Optimal control for stochastic Volterra equations with multiplicative Lévy noise

Stefano Bonaccorsi and Fulvia Confortola

Abstract. This paper is devoted to the analysis of an optimal control problem for stochastic integro-differential equations driven by a non-gaussian Lévy noise. The memory effect in the equation is driven by a completely monotone kernel (thus covering, for instance, the class of fractional time derivative of order less than 1).

We suppose that the control acts on the jump rate of the noise. We show that this allows to tackle the problem through a backward stochastic differential equations approach, since the structure condition required by this approach is naturally satisfied. We solve the optimal control problem of minimizing a cost functional on a finite time horizon, with both running and final costs. We finally prove the existence of a weak solution of the closed-loop equation and we construct an optimal feedback control.

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1. Introduction. Statement of the problem

The main aim of this paper is to extend the theory developed in the context of classical optimal control for diffusion processes -constructed as solutions to stochastic differential equations of Ito type driven by Browian motion- to the framework of the processes costructed as solutions of stochastic integro-differential equations driven by a non-gaussian Lévy noise. The majority of those results requires that only the drift coefficient of the stochastic equation depends on the control parameter, see e.g. [29]. Generally, in this case the laws of the corresponding controlled processes are all absolutely continuous

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with respect to the law of a given, uncontrolled process, so that they form a dominated model.

In the present paper we adopt a similar framework. We start by describing our setting in an informal way.

Let H be a real separable Hilbert space endowed with the Borel σ-field $\mathcal{H}$. Let $\{L_t,\ t \ge 0\}$ be a *pure jump* Lévy process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ where $\{\mathcal{F}_t\}_{t\ge 0}$ is the completed, right-continuous filtration generated by the Lévy process (see Section 2). Denote with $\tilde\pi(\mathrm{d}t, \mathrm{d}\xi) := \pi(\mathrm{d}t, \mathrm{d}\xi)(\omega) - \nu(\mathrm{d}\xi)\,\mathrm{d}t$ its compensated Poisson random measure under $\mathbb{P}$ with compensator given by $\nu(\mathrm{d}\xi)\,\mathrm{d}t$.

We denote with U the space of controls, which we assume be equal to a countable union of compact metrizable subsets of itself, for example $\mathbb{R}$ or $\mathbb{N}$. We write $\mathcal{A}$ for the space of the admissible controls γ which are $\{\mathcal{F}_t\}_{t\ge 0}$-predictable processes taking values in U.

We introduce a function $r : [0,T] \times H \times H \times U \to]0, \infty[$, and define the measure $\mathbb{P}^\gamma$ through the Dolans-Dade exponential

$$\frac{d\mathbb{P}^\gamma}{d\mathbb{P}} = \exp\Bigg[\int_0^t \int_{H\setminus\{0\}} (r(s, u_s, \xi, \gamma_s) - 1)\,[\pi(\mathrm{d}s, \mathrm{d}\xi) - \nu(\mathrm{d}\xi)\,\mathrm{d}s] \\ - \int_0^t \int_{H\setminus\{0\}} (r(s, u_s, \xi, \gamma_s) - 1 - \ln(r(s, u_s, \xi, \gamma_s)))\,\pi(\mathrm{d}s, \mathrm{d}\xi)\Bigg]$$

Under suitable assumptions on r (for instance if r is positive and uniformly bounded), this defines a true probability measure $\mathbb{P}^\gamma$ equivalent to $\mathbb{P}$. By Girsanov's theorem the compensator of π under $\mathbb{P}^\gamma$ is given by $\nu^\gamma(\mathrm{d}\xi, \mathrm{d}t) := r(t, u_t, \xi, \gamma_t)\nu(\mathrm{d}\xi)\,\mathrm{d}t$.

We are concerned with the following controlled stochastic integral Volterra equation under $\mathbb{P}^\gamma$

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s \\ \qquad = Au(t) + \displaystyle\int_{H\setminus\{0\}} \psi(t, \xi, u(t))\,(r(t, u_t, \xi, \gamma_t) - 1)\,\nu(\mathrm{d}\xi) \\ \qquad\quad + \displaystyle\int_{H\setminus\{0\}} \psi(t, \xi, u(t))\,\tilde\pi^\gamma(\mathrm{d}t, \mathrm{d}\xi), \qquad t > 0 \\ u(t) = u_0(t), \qquad t < 0 \end{cases} \tag{1.1}$$

where a is a completely monotone kernel and A is a m-dissipative operator defined on a domain $D(A) \subset H$ and

$$\tilde\pi^\gamma(\mathrm{d}t, \mathrm{d}\xi) := \pi(\mathrm{d}t, \mathrm{d}\xi) - r(t, u_t, \xi, \gamma_t)\,\nu(\mathrm{d}\xi)\,\mathrm{d}t$$

is the compensated Poisson random measure under $\mathbb{P}^\gamma$.

In the equation, the function ψ is a Lipschitz continuous function in u uniformly in $t \in [0,T]$ and $\xi \in H$. The past of the system up to time $t = 0$ is a given function $u_0(s)$ which satisfies certain regularity conditions; a precise statement of all the assumptions concerning the memory terms is given in Section 3.

The aim is to choose a control process γ, within a set of admissible controls, in such way to minimize a cost functional of the form

$$\mathbb{J}(u_0,\gamma) = \mathbb{E}^\gamma \left[\int_0^T l(t,u(t),\gamma(t))\,\mathrm{d}t + g(u(T)) \right] \tag{1.2}$$

where $\mathbb{E}^\gamma$ denotes the expectation under $\mathbb{P}^\gamma$, $T > 0$ is a given deterministic finite horizon and l, g are given real function, representing the running cost and the terminal cost, respectively.

Remark 1.1. Under the probability $\mathbb{P}$, equation (1.1) becomes

$$\begin{cases} \frac{d}{dt}\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = Au(t) \\ \qquad + \int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,(r(t,u_t,\xi,\gamma_t)-1)\,\nu(\mathrm{d}\xi)\,\mathrm{d}t \\ \qquad + \int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,(\pi(\mathrm{d}t,\mathrm{d}\xi) - r(t,u_t,\xi,\gamma_t))\,\nu(\mathrm{d}\xi)\,\mathrm{d}t, \\ = Au(t) + \int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,\tilde{\pi}(\mathrm{d}t,\mathrm{d}\xi) \qquad t>0 \\ u(t) = u_0(t), \qquad t<0 \end{cases} \tag{1.3}$$

This means that we have a *reference dynamics* for u under the *reference probability* $\mathbb{P}$ and we can add a drift to the dynamics of u, in probabilistic weak sense. Using Girsanov's theorem, this corresponds to a change of measure. So we can formulate the control problem in an alternative way by considering a control which affects the probability measure directly. The function r, the controller, modifies the measure $\mathbb{P}$ under which our system evolves, replacing it with the measure $\mathbb{P}^\gamma$. By Girsanov's theorem, the compensator of π under $\mathbb{P}^\gamma$ is given by $\nu^\gamma(\mathrm{d}\xi,\mathrm{d}t) := r(t,u_t,\xi,\gamma_t)\nu(\mathrm{d}\xi)\,\mathrm{d}t$. Therefore, our controller is effectively modifying the solution u by modifying the drift and the jumps (by controlling both the intensity and the amplitude) of the noise.

There has been recently a large interest in the analysis of Kolmogorov's equations associated to path-dependent stochastic differential equations, mainly motivated by the applications to optimal control problems [21, 22, 31, 23]. These papers deal with *functional* Kolmogorov's equations associated to forward-backward stochastic functional equations, i.e., equations with path-dependent coefficients. This approach allows to study non-Markovian control problems, where the state of the system and the control are allowed to depend on the past [22].

However, in the analysis of equation (1.1), it is possible to associate to the solution a Markovian system, thus the associated Kolmogorov's equation can be solved via a (classical) system of forward-backward stochastic differential equations. This Markovian representation of the solution, that is called *state space setting*, follows the classical ideas first introduced in [40, 28] and recently revised, for the stochastic case, in [32, 10, 7]. According to this approach, the original equation (1.1) is shown to be equivalent to an equation in a (different) Hilbert space X:

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t+ \\ \qquad + (I-B)P\displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,(r(t,Jv(t),\xi,\gamma(t))-1)\,\nu(\mathrm{d}\xi)\mathrm{d}t \\ \qquad (I-B)P\displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde{\pi}^\gamma(\mathrm{d}t,\mathrm{d}\xi), \\ v(0) = v_0 \\ u(t) = Jv(t) \qquad t\in(0,T]. \end{cases} \tag{1.4}$$

As opposite to other reformulations of functional equations, the state space setting does not have an intuitive interpretation of the transformed equation.

On the Hilbert space X, the internal state of the system at time t is recorded into a random variable $v(t)$. The operator $B : D(B) \subset X \to X$ which governs the evolution from past to future, is related to the Laplace transform of the state equation rather than its value in the space H. What makes this approach more powerful is that B generates an analytic, strongly continuous semigroup on X, which allows the use of the tools of interpolation theory. $P : H \to X$ is a linear operator which acts as a sort of projection into the state space X. The core of this approach is the existence of the linear operator $J : D(J) \subset X \to H$ which maps the solution of the state equation (1.4) into the (unique) solution of (1.1). A precise statement of the definitions and results that we need will be given in Section 3; for further details see [32, 10]. Therefore, once we prove the existence of a solution of the state equation, with a little effort we also get the existence of a solution of the Volterra equation (1.1).

Remark 1.2. There is a huge literature concerning integro-differential equations; see, for instance, the monograph [47]. Other semigroup approaches have been developed, too, among them the history function approach, which also appears in the study of stochastic equations (see, for instance, [9, 19]). In general, stochastically perturbed problems have been treated with the aid of the resolvent operator; compare, for instance, the paper [13]. However, this approach misses the semigroup property and cannot be applied here.

The existence and uniqueness of the solution of the equation (1.4), which, by the equivalence of the formulation between the two problems, translate into an analog result for the equation (1.1), are contained in Section 4. Let us remark that equation (1.4) contains an unbounded operator in the stochastic term. This introduces some further technical difficulties; here, we follow the approach of [7, 8, 20] to the case of a pure jump Lévy noise.

Remark 1.3. Classically, in the literature, diffusion term driven by unbounded operators arise in very distant fields, e.g., the analysis of Zakai equation [39, 43], or stochastic equations with boundary noise [25, Chapter 11].

Given the Markovian solution of (1.4), we can translate the original control problem in the state space setting. Here we have to minimize the cost functional

$$\mathbb{J}(v_0, \gamma) = \mathbb{E}^\gamma \left[\int_0^T l(t, Jv(t), \gamma(t)) \, \mathrm{d}t + g(Jv(T)) \right] \tag{1.5}$$

where we use the linear operator J in order to express the cost functional (1.2) in terms of the state v.

We can proceed to find the solution of the optimal control problem by using the forward-backward system approach, well-known in the diffusive case (see e.g. [48]. It is worth noting that the connection between BSDEs and optimal control has been fundamental since the early work of Bismut [5].)

On the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ we consider the equation

$$\begin{cases} \mathrm{d}v(t) = Bv(t) \, \mathrm{d}t + (I - B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t, \xi, v(t)) \, \tilde{\pi}(\mathrm{d}t, \mathrm{d}\xi), \\ v(0) = v_0 \\ u(t) = Jv(t) \qquad t \in (0, T]. \end{cases} \tag{1.6}$$

to which we associate the backward stochastic differential equation $\mathbb{P}$-a.s.

$$Y_t + \int_t^T \int_{H\setminus\{0\}} Z_r(\xi) \, \tilde{\pi}(\mathrm{d}s, \mathrm{d}\xi) = g(v_T) + \int_t^T f(s, Jv_s, Z_s(\cdot)) \, \mathrm{d}s, \quad t \in [0, T], \tag{1.7}$$

with unknown processes $(Y_t, Z_t)_{t\in[0,T]}$. The generator f is the Hamiltonian function related to the above problem (1.5), defined in the usual way (see e.g. [15], Chapter 21)

$$f(s, v, z(\cdot)) = \inf_{\gamma\in U} \left\{ l(s, Jv, \gamma) + \int_{H\setminus\{0\}} z(\xi) \, (r(s, Jv, \xi, \gamma) - 1) \, \nu(\mathrm{d}\xi) \right\}. \tag{1.8}$$

Remark 1.4. This type of equation was first considered by Pardoux and Peng [44] in the diffusion case with a finite-dimensional Brownian motion. Since then, many generalizations have been considered, see in particular [29, 30, 11] where the Wiener process was replaced by more general continuous processes. Also the case where both the diffusion and jump term are present has been considered, see e.g. [53, 4]. A notable extension to the case of a probability space endowed with a general filtration (instead of the filtration generated by the Brownian motion) has been presented in [14].

Under our assumptions on l, g and r, the Hamiltonian function is Lipschitz continuous and the backward stochastic differential equation (1.7) admits a unique solution (see Proposition 5.4) (Y, Z) which are Borel measurable functions of (t, v), as they come from the solution of a Markovian backward stochastic differential equation. Our main result, Theorem 5.5, proves that under appropriate assumptions the optimal control problem has a solution and that the value function and the optimal control can be represented by means of the solution to the backward stochastic differential equation (1.7).
Finally we address the problem of finding a weak solution to the so-called *closed loop equation*. If we assume that the infimum in (1.8) is attained, we can prove that there exists a measurable function $\bar{\gamma}$ of t, v such that

$$f(t,v,z) = l(t, J(v), \bar{\gamma}(t,v)) + \int_{H\setminus\{0\}} z(\xi)\,(r(s, Jv, \xi, \bar{\gamma}(t,v)) - 1)\,\nu(d\xi).$$

The closed loop equation is

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + \\ \qquad + (I-B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,(r(t, Jv(t), \xi, \bar{\gamma}(t, v_t)) - 1)\,\nu(\mathrm{d}\xi)\mathrm{d}t \\ \qquad\qquad + (I-B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde{\pi}^{\bar{\gamma}}(\mathrm{d}t, \mathrm{d}\xi), \\ v(0) = v_0 \\ u(t) = Jv(t) \qquad t \in (0,T]. \end{cases} \tag{1.9}$$

We prove that there exists a weak solution of the closed loop equation and we construct an optimal feedback control: if v is a solution to (1.9) and we set $\gamma^*_s = \bar{\gamma}(s, v(s))$, then $J(v_0, \gamma^*) = Y_0$, and consequently v is an optimal state, γ^*_s is an optimal control, and $\bar{\gamma}$ is an optimal feedback.

1.1. Motivation of the problem

The analysis of problems which take into consideration the past history of the system have been stimulated by the applications to many physical phenomena, in viscoelasticity, heat conduction in materials with memory, electrodynamics with memory [47]; further, memory terms naturally enter in the description of other natural (e.g., epidemiological models [49]), social (e.g., population dynamics [34]) and economical (e.g., financial markets with stochastic volatility [12]) phenomena. Of a similar shape, but arising from different motivations, are fractional diffusion equations $D^\alpha u = Au + f(u)$, which extends the parabolic diffusion equation to model long range dependence of the solution [2]. Since uncertainty is often an important characteristic of the models, the introduction of a stochastic term is a necessary step in their formulation. Empirical data from many phenomena suggest that Brownian motion is often not an effective process to use in their models. For instance, the classical Black-Scholes model does not fit well with the data, and it is necessary to adjust any price obtained from the Black-Scholes model in order to be realistic. There are several models based on Lévy processes that offer better model fit, since Lévy processes give a very wide modeling freedom [51]. In the applications, an important procedure in the analysis of a system is the introduction of an optimal control (for several examples of application in the study of heating processes, see the standard monographs by Lions [36] or Troeltzsch [54].) Despite the interest in modeling the noise in such systems through a Lévy-type one instead of a Gaussian one, which has motivations in all the areas which have been mentioned, apparently there were no efforts in the literature to study this problem.

Example 1.5. In this example, we discuss a model of cash flow with consumption. Let $u(t)$ denotes the cash amount at time t. If we assume the presence of memory effects in the economic process, a typical model in the literature requires the introduction of fractional order derivatives [50, 35]. Here we consider the left sided Riemann-Liouville fractional derivative:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s$$

where n is the smallest integer larger than α and $\Gamma(z)$ is the Gamma function. We assume that $u(t)$ follows the stochastic fractional equation

$$D_t^\alpha u(t) = A(t)u(t) + \int_{\mathbb{R}\setminus\{0\}} C(t,z)\tilde{N}(\mathrm{d}t,\mathrm{d}z)$$

(for similar models, compare [41, 42]). Suppose that the consumption rate $\gamma(t)$ influences the growth of $u(t)$ by affecting the intensity measure of the jumps. By Girsanov's theorem, we know that this is equivalent (in the sense of weak solutions) to add a drift to the dynamics of $u(t)$. In this setting, the consumer may want to maximize the combined utility of the consumption up to the terminal time T and the terminal wealth. Then the problem is to find $\gamma(t)$ such that the expected utility

$$\mathbb{J}(\gamma) = \mathbb{E}^\gamma \left[\int_0^T l(t,\gamma(t))\,\mathrm{d}t + \; g(u(T)) \right]$$

is maximal (l and g are given utility functions).

2. Stochastic integration for Lévy processes

In this section we recall some basic results concerning Lévy processes. For a thorough presentation and additional material on Lévy processes we refer to [45].

We are given a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t,\ t \geq 0\}, \mathbb{P})$ that satisfies the so called usual assumptions: $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$, for every $t \geq 0$, and the filtration is right-continuous. The σ field of $\mathcal{F}_s-$predictable sets in $[0,T] \times \Omega$ will be denoted by $\mathcal{P}$.

Recall that by definition $L = \{L(t),\ t \ge 0\}$ is a (càdlàg) Lévy process on the Hilbert space H if $L(0) = 0$, L is *stochastically continuous* and it has *stationary*, *independent increments*. It is known (compare for instance [45, Theorem 4.3]) that every Lévy process has a càdlàg modification. Notice that Brownian motion with drift is the only Lévy process with continuous paths.
We associate to $L(t)$ the process of jumps $X_t = \Delta L(t) = L(t) - L(t^-) = L(t) - \lim_{s\uparrow t} L(s)$. The random counting measure

$$N_t(B) = \pi((0,t] \times B) = \sum_{s \le t} 1_B(\Delta L(s)), \qquad B \in \mathcal{B}(H \setminus \{0\})$$

is called the *Poisson random measure* of the process L. The *intensity measure* ν associated to the process L under the probability $\mathbb{P}$ is the measure

$$\nu(B) = \mathbb{E}[N_1(B)].$$

ν is a measure that is finite on sets separated from 0 and satisfies (see [46, Theorem 4.23])

$$\int_H (|y|^2 \wedge 1)\, \nu(\mathrm{d}y) < \infty.$$

The (family of) random measures

$$\tilde\pi([0,t] \times B) = \pi([0,t] \times B) - t\,\nu(B), \qquad t \ge 0,\ B \in \mathcal{B}(H \setminus \{0\}),$$

is called the *compensated Poisson random measure.*

From this point on we shall assume that

$$L \text{ is a pure jump process with finite first moment } \mathbb{E}[|L(t)|] < \infty \text{ and intensity measure } \nu \text{ that is supported by the ball } B_R(0), \tag{2.1}$$

hence its Lévy exponent is given by

$$\psi(x) = \int_{\{|y|<R\}} \left(1 - e^{i\langle x,y\rangle} + i\langle x, y\rangle\right)\, \nu(\mathrm{d}y)$$

and the process is identified by the formula

$$L(t) = \int_0^t \int_{H\setminus\{0\}} \xi\, \tilde\pi(\mathrm{d}s, \mathrm{d}\xi).$$

In order to define stochastic integrals with respect to the compensated Poisson random measure $\tilde\pi(\mathrm{d}t, \mathrm{d}\xi)$, we introduce the space

$$\mathcal{L}^2_{\nu,T} = L^2([0,T] \times \Omega \times H, \mathcal{P} \otimes \mathcal{H}, \mathrm{d}t\, \mathrm{d}\mathbb{P}\, \mathrm{d}\nu)$$

which is the space of $\mathcal{P} \otimes \mathcal{H}$-measurable functions $\psi(\omega, s, \xi)$ on $\Omega \times [0,T] \times H$ with values in H such that

$$\mathbb{E} \int_{[0,T]} \int_{H\setminus\{0\}} |\psi(t,\xi)|^2\, \nu(\mathrm{d}\xi)\mathrm{d}t < \infty.$$

Then the Itô stochastic integral

$$\int_{[0,T]} \int_{H\setminus\{0\}} \psi(t,\xi)\, \tilde{\pi}(\mathrm{d}t, \mathrm{d}\xi) := \int_{[0,T]} \int_{H\setminus\{0\}} \psi(t,\xi)\, \pi(\mathrm{d}t, \mathrm{d}\xi)$$

$$- \int_{[0,T]} \int_{H\setminus\{0\}} \psi(t,\xi)\, \nu(\mathrm{d}\xi) dt$$

is a linear bounded operator from $\mathcal{L}^2_{\nu,T}$ into $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

Remark 2.1. The stochastic integral is defined as the difference of ordinary integral with respect to $\pi(\mathrm{d}t, \mathrm{d}\xi)$ and $\nu(\mathrm{d}\xi)dt$.

The following Bichteler-Jacod inequality for Poisson integrals (see for instance [38, Lemma 3.1]) will result useful in the sequel.

Proposition 2.2. *If $\psi \in \mathcal{L}^2_{\nu,T}$, then one has*

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_{(0,t]} \int_{H\setminus\{0\}} \psi(s,\xi)\, \tilde{\pi}(\mathrm{d}s, \mathrm{d}\xi) \right|^2 \le C\, \mathbb{E} \int_{(0,T]} \int_{H\setminus\{0\}} |\psi(s,\xi)|^2 \nu(\mathrm{d}\xi)\, \mathrm{d}s.$$

2.1. Girsanov's change of probability formula

In the sequel, we shall be dealing with more than a probability measure on the filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$. For the sake of clearness, we recall some basic facts about change of measure theorem.
Let $Z = \{Z(t),\ t \in [0,T]\}$ be a positive martingale of the form

$$Z(t) = \exp\Bigg[\int_0^t \int_{H\setminus\{0\}} (Y(s,\xi) - 1) \left[\pi(\mathrm{d}s, \mathrm{d}\xi) - \nu(\mathrm{d}\xi)\, \mathrm{d}s\right]$$

$$- \int_0^t \int_{H\setminus\{0\}} (Y(s,\xi) - 1 - \ln(Y(s,\xi)))\, \pi(\mathrm{d}s, \mathrm{d}\xi) \Bigg]$$

for a measurable, non-negative, deterministic process Y satisfying

$$\int_0^t \int_{H\setminus\{0\}} |\xi(Y(s,\xi) - 1)|\ \nu(\mathrm{d}\xi)\, \mathrm{d}s < \infty, \qquad 0 \le t \le T. \tag{2.2}$$

Then we can define a measure $\mathbb{P}^Y$ on the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ using the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}^Y}{\mathrm{d}\mathbb{P}} = Z(T).$$

The measure $\mathbb{P}^Y$ is equivalent to $\mathbb{P}$ on the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$, i.e., $\mathbb{P}^Y(A) = 0$ if and only if $\mathbb{P}(A) = 0$ for all $A \in \mathcal{F}$.

Recall that under the measure $\mathbb{P}$, the process L has the canonical form

$$L(t) = \int_0^t \int_{H\setminus\{0\}} \xi\ \left[\pi(\mathrm{d}s, \mathrm{d}\xi) - \nu(\mathrm{d}\xi)\, \mathrm{d}s\right].$$

Then we have that

$$\nu^Y(\mathrm{d}s, \mathrm{d}\xi) = Y(s, \xi)\, \nu(\mathrm{d}\xi)\, \mathrm{d}s$$

is the $\mathbb{P}^Y$ compensator of $\pi(\mathrm{d}s, \mathrm{d}\xi)$ and L has the following canonical decomposition under $\mathbb{P}^Y$:

$$L(t) = \int_0^t \int_{H\setminus\{0\}} \xi(Y(s, \xi) - 1)\, \nu(\mathrm{d}\xi)\mathrm{d}s \\ + \int_0^t \int_{H\setminus\{0\}} \xi \left[\pi(\mathrm{d}s, \mathrm{d}\xi) - \nu^Y(\mathrm{d}s, \mathrm{d}\xi)\right]. \quad (2.3)$$

Remark 2.3. We can give a heuristical interpretation of formula (2.3). If the original Lévy process $L(t)$ experiences a jump ξ at time t with probability $\frac{\nu(\mathrm{d}\xi)}{\nu(H\setminus\{0\})}$, then the transformed process jumps at the same time t and the size of the jump is ξ but with probability $\frac{Y(s,\xi)\nu(\mathrm{d}\xi)}{\int_{H\setminus\{0\}} Y(s,\xi)\nu(\mathrm{d}\xi)}$.

Notice that the random measure of jumps of L did not change under the change of measure from $\mathbb{P}$ to $\mathbb{P}^Y$. That happens because π is a paths property of the process and do not change under an equivalent change of measure. Intuitively speaking, the paths do not change, the probability of certain paths occurring changes.

3. The state space setting. Statement of the results

In the past years an important approach to the analysis of equations with memory was provided by Desch and Miller [28] for deterministic Volterra equations arising in linear viscoelasticity, see also [27, 52], and further developed by several authors also in the stochastic case, see [32, 7, 8, 10].

This approach introduces a state $v(t)$ of the system, which contains all information about the solution up to time t required to predict the future development. The state of the system is then governed by an abstract differential equation in a large Hilbert space.

In the following, we shall apply this technique to the analysis of problem (1.1). In order to avoid some technicalities, we discuss in this section the equation

$$\begin{cases} \dfrac{d}{dt} \displaystyle\int_{-\infty}^t a(t-s)u(s)\, \mathrm{d}s = Au(t) + f(t), & t > 0 \\ u(t) = u_0(t), & t < 0 \end{cases} \quad (3.1)$$

Here we state the assumptions on the kernel a, the operator A and the initial condition u_0 which also apply to equation (1.1).

Hypothesis 3.1. *$A : D(A) \subset H \to H$ is a sectorial operator in H. Thus A generates an analytic semigroup e^{tA}. Interpolation and extrapolation spaces H_γ of H will always be constructed with respect to A.*

Hypothesis 3.2. *The convolution kernel* $a : (0, +\infty) \to \mathbb{R}$ *is completely monotone with Bernstein measure* ν. *Furthermore,* a *satisfies*

1. $a(0+) = \infty$, *equivalently* $\int_{[0,\infty)} \nu(\mathrm{d}\kappa) = \infty$;
2. $\int_0^T a(t)\,\mathrm{d}t < \infty$ *for all* $T \in (0, \infty)$, *equivalently:* $\int_{[0,\infty)} (\kappa + 1)^{-1}\,\nu(\mathrm{d}\kappa) < \infty$;
3. *there exists* $\omega_a \in (0, \pi - \omega_A)$ *such that for all* $s \in \mathbb{C}^+$ *we have* $s\hat{a}(s) \in \Sigma_{\omega_a}$;
4. *the coefficient* $\alpha(a)$ *(first introduced in* [32] *to measure the singularity of* a *at* $t = 0+$*)*

$$\alpha(a) = \sup\left\{\rho \le 1 \,:\, \int_1^\infty s^{\rho-2} \frac{1}{\hat{a}(s)}\,\mathrm{d}s < \infty\right\} \tag{3.2}$$

satisfies

$$\alpha(a) > \frac{1}{2}. \tag{3.3}$$

Remark 3.3. A classical example of equations arising from physical application having the form (3.1) are those governed by a fractional order time-derivative. Recall that the Riemann-Liouville derivative of order $\alpha \in (0, 1)$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(\tau)\,\mathrm{d}\tau.$$

The kernel $a(t) = \frac{1}{\Gamma(1-\alpha)} t^{-\alpha}$ satisfies Hypothesis 3.2: it is completely monotone, and since $\hat{a}(s) = s^{\alpha-1}$ it follows that $\alpha(a) = \alpha$, provided we choose $\alpha > \frac{1}{2}$ (in order to satisfy condition (4)).

Hypothesis 3.4. *The initial condition* u_0 *belongs to the space of* admissible initial conditions $\tilde{X}$

$$\tilde{X} = \Big\{\varphi : (-\infty, 0) \to H, \quad \text{there exist } M > 0 \text{ and } \omega > 0 \text{ such that } |\varphi(t)| \le M e^{-\omega t}\Big\}.$$

We shall further assume that there exist $\eta > \frac{1-\alpha(a)}{2}$, $\epsilon > \frac{\eta}{2\alpha(a)}$ *and* $\rho > 2\eta$ *such that* $u_0(0)$ *belongs to* H_ϵ *and there exist* $M_2 > 0$ *such that* $|u_0(-t) - u_0(0)| \le M|t|^{2\rho}$ *for all* $t \in [-1, 0]$.

Definition 3.5. A function $\{u(t),\ t \in [0, T]\}$ is a weak solution of (3.1) if it belongs to $L^2([0, T]; H)$ and for any $z \in D(A^*)$ it satisfies the identity

$$\int_{-\infty}^t \langle a(t-s)u(s), z\rangle_H ds = \langle \int_{-\infty}^0 a(-s)u_0(s)ds, z\rangle_H + \int_0^t \langle A^* z, u(s)\rangle_H \mathrm{d}s + \int_0^t \langle z, f(s)\rangle_H\,\mathrm{d}s. \tag{3.4}$$

We quote from [10] the main result concerning the state space setting for stochastic Volterra equations in infinite dimensions.

Theorem 3.6 (State space setting). *Let A, a, $\alpha(a)$ be given above and satisfying Hypotheses* 3.1*,* 3.2*,* 3.4*; choose numbers $\eta \in (0,1)$, $\theta \in (0,1)$ such that*

$$\eta > \frac{1}{2}(1-\alpha(a)), \qquad \theta < \frac{1}{2}(1+\alpha(a)), \qquad \theta - \eta > \frac{1}{2} \tag{3.5}$$

Then there exist

1. *a separable Hilbert space X (the* state space*) and an isometric isomorphism $Q : \tilde{X} \to X$,*
2. *a densely defined sectorial operator $B : D(B) \subset X \to X$ generating an analytic semigroup e^{tB} with growth bound ω_0,*
3. *its real interpolation spaces $X_\rho = (X, D(B))_{(\rho,2)}$ with their norms $|\cdot|_\rho$,*
4. *linear operators $P : H \to X_\theta$, $J : X_\eta \to H$*

such that the following holds: the analysis of problem (3.1) *can be reduced to that of the evolution equation*

$$\begin{cases} v'(t) = Bv(t) + (I-B)Pf(t) \\ v(0) = v_0 = Qu_0 \end{cases} \tag{3.6}$$

in the sense that if $u_0 \in \tilde{X}$ and $v(t; v_0)$ is the weak solution to Equation (3.6) *with $v_0 = Qu_0$, then $u(t; u_0) = Jv(t; v_0)$ is the unique weak solution to Problem* (3.1)*.*

Remark 3.7. According to Lemma 3.16 in [10], the second part of Hypothesis 3.4 implies that $v_0 = Qu_0 \in X_\eta$.

4. The state equation

In this section we prove the relevant existence result for the solution of equation (1.1)

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-\infty}^{t} a(t-s)u(s)\,\mathrm{d}s = Au(t) + \int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,\mathrm{d}L(t), & t > 0 \\ u(t) = u_0(t), \qquad t < 0 \end{cases} \tag{1.1}$$

in the framework described above. Thus, we let L be a pure jump Lévy process with compensated random measure $\tilde{\pi}(\mathrm{d}t, \mathrm{d}\xi) := \pi(\mathrm{d}t, \mathrm{d}\xi)(\omega) - \nu(\mathrm{d}\xi)\mathrm{d}t$ defined on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. From now on we suppose that $\{\mathcal{F}_t\}_{t\geq 0}$ is the completed, right-continuous filtration generated by the Lévy process.

The operator A and the convolution kernel a satisfy Hypothesis 3.1 and 3.2 respectively. On the coefficient ψ we formulate the following

Hypothesis 4.1. *$\psi : [0,T] \times H \times H \to H$ is a well defined, continuous and Lipschitz continuous mapping.*

The following definition extends in a natural way the notion of weak solution for Volterra equations with stochastic forcing term from the one in the deterministic case given in Definition 3.5.

Definition 4.2. A process $\{u(t),\ t \in [0,T]\}$ is a weak solution of (1.1) if it belongs to $L^2_{ad}(\Omega \times [0,T]; H)$ and for any $z \in D(A^*)$ it satisfies $\mathbb{P}$-a.s. the identity

$$\int_{-\infty}^{t} \langle a(t-s)u(s), z\rangle_H \,\mathrm{d}s = \langle \int_{-\infty}^{0} a(-s)u_0(s)\,\mathrm{d}s, z\rangle_H + \int_0^t \langle A^* z, u(s)\rangle_H \,\mathrm{d}s \\ + \langle z, \int_0^t \int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,\mathrm{d}L(s)\rangle_H. \quad (4.1)$$

If we apply to problem (1.1) the machinery introduced in previous section, we obtain that in the state space setting, on the separable Hilbert space X, the state $v(t)$ and the solution $u(t)$ of (1.1) will be governed by a system of the form

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + (I-B)P \int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde{\pi}(\mathrm{d}t,\mathrm{d}\xi), & t \in [0,T] \\ v(0) = v_0 \\ u(t) = Jv(t), \end{cases} \quad (4.2)$$

with an initial condition $v_0 \in X_\eta$. is an unbounded operator, where the interpolation space $(X, D(B))_{\eta,2} \subset D(J)$ if η is sufficiently large. Recall that B is the generator of an analytic semigroup $S(t)$ on X; the coefficient ψ_J is defined by $\psi_J(v) = \psi(J(v))$, where $J : X_\eta \to H$ is defined in Theorem 3.6.

Lemma 4.3. *Assume Hypothesis* 4.1. *Let us define*

$$G(t,\xi,v) = (I-B)P\psi_J(t,\xi,v) \qquad (t \in [0,T],\ \xi \in H,\ v \in X_\eta).$$

Then $G : [0,T] \times H \times X_\eta \to X_{\theta-1}$ is a Lipschitz continuous mapping with respect to the relevant norms.

The stochastic equation in (4.2) is only formal in X_η since the coefficients do not belong to the state space. We shall deal with the following concept of solution for equation (4.2).

Definition 4.4. A predictable process $v : [0,T] \to X_\eta$ is a *mild solution* for (4.2) if it satisfies

$$v(t) = e^{tB}v_0 + \int_0^t \int_H e^{(t-s)B}(I-B)P\psi_J(s,\xi,v(s))\,\tilde{\pi}(\mathrm{d}s,\mathrm{d}\xi) \quad (4.3)$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$, where at the same time we assume that the stochastic integral on the right-hand side exists.

For each $p \geq 2$ we shall denote $\mathcal{H}_p(0,T;X_\eta)$ the spaces of all predictable processes $v : [0,T] \to X_\eta$ such that

$$\|v\|_p = \left(\sup_{t\in[0,T]} \mathbb{E}|v(t)|_\eta^p \right)^{1/p} < \infty.$$

The following well-posedness result in $\mathcal{H}_2(0,T;X_\eta)$ is our main result in this section. We follow the approach in [38], see also [1, 45], with suitable modifications in order to handle the unboundedness of the coefficients in (4.3).

Remark 4.5. As opposite to [38], we are not able to prove well-posedness of the stochastic convolution process in the smaller space $\mathcal{S}_p(0,T;X_\eta)$ of all predictable processes $v : [0,T] \to X_\eta$ such that the following norm is finite:

$$|\!|\!| v |\!|\!|_p = \left(\mathbb{E} \sup_{t\in[0,T]} |v(t)|_\eta^p \right)^{1/p} < \infty,$$

due to the unboundedness of the coefficient involved in the stochastic convolution integral.

Theorem 4.6. *Assume that Hypotheses* 3.1*,* 3.2*,* 3.4 *and* 4.1 *are satisfied; there exists a unique mild solution of problem* (4.2) *in the space* $\mathcal{H}_2(0,T;X_\eta)$*.*

Proof. We shall use a fixed point argument in the space $\mathcal{H}_2(0,T;X_\eta)$ where we introduce the following equivalent norm

$$\|v\|_{\{2,\beta\},\eta} = \left(\sup_{t\in[0,T]} \mathbb{E} e^{-2\beta t} |v(t)|_\eta^2 \right)^{1/2},$$

for some $\beta > 0$ to be chosen later.
Let us introduce the mapping

$$\mathcal{K} : \mathcal{H}_2 \to \mathcal{H}_2, \qquad \mathcal{K}(v)(t) = \int_0^t \int_H e^{(t-s)B} G(s,\xi,v(s)) \, \tilde{\pi}(\mathrm{d}s, \mathrm{d}\xi)$$

where we define as before

$$G(t,\xi,v) = (I-B)P\psi_J(t,\xi,v).$$

In particular, we want to prove that this mapping is a well-defined contraction on $\mathcal{H}_2(0,T;X_\eta)$.
The key point is the following inequality, that is a consequence of Itô formula (see for instance [38, Lemma 3.1])

$$\begin{aligned}
&\mathbb{E} \left| e^{-\beta t} (\mathcal{K}(v_1)(t) - \mathcal{K}(v_2)(t) \right|_\eta^2 \\
&\quad = \mathbb{E} \left| \int_0^t \int_{H\setminus\{0\}} e^{-2\beta t} e^{(t-r)B} [G(r,\xi,v_1(r)) - G(r,\xi,v_2(r))] \, \tilde{\pi}(\mathrm{d}r, \mathrm{d}\xi) \right|_\eta^2 \\
&\quad \leq C \, \mathbb{E} \int_0^t \int_{H\setminus\{0\}} \left| e^{-2\beta t} e^{(t-r)B} [G(r,\xi,v_1(r)) - G(r,\xi,v_2(r))] \right|_\eta^2 \nu(\mathrm{d}\xi) \, \mathrm{d}r
\end{aligned}$$

since B is the generator of an analytic semigroup, we can use standard estimates from [37]: $|e^{tB}g|_\eta \le t^{\theta-\eta}|g|_\theta$, where $|x|_\theta$ (resp., $|x|_\eta$) stands for the norm in X_θ (in X_η). Then $\left|e^{(t-r)B}[G(r,\xi,v_1(r)) - G(r,\xi,v_2(r))]\right|_\eta \le (t-r)^{\theta-\eta-1}\,|G(r,\xi,v_1(r)) - G(r,\xi,v_2(r))|_{\theta-1}$ and, by using the Lipschitz continuity of G, this quantity is bounded by $L_G(t-r)^{\theta-\eta-1}\,|v_1(r)-v_2(r)|_\eta$

$$\mathbb{E}\left|e^{-\beta t}(\mathcal{K}(v_1)(t) - \mathcal{K}(v_2)(t)\right|_\eta^2 \\ \le C\,\mathbb{E}\int_0^t\int_{H\setminus\{0\}}\left|e^{-2\beta t}(t-r)^{\theta-\eta-1}[v_1(r)-v_2(r)]\right|_\eta^2\,\nu(\mathrm{d}\xi)\,\mathrm{d}r$$

Then, using $\nu(H\setminus\{0\}) < +\infty$, and Young's inequality, we get

$$\mathbb{E}\left|e^{-\beta t}\mathcal{K}(v_1(t) - \mathcal{K}(v_2)(t)\right|_\eta^2 \\ = C\left(\int_0^t\left|e^{-2\beta(t-r)}(t-r)^{\theta-\eta-1}\right|_\eta^2\,\mathrm{d}r\right)\mathbb{E}\int_0^t\left|e^{-2\beta r}[v_1(r)-v_2(r)]\right|_\eta^2\,\mathrm{d}r \\ \le C\,\frac{1}{\Gamma[2(\theta-\eta)-1]}\,\frac{1}{\beta^{2(\theta-\eta)-1}}\,\|v_1-v_2\|_{\{2,\beta\},\eta}$$

and we conclude that

$$\|e^{-\beta t}\mathcal{K}(v_1)(t) - \mathcal{K}(v_2)(t)\|_{\{2,\beta\},\eta} \le C\frac{1}{\beta^{2(\theta-\eta)-1}}\|v_1-v_2\|_{\{2,\beta\},\eta}. \tag{4.4}$$

Then, by taking β large enough, the mapping $\mathcal{K}$ above defines a contraction in the space $\mathcal{H}_2(0,T;X_\eta)$ and the theorem is proved. □

Now we deal with existence and uniqueness of the stochastic Volterra equation. The following result extends to the stochastic case Theorem 3.6. We don't give the proof here, since it is a direct extension of the result given in [10, Theorem 1.23], see also the bibliography quoted before.

Proposition 4.7. *Assume that Hypotheses* 3.1*,* 3.2*,* 3.4 *and* 4.1 *hold; choose numbers* $\eta\in(0,1)$*,* $\theta\in(0,1)$ *such that* (3.5) *holds. Given the process*

$$v(t) = e^{tB}v_0 + \int_0^t\int_H e^{(t-s)B}(I-B)P\psi_J(s,\xi,v(s))\,\tilde{\pi}(\mathrm{d}s,\mathrm{d}\xi)$$

we define the process

$$u(t) := \begin{cases} Jv(t), & t\ge 0\\ u_0, & t\le 0\end{cases}$$

Then $u(t)$ *is a weak solution to problem*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = Au(t) + \int_{H\setminus\{0\}}\psi(t,\xi,u(t))\,\mathrm{d}L(t), & t>0\\ u(t) = u_0(t), & t<0\end{cases}$$

5. The control problem

In this section we address the optimal control problem associated with the solution of a stochastic integral Volterra equation.

We shall denote $T > 0$ the *time horizon* of the control problem. The data specifying the optimal control problem that we will address are a measurable space U, called the action (or decision) space, endowed with a σ-field $\mathcal{U}$; a running cost function l; a terminal cost function g, and the function r specifying the effect of the control process.

We define an admissible control process, or simply a control, as an $\{\mathcal{F}_t\}$-predictable process γ with values in U. The set of admissible control processes is denoted by $\mathcal{A}$. We will make the following assumptions.

Hypothesis 5.1.

1. $(U, \mathcal{U})$ *is a topological space which is the union of countably many compact metrizable subsets of itself.*
2. $r : [0,T] \times H \times H \times U \to \mathbb{R}$ *is* $\mathcal{B}([0,T]) \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{U}$*-measurable and there exists a constant* $C_r > 0$ *such that*
$$0 < r(t,x,\xi,\gamma) \le C_r, \qquad t \in [0,T],\ x,\xi \in H, \gamma \in U. \tag{5.1}$$
3. $g : H \to \mathbb{R}$ *is* $\mathcal{H}$*-measurable and*
$$\mathbb{E}|g(u(T))|^2 < \infty. \tag{5.2}$$
4. $l : [0,T] \times H \times U \to \mathbb{R}$ *is* $\mathcal{B}([0,T]) \otimes \mathcal{H} \otimes \mathcal{U}$*-measurable, and there exists* $\alpha > 1$ *such that for every* $t \in [0,T]$*,* $x \in H$ *and any admissible control* $\gamma(\cdot)$ *we have*
$$\inf_{\chi \in U} l(t,x,\chi) > -\infty, \qquad \mathbb{E}\int_0^T |\inf_{\chi\in U} l(s,u(s),\chi)|^2\,\mathrm{d}s < \infty, \tag{5.3}$$
$$\mathbb{E}\left(\int_0^T |l(s,u(s),\gamma(s))|\,\mathrm{d}s\right)^\alpha < \infty. \tag{5.4}$$

Remark 5.2. We note that the cost functions g and l need not be bounded. Clearly, (5.4) follows from the other assumptions if we assume for instance that
$$\mathbb{E}\int_0^T |\sup_{\chi\in U} l(s,u(s),\chi)|\,\mathrm{d}s < \infty.$$

Using the function r, for each control process γ, we define the measure $\mathbb{P}^\gamma$ through the Dolans-Dade exponential

$$\frac{d\mathbb{P}^\gamma}{d\mathbb{P}} = \exp\left[\int_0^t \int_{H\setminus\{0\}} (r(s,u(s),\xi,\gamma(s)) - 1)\,[\pi(\mathrm{d}s,\mathrm{d}\xi) - \nu(\mathrm{d}\xi)\,\mathrm{d}s]\right.$$
$$\left. - \int_0^t \int_{H\setminus\{0\}} (r(s,u(s),\xi,\gamma(s)) - 1 - \ln(r(s,u(s),\xi,\gamma(s))))\,\pi(\mathrm{d}s,\mathrm{d}\xi)\right].$$

For ease of notation, we define

$$\Lambda_t^\gamma := \left.\frac{d\mathbb{P}^\gamma}{d\mathbb{P}}\right|_{\mathcal{F}_t}.$$

It is a well-known result that Λ^γ is a nonnegative supermartingale relative to $\mathbb{P}$ and $\mathcal{F}_t$ (see [33, Proposition 4.3] or [6]). Since the function r is uniformly bounded, which guarantees this defines a true probability measure $\mathbb{P}^\gamma$ equivalent to $\mathbb{P}$, the process Λ^γ is a strictly positive martingale (relative to $\mathbb{P}$ and $\mathcal{F}_t$). By applying Girsanov's theorem we see that while the compensator of π under $\mathbb{P}^\gamma$ is given by $\nu^\gamma(\mathrm{d}\xi, \mathrm{d}t) := r(t, \xi, \gamma(t))\nu(\mathrm{d}\xi)\,\mathrm{d}t$.

We are concerned with the following controlled stochastic integral Volterra equation under $\mathbb{P}^\gamma$

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-\infty}^t a(t-s)u(s)\,\mathrm{d}s = Au(t) \\ \qquad\qquad + \displaystyle\int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,(r(t,u_t,\xi,\gamma_t)-1)\,\nu(\mathrm{d}\xi)\,\mathrm{d}t \\ \qquad\qquad + \displaystyle\int_{H\setminus\{0\}} \psi(t,\xi,u(t))\,\tilde{\pi}^\gamma(\mathrm{d}t,\mathrm{d}\xi), \qquad t>0 \\ u(t) = u_0(t), \qquad t<0 \end{cases} \tag{5.5}$$

where $\tilde{\pi}^\gamma(\mathrm{d}t, \mathrm{d}\xi) := \pi(\mathrm{d}t, \mathrm{d}\xi) - r(t, u_t, \xi, \gamma_t)\,\nu(\mathrm{d}\xi)\,\mathrm{d}t$ is the compensated Poisson random measure under $\mathbb{P}^\gamma$.

According to the results in Section 4, there exists a unique solution $u = \{u(t),\ t \in [0,T]\}$ to it.

If the control $\gamma \in \mathcal{A}$ is used the total expected cost is given by the functional

$$\mathbb{J}(u_0,\gamma) = \mathbb{E}^\gamma\left[\int_0^T l(t,u(t),\gamma(t))\,\mathrm{d}t + g(u(T))\right] \tag{1.2}$$

where $\mathbb{E}^\gamma$ denotes the expectation with respect to $\mathbb{P}^\gamma$.

Taking into account (5.2), (5.4), and using the Hölder inequality it is easily seen that the cost is finite for every admissible control. The stochastic optimal control problem consists in minimizing $\mathbb{J}(u_0, \gamma)$ over all $\gamma \in \mathcal{A}$.

Likewise as in Section 4, we associate to the stochastic Volterra equation (5.5) the state equation

$$\begin{cases} \mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + \\ \qquad + (I-B)P\displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,(r(t,Jv(t),\xi,\gamma_t)-1)\,\nu(\mathrm{d}\xi)\mathrm{d}t \\ \qquad\qquad + (I-B)P\displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde{\pi}^\gamma(\mathrm{d}t,\mathrm{d}\xi), \\ v(0) = v_0 \\ u(t) = Jv(t) \qquad t\in(0,T]. \end{cases} \tag{5.6}$$

We notice that the process L^γ associated to the random measure π^γ is a Lévy process relatively the probability $\mathbb{P}^\gamma$.

Theorem 5.3. *For any fixed control* $\gamma \in \mathcal{A}$ *the state equation* (5.6) *admits a unique mild solution (in the analytical sense), in weak probabilistic sense on the probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^\gamma)$*, compare Definition 18.0.5 in* [15].

Proof. We can use the same reasonings outlined in the proof of Theorem 4.6, the only difference here being that there is a second term to be treated. Let

$$\mathcal{L} : \mathcal{H}_2 \to \mathcal{H}_2, \qquad \mathcal{L}(v)(t) = \int_0^t \int_H e^{(t-s)B} \Gamma(s, \xi, v(s))\, \nu(\mathrm{d}\xi)\, \mathrm{d}s$$

where we set

$$\Gamma(s, \xi, v) = (I - B) P \psi_J(t, \xi, v)\, (r(s, Jv, \xi, \gamma_s) - 1), \qquad s \in [0, T],\ \xi \in H,\ v \in \mathcal{H}_2.$$

We have that

$$\mathbb{E}|e^{-\beta t}(\mathcal{L}(v_1) - \mathcal{L}(v_2))(t)|_\eta^2 \leq C \int_0^t \int_H \left| e^{-\beta t} e^{(t-s)B} [\Gamma(s, \xi, v_1(s)) - \Gamma(s, \xi, v_2(s))] \right|_\eta^2 \nu(\mathrm{d}\xi)\, \mathrm{d}s$$

and inequalities similar to those leading to (4.4) implies that the bound

$$\|e^{-\beta t} \mathcal{L}(v_1)(t) - \mathcal{L}(v_2)(t)\|_{\{2,\beta\},\eta} \leq C \frac{1}{\beta^{2(\theta-\eta)-1}} \|v_1 - v_2\|_{\{2,\beta\},\eta} \tag{5.7}$$

holds as well.

Then, by taking β large enough, the mapping $\mathcal{K} + \mathcal{L}$ ($\mathcal{K}$ was defined in the proof of Theorem 4.6) defines a contraction in the space $\mathcal{H}_2(0, T; X_\eta)$ and the theorem is proved.

□

The original control problem can be translated in the state space setting: we have to minimize over all $\gamma \in \mathcal{A}$ the cost functional

$$\mathbb{J}(v_0, \gamma) = \mathbb{E}^\gamma \left[\int_0^T l(t, Jv(t), \gamma(t))\, \mathrm{d}t + g(Jv(T)) \right] \tag{5.8}$$

where the state v of the system evolves, under $\mathbb{P}^\gamma$, according to the equation (5.6). We next provide the solution of the optimal control problem formulated above. As stated in the introduction, in order to solve the control problem in the state space setting we associate to this equation a backward stochastic equation and we try to solve the control problem via this forward-backward system. Nonlinear BSDEs were first introduced by Pardoux and Peng [44] and are currently used in the field of the stochastic control theory : see, e.g., [29, 55]. Recently BSDEs driven by random measures have been introduced to solve optimal control problem for marked point processes [16, 17, 20, 18, 3].

We define in a classical way the Hamiltonian function relative to the above problem: for all $s \in [0,T], v \in X_\eta, z \in L^2(H, \mathcal{H}, \nu(\mathrm{d}\xi))$

$$f(s,v,z) = \inf_{\chi \in U} \left\{ l(s,Jv,\chi) + \int_{H\setminus\{0\}} z(\xi)\,(r(s,Jv,\xi,\chi)-1)\,\nu(\mathrm{d}\xi) \right\}. \quad (5.9)$$

The (possibly empty) set of minimizers will be denoted

$$\Gamma(s,v,z) = \Bigg\{ \chi \in U \ :$$

$$f(s,v,z) = l(s,Jv,\chi) + \int_{H\setminus\{0\}} z(\xi)\,(r(s,Jv,\xi,\chi)-1)\,\nu(\mathrm{d}\xi) \Bigg\}. \quad (5.10)$$

Let us consider, on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, the forward equation

$$v(t) = e^{tB}v_0 + \int_0^t \int_{H\setminus\{0\}} e^{(t-s)B}(I-B)P\psi_J(s,\xi,v(s))\,\tilde{\pi}(\mathrm{d}s,\mathrm{d}\xi). \quad (5.11)$$

The solution of (5.11) is a $\{\mathcal{F}_t\}$-predictable process, which exists and is unique by the results in Section 4. Next we consider the associated backward equation of parameters $(f, T, g(J(v(T)))$ and driven by the random measure $\tilde{\pi}(\mathrm{d}s,\mathrm{d}\xi)$ associated to the pure jump Lévy process L

$$Y_t + \int_t^T \int_{H\setminus\{0\}} Z_s(\xi)\,\tilde{\pi}(\mathrm{d}s,\mathrm{d}\xi) = g(Jv(T)) + \int_t^T f(s,v(s),Z_s)\,\mathrm{d}s. \quad (5.12)$$

BSDEs of this type can be considered as a case of BSDE driven by a general Lévy processes (see [53] or [15, Chapter 19]).

Proposition 5.4. *Under Hypothesis* 3.1, 3.2, 3.4, 4.1, 5.1 *there exists a unique pair* (Y,Z) *which solves equation* (5.12) *such that* Y *is real-valued, càdlàg and adapted,* $Z : \Omega \times [0,T] \times H \to \mathbb{R}$ *is* $\mathcal{P} \otimes \mathcal{H}$*-measurable and*

$$\mathbb{E}\int_0^T |Y_s|^2\,\mathrm{d}s + \mathbb{E}\int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|^2 \nu(\mathrm{d}\xi)\,\mathrm{d}s < \infty.$$

Moreover there exists a deterministic function $V : [0,T] \times H \to \mathbb{R}$ *such that*

$$Y_s = V(s,Jv(s)) \ \textit{and} \ Z_s(\xi) = V(s,\xi) - V(s,Jv(s-)). \quad (5.13)$$

Proof. We start by showing that, under our assumptions, the Hamiltonian function is Lipschitz continuous in the last variable, i.e., there exist $L \geq 0$ such that for every $s \in [0,T]$, $v \in X_\eta$, $z, z' \in L^2(H, \mathcal{H}, \nu(\mathrm{d}\xi))$:

$$|f(s,v,z) - f(s,v,z')| \leq L \left(\int_{H\setminus\{0\}} |z(\xi) - z'(\xi)|^2\,\nu(\mathrm{d}\xi) \right)^{1/2}. \quad (5.14)$$

The boundedness assumption (5.1) implies that for every $s \in [0,T]$, $v \in X_\eta$, $z, z' \in L^2(H, \mathcal{H}, \nu(\mathrm{d}\xi))$, $\gamma \in U$,

$$\begin{aligned}\int_{H\setminus\{0\}} & z(\xi)\,(r(s,Jv,\xi,\gamma)-1)\,\nu(\mathrm{d}\xi)\\ &\le \int_{H\setminus\{0\}} |z(\xi)-z'(\xi)|\,(r(s,Jv,\xi,\gamma)-1)\,\nu(\mathrm{d}\xi)\\ &\quad + \int_{H\setminus\{0\}} z'(\xi)\,(r(s,Jv,\xi,\gamma)-1)\,\nu(\mathrm{d}\xi)\\ &\le (C_r+1)\,\nu(H\setminus\{0\})^{1/2}\left(\int_{H\setminus\{0\}} |z(\xi)-z'(\xi)|^2\,\nu(\mathrm{d}\xi)\right)^{1/2}\\ &\quad + \int_{H\setminus\{0\}} z'(\xi)\,(r(s,Jv,\xi,\gamma)-1)\,\nu(\mathrm{d}\xi),\end{aligned}$$

so that adding $l(s,Jv,\gamma)$ to both sides and taking the infimum over $\gamma \in U$ it follows that

$$f(s,v,z) \le L\left(\int_{H\setminus\{0\}} |z(\xi)-z'(\xi)|^2\,\nu(\mathrm{d}\xi)\right)^{1/2} + f(s,v,z')$$

where $L = (C_r+1)\,\nu(H\setminus\{0\})^{1/2} < \infty$; exchanging z and z' we obtain (5.14).

We conclude the proof by applying Lemma 19.1.5 in [15], which can be applied here since we are assuming that (5.2) and (5.3) hold and we have proved the estimate in (5.14). Since the backward stochastic differential equation (5.12) is Markovian in v, the connection with a deterministic function is given by Theorem 19.4.5 in [15]. □

We are now able to state our main result.

Theorem 5.5. *Suppose that Hypotheses* 3.1, 3.2, 3.4, 4.1, 5.1 *hold. Let* (Y,Z) *be the unique solution to the BSDE (*5.12*). For any admissible control* $\gamma \in \mathcal{A}$ *and for the corresponding trajectory* v *starting at* v_0*, we have* $Y_0 \le J(v_0,\gamma)$ *and the equality holds if and only if the following feedback law is verified by* γ *and* v*,* $\mathbb{P}$*-a.s. for almost every* $t \in [0,T]$*:*

$$\gamma(t) \in \Gamma(t, v(t-), z), \qquad \text{where } z(\xi) = V(t,\xi) - V(t, Jv(t-)), \quad \xi \in H. \tag{5.15}$$

Suppose in addition that the minimizer sets $\Gamma(t,v,z)$ *introduced in* (5.10) *are non empty, for* $\mathrm{d}\mathbb{P}\times\mathrm{d}t$*-almost all* (ω,t) *and all* $v \in X_\eta$ *and* $z \in L^2(H,\mathcal{H},\nu)$*, i.e., there exists* $\gamma' \in U$ *such that*

$$\begin{aligned} f(t,v,z) =& l(s,Jv,\gamma') + \int_{H\setminus\{0\}} z(\xi)\,(r(s,Jv,\xi,\gamma')-1)\,\nu(\mathrm{d}\xi)\\ &= \inf_{\gamma\in U}\left\{ l(s,Jv,\gamma) + \int_{H\setminus\{0\}} z(\xi)\,(r(s,Jv,\xi,\gamma)-1)\,\nu(\mathrm{d}\xi)\right\}. \end{aligned} \tag{5.16}$$

Then there exists a feedback control, that is, a map $\bar\gamma : [0,T] \times X_\eta \to U$ such that the process $\gamma^*_t = \bar\gamma(t, v(t-))$ is optimal among all predictable controls.

Finally the closed loop equation

$$
\begin{cases}
\mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + \\
\qquad + (I-B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,(r(t,Jv(t),\xi,\bar\gamma(t,v_t)) - 1)\,\nu(\mathrm{d}\xi)\mathrm{d}t \\
\qquad\quad + (I-B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde\pi^{\bar\gamma}(\mathrm{d}t,\mathrm{d}\xi), \\
v(0) = v_0 \\
u(t) = Jv(t) \qquad t \in (0,T].
\end{cases}
\tag{5.17}
$$

admits a solution in weak probabilistic sense, compare Definition 18.0.5 in [15].

Remark 5.6. The existence of an element $\gamma' \in U$ satisfying (5.16) is crucial in order to apply the theorem and solve the optimal control in a satisfactory way. It is possible to formulate general sufficient condition for the existence of γ'. For instance, if $r(s,x,\xi,\cdot), l(s,x,\cdot) : U \to \mathbb{R}$ are continuous for every $s \in [0,T], x \in H$ and U is a compact metric space with its Borel σ-algebra $\mathcal{U}$, then condition (5.16) is immediately satisfied.

Using the structure of the FBSDE system, we can then see that, if an optimal control exists, then an optimal feedback control exists, that is, the optimal control depends only on the current values of the state variables $(t, v(t))$.

Proof. Under the reference probability $\mathbb{P}$ the process L is a pure jump $\{\mathcal{F}_t\}$-adapted Lévy process with compensated random measure $\tilde\pi$ and relatively to L the dynamic (5.6) of v can be rewritten

$$
\begin{cases}
\mathrm{d}v(t) = Bv(t)\,\mathrm{d}t + (I-B)P \displaystyle\int_{H\setminus\{0\}} \psi_J(t,\xi,v(t))\,\tilde\pi(\mathrm{d}t,\mathrm{d}\xi), \\
v(0) = v_0 \\
u(t) = Jv(t) \qquad t \in (0,T]
\end{cases}
\tag{5.18}
$$

The process v is adapted to the filtration $\{\mathcal{F}_t\}$ generated by $\tilde\pi$ and completed in the usual way. In the space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, we can consider the system of forward-backward equations

$$
\begin{cases}
v(t) = e^{tB} v_0 + \displaystyle\int_0^t \int_{H\setminus\{0\}} e^{(t-s)B}(I-B)P\psi_J(s,\xi,v(s))\,\tilde\pi(\mathrm{d}s,\mathrm{d}\xi) \\
Y_t + \displaystyle\int_t^T \int_{H\setminus\{0\}} Z_s(\xi)\,\tilde\pi(\mathrm{d}s,\mathrm{d}\xi) = g(v(T)) + \int_t^T f(s, v(s), Z_s(\cdot))\,\mathrm{d}s
\end{cases}
\tag{5.19}
$$

where the generator f is the Hamiltonian function associated to the control problem and defined in (5.9). Writing the backward equation in (5.19) for

$t = 0$ and with respect to the $\mathbb{P}^\gamma$ probability we obtain

$$Y_0 + \int_0^T \int_{H\setminus\{0\}} Z_s(\xi)\, \tilde{\pi}^\gamma(\mathrm{d}s, \mathrm{d}\xi)$$
$$+ \int_0^T \int_{H\setminus\{0\}} Z_s(\xi)(r(s,\xi,Jv_s,\gamma_s)-1)\nu(\mathrm{d}\xi)\,\mathrm{d}s = g(v_T) + \int_0^T f(s, v_s, Z_s(\cdot))\,\mathrm{d}s \tag{5.20}$$

We observe that the stochastic integral has mean zero with respect to $\mathbb{P}^\gamma$. In fact

$$\begin{aligned}
\mathbb{E}^\gamma &\int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|\, r(s,\xi,Jv(s),\gamma_s)\nu(\mathrm{d}\xi)\,\mathrm{d}s \\
&\le C_r\, \mathbb{E}^\gamma \int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|\, \nu(\mathrm{d}\xi)\,\mathrm{d}s \\
&= C_r\, \mathbb{E}\left[\Lambda_T^\gamma \int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|\, \nu(\mathrm{d}\xi)\,\mathrm{d}s\right] \\
&\le C_r\, \left(\mathbb{E}|\Lambda_T^\gamma|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left|\int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|\, \nu(\mathrm{d}\xi)\,\mathrm{d}s\right|^2\right)^{\frac{1}{2}} \\
&\le C_r\, \left(\mathbb{E}|\Lambda_T^\gamma|^2\right)^{\frac{1}{2}} \left(T\nu(H\setminus\{0\})\mathbb{E}\int_0^T \int_{H\setminus\{0\}} |Z_s(\xi)|^2\, \nu(\mathrm{d}\xi)\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

So, if we take in (5.20) the expectation with respect $\mathbb{P}^\gamma$, adding and subtracting $\mathbb{E}^\gamma \int_0^T l(s, Jv_s, \gamma_s)\,\mathrm{d}s$ and recalling (5.13), we conclude

$$\begin{aligned}
Y_0 = \mathbb{E}^\gamma J(v_0,\gamma) + \mathbb{E}^\gamma &\int_0^T f(s, v(s), V(s,\cdot) - V(s, Jv(s)))\,\mathrm{d}s \\
&+ \mathbb{E}^\gamma \int_0^T \Bigg\{ -l(s, Jv(s), \gamma_s) \\
- \int_{H\setminus\{0\}} (V(s,\xi) - V(s,Jv(s)))&\,(r(s,\xi,Jv(s),\gamma_s) - 1)\,\nu(\mathrm{d}\xi)\Bigg\}\,\mathrm{d}s.
\end{aligned} \tag{5.21}$$

The above equality is sometimes called the fundamental relation and immediately implies that $Y_0 \le J(v_0, \gamma)$ and that the equality holds if and only if (5.15) holds.

Assume now that the set Γ defined in (5.10) is not empty. Using Filippov's implicit function theorem (see Theorem 21.3.4 in [15]) and taking into account (5.13) we see that there is a $\mathcal{B}([0,T] \times X_\eta)$-measurable map $\bar{\gamma}(t, v)$

such that

$$l(t, Jv, \bar{\gamma}(t, v)) + \int_{H\setminus\{0\}} (V(t, \xi) - V(t, Jv))\, (r(t, \xi, Jv, \bar{\gamma}(t, v)) - 1)\, \nu(\mathrm{d}\xi)$$
$$= \inf_{\gamma} \{l(t, Jv, \gamma) + \int_{H\setminus\{0\}} (V(t, \xi) - V(t, Jv))\, (r(t, \xi, Jv, \gamma) - 1)\, \nu(\mathrm{d}\xi)$$

for all $t \in [0, T]$ and $v \in X_\eta$. Then the process $\gamma_t^* = \bar{\gamma}(t, v(t-))$ satisfies (5.15), hence it is optimal.

Finally the existence of a solution weak in probabilistic sense (compare Definition 18.0.5 in [15]) to equation (5.17) is again a consequence of the Girsanov's theorem. Namely let v be the mild solution of (5.11) and $\mathbb{P}^{\bar{\gamma}}$ be the probability under which the compensator of the random measure $\tilde{\pi}$ is given by $r(t, \xi, Jv(t), \bar{\gamma}(t, v(t)))\, \nu(\mathrm{d}\xi)\, \mathrm{d}t$. Then v is the mild solution of equation (5.17) relatively to the probability $\mathbb{P}^{\bar{\gamma}}$ and the Lévy process $L^{\bar{\gamma}}$. □

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Stefano Bonaccorsi
Dipartimento di Matematica, Università di Trento
via Sommarive 14, 38123 Povo, Trento, Italia
e-mail: stefano.bonaccorsi@unitn.it

Fulvia Confortola
Politecnico di Milano, Dipartimento di Matematica F. Brioschi,
Piazza Leonardo da Vinci 32, 20133 Milano, Italia
e-mail: fulvia.confortola@polimi.it