A new mathematical explanation of what triggered the catastrophic torsional mode of the Tacoma Narrows Bridge

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Abstract

The spectacular collapse of the Tacoma Narrows Bridge has attracted the attention of engineers, physicists, and mathematicians in the last 74 years. There have been many attempts to explain this amazing event, but none is universally accepted. It is however well established that the main culprit was the unexpected appearance of torsional oscillations. We suggest a mathematical model for the study of the dynamical behavior of suspension bridges which provides a new explanation for the appearance of torsional oscillations during the Tacoma collapse. We show that internal resonances, which depend on the bridge structure only, are the source of torsional oscillations.

1 Introduction

The collapse of the Tacoma Narrows Bridge (TNB), which occurred on November 7, 1940, is certainly the most celebrated structural failure of all times, both because of the impressive video [38] and because of the huge number of studies that it has generated. In the Appendix (Section 7.1) we quote some testimony of witnesses and questions raised by the collapse. Soon after the TNB accident, three engineers were assigned to investigate the collapse and report to the Public Works Administration. Their Report [1] considers *...the crucial event in the collapse to be the sudden change from a vertical to a torsional mode of oscillation*, see [35, p.63]. In 1978, Scanlan [33, p.209] writes that *The original Tacoma Narrows Bridge withstood random buffeting for some hours with relatively little harm until some fortuitous condition "broke" the bridge action over into its low antisymmetrical torsion flutter mode*. In 2001, Scott [35] writes that *Opinion on the exact cause of the Tacoma Narrows Bridge collapse is even today not unanimously shared.* After more than seventy years, a full explanation of the reasons of the collapse is not available: in particular, the main question which arises is

why did torsional oscillations appear suddenly? (**Q**)

Some explanations attribute the failure to a structural problem, some others to a resonance between the frequency of the wind and the oscillating modes of the bridge. Further explanations involve vortices, due both to the particular shape of the bridge and to the angle of attack of the wind. Finally, we mention explanations based on flutter theory and self-excited oscillations due to the flutter speed of the wind. These theories differ as to what caused the torsional oscillation of the bridge, but they all agree that the extreme flexibility, slenderness, and lightness of the TNB allowed these oscillations to grow until they destroyed it. In [5] we discuss in detail all these theories and explain why they fail to answer to **(Q)**. A convincing answer to **(Q)** needs the background of a reliable mathematical model well describing the behavior of suspension bridges. In Section 7.2 we quickly revisit some models considered in literature.

It is our purpose to introduce a new mathematical model for suspension bridges and to give a satisfactory answer to **(Q)**. In order to view the torsional oscillations, it appears natural to consider the cross section of the roadway as a rod having two degrees of freedom: as far as we are aware, this was first suggested by Rocard [32, p.121]. The degrees of freedom are the vertical displacement y of its barycenter with respect to equilibrium and the angle of deflection from the horizontal position θ. The rod is linked at its endpoints to two hangers C_1 and C_2, as in Figure 1. A crucial issue is the choice of the restoring force applied by the

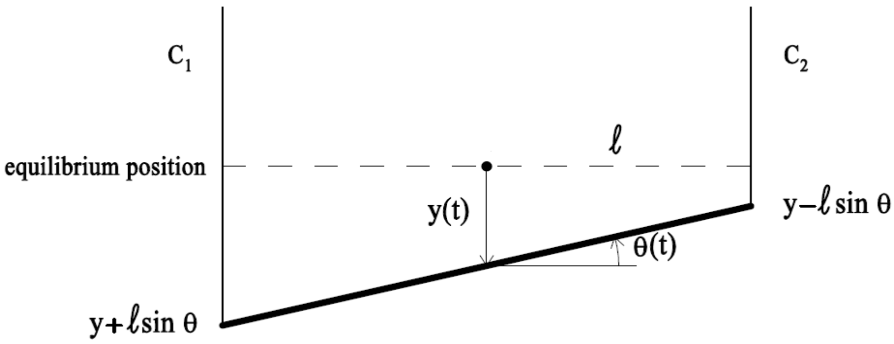

Figure 1: Vertical (y) and torsional (θ) displacements of a cross section of the bridge.

hangers: due to the elastic behavior of steel and to the action of the sustaining cable, the force should be taken nonlinear.

Inspired by the celebrated Fermi-Pasta-Ulam model [13], and also by a model previously studied by us [2, 3, 6], we consider the bridge as finitely many cross sections (seen as rods acting as oscillators) linked by linear forces. This discretization views a suspension bridge as in Figure 2, where the red cross sections are the oscillators linked to the hangers (which act as nonlinear springs) while the grey part is a membrane connecting two adjacent oscillators.

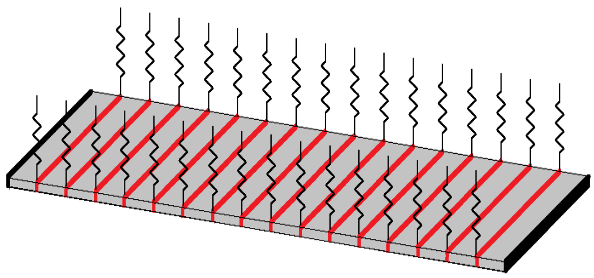

Figure 2: The discretized suspension bridge.

As pointed out in [25], *the torsional oscillations that preceded the collapse were never observed until the day of the collapse*. Our model explains why torsional oscillations may be seen, or may be hidden, or may even not appear, independently of the force applied to the bridge.

It is unlikely for an irregular wind to generate regular torsional oscillations or resonances which would require the matching of its frequency with an internal frequency of the bridge. Hence, the answer to **(Q)** should not be sought in the behavior of the wind; one should instead study very carefully what happens **inside the bridge**. For this reason our model represents an isolated system, without any damping or forcing, which therefore conserves energy. Nowadays, the dominant explanation of the Tacoma collapse relies on the so-called aerodynamic forces generated by the wind-structure interaction, see Billah-Scalan [8]. These forces act in several different ways according to how far is the structure from equilibrium, in particular how large is the torsional angle (see (12) below), and may generate self-excitation and negative damping effects. So the attention should now be focused on the cause of wide torsional oscillations. In this paper we emphasize a **structural instability**, large vertical oscillations may instantaneously switch to the more destructive torsional ones even in isolated systems. In order to achieve this task, we need to strip the model of any interaction with external effects such as the action of the wind, damping and dissipation, and aerodynamic forces. This enables us to show that, in an ideally isolated bridge in vacuum, a structural instability may occur provided enough energy is initially put inside the structure. The wind and vortex shedding are usually responsible for introducing energy within the structure and our analysis starts after the energy is inserted. The structural instability highlighted in the present paper should then be combined with the well-known aerodynamic effect.

The model represented in Figure 1 views the cross section of the bridge as two coupled oscillators (vertical and torsional). It is well-known that nonlinearly coupled oscillators can **transfer energy** between each other if they are in resonance and this may occur only at certain energy levels. We call this phenomenon **internal resonance** and we call **critical energy threshold** the minimal energy level where this occurs. The critical threshold depends on all the parameters involved such as the explicit form of the nonlinear force, the length of the cross section and the coupling constants. The transfer of energy may occur suddenly and implies that the amplitude of the oscillations of the vertical oscillator decreases while that of the torsional oscillator increases.

In Section 3 we exhibit some numerical results which highlight a transfer of energy between vertical and torsional oscillators within the system describing the model in Figure 2. A substantial energy transfer occurs only at certain energy levels which we call again **critical energy thresholds**. Our model reproduces fairly well what was observed the day of the TNB collapse, see the experiments in Section 3 and the movies at [4]. These observations are purely numerical and lack both an explanation and some procedure to compute the critical energy thresholds.

For this reason, in Section 4 we study in detail the simpler model described by the double oscillator represented in Figure 1: we show that the critical energy threshold can be computed by analyzing the eigenvalues of the linearization of the Poincaré map obtained by taking a section of the energy hypersurface. This analysis confirms that, when raising the total internal energy, there is a sudden switch between the regime where the two oscillators behave almost independently and the regime where they are strongly coupled. The starting spark for torsional oscillations is an internal resonance which creates a bifurcation of the Poincaré map and occurs when a certain amount of energy is present in the rod.

In Section 5 we take advantage of this analysis. Although the full bridge model represented in Figure 2 is described by a system with many degrees of freedom, we are able to determine the critical energy thresholds by analyzing the eigenvalues of the linearization of an evolution map. We show that, when these thresholds are reached, there is a sudden transfer of energy within the different fundamental vibrations of the bridge, just as observed at the TNB, see (11) below. This enables us to conclude that

**the bridge behaves driven by its own internal features,
independently of the angle of attack and of the frequency of the wind.**

And the above discussion yields the following answer to **(Q)**:

the sudden appearance of torsional oscillations is due to internal resonances which arise when a certain amount of energy is present into the structure.

Hopefully, this answer will give hints on how to plan future bridges in order to prevent destructive torsional oscillations without excessive costs for stiffening trusses.

This paper is organized as follows. In Section 2 we describe the model and in Section 3 we give the results of some numerical experiments. In Section 4 we consider the simpler model consisting of a single cross section and we provide an explanation of the results of Section 3. In Section 5 we extend the results of Section 4 to the full model. In Section 6 we draw our conclusions and explain in detail why our results give a satisfactory answer to **(Q)**. Finally, the Appendix provides some details on the collapse of the TNB and on prior mathematical models.

2 Description of the model

The rod represented in Figure 1 has mass m and length 2ℓ, it is free to rotate about its center with angular velocity $\dot\theta$ and therefore has torsional kinetic energy $m\ell^2\dot\theta^2/6$. The center of the cross section behaves as an oscillator where the forces are exerted by the two lateral hangers C_1 and C_2, and are denoted respectively by $f(y+\ell\sin\theta)$ and $f(y-\ell\sin\theta)$; these terms take into account also the gravity force. Newton's equation, describing the vertical-torsional oscillations of the rod, was first derived by McKenna [24] and reads

$$\frac{m\ell^2}{3}\,\ddot\theta = \ell\cos\theta\Big(f(y+\ell\sin\theta)-f(y-\ell\sin\theta)\Big)\,,\qquad m\,\ddot y = f(y-\ell\sin\theta)+f(y+\ell\sin\theta)\,. \tag{1}$$

If f is linear, then (1) decouples and describes two independent oscillators. Note also that, by rescaling the time $t\mapsto\sqrt{m}t$, we can set $m=1$.

In order to model the length of the bridge we consider n parallel rods labeled by $i=1,\dots,n$ and we assume that each rod interacts with the two adjacent ones by means of attractive linear forces. For the i-th cross section ($i=1,...,n$), we denote by y_i the downwards displacement of its midpoint and by θ_i its angle of deflection from horizontal. We assume that the mass of each beam modeling a cross section is $m=1$ and its half-length is $\ell=1$. We set $y_0=y_{n+1}=\theta_0=\theta_{n+1}=0$ to model the connection between the bridge and the ground. We have the following system of $2n$ equations:

$$\begin{cases}\ddot\theta_i+3\dfrac{\partial U}{\partial\theta_i}(\Theta,Y)=0\\[2ex] \ddot y_i+\dfrac{\partial U}{\partial y_i}(\Theta,Y)=0\end{cases}\qquad (i=1,...,n)\,, \tag{2}$$

where $(\Theta,Y)=(\theta_1\dots,\theta_n,y_1,\dots,y_n)\in\mathbb{R}^{2n}$ and

$$U(\Theta,Y)=\sum_{i=1}^n\Big[F(y_i+\sin\theta_i)+F(y_i-\sin\theta_i)\Big]+\frac12\sum_{i=0}^n\Big[K_y(y_i-y_{i+1})^2+K_\theta(\theta_i-\theta_{i+1})^2\Big]\,.$$

The constants $K_y,K_\theta>0$ represent the vertical and torsional stiffness of the bridge while $F(s)=-\int_0^s f(\tau)d\tau$. The conserved total energy of the system is given by

$$\mathcal{E}(\dot\Theta,\dot Y,\Theta,Y)=\frac{|\dot\Theta|^2}{6}+\frac{|\dot Y|^2}{2}+U(\Theta,Y)\,. \tag{3}$$

The choice of the nonlinear restoring force $f=-F'$ is delicate. Here we take

$$f(s)=-(s+s^2+s^3)\quad\text{and then}\quad F(s)=\frac{s^2}{2}+\frac{s^3}{3}+\frac{s^4}{4}\,, \tag{4}$$

see Figure 3, and there are several reasons for this choice. First of all, the same kind of force was used in

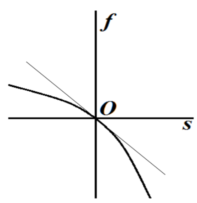

Figure 3: Graph of the restoring force f in (4) close to the origin.

the famous Fermi-Pasta-Ulam experiment [13] and more recently in engineering literature, see e.g. the work by Plaut-Davis [30, Section 3.5], as a simple example of nonlinearity. It reproduces the linear Hooke law for small displacements, with elasticity constant equal to 1 (the corresponding linear behavior is represented in Figure 3 with a thin line tangent to the graph of f). Furthermore, the function f is concave at the origin, that is $f''(0) < 0$, which means that the rate of increase of the restoring force of the prestressed hangers grows with extension. Finally, the s^3-term in f guarantees that the potential F is bounded from below. The slackening of the hangers occurs for some $s < 0$ due to the fact that f also includes gravity. Brownjohn [10, p.1364] explicitly writes that slackening does not have an instantaneous effect: *The hangers are critical elements in a suspension bridge and for large-amplitude motion their behaviour is not well modelled by either simple on/off stiffness or invariant connections*. This justifies the choice of a smooth f. Summarizing, the choice (4) satisfies the minimal requirements for f to be an asymmetric perturbation of a linear force with positive potential energy, even if it is not necessarily expected to yield accurate quantitative information. We tested a fairly wide class of different nonlinearities and we saw that the qualitative behavior of the system is not affected by the specific choice of the parameters in the nonlinearity.

We now define the nonlinear normal modes of the system (2). Let $\mathtt{dst} : \mathbb{R}^n \to \mathbb{R}^n$ be the discrete sine transform, that is, the linear invertible map defined for all $x \in \mathbb{R}^n$ by

$$x_i = \frac{2}{n+1}\sum_{j=1}^{n}(\mathtt{dst}x)_j \sin\left(\frac{\pi i j}{n+1}\right) \qquad \text{and} \qquad (\mathtt{dst}x)_j = \sum_{i=1}^{n} x_i \sin\left(\frac{\pi i j}{n+1}\right),$$

and note that, for any given $k \in \{1, \dots, n\}$ and $E_0 > 0$, there exists a unique $\alpha = \alpha(k, E_0) > 0$ such that $\mathcal{E}\left(0, \alpha(k, E_0)\mathtt{dst}(e_k), 0, 0\right) = E_0$, where e_k is the $k-$th element of the canonical basis of $\mathbb{R}^n$. If f were linear, then the initial condition $(\dot\Theta(0), \dot Y(0), \Theta(0), Y(0)) = (0, \alpha(k, E_0)\mathtt{dst}(e_k), 0, 0)$ would raise a periodic solution to (2) for all k, E_0; such solution is usually called a (linear) normal mode of the system. If f is nonlinear, e.g. as in (4), by a minimization algorithm we can compute numerically $Y^0(k, E_0), Y^1(k, E_0) \in \mathbb{R}^n$ such that $|Y^0(k, E_0)|$ and $|Y^1(k, E_0) - \alpha(k, E_0)\mathtt{dst}(e_k)|$ are small (and tends to 0 as $E_0 \to 0$) and $\big(0, Y^1(k, E_0), 0, Y^0(k, E_0)\big)$ lies on the orbit of a periodic solution to the nonlinear problem (2).

Definition 1. (NONLINEAR NORMAL MODES)
We call the periodic solution of (2) with initial data $(\dot\Theta(0), \dot Y(0), \Theta(0), Y(0)) = (0, Y^1(k, E_0), 0, Y^0(k, E_0))$ *the k-th nonlinear normal mode of (2) at energy E_0.*

Our purpose is to study the stability of the nonlinear normal modes under small perturbations of the null torsional initial data.

3 Numerical results

We consider (2) with $n = 16$ and $K_y = K_\theta = 320$. Let $Y^0(k, E_0), Y^1(k, E_0) \in \mathbb{R}^n$ be as in Section 2: Figure 4 represents the solutions to the system (2) with initial conditions

$$\Big(\dot\Theta(0), \dot Y(0), \Theta(0), Y(0)\Big) = (\Theta^1, Y^1(k, E_0), 0, Y^0(k, E_0)) \tag{5}$$

where $\Theta^1 \in \mathbb{R}^{16}$ is a random vector whose components lie in the interval $[-5 \cdot 10^{-6}, 5 \cdot 10^{-6}]$, and with $(k, E_0) = (1, 516), (2, 500), (3, 6000)$, each one on a line from the first to the third.

In all the pictures the black and grey plots represent $\theta_i(t)$ and $y_i(t)$ respectively for $i = 1 \dots, 8$ going from left to right; the first one is the closest to a tower whereas the eighth one is in the center of the span. We only display the first 8 plots of (θ_i, y_i) because $(\theta_i, y_i) \approx (\theta_{17-i}, y_{17-i})$ if k is odd and $(\theta_i, y_i) \approx -(\theta_{17-i}, y_{17-i})$ if k is even. In order to better explain the phenomenon, we enlarge the first picture of the second line, see Figure 5. We observe that the (black) torsional oscillations are initially negligible with respect to the (grey) vertical oscillations but, suddenly, they become visible, and then larger and larger, and their maximum amplitude would suffice for an actual bridge to collapse. Even if the amount of energy E is large enough to generate instability, the energy transfer does not occur instantaneously, it takes some time $T = T(E) > 0$. This delay in time may be seen as a structural version of the Wagner effect [39] which was originally

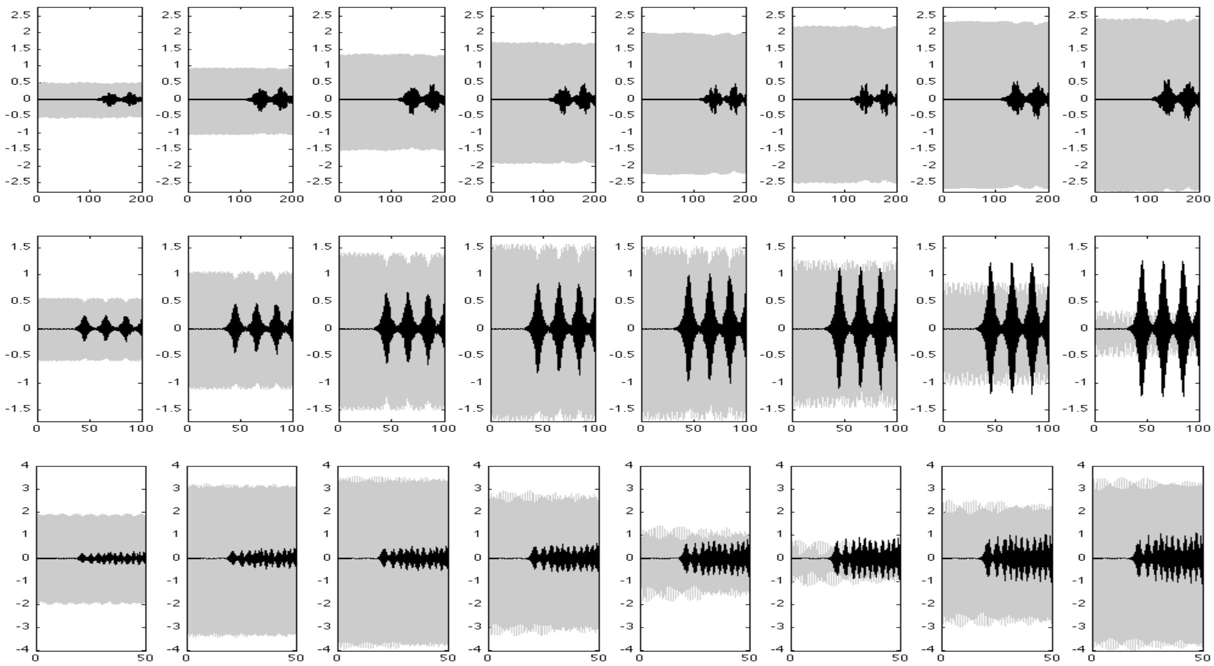

Figure 4: Unstable torsional oscillations (black) and vertical oscillations (grey) of the cross sections.

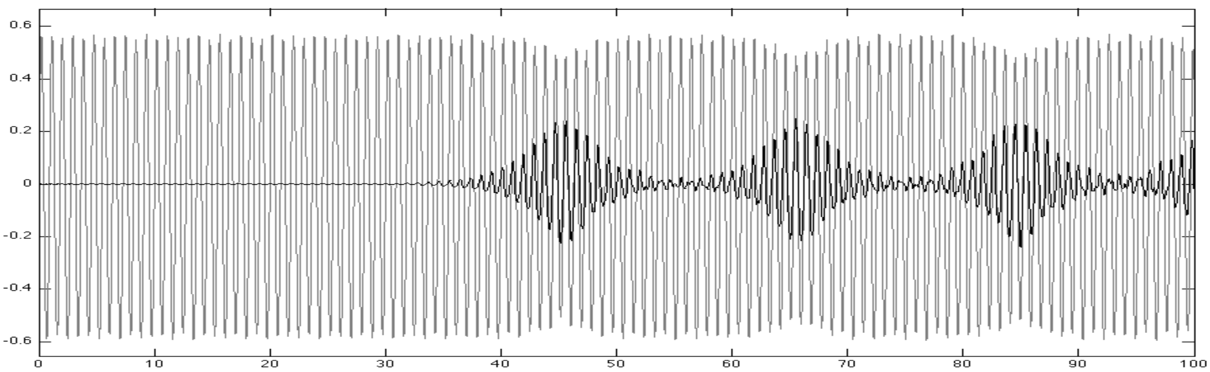

Figure 5: Sudden appearance of torsional oscillations.

discovered in the aerodynamic action of the wind on the airfoil of an aircraft: if the incidence of a wing changes suddenly, the new lifting force resulting from the change in incidence is not set up instantaneously. However, as soon as the instability is apparent, the oscillations suddenly grow up. In this respect, Billah-Scalan [8] remark that when instability (or flutter) occurs for suspension bridges without a streamlined deck, it tends to be very precipitate. Note also that it appears clearly in Figure 5 that the vertical (grey) oscillations decrease in amplitude when the torsional (black) oscillations increase: this is what we call an energy transfer between oscillating modes. Numerical experiments show that the sudden appearance of wide torsional oscillations as in Figures 4 and 5 can be seen only if the energy is sufficiently large, that is, above a critical energy threshold. Above this threshold a very small perturbation of the k-th mode in any θ_i-variable can lead to a significant torsional motion, while below the threshold small initial torsional oscillations remain small for all time. In the latter case, the amplitudes of both the torsional and the vertical modes are constant, therefore the pictures are trivial and we do not display them.

We have also produced a dynamic representation of the solutions (Θ, Y) to (2)-(5). The movies available at [4] display the dynamics of the discretized bridge, and in particular the similarity to the original movie of the Tacoma Narrows Bridge collapse [38].

4 The single cross section model

In order to explain the results described in the previous section, we consider (2) with $n = 1$ and $K_y = K_\theta = 1$. We omit here the (redundant) index referring to the cross section. Since the energy (3) of the system is a constant of motion, for any $E_0 > 0$ the 3-dimensional submanifold $\mathcal{E}(\dot\theta, \dot y, \theta, y) = E_0$ of the phase space $\mathbb{R}^4$ is flow-invariant, that is, the motion is confined to this 3-dimensional energy surface. We study a 2-dimensional section of this surface, the so-called Poincaré section (see e.g. [16, Section 11.5] or [19, Section 1.4]), whose construction adapted to the problem at hand we now give in detail.

First observe that

$$\text{the plane } \dot\theta = \theta = 0 \text{ is flow-invariant;} \tag{6}$$

in particular, for all $y^1 \in \mathbb{R}$, the initial data $(\dot\theta(0), \dot y(0), \theta(0), y(0)) = (0, y^1, 0, 0)$ yields a periodic solution of system (2). We wish to study the stability of this solution.

Now fix $E_0 > 0$ and consider the bounded set

$$\mathcal{U}_{E_0} := \{(\theta^1, \theta^0) \in \mathbb{R}^2;\, \mathcal{E}(\theta^1, 0, \theta^0, 0) < E_0\}\,.$$

For all $(\theta^1, \theta^0) \in \mathcal{U}_{E_0}$ define

$$y^1 = y^1(E_0, \theta^1, \theta^0) := \sqrt{2[E_0 - \mathcal{E}(\theta^1, 0, \theta^0, 0)]} > 0\,,$$

namely the unique positive value of y^1 that satisfies $\mathcal{E}(\theta^1, y^1, \theta^0, 0) = E_0$. It is easily shown that there exists a first $T = T(\theta^1, \theta^0) > 0$ such that the solution of (2) with initial data

$$(\dot\theta(0), \dot y(0), \theta(0), y(0)) = (\theta^1, y^1(E_0, \theta^1, \theta^0), \theta^0, 0) \tag{7}$$

satisfies $y(T) = 0$ and $\dot y(T) > 0$. The Poincaré map $P_{E_0} : \mathcal{U}_{E_0} \to \mathbb{R}^2$ is then defined by

$$P_{E_0}(\theta^1, \theta^0) := (\dot\theta(T), \theta(T)) \tag{8}$$

where $(\theta(t), y(t))$ is the solution to the system (2) with initial data (7). Note that, for such solution, one has $\mathcal{E} = E_0$. In view of (6), the origin $(0, 0) \in \mathcal{U}_{E_0}$ is a fixed point for the map P_{E_0} for any $E_0 > 0$. In Figure 6 we represent some iterates of the map P_{E_0} for system (2) with different initial data $(\dot\theta(0), \theta(0)) \in \mathcal{U}_{E_0}$. The energies considered are $E_0 = 3.4$, 3.5, 3.6, 3.8 and we observe a change of behavior between $E_0 = 3.5$ and $E_0 = 3.6$ so that the critical energy threshold of (2) (case $n = 1$) satisfies

$$3.5 < \overline{E} < 3.6\,.$$

Finer experiments yield $\overline{E} \approx 3.56$.

What we have just observed has a clear explanation in the theory of dynamical systems. The pictures in Figure 6 show that if $\mathcal{E} < \overline{E}$, then any initial condition with small $(\dot\theta(0), \theta(0))$ leads to solutions with small $(\dot\theta(t), \theta(t))$ for all t, while if $\mathcal{E} > \overline{E}$, any initial condition with small $(\dot\theta(0), \theta(0))$ leads to large values of $(\dot\theta(t), \theta(t))$ for some t. Hence, if $E_0 < \overline{E}$ then the origin is a stable fixed point of P_{E_0}, whereas if $E_0 > \overline{E}$ the origin is unstable, that is, the system undergoes a *bifurcation* at $E_0 = \overline{E}$. The stability of the origin can be determined by the eigenvalues λ_1 and λ_2 of the Jacobian $JP_{E_0}(0, 0)$ of P_{E_0} at $(0, 0)$. Since the system (2) is conservative, $JP_{E_0}(0, 0)$ has determinant equal to 1, therefore one of the following cases applies:

(i) $|\lambda_1| = |\lambda_2| = 1$ and $\lambda_2 = \overline{\lambda_1}$, in which case $(0, 0)$ is stable for P_{E_0};

(ii) $\lambda_1, \lambda_2 \in \mathbb{R}$ and $0 < |\lambda_1| < 1 < |\lambda_2|$, in which case $(0, 0)$ is unstable for P_{E_0}.

Since the eigenvalues depend continuously on E_0, the bifurcation, i.e. the loss of stability, may only occur when $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$. In the former case the Jacobian of $P_{E_0} - I$ at $(0, 0)$ is not invertible, therefore the fixed point is not guaranteed to be locally unique, and indeed two new stable fixed points are created: this kind of bifurcation is called *pitchfork*. In the latter case the Jacobian of $P_{E_0} - I$ at $(0, 0)$ is invertible, but the Jacobian of $P_{E_0}^2 - I$ is not; then, by the implicit function theorem the fixed point is locally

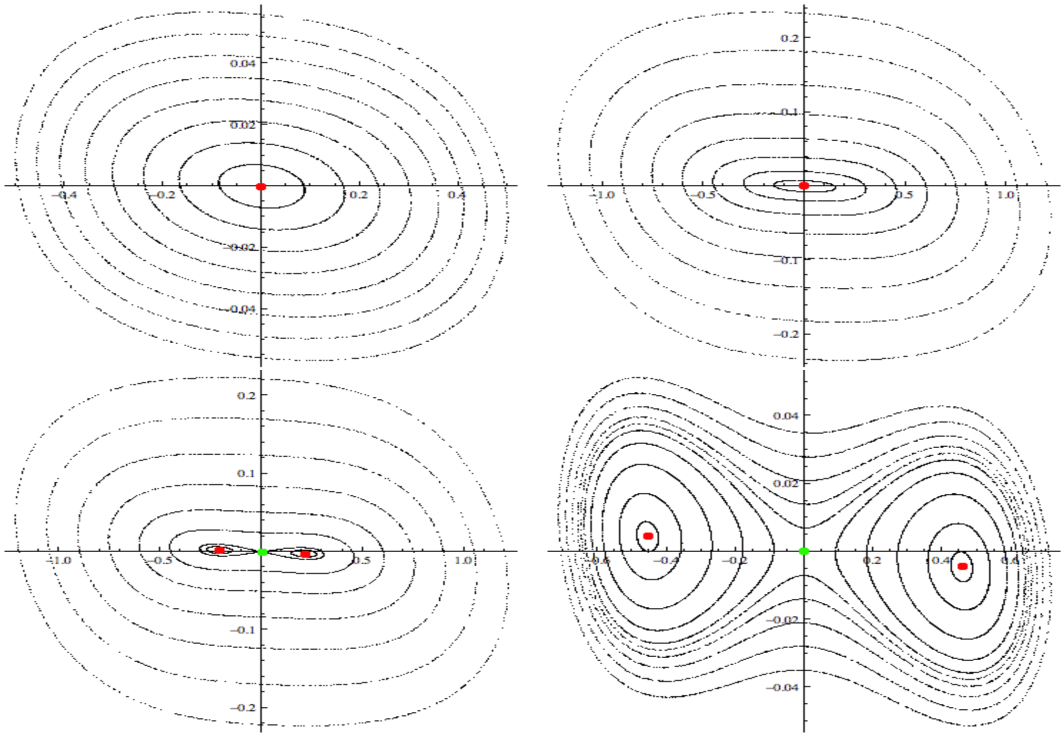

Figure 6: The Poincaré map for (2) in the plane $(\dot{\theta}, \theta)$: from left to right and top to bottom, the values of E_0 are 3.4, 3.5, 3.6, 3.8. Stable fixed points are red and unstable fixed points are green.

unique, but periodic points of period 2 are not, and indeed two such points are created at the bifurcation; this kind of bifurcation is called *period doubling*.

The experiments displayed in Figure 6 show that the bifurcation generates two new stable fixed points (red color), therefore we have a pitchfork bifurcation. Since at the bifurcation point both eigenvalues of the Jacobian of P_{E_0} at $(0, 0)$ are equal to 1, then $P_{\overline{E}}(\theta^1, \theta^0) = (\theta^1, \theta^0) + o(\theta^1, \theta^0)$, so that small initial data (θ^1, θ^0) yield solutions $\theta(t)$ to (2) being close to a periodic solution having the same period of $y(t)$. This is what we call an **internal resonance**. In our experiments we observe that, as E_0 increases from 0, the eigenvalues λ_1 and λ_2 move on the unit circle of the complex plane and meet at the point $(1, 0)$ when $E_0 = \overline{E}$. When $E_0 > \overline{E}$ the eigenvalues move along the real line in opposite directions and the two new stable (red) fixed points represent periodic solutions $\theta(t)$ having the same period of $y(t)$.

In the case of a period doubling bifurcation we would have $P_{\overline{E}}(\theta^1, \theta^0) = -(\theta^1, \theta^0) + o(\theta^1, \theta^0)$, so that small initial data (θ^1, θ^0) would yield solutions $\theta(t)$ to (2) being close to a periodic solution having period equal to the double of the period of $y(t)$. This is another kind of **internal resonance**, which does not happen in the experiment that we describe here, but can be observed with other nonlinearities, e.g. with $f(s) = -(s + s^2)$. If $E_0 > \overline{E}$, then P_{E_0} has two periodic points of period 2, corresponding to periodic solutions $\theta(t)$ having the double of the period of $y(t)$.

Summarizing, a necessary condition for a bifurcation to occur is a resonance between the oscillators; in absence of resonance the double oscillator is torsionally stable and small initial torsional data remain small for all time. What we have just explained leads us to the following definition and criterion:

Definition 2. (TORSIONAL STABILITY)
We say that the system (2) (for $n = 1$) is torsionally stable at energy $E_0 > 0$ if the origin $(0, 0) \in \mathbb{R}^2$ is stable for the Poincaré map P_{E_0}. Otherwise, we say that the system is torsionally unstable.

Criterion 3. *Let $\lambda_1 = \lambda_1(E_0)$ and $\lambda_2 = \lambda_2(E_0)$ be the complex eigenvalues of the Jacobian of the Poincaré map P_{E_0} at the origin $(0, 0) \in \mathbb{R}^2$. Then $\lambda_1\lambda_2 = 1$ and two cases may occur:*

(S) *if* $|\lambda_1| = |\lambda_2| = 1$ *and* $\lambda_2 = \overline{\lambda_1}$*, then the system (2) is torsionally stable;*
(U) *if* $\lambda_1, \lambda_2 \in \mathbb{R}$ *and* $0 < |\lambda_1| < 1 < |\lambda_2|$*, then the system (2) is torsionally unstable.*

We have observed above that equal eigenvalues yield a resonance between the oscillators, which may occur only at particular values of the energy. When the energy is larger than such value, the system is in the unstable regime, while when the energy is smaller the system is in the stable regime. In principle it may happen that also at higher energy the system undergoes another bifurcation and the origin becomes again stable. Then, we have shown that there exists $\overline{E} > 0$ such that the system (2) is stable (case (S)) whenever $0 < \mathcal{E} < \overline{E}$ whereas the system (2) is unstable (case (U)) whenever $\overline{E} < \mathcal{E} < \overline{E} + \delta$ for some $\delta > 0$.

Definition 4. (CRITICAL ENERGY THRESHOLD)
When $n = 1$ *we call*

$$\overline{E} := \inf \{E_0 > 0;\ \max\{|\lambda_1(E_0)|, |\lambda_2(E_0)|\} > 1\}$$

the critical energy threshold of (2).

What we have seen in this section enables us to conclude that

the bifurcation is caused by a resonance between the nonlinear oscillators and generates torsional instability.

Moreover, the Poincaré maps show that

the onset of torsional instability is generated by internal resonances.

This simple description is possible because we are dealing with a model with two degrees of freedom only. In the next section we provide a suitable generalization of these results for the full bridge model.

5 Stability of the multiple beam system

We extend here the results obtained in the previous section to the full bridge model where $n > 1$. For any mode $k \in \{1, ..., n\}$ and any energy $E_0 > 0$ let $Y^1(k, E_0)$ and $Y^0(k, E_0)$ be as in Section 2. Let $T(k, E_0) > 0$ be the period of the k-th nonlinear normal mode of (2) at energy E_0 and let $\Psi^k_{E_0} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the evolution map defined by

$$\Psi^k_{E_0}(\Theta^1, \Theta^0) = \Big(\dot{\Theta}(T(k, E_0)), \Theta(T(k, E_0))\Big)\ , \tag{9}$$

where $(\Theta(t), Y(t))$ is the solution to (2) with initial conditions

$$\Big(\dot{\Theta}(0), \dot{Y}(0), \Theta(0), Y(0)\Big) = (\Theta^1, Y^1(k, E_0), \Theta^0, Y^0(k, E_0))\,.$$

We remark that the origin is a fixed point of $\Psi^k_{E_0}$ and that $\Psi^k_{E_0}$ is not a Poincaré map; in particular, the iteration time T does not depend on (Θ^1, Θ^0). One could compute a Poincaré map even in the full model case, but its construction is theoretically and computationally more complicated, and, due to the higher dimensionality of the problem, it would not provide any additional insights. The maps $\Psi^k_{E_0}$ enable us to generalize Definition 2 as follows:

Definition 5. (TORSIONAL STABILITY)
We say that the k*-th nonlinear normal mode of (2) at energy* $E_0 > 0$ *is torsionally stable if the origin* $(0, 0) \in \mathbb{R}^{2n}$ *is stable for the evolution map* $\Psi^k_{E_0}$*. Otherwise, we say that it is torsionally unstable.*

In order to evaluate the stability of the k-th nonlinear normal mode, we study the Jacobian $J\Psi^k_{E_0}(0,0)$ of $\Psi^k_{E_0}$ at $(\Theta^1,\Theta^0)=(0,0)$. To compute the derivatives of this map, we linearize (2) at $(\Theta,Y)=(0,\overline{Y}_k)$, where $\overline{Y}_k(t)$ is the k-th nonlinear normal mode of (2) at energy E_0. We are led to solve the system

$$\ddot{\xi}_i+3\sum_{j=1}^n\frac{\partial^2U}{\partial\theta_i\partial\theta_j}(0,\overline{Y}_k(t))\xi_j=0\qquad (i=1,...,n)\,,\tag{10}$$

where $\Xi(t)=(\xi_1(t),\dots,\xi_n(t))$ is the variation of $\Theta\equiv 0$. The l-th column of $J\Psi^k_{E_0}(0,0)$ is the solution $(\dot{\Xi}(T),\Xi(T))$ at time $T(k,E_0)$ of (10) with initial conditions $(\dot{\Xi}(0),\Xi(0))=\eta_l$ ($l=1,...,2n$), where η_l is the l-th element of the canonical basis of $\mathbb{R}^{2n}$.

In principle, when $n>1$ one cannot infer the full stability of the nonlinear normal mode $\overline{Y}_k$ from its linear stability, that is, when all the eigenvalues of $J\Psi^k_{E_0}(0,0)$ have modulus 1. On the other hand, we have numerical evidence that the model is torsionally stable if and only if all the eigenvalues of $J\Psi^k_{E_0}(0,0)$ lie on the unit circle. This leads to the following:

Criterion 6. *Let $\lambda_i=\lambda_i(k,E_0)$ ($i=1,...,2n$) be the eigenvalues of $J\Psi^k_{E_0}(0,0)$. Then:*
(S) *if $\max_i|\lambda_i|=1$, then the k-th nonlinear normal mode of (2) at energy E_0 is torsionally stable;*
(U) *if $\max_i|\lambda_i|>1$, then the k-th nonlinear normal mode of (2) at energy E_0 is torsionally unstable.*

We take again (4) while we fix $n=16$ and $K_y=K_\theta=320$. In the graphs in Figure 7 we display the largest modulus of the eigenvalues of $J\Psi^k_{E_0}(0,0)$ as a function of the energy E_0, with $k=1,2,3$. It

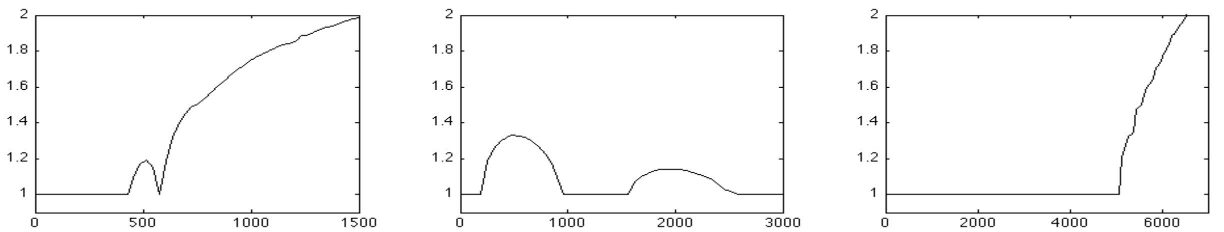

Figure 7: Largest modulus of the eigenvalues of $J\Psi^k_{E_0}(0,0)$ versus the energy E_0, $k=1,2,3$.

appears that for all such k there exists a largest $\overline{E}_k>0$ such that the k-th mode of (2) is torsionally stable whenever $\mathcal{E}<\overline{E}_k$. It turns out that, for higher levels of energy, a mode may become stable again, but this has a purely theoretical (mathematical) relevance since, in order to ensure that the bridge is safe, one should consider only the lower energy threshold $\overline{E}_k$; here we computed $\overline{E}_1\approx 450$, $\overline{E}_2\approx 200$, $\overline{E}_3\approx 5100$. All our experiments have shown that a nonlinear normal mode is stable, that is small initial torsional data yield small torsional oscillations for all time, whenever it is linearly stable.

Criterion 6 enables us to provide a rigorous definition of the critical energy threshold of each mode.

Definition 7. (CRITICAL ENERGY THRESHOLD)
We call critical energy threshold $\overline{E}_k$ of the k-th nonlinear normal mode of (2) the positive number

$$\overline{E}_k:=\inf\left\{E_0>0;\ \max_i|\lambda_i(k,E_0)|>1\right\}\,.$$

Figure 7 shows that $\overline{E}_k$ depends on k and the effective critical energy threshold $\overline{E}$ of the bridge satisfies

$$\overline{E}\ \le\ \min_{1\le k\le n}\ \overline{E}_k\,.$$

In order to show further that our model well explains the behavior of suspension bridges and is able to reproduce the collapse of the TNB, we revisit our results trying to match the phenomenon described by Farquharson [12] concerning the sudden appearance of torsional oscillations that contemporaneously changed the vertical oscillations

$$\textit{...which a moment before had involved nine or ten waves, had shifted to two.} \tag{11}$$

Let us first introduce a new definition.

Definition 8. (FUNDAMENTAL VIBRATIONS)
For all $j = 1, ..., n$ we call j-th vertical (respectively, torsional) fundamental vibration of a solution $(\Theta(t), Y(t))$ of (2) the function $t \mapsto (\mathtt{dst}Y(t))_j$ (respectively, the function $t \mapsto (\mathtt{dst}\Theta(t))_j$), that is, the j-th component of the discrete sine transform.

In Figure 8 we plot a simple moving average of the first four (vertical and torsional) fundamental vibrations of a solution to (2) in the case $(k, E_0) = (2, 500)$. The graphs show that initially most of the dynamics is concentrated on the second vertical fundamental vibration, but at time $t \approx 50$ part of the energy is transferred to the first torsional fundamental vibration; then the second vertical and the first torsional fundamental vibrations begin a somehow periodic exchange of energy. All the other fundamental vibrations, vertical and torsional, appear to be almost unaffected. The testimony (11) tells us that, at the TNB, the appearance of torsional oscillations had changed the vertical oscillations form the ninth to the second fundamental vibration.

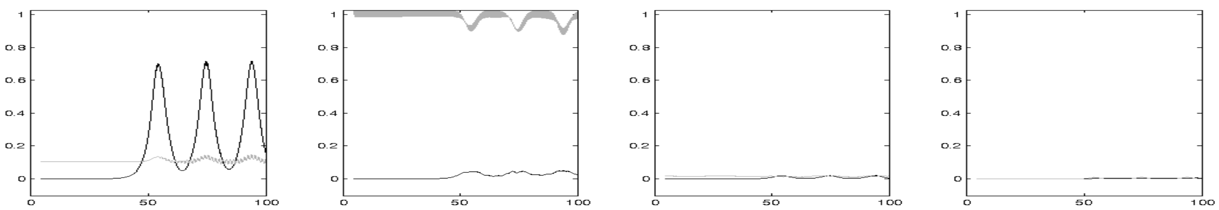

Figure 8: The first four torsional (black) and vertical (grey) FV's of a solution to (2).

The next step should be an accurate analysis of the nonlinear modes of a suspension bridge and of the fundamental vibrations of its oscillations, see [7] for the linear case. Combined with our analysis, this could give precise suggestions on how to plan bridges in order to higher $\overline{E}_k$, at least for small k.

6 Our explanation of the Tacoma collapse

In [5] we analyze the collapses of the Broughton Suspension Bridge and of the Angers Bridge, which were caused by a light and periodic external forcing that gave rise to a resonance with the natural frequencies of the bridges. This should not be confused with the violent and disordered behavior of the wind at the TNB. By no means, one may expect that a random and variable wind might match the natural frequency of a bridge.

The model we suggest here views the bridge as an elastic structure formed by many coupled nonlinear oscillators whose frequencies may synchronize, creating internal resonances. Since the model is nonlinear, the frequencies depend on the energy involved and resonances may occur only if a certain amount of energy is present into the structure, that is,

if the total energy within the structure is small, then the oscillators weakly interact and only a negligible part of the energy of the vertical oscillators can be transferred to a torsional fundamental vibration, whereas if the total energy is sufficiently large then the oscillators are in resonance and tiny torsional oscillations may suddenly become wide.

An external action inserting energy inside the structure may exceed the critical thresholds of the bridge and give rise to uncontrolled oscillations. The critical energy thresholds of the nonlinear normal modes of the bridge depend only on its structural parameters such as width, length, rigidity, mass, elasticity, stiffness, and distance between hangers. Future bridges should be planned with structural parameters yielding very large critical energy thresholds.

We believe that the TNB has collapsed because, on November 7, 1940, the wind inserted enough energy to overcome the critical energy threshold of its 9th nonlinear normal mode. This gave rise to internal resonances that were the onset of the destructive torsional oscillations. Then the aerodynamic forces self-excited these oscillations until the collapse of the bridge. Is this the final answer to **(Q)**?

7 Appendix

7.1 The Tacoma Narrows Bridge collapse

The TNB was considered very light and flexible. Not only this was apparent to traffic after the opening, but also it was felt during the construction. According to [35, pp.46-47], *...during the final stages of work, an unusual rhythmic vertical motion began to grip the main span in only moderate winds ... these gentle but perceptible undulations were sufficient to induce both bridgeworker nausea and engineering concern.* The undulatory motion of the span attracted the local interest and *...motorists ventured onto the TNB to observe vehicles ahead of them slowly disappearing in the trough of a wave.* So, it was not surprising that vertical oscillations were visible on the day of the collapse. The wind was blowing at approximately 80 km/h and, apparently, *the oscillations were considerably less than had occurred many times before*, see [35, p.49]. Hence, although the wind was the strongest so far since the bridge had been built, the motions were in line with what had been observed earlier. However, a sudden change in the motion was alarming. Without any intermediate stage, a violent destructive torsional movement started: the oscillation changed from nine or ten smaller waves to the two dominant twisting waves. A witness to the collapse was Farquharson, the man escaping in the video [38]. According to his detailed description [12] *...a violent change in the motion was noted. This change appeared to take place without any intermediate stages and with such extreme violence that the span appeared to be about to roll completely over.*

Leon Moisseiff (1873-1943), who was charged with the project, had an eye to economy and aesthetics, but he was not considered guilty for the TNB failure. For instance, Steinman-Watson [37] wrote that *...the span failure is not to be blamed on him; the entire profession shares in the responsibility. It is simply that the profession had neglected to combine, and apply in time, the knowledge of aerodynamics and of dynamic vibrations with its rapidly advancing knowledge of structural design.* The reason of this discharge probably relies on forgotten similar collapses previously occurred: one should compare the torsional motion prior to the collapse of the Brighton Chain Pier in 1836, as painted by William Reid [31, p.99], and torsional motions prior to the TNB collapse, see Figure 9.

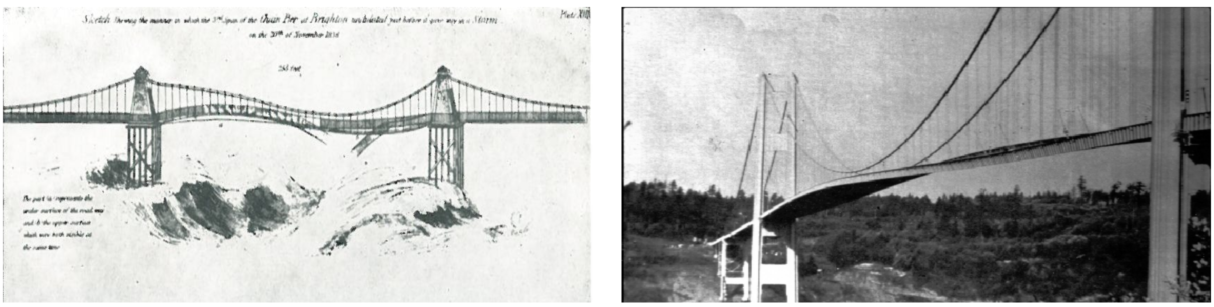

Figure 9: Torsional motion in the Brighton Chain Pier and in the Tacoma Narrows Bridge.

Similar torsional behaviors were displayed in several other bridges before 1940, see [35, Section 4.3], [18, p.75], and also [14, Section 2]. So, it seems that torsional oscillations are to be expected in flexible suspension bridges. Still, the natural questions which arise from the TNB collapse and from similar failures

are the following [35, p.53]:
- how could a span designed to withstand 161 km/h winds and a static horizontal wind pressure of 146 kg/m^2 succumb under a wind of less than half that velocity imposing a static force one-sixth the design limit?
- how could horizontal wind forces be translated into dynamic vertical and torsional motion?

In fact, the answers to these questions are strictly linked. The bridge was ready to withstand 161 km/h wind provided that the oscillation would have been vertical. But since unexpected torsional oscillations appeared, this considerably lowered the critical speed of the wind. Therefore, the two above questions reduce to the main question **(Q)**.

7.2 Previous mathematical models

The celebrated report by Navier [29] has been for about one century the only mathematical treatise of suspension bridges. The second milestone contribution is certainly the monograph by Melan [27]. After the TNB collapse, the scientific community felt the necessity to find accurate equations in order to attempt explanations of what had occurred. In this respect, historical sources are [9, 36] which consider the function representing the amplitude of the vertical oscillation as unknown but do not consider torsional oscillations.

In a model suggested by Scanlan-Tomko [34], the angle of twist α of the torsional oscillator (bridge deck section) is assumed to satisfy the equation

$$I\,[\ddot{\alpha} + 2\zeta_\alpha\omega_\alpha\dot{\alpha} + \omega_\alpha^2\alpha] = A\dot{\alpha} + B\alpha\,, \tag{12}$$

where I, ζ_α, ω_α are, respectively, associated inertia, damping ratio, and natural frequency. The aerodynamic force (the r.h.s. of (12)) is assumed to depend linearly on both $\dot{\alpha}$ and α with the positive constants A and B depending on several parameters of the bridge. Constant coefficient second order linear equations such as (12) have elementary solutions. Roughly speaking, one can say that chaos manifests itself as an unpredictable behavior of the solutions in a dynamical system. With this characterization, there is no doubt that chaos was somehow present in the dynamic of the TNB. From [16, Section 11.7] we recall that *neither linear differential equations nor systems of less than three first-order equations can exhibit chaos*. Since (12) may be reduced to a two-variables first order linear system, it cannot be suitable to fully describe the disordered behavior of the bridge. In order to have a description of the bridge obeying the two rules for chaos, the fourth order nonlinear ODE $w'''' + kw'' + f(w) = 0$ ($k \in \mathbb{R}$) was studied in [15] and it was proved that solutions to this equation blow up in finite time with self-excited oscillations appearing suddenly, without any intermediate stage. Following these general rules, we propose the explanation that the structural instability generated the sudden excitation of the torsional mode and, once the torsional mode was activated, (12) explains how aerodynamic forces led to the Tacoma collapse.

In [10, 14, 19] one may find further evidence that some nonlinearity should appear in any model aiming to describe suspension bridges. Furthermore, it was recently confirmed by Luco-Turmo [22] that the flexibility of the hangers is generally negligible so that their nonlinear behavior is mainly due to the cable; see also [17] for more details on the behavior of hangers. For large displacements one cannot apply the linear Hooke law of elasticity. This is also the opinion of McKenna [24, p.16]: *We doubt that a bridge oscillating up and down by about 10 meters every 4 seconds obeys Hooke's law*. Moreover, McKenna [24, p.4] comments (12) by writing *This is the point at which the discussion of torsional oscillation starts in the engineering literature*. He then claims that a key error in previous models was the linearization of (1) with respect to θ, the term $\sin\theta$ was usually replaced by θ whereas $\cos\theta$ was replaced by 1: this is reasonable for small θ, but appears inaccurate for large deflections. McKenna concludes by noticing that *Even in recent engineering literature ... this same mistake is reproduced*. And indeed, [24, 26] show that numerical solutions to (1) starting with large initial data die down in the linear model while they do not for the nonlinear model where large oscillations continue for all time until the eventual collapse of the bridge: by linear model we mean here that the approximation $\sin\theta \approx \theta$ is made, whereas nonlinear means that $\sin\theta$ is maintained. In these experiments the restoring force is assumed to be piecewise linear. Supported by numerical experiments, our opinion is that, although the approximation $\sin\theta \approx \theta$ is inaccurate, it does not hide the main phenomenon, while we believe that the nonlinearity in the restoring force f plays the major role. Indeed, the nonlinearity of f plays a significant role even when θ is small, that is when $\sin\theta \approx \theta$ is acceptable.

The interaction between different components of the bridge is the most delicate part of any model. Lazer-McKenna [20, Section 3.4] introduce a system describing the coupled motion of the roadway and the sustaining cable, see also [21] for the same system with different external sources. This model views the roadway as a one-dimensional beam and, therefore, it cannot display torsional oscillations.

A model suggested by Moore [28] extends (1) to the entire length L of the roadway. Assuming that the restoring forces are piecewise linear, the following generalization of problem (1) is obtained

$$\begin{cases} \theta_{tt} - \varepsilon_1 \theta_{xx} = \frac{3K}{m\ell}\cos\theta[(y - \ell\sin\theta)^+ - (y + \ell\sin\theta)^+] - \delta\theta_t + h_1(x,t) \\ y_{tt} + \varepsilon_2 y_{xxxx} = -\frac{K}{m}[(y - \ell\sin\theta)^+ + (y + \ell\sin\theta)^+] - \delta y_t + g + h_2(x,t) \\ \theta(0,t) = \theta(L,t) = y(0,t) = y(L,t) = y_{xx}(0,t) = y_{xx}(L,t) = 0\,, \end{cases} \tag{13}$$

where ε_1, ε_2 are physical constants related to the flexibility of the beam, $\delta > 0$ is the damping constant, h_1 and h_2 are external forces, and g is the gravity acceleration. Multiple periodic solutions to (13) are determined in [28]. The system (13) was complemented by Matas-Očenášek [23] with two further equations governing the displacements of the lateral cables; then the displacements of the cables also appear in the equations in (13). We believe that (13) is a nice reliable model which, however, may be improved in several aspects. First, the restoring force needs not to be piecewise linear; second, it does not act on the whole length of the roadway but only in those points where the hangers are present. As we have seen, the answer to **(Q)** is hidden in a generalized version of system (1) independently of external forces.

Any model aiming to describe the complex behavior of a bridge has to face the difficult choice between accurate but complicated equations on one hand and simplified but less reliable equations on the other hand. Excessive linearizations lead to inaccurate models (such as (12)) and to unreliable responses. The model suggested in the present paper seems to be a good compromise between these two choices. In particular, it gives a satisfactory answer to **(Q)**. We refer to [11] and to [14, Section 3.2] for a detailed story of further mathematical models which, however, could not lead even to partial answers to **(Q)**.

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