(S)$ with a non-maximal representation $\omega = \sum_{i=2}^{\nu} \lambda_i g_i$, where $\lambda_i \leq \gamma_i$ (for every $i = 2, \dots, \nu$). Notice that, as we have already seen for the case $\text{Ap}(S) = B$, if $\text{Ap}(S) = \Gamma$, then $f + m = \sum_{i=2}^{\nu} \gamma_i g_i$. In particular, up to substituting the non-maximal representation of ω in $f + m = \sum_{i=2}^{\nu} \gamma_i g_i$, we may assume $\omega = f + m$ and hence $\lambda_i = \gamma_i$, for each i .

If we take a maximal representation (μ_2, \dots, μ_ν) of $f + m$, then we have the strict inequality

$$(2.3) \quad (\gamma_2, \dots, \gamma_\nu) < (\mu_2, \dots, \mu_\nu)$$

with respect to **grlex** (since the sums of the respective coefficients are different). In particular (2.3) holds if we choose (μ_2, \dots, μ_ν) to be the maximum in the set of all the representations of $f + m$ with respect to **grlex**. By Remark 2.8 we have

$$(\gamma_2, \dots, \gamma_\nu) < (\mu_2, \dots, \mu_\nu) \leq (\gamma_2, \dots, \gamma_\nu)$$

with respect to **grlex**, and thus we reach a contradiction. \square

Lemma 2.20. *Let S be a semigroup with γ -rectangular Apéry set. Then each $\omega \in \text{Ap}(S)$ has a unique representation of the form $\omega = \lambda_2 g_2 + \dots + \lambda_\nu g_\nu$, with $\lambda_i \leq \gamma_i$, for every $i = 2, \dots, \nu$.*

Proof. Assume, by absurd, that there are two distinct representations of $\omega \in \text{Ap}(S)$

$$\omega = \sum_{i=2}^{\nu} \lambda_i g_i = \sum_{i=2}^{\nu} \mu_i g_i, \text{ with } \lambda_i, \mu_i \leq \gamma_i.$$

By Lemma 2.19 both representations are maximal, and in particular $\sum_{i=2}^{\nu} \lambda_i = \sum_{i=2}^{\nu} \mu_i$; since they are distinct we have, for instance,

$$(\lambda_2, \dots, \lambda_\nu) < (\mu_2, \dots, \mu_\nu)$$

with respect to **lex**. By adding the tuple $(\gamma_2 - \lambda_2, \dots, \gamma_\nu - \lambda_\nu)$ to both sides of the last equality we have two maximal representations of $f + m$:

$$(\gamma_2, \dots, \gamma_\nu) < (\mu_2 + \gamma_2 - \lambda_2, \dots, \mu_\nu + \gamma_\nu - \lambda_\nu)$$

with respect to **lex**. We reach an absurd by Lemma 2.6. \square

Corollary 2.21. *If S has γ -rectangular Apéry set, then it is M -pure symmetric.*

Proof. Let $\omega \in \text{Ap}(S)$. By Lemma 2.6, $\omega = \sum_{i=2}^{\nu} \lambda_i g_i$, with $\lambda_i \leq \gamma_i$, and, by Lemma 2.19, this representation is maximal; therefore $\text{ord}(\omega) = \sum_{i=2}^{\nu} \lambda_i$. Since this is valid for an arbitrary $\omega \in \text{Ap}(S)$, we easily have $\omega \preceq_M \sum_{i=2}^{\nu} \gamma_i g_i = f + m$ and the thesis follows by Proposition 2.2. \square

We are ready to characterize semigroups with γ -rectangular Apéry set.

Theorem 2.22. *The following conditions are equivalent:*

- (i) $\text{Ap}(S)$ is γ -rectangular;
- (ii) $f + m = \sum_{i=2}^{\nu} \gamma_i g_i$;
- (iii) $m = \prod_{i=2}^{\nu} (\gamma_i + 1)$.

Proof. (i) \Rightarrow (ii) Clear, as $f + m$ is the biggest element in $\text{Ap}(S)$.

(ii) \Rightarrow (i) It follows by Lemma 1.1.

(i) \Rightarrow (iii) It follows by Lemma 2.20.

(iii) \Rightarrow (i) Since $\text{Ap}(S) \subseteq \Gamma$ and $|\text{Ap}(S)| = m = \prod_{i=2}^{\nu} (\gamma_i + 1) \geq |\Gamma|$, the inclusion must be an equality. \square

Notice that we do not obtain a full analogous result of Theorem 2.16; more precisely, we cannot recover conditions (ii) and (iii). The closest analogous to those conditions is actually expressed by Lemma 2.20. If we look at the second semigroup in Examples 2.12, $S = \langle 8, 10, 11, 12 \rangle$ (with $\text{Ap}(S) = \{0, 10, 11, 12, 21, 22, 23, 33\}$ and $\gamma_i = 1$, for each $i = 2, 3, 4$), we notice that both 22 and 33 have two maximal representations, but only one in Γ .

3 The Main Theorem

In this section, we apply the results contained in the previous section in order to give a characterization of the numerical semigroup rings whose associated graded ring is Complete Intersection.

We recall that in the first section we defined $\overline{R} = R/(t^m)$ and $\overline{G} = \text{gr}_{\overline{\mathfrak{m}}}(\overline{R})$, where $\overline{\mathfrak{m}}$ is the maximal ideal of \overline{R} .

We also need to introduce two more invariants associated to R . The ideal $Q = (t^m)$ is a principal reduction of the maximal ideal \mathfrak{m} , that is a principal ideal $Q \subseteq \mathfrak{m}$ such that $Q\mathfrak{m}^h = \mathfrak{m}^{h+1}$ for some non-negative integer h . The *reduction number* is the integer $r = r_Q(\mathfrak{m}) = \min\{h \in \mathbb{N}, Q\mathfrak{m}^h = \mathfrak{m}^{h+1}\}$, while

the *index of nilpotency* is defined as $s = s_Q(\mathfrak{m}) = \min\{h \in \mathbb{N}, \mathfrak{m}^{h+1} \subseteq Q\}$. In the case of numerical semigroup rings we have $r = \min\{h \in \mathbb{N}, m + hM = (h+1)M\}$ and $s = \max\{\text{ord}(\omega_i) \mid \omega_i \in \text{Ap}(S)\}$; from the last two equalities it is easy to see that $s \leq r$.

We will also need the following result of Bryant.

Theorem 3.1 ([3], Theorem 3.14). *Under the above notation, we have:*

- (1) S is M -pure symmetric if and only if \overline{G} is Gorenstein;
- (2) if $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, then $s = r$. The converse holds if S is M -pure;
- (3) $\text{gr}_{\mathfrak{m}}(R)$ is Gorenstein if and only if S is M -pure symmetric and $s = r$.

To prove the main result of the paper we have to be more precise about the generators of the ideal J defined in Discussion 1.5.

Discussion 3.2. Using the terminology introduced in the previous section, in Discussion 1.5 we have shown that J is generated by all the the binomials of the form

$$(+) \quad x_2^{j_2} \cdots x_\nu^{j_\nu} - x_2^{h_2} \cdots x_\nu^{h_\nu},$$

where $j_2g_2 + \cdots + j_\nu g_\nu = h_2g_2 + \cdots + h_\nu g_\nu \in \text{Ap}(S)$ are two maximal representations, and by all the monomials of the form

$$(++) \quad x_2^{j_2} \cdots x_\nu^{j_\nu},$$

where either $j_2g_2 + \cdots + j_\nu g_\nu \notin \text{Ap}(S)$ or $j_2g_2 + \cdots + j_\nu g_\nu \in \text{Ap}(S)$ and it is not a maximal representation.

By definition of β_i it follows that $x_i^{\beta_i+1} \in J$. In fact, if $\text{ord}(\beta_i+1)g_i > \beta_i+1$, then $(tg_i)^{\beta_i+1} \in \mathfrak{m}^{\beta_i+2}$ and therefore $(\overline{tg_i})^{\beta_i+1} = 0$ in $\text{gr}_{\mathfrak{m}}(R)$ and hence in \overline{G} . On the other hand, if $(\beta_i+1)g_i \notin \text{Ap}(S)$, then $(\overline{tg_i})^{\beta_i+1} = 0$ in \overline{R} and hence in \overline{G} . Moreover, by definition of β_i and by Discussion 1.5 it is clear that $x_i^{\beta_i} \notin J$ for every index i .

On the other hand, by definition of γ_i we have that, $\gamma_i < \beta_i$ if and only if $(\gamma_i+1)g_i \in \text{Ap}(S)$ is a maximal representation, but it is not unique. Hence $(\gamma_i+1)g_i = \sum_{j \neq i} \lambda_j g_j$ and $\gamma_i+1 = \sum_{j \neq i} \lambda_j$; equivalently $x_i^{\gamma_i+1} - \prod_{j \neq i} x_j^{\lambda_j} \in J$.

Notice that, for some $h \leq \beta_i$ (hence, also for some $h \leq \gamma_i$), it could happen that $hg_i = \sum_{j \neq i} \lambda_j g_j$ and $h > \sum_{j \neq i} \lambda_j$; in this case, $\prod_{j \neq i} x_j^{\lambda_j} \in J$.

Hence the smallest pure power of x_i appearing in a monomial or a binomial of J is $x_i^{\gamma_i+1}$.

Finally, it is clear that, if a binomial (+) is in J , then we can cancel all the common factors: in fact, by Proposition 1.1, if $\omega \in \text{Ap}(S)$ and $u \in S$ is such that $u \preceq \omega$, then $u \in \text{Ap}(S)$.

In the next corollary we summarize the results of Discussions 1.5 and 3.2, that we will need in the rest of the paper.

Corollary 3.3. *For every $i = 2, \dots, \nu$, we have:*

- (i) $x_i^{\beta_i+1} \in J$ and $x_i^{\beta_i} \notin J$;
- (ii) $\gamma_i < \beta_i \iff (\gamma_i + 1)g_i = \sum_{j \neq i} \lambda_j g_j$ and $\gamma_i + 1 = \sum_{j \neq i} \lambda_j \iff x_i^{\gamma_i+1} - \prod_{j \neq i} x_j^{\lambda_j} \in J$;
- (iii) *the smallest pure power of x_i appearing in a monomial or a binomial of J is $x_i^{\gamma_i+1}$.*
- (iv) $x_2^{\lambda_2} \dots x_\nu^{\lambda_\nu} \notin J \iff \sum_{j=2}^\nu \lambda_j g_j \in \text{Ap}(S)$ and $\sum_{j=2}^\nu \lambda_j g_j$ is a maximal representation.

The next result is a key step in order to prove the main theorem.

Lemma 3.4. *The ring \overline{G} is Complete Intersection if and only if the defining ideal J is of the following form*

$$J = (x_i^{\gamma_i+1} - \rho_i \prod_{j \neq i} x_j^{\lambda_j} : i = 2, \dots, \nu),$$

where $\rho_i = 0$, if $\beta_i = \gamma_i$ and 1 otherwise; as soon as $\rho_i = 1$, $(\gamma_i + 1)g_i = \sum_{j \neq i} \lambda_j g_j$ and $\gamma_i + 1 = \sum_{j \neq i} \lambda_j$.

Proof. We have that $\overline{G} \cong \mathbb{k}[x_2, x_3, \dots, x_\nu]/J$; hence, if J is of the form described in the statement, $\mu(J) = \nu - 1$ and \overline{G} is Complete Intersection.

Conversely, assume that \overline{G} is Complete Intersection, that is $\mu(J) = \nu - 1$. By Corollary 3.3 (i), we know that $(x_2^{\beta_2+1}, \dots, x_\nu^{\beta_\nu+1}) \subseteq J$ and that $x_i^{\beta_i} \notin J$ for every index i .

Hence, if $J \supsetneq (x_2^{\beta_2+1}, \dots, x_\nu^{\beta_\nu+1})$, for every index i such that $x_i^{\beta_i+1}$ is not a minimal generator, there exists a unique binomial of the form $x_i^h - \prod_{j \neq i} x_j^{h_j}$, which is a minimal generator. By Corollary 3.3 (ii) and (iii), we have that

$\gamma_i < \beta_i$ and the generator is $x_i^{\gamma_i+1} - \prod_{j \neq i} x_j^{\lambda_j}$ with $\gamma_i + 1 = \sum_{j \neq i} \lambda_j$, since there is not any binomial in J involving a pure power of x_i with exponent smaller than $\gamma_i + 1$. \square

Remarks 3.5. (1) Let $\tilde{J} = (x_i^{\gamma_i+1} - \rho_i \prod_{j \neq i} x_j^{\lambda_j} : i = 2 \dots, \nu)$ (with the same notation of the previous lemma); by Corollary 3.3, it is clear that we always have the inclusion $J \supseteq \tilde{J}$.

(2) We can assume that in the binomials appearing as generators of \tilde{J} every exponent λ_j is less than or equal to γ_j : choose the biggest $(\nu - 2)$ -tuple $(\lambda_2, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_\nu)$ with respect to lex , among all the possible $(\nu - 2)$ -tuples corresponding to the maximal representations of $(\gamma_i + 1)g_i$ and then argue in a similar way as in Lemma 2.6.

(3) Since $g_2 < g_3 < \dots < g_\nu$, necessarily $\rho_2 = \rho_\nu = 0$, that is $x_2^{\gamma_2+1}$ and $x_\nu^{\gamma_\nu+1}$ are minimal generators of \tilde{J} .

Theorem 3.6. *The following conditions are equivalent:*

- (i) $\text{gr}_m(R)$ is Complete Intersection;
- (ii) $\text{Ap}(S)$ is γ -rectangular and $\text{gr}_m(R)$ is Cohen-Macaulay;
- (iii) $\text{Ap}(S)$ is γ -rectangular and $r = s$;
- (iv) $\text{Ap}(S)$ is γ -rectangular and $\text{gr}_m(R)$ is Buchsbaum;
- (v) $\text{Ap}(S)$ is γ -rectangular and $\text{gr}_m(R)$ is Gorenstein.

Proof. (i) \Rightarrow (ii) Clearly $\text{gr}_m(R)$ is Cohen-Macaulay. By Remark 1.3 we know that \overline{G} is Complete Intersection. Using the previous lemma, we know that J is generated by $\nu - 1$ forms of degree $\gamma_i + 1$, for every $i = 2, \dots, \nu$.

Since $\overline{G} \cong \mathbb{k}[x_2, \dots, x_\nu]/J$, we have $\dim_{\mathbb{k}}(\overline{G}) = \prod_{i=2}^{\nu} (\gamma_i + 1)$, hence every monomial $x_2^{\lambda_2} \dots x_\nu^{\lambda_\nu}$, with $\lambda_i \leq \gamma_i$, does not belong to J and their images are pairwise different in \overline{G} . By Corollary 3.3 (iv), the corresponding elements $\lambda_2 g_2 + \dots + \lambda_\nu g_\nu$ of S belong to $\text{Ap}(S)$ (and all of these are maximal representations). Hence we obtain $\Gamma = \{\lambda_2 g_2 + \lambda_3 g_3 + \dots + \lambda_\nu g_\nu \mid \lambda_i = 0, \dots, \gamma_i, i = 2, \dots, \nu\} = \text{Ap}(S)$, that is $\text{Ap}(S)$ is γ -rectangular.

(ii) \Rightarrow (i) $\text{gr}_m(R)$ is Cohen-Macaulay and, by [10, Theorem 7], $(t^m)^*$ is regular. Hence, by Remark 1.3, G is Complete Intersection if and only if \overline{G} is Complete Intersection.

We know that, as \mathbb{k} -vector space, $\overline{G} = \langle \overline{t^{\omega_i}} \mid \omega_i \in \text{Ap}(S) \rangle_{\mathbb{k}}$. Moreover, by Remarks 3.5 (1), it is clear that $J \supseteq \tilde{J}$, hence

$$m = |\text{Ap}(S)| = \dim_{\mathbb{k}}(\overline{G}) \leq \dim_{\mathbb{k}}(\mathbb{k}[x_2, \dots, x_\nu]/\tilde{J}) = \prod_{i=2}^{\nu} (\gamma_i + 1).$$

Since $\text{Ap}(S)$ is γ -rectangular, by Theorem 2.22, $m = \prod_{i=2}^{\nu} (\gamma_i + 1)$; thus in the above chain we have all equalities and, therefore, $J = \tilde{J}$, that is \overline{G} is Complete Intersection.

(ii) \Leftrightarrow (iii) By Corollary 2.21, S is M -pure; under this hypothesis G is Cohen-Macaulay if and only if $r = s$ (Theorem 3.1).

(ii) \Leftrightarrow (iv) By [6, Proposition 5.5].

(iv) \Leftrightarrow (v) By [6, Corollary 5.6]. \square

Remark 3.7. Using the last theorem, in order to know if $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection we have to check if $\text{Ap}(S)$ is γ -rectangular and if $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay (or Buchsbaum, or Gorenstein). In [2, Theorem 2.6] there is a characterization of the Cohen-Macaulayness of $\text{gr}_{\mathfrak{m}}(R)$ that has been strengthened in [6, Proposition 5.1]; in particular, in case $\text{Ap}(S)$ is γ -rectangular, one has to compute the integers a_i and b_i defined in these characterizations, only for $\omega_i = f + m$, which is the only element in $\max\text{Ap}_{\mathfrak{M}}(S)$. Notice also that, in this case, to check the Buchsbaumness of $\text{gr}_{\mathfrak{m}}(R)$ it is not easier than to check the Cohen-Macaulayness (cf. [6, Proposition 3.6]).

Examples 3.8. Let us consider the semigroups (2) and (3) of the Examples 2.12.

(2) $S = \langle 8, 10, 11, 12 \rangle$: here $\text{Ap}(S)$ is γ -rectangular (and not β -rectangular) and $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay, since $r = 3 = \text{ord}(33)$. Hence $\text{gr}_{\mathfrak{m}}(R)$ is Complete Intersection. Computing the defining ideals we get: $I = (x_2^2 - x_1x_4, x_3^2 - x_2x_4, x_1^3 - x_4^2)$, $I^* = (x_2^2 - x_1x_4, x_3^2 - x_2x_4, x_4^2)$ and $J = (x_2^2, x_3^2 - x_2x_4, x_4^2)$.

(3) $S = \langle 5, 6, 9 \rangle$: here $\text{Ap}(S)$ is not γ -rectangular and $\text{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay (as can be checked using [6, Proposition 5.1]); S is symmetric, but not M -pure (since $9 \not\leq_M 18$). Therefore, $\text{gr}_{\mathfrak{m}}(R)$ is not Complete Intersection (nor Gorenstein). Computing the defining ideals we obtain: $I = (x_1^3 - x_2x_3, x_2^3 - x_3^2)$, $I^* = (x_2x_3, x_3^2, x_4^2 - x_1^3x_3)$ and $J = (x_3^2, x_2x_3, x_4^2)$.

Remark 3.9. The proof of equivalence (i) \Leftrightarrow (ii) of Theorem 3.6 shows that $\overline{G} = \text{gr}_{\overline{\mathfrak{m}}}(\overline{R})$ is Complete Intersection if and only if $\text{Ap}(S)$ is γ -rectangular.

If we want to extend the result to $\text{gr}_m(R)$ the hypothesis of the Cohen-Macaulayness is indispensable, that is

$\text{Ap}(S)$ is γ -rectangular $\not\Rightarrow \text{gr}_m(R)$ is Complete Intersection.

Indeed, let $S = \langle 6, 7, 15 \rangle$. Here $\text{Ap}(S)$ is β -rectangular (hence γ -rectangular), since $f = 23$, $\beta_2 = 2$, $\beta_3 = 1$ and $f + m = \beta_2 g_2 + \beta_3 g_3$. It follows that \overline{G} is Complete Intersection and $J = (x_3^2, x_2^3)$. However, in this case, G is not Complete Intersection, since $\text{gr}_m(R)$ is not Cohen-Macaulay, because $s = 3$ and $r = 6$. It is not difficult to compute that $I = (x_3^2 - x_1^5, x_2^3 - x_1 x_3)$ (S is symmetric, hence R is Gorenstein that, if the embedding dimension is 3, implies Complete Intersection) and that $I^* = (x_3^2, x_1 x_3, x_2^6)$.

Remark 3.10. We note that:

$\text{gr}_m(R)$ is Gorenstein and R is Complete Intersection $\not\Rightarrow$
 $\text{gr}_m(R)$ is Complete Intersection.

In fact, let $S = \langle 16, 18, 21, 27 \rangle$; we have $\text{Ap}(S) = \{0, 18, 21, 27, 36, 39, 42, 45, 54, 57, 60, 63, 72, 78, 81, 99\}$. Let us compute the integer γ_3 : $2 \cdot 21 = 42 \in \text{Ap}(S)$ and it has a unique representation, while $3 \cdot 21 = 2 \cdot 18 + 27$ are two maximal representations of 63; hence $\gamma_3 = 2$. The multiplicity of S is $m = 16$ and $\gamma_3 + 1 = 3$ does not divide m ; therefore, by Theorem 2.22, $\text{Ap}(S)$ is not γ -rectangular (hence $\text{gr}_m(R)$ is not Complete Intersection).

On the other hand, it is not difficult to check that S is symmetric, M -pure and that $\text{ord}(99) = 5 = r$; hence, by Theorem 3.1 (3), $\text{gr}_m(R)$ is Gorenstein. Finally, R is Complete Intersection, since $I = (x_1^3 - x_3 x_4, x_2^3 - x_4^2, x_2^2 x_4 - x_3^3)$.

Remark 3.11. If we substitute the condition “ γ -rectangular Apéry set” with the condition “ β -rectangular Apéry set” in the previous Theorem 3.6 we obtain not only that $\text{gr}_m(R)$ is Complete Intersection, but also that J is monomial. However this fact does not implies that the defining ideal I^* of $\text{gr}_m(R)$ is monomial. For example, let us consider the semigroup of the Example 2.17: $S = \langle 12, 14, 16, 23 \rangle$; its Apéry set is β -rectangular and $r = s = 4$, hence $\text{gr}_m(R)$ is Complete Intersection. In this case the defining ideals are the following: $I = (x_2^2 - x_1 x_3, x_4^2 - x_2 x_3^2, x_3^3 - x_1^4)$, $I^* = (x_2^2 - x_1 x_3, x_4^2, x_3^3)$ and $J = (x_2^2, x_3^3, x_4^2)$.

As a corollary to Theorem 3.6, we get the following corollary, that can be also obtained easily by Bézout’s theorem applied to the algebraic variety defined by G . Given a positive integer x , we call $\ell(x)$ the length of its unique factorization i.e. the number of (possibly equal) prime factors of x .

Corollary 3.12. *Let $\text{gr}_m(R)$ be Complete Intersection, then we have:*

- (1) $\nu \leq \ell(m) + 1$;
- (2) if m is a prime number then $\nu = 2$.

Proof. (1) By Theorem 3.6, $\text{Ap}(S)$ is γ -rectangular. Thus we have $m = |\text{Ap}(S)| = \prod_{i=2}^{\nu} (\gamma_i + 1)$ and the thesis follows as $\gamma_i \geq 1$.
(2) If m is prime, we get $\nu \leq \ell(m) + 1 = 2$ by (1); but $\nu > 1$, otherwise $m = 1$. \square

Using our method, we also obtain an alternative proof of a result from [2]:

Corollary 3.13. *Let $n_{\nu} < \dots < n_1$ be pairwise relatively prime positive integers, $N = \prod_{i=1}^{\nu} n_i$. Let $g_i = \frac{N}{n_i}$ and $S = \langle g_1, g_2, \dots, g_{\nu} \rangle$. Then $\text{gr}_m(R)$ is Complete Intersection.*

Proof. As proved in [2, Proposition 3.6], $\text{gr}_m(R)$ is Cohen-Macaulay; we only need to prove that $\text{Ap}(S)$ is γ -rectangular. We easily have $\gamma_i \leq n_i - 1$ and hence $\prod_{i=2}^{\nu} (\gamma_i + 1) \leq \prod_{i=2}^{\nu} n_i = g_1 = m$. Since $m = |\text{Ap}(S)| \leq |\Gamma| \leq \prod_{i=2}^{\nu} (\gamma_i + 1)$, the equality $m = \prod_{i=2}^{\nu} (\gamma_i + 1)$ holds and, by Theorem 2.22, we get the thesis. \square

We finish the paper studying the case $\nu(S) = 3$. By Remarks 3.5 (3), it follows immediately that $\text{Ap}(S)$ is γ -rectangular if and only if it is β -rectangular. However, we can prove something more.

Theorem 3.14. *Let $S = \langle g_1, g_2, g_3 \rangle$ be a three-generated semigroup (with $g_1 < g_2 < g_3$). The following conditions are equivalent:*

- (i) $\text{Ap}(S)$ is β -rectangular;
- (ii) $\text{Ap}(S)$ is γ -rectangular;
- (iii) S is M -pure symmetric.

Proof. By Corollary 2.10 and Corollary 2.21 we only need to show the implication (iii) \Rightarrow (i). Assume that S is M -pure symmetric, that is $\omega \preceq_M f + m$ for each $\omega \in \text{Ap}(S)$. If we have two maximal representations of $f + m$:

$$f + m = \lambda_2 g_2 + \lambda_3 g_3 = \mu_2 g_2 + \mu_3 g_3, \quad \text{with } \lambda_2 + \lambda_3 = \mu_2 + \mu_3$$

then it follows that

$$(\lambda_2 - \mu_2)g_2 = (\mu_3 - \lambda_3)g_3, \quad \text{with } \lambda_2 - \mu_2 = \mu_3 - \lambda_3$$

and, since $g_2 < g_3$, $\lambda_2 - \mu_2 = \lambda_3 - \mu_3 = 0$. Thus $f + m$ has a unique maximal representation and the thesis follows by Theorem 2.16 (iii). \square

As a consequence of the last result, we get the well known fact that Gorensteinness and Complete Intersection are equivalent in codimension two (i.e., in our hypotheses, in embedding dimension three).

Corollary 3.15. *Let R be a numerical semigroup ring with $\nu(R) = 3$. If $\text{gr}_m(R)$ is Gorenstein, then it is Complete Intersection (and the ideal J is monomial).*

Proof. Apply Theorems 3.1, 3.14 and 3.6. \square

Acknowledgments. The authors are grateful to Yi Huang Shen for some helpful remarks on a preliminary version of the paper that allowed to define properly the γ -rectangularity and to the referee for several suggestions and comments that improved the paper.

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