

## Surface superconductivity in presence of corners

Michele Correggi<sup>\*,†</sup> and Emanuela L. Giacomelli<sup>\*,‡</sup>

*\*Dipartimento di Matematica “G. Castelnuovo”,  
 Università Degli Studi di Roma “La Sapienza”,  
 P.le Aldo Moro 5, 00185, Roma, Italy*

<sup>†</sup>*michele.correggi@gmail.com*

<sup>‡</sup>*giacomelli@mat.uniroma1.it*

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We consider an extreme type-II superconducting wire with non-smooth cross section, i.e. with one or more corners at the boundary, in the framework of the Ginzburg–Landau theory. We prove the existence of an interval of values of the applied field, where superconductivity is spread uniformly along the boundary of the sample. More precisely, the energy is not affected to leading order by the presence of corners and the modulus of the Ginzburg–Landau minimizer is approximately constant along the transversal direction. The critical fields delimiting this surface superconductivity regime coincide with the ones in absence of boundary singularities.

*Keywords:* Superconductivity; Ginzburg–Landau theory; corners.

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## 1. Introduction and Main Result

The success of the Ginzburg–Landau (GL) theory in predicting and modeling the response of a superconductor to an external applied field cannot be overlooked. Since the ‘50s, when it was proposed by Ginzburg and Landau [18] as a phenomenological theory to describe superconductivity near the critical temperature marking the transition to the superconducting state, several investigations have been performed within the GL theory from both the physical and mathematical view points (see, e.g., the monographs [17, 24] and references therein).

Here we are mainly concerned with the phenomenon of *surface superconductivity*, which was predicted and observed in experiments in the ‘60s [25]. (See also [21] for recent experimental data.) When the applied field becomes larger than some critical value (*second critical field*), the external magnetic field penetrates the sample and destroys superconductivity everywhere but in a narrow region close to the sample boundary.

The GL free energy of a type-II superconductor confined to an infinite cylinder of cross section  $\Omega \subset \mathbb{R}^2$  is given by

$$\begin{aligned} \mathcal{G}_\kappa^{\text{GL}}[\psi, \mathbf{A}] = & \int_\Omega \mathbf{d}\mathbf{r} \left\{ |(\nabla + ih_{\text{ex}}\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{1}{2}\kappa^2|\psi|^4 \right\} \\ & + h_{\text{ex}}^2 \int_{\mathbb{R}^2} \mathbf{d}\mathbf{r} |\text{curl}\mathbf{A} - 1|^2, \end{aligned} \quad (1.1)$$

where  $\psi : \Omega \rightarrow \mathbb{C}$  is the order parameter,  $h_{\text{ex}}\mathbf{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the vector potential, generating the induced magnetic field  $h_{\text{ex}}\text{curl}\mathbf{A}$ . The applied magnetic field is thus of uniform intensity  $h_{\text{ex}}$  along the superconducting wire. The parameter  $\kappa > 0$  is a physical quantity (inverse of the penetration depth), which is typical of the material and an extreme type-II superconductor is identified by the condition  $\kappa \gg 1$ . We also recall the physical meaning of the order parameter  $\psi$ , i.e.  $|\psi|^2$  is a measure of the relative density of superconducting Cooper pairs:  $|\psi|$  varies between 0 and 1 and  $|\psi| = 0$  in a certain region means that there are no Cooper pairs there and thus a loss of superconductivity, whereas if  $|\psi| = 1$  somewhere then all the electrons are superconducting in the region. The cases  $|\psi| \equiv 0$  and  $|\psi| \equiv 1$  everywhere in  $\Omega$  correspond to the *normal* and to the *perfectly superconducting states*, respectively.

In the present paper, we put the focus on *non-smooth* domains  $\Omega$ , i.e. more precisely we assume that  $\Omega$  is piecewise smooth but contains finitely many *corners*. This is expected to generate a richer physics for large applied magnetic fields (see below), since the corners can act as attractors for Cooper pairs and, before eventually disappearing, superconductivity might survive close to them.

The equilibrium state of the superconductor given by the configuration  $(\psi_\kappa^{\text{GL}}, \mathbf{A}_\kappa^{\text{GL}}) \in \mathcal{D}^{\text{GL}}$  is a minimizing pair for the functional  $\mathcal{G}_\kappa^{\text{GL}}$ . The minimal energy will be denoted by  $E_\kappa^{\text{GL}}$  and the minimization domain can be taken to be

$$\mathcal{D}^{\text{GL}} = \{(\psi, \mathbf{A}) \in H^1(\Omega) \times H_{\text{loc}}^1(\mathbb{R}^2; \mathbb{R}^2) \mid \text{curl}\mathbf{A} - 1 \in L^2(\mathbb{R}^2)\}. \quad (1.2)$$

Any critical point  $(\psi, \mathbf{A})$  of the GL functional solves, at least weakly, the GL equations

$$\begin{cases} -(\nabla + ih_{\text{ex}}\mathbf{A})^2 \psi = \kappa^2 (1 - |\psi|^2) \psi, & \text{in } \Omega, \\ -h_{\text{ex}}\nabla^\perp \text{curl}\mathbf{A} = \mathbf{j}_{\mathbf{A}}[\psi] \mathbf{1}_\Omega, & \text{in } \mathbb{R}^2, \\ \mathbf{n} \cdot (\nabla + ih_{\text{ex}}\mathbf{A}) \psi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where the last boundary condition has to be interpreted in trace sense and

$$\mathbf{j}_{\mathbf{A}}[\psi] := \frac{i}{2} [\psi (\nabla - ih_{\text{ex}}\mathbf{A}) \psi^* - \psi^* (\nabla + ih_{\text{ex}}\mathbf{A}) \psi], \quad (1.4)$$

is the superconducting current. Any minimizing configuration  $(\psi_\kappa^{\text{GL}}, \mathbf{A}_\kappa^{\text{GL}})$  is a weak solution of the above system too.

The existence of such a minimizing pair and the fact that it solves the GL equations (1.3) is a rather standard result [17, Chap. 15] and we skip the discussion here. Note however that, because of the presence of a non-smooth boundary, it is more convenient to consider a GL functional where the last term is integrated all over the plane  $\mathbb{R}^2$ . In the smooth boundary case, there is a one-to-one correspondence between minimizers on  $\mathbb{R}^2$  and minimizing configurations of the energy restricted to  $\Omega$  (see, e.g., [24, Proposition 3.4]) and therefore the two settings are perfectly equivalent. For non-smooth boundaries, such a correspondence is difficult to state because of possible boundary singularities of the solution. It is easy to prove however [17, Lemma 15.3.2] that

$$\text{curl}\mathbf{A} = 1, \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \quad (1.5)$$

for any  $\mathbf{A}$  weak solution of the GL equations (1.3).

As the intensity of the applied field increases, one observes three subsequent transitions in an extreme type-II superconductor with smooth boundary and correspondingly three critical values of  $h_{\text{ex}}$  can be identified, i.e. three *critical fields*:

- the first critical field  $H_{c1} \propto \log \kappa$  is associated to the nucleation of vortices;
- at  $H_{c2} = \kappa^2$  the transition from bulk to boundary superconductivity occurs;
- above  $H_{c3} = \Theta_0^{-1} \kappa^2 + O(1)$  the normal state is the unique minimizer of the GL functional and superconductivity is lost everywhere.

In presence of boundary singularities the above picture might require some modification (see [16, 23] for a discussion of the physics) and, although the first transition is untouched, being a bulk phenomenon, the two other transitions can be affected by the corners. More precisely it is known [6] that, under a suitable unproven conjecture about the ground state behavior of a magnetic Schrödinger operator in an infinite sector, the third critical field  $H_{c3}$  can be strictly larger than the one in the case of smooth domains, although of the same order  $\kappa^2$ . A more refined analysis reveals [6, Theorem 1.4] that, at least if one of the corners has an acute angle, the expected value of  $H_{c3}$  is

$$\mu(\alpha)^{-1} \kappa^2, \quad (1.6)$$

$\mu(\alpha)$  standing for the ground state of  $-(\nabla + \frac{1}{2}i\mathbf{x}^\perp)^2$ , i.e. a Schrödinger operator with uniform magnetic field of unit strength in an infinite sector of opening angle  $0 < \alpha < \pi/2$ , and with  $\alpha$  equal to the smallest angle of the corners. The unproven assumption is that for any acute angle  $0 < \alpha < \pi$ , the eigenvalue  $\mu(\alpha)$  is strictly smaller than  $\Theta_0$ . For the discussion of this and many more questions concerning the Schrödinger operator with magnetic field in a corner domain, we refer to [2–5].

When (and if)  $\mu(\alpha) > \Theta_0$  thus the third critical field changes value and, in fact, it is conjectured that another transition might take place below  $H_{c3}$ : as proven by usual Agmon estimate superconductivity can survive only at the boundary, if the field is above  $H_{c2}$ ; however one might distinguish between a boundary state distributed along the boundary and one concentrated near the corner of smallest opening angle. Under the aforementioned conjecture, this is indeed the structure of the GL ground state before the transition to the normal state takes place. As strongly suggested by the modified Agmon estimates proven in [6, Theorem 1.6] one should introduce an additional critical field  $H_{cc}$  so that

$$H_{c2} < H_{cc} \leq H_{c3} \tag{1.7}$$

marking such a phase transition from *boundary* to *corner superconductivity*. The order of magnitude of  $H_{cc}$  is clearly of order  $\kappa^2$ .

In the present paper, we prove that, if  $\Omega$  contains finitely many corners, there is uniform surface superconductivity for applied fields satisfying asymptotically

$$1 < \frac{h_{\text{ex}}}{\kappa^2} < \Theta_0^{-1}, \tag{1.8}$$

and the energy expansion is the same as in absence of corners, at least to leading order. As a consequence we infer the asymptotic estimate

$$\Theta_0^{-1} \leq \frac{H_{cc}}{\kappa^2} \leq \frac{H_{c3}}{\kappa^2}, \tag{1.9}$$

which must hold true also in presence of corners. The physics relevance of the result can be better understood in a comparison with the experimental data provided in [21]: the nucleation of surface superconductivity is observed in a very thin island of material, whose boundary is not completely determined and might as well contain corners. It would then be useful to know that the surface behavior is stable with respect to the presence of mild boundary singularities, as it is proven to some extent here.

Before going deeper in the discussion of our result, we first make a change of units, which is mostly convenient in the surface superconductivity regime, i.e. we assume that the applied field  $h_{\text{ex}}$  is of order  $\kappa^2$ ,

$$h_{\text{ex}} = b\kappa^2, \tag{1.10}$$

for some  $0 < b = \mathcal{O}(1)$ , and set

$$\varepsilon := b^{-\frac{1}{2}}\kappa^{-1} \ll 1. \tag{1.11}$$

We then study the asymptotic  $\varepsilon \rightarrow 0$  of the minimization of the GL functional, which in the new units reads

$$\begin{aligned} \mathcal{G}_\varepsilon^{\text{GL}}[\psi, \mathbf{A}] &= \int_\Omega d\mathbf{r} \left\{ \left| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi|^2 - |\psi|^4) \right\} \\ &\quad + \frac{1}{\varepsilon^4} \int_{\mathbb{R}^2} d\mathbf{r} |\text{curl} \mathbf{A} - 1|^2. \end{aligned} \quad (1.12)$$

We also set

$$E_\varepsilon^{\text{GL}} := \min_{(\psi, \mathbf{A}) \in \mathcal{D}^{\text{GL}}} \mathcal{G}_\varepsilon^{\text{GL}}[\psi, \mathbf{A}], \quad (1.13)$$

and denote by  $(\psi^{\text{GL}}, \mathbf{A}^{\text{GL}})$  any minimizing pair.

In the new units, the surface superconductivity regime coincides with the parameter region

$$1 < b < \Theta_0^{-1}, \quad (1.14)$$

at least for smooth boundaries. The key features of the surface superconductivity phase are listed below:

- the GL order parameter is concentrated in a thin boundary layer of thickness  $\simeq \varepsilon$  and exponentially small in  $\varepsilon$  in the bulk;
- the induced magnetic field is very close to the applied one, i.e.  $\text{curl} \mathbf{A} \simeq 1$ ;
- up to an appropriate choice of gauge and a mapping to boundary coordinates, the ground state of the theory can be approximated by an effective 1D energy functional in the direction normal to the boundary.

More precisely, in [11], it was proven<sup>a</sup> that, if  $\Omega$  is a smooth and simply connected domain, as  $\varepsilon \rightarrow 0$ ,

$$E_\varepsilon^{\text{GL}} = \frac{|\partial\Omega| E_\star^{\text{1D}}}{\varepsilon} + \mathcal{O}(1), \quad (1.15)$$

where  $E_\star^{\text{1D}}$  is the infimum of the functional

$$\mathcal{E}_\alpha^{\text{1D}}[f] := \int_0^{+\infty} dt \left\{ |\partial_t f|^2 + (t + \alpha)^2 f^2 - \frac{1}{2b} (2f^2 - f^4) \right\}, \quad (1.16)$$

both with respect to the real function  $f$  and to the number  $\alpha \in \mathbb{R}$ , i.e.

$$E_\star^{\text{1D}} := \inf_{\alpha \in \mathbb{R}} \inf_{f \in \mathcal{D}^{\text{1D}}} \mathcal{E}_\alpha^{\text{1D}}[f], \quad (1.17)$$

with  $\mathcal{D}^{\text{1D}} = \{f \in H^1(\mathbb{R}^+; \mathbb{R}^+) \mid tf(t) \in L^2(\mathbb{R}^+)\}$ . The corresponding minimizing  $\alpha$  will be denoted by  $\alpha_\star$ , while  $f_\star$  will stand for the function minimizing  $E_{\alpha_\star}^{\text{1D}}$ .

<sup>a</sup>Note that, for the sake of clarity, we have changed the notation with respect to [11] and denoted the 1D ground state energy  $E_\star^{\text{1D}}$ , instead of  $E_0^{\text{1D}}$  (the corresponding minimizer is then denoted by  $f_\star$  instead of  $f_0$ ). The reason is that the label 0 in the latter was actually referring to the dependence on the curvature, that we do not take into account in the present investigation.

The 1D functional (1.16) was known to provide a model problem for surface superconductivity since [22], but only in [11] (for disc-like samples) and [12] (see also [13] for further results in this direction) it was proven that  $f_\star$  is a good approximation of  $|\psi^{\text{GL}}|$ . The main idea is indeed that any minimizing GL order parameter has the structure

$$\psi^{\text{GL}}(\mathbf{r}) \simeq f_\star(\varepsilon t) \exp(-i\alpha_\star s) \exp\{i\phi_\varepsilon(s, t)\} \quad (1.18)$$

where  $(s, t)$  are rescaled tubular coordinates (tangent and normal coordinates, respectively, both rescaled by  $\varepsilon$ ) defined in a neighborhood of  $\partial\Omega$  (see, e.g., [17, Appendix F] and Sec. 2.3) and  $\phi_\varepsilon$  is a gauge phase factor which plays an important role in the estimate of the magnetic field  $\mathbf{A}^{\text{GL}}$ . By the above expression we mean that  $\psi^{\text{GL}}$  can be approximated by the right-hand side at least in a weak sense, e.g., in the  $L^2$  norm, once the gauge has been suitably chosen.

Before stating our main result, we specify the assumptions on the boundary domain: we consider a bounded domain  $\Omega \subset \mathbb{R}^2$  open and simply connected with Lipschitz boundary  $\partial\Omega$  such that the unit inward normal to the boundary,  $\boldsymbol{\nu}$ , is well defined on  $\partial\Omega$  with the possible exception of a finite number of points — the *corners* of  $\Omega$ . We refer to the monographs [15, 19] for a complete discussion of domains with non-smooth boundaries. More precisely we assume that the boundary  $\partial\Omega$  is a curvilinear polygon of class  $C^\infty$  in the following sense (see also [19, Definition 1.4.5.1]):

**Assumption 1.1 (Piecewise Smooth Boundary).** Let  $\Omega$  a bounded open subset of  $\mathbb{R}^2$ , we assume that  $\partial\Omega$  is a smooth curvilinear polygon, i.e. for every  $\mathbf{r} \in \Gamma$  there exists a neighborhood  $U$  of  $\mathbf{r}$  and a map  $\Phi : U \rightarrow \mathbb{R}^2$ , such that

- (1)  $\Phi$  is injective;
- (2)  $\Phi$  together with  $\Phi^{-1}$  (defined from  $\Phi(U)$ ) are smooth;
- (3) the region  $\Omega \cap U$  coincides with either  $\{\mathbf{r} \in \Omega \mid (\Phi(\mathbf{r}))_1 < 0\}$  or  $\{\mathbf{r} \in \Omega \mid (\Phi(\mathbf{r}))_2 < 0\}$  or  $\{\mathbf{r} \in \Omega \mid (\Phi(\mathbf{r}))_1 < 0, (\Phi(\mathbf{r}))_2 < 0\}$ , where  $(\Phi)_j$  stands for the  $j$ th component of  $\Phi$ .

**Assumption 1.2 (Boundary with Corners).** We assume that the set  $\Sigma$  of corners of  $\partial\Omega$ , i.e. the points where the normal  $\boldsymbol{\nu}$  does not exist, is nonempty but finite and we denote by  $\alpha_j$  the angle of the  $j$ th corner (measured towards the interior).

We are now able to formulate the main result proven in this article:

**Theorem 1.1 (GL Asymptotics).** *Let  $\Omega \subset \mathbb{R}^2$  be any bounded simply connected domain satisfying the above Assumptions 1.1 and 1.2. Then for any fixed*

$$1 < b < \Theta_0^{-1}, \quad (1.19)$$

as  $\varepsilon \rightarrow 0$ , it holds

$$E^{\text{GL}} = \frac{|\partial\Omega| E_\star^{1\text{D}}}{\varepsilon} + \mathcal{O}(|\log \varepsilon|^2), \quad (1.20)$$

and

$$\begin{aligned} & \left\| |\psi^{\text{GL}}(\mathbf{r})|^2 - f_\star^2(\text{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} \\ &= \mathcal{O}(\varepsilon |\log \varepsilon|) \ll \left\| f_\star^2(\text{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)}. \end{aligned} \quad (1.21)$$

**Remark 1.1 (Order Parameter Asymptotics).** The convergence stated in (1.21) implies that  $f_\star$  provides a good approximation of  $|\psi^{\text{GL}}|$  in the boundary layer, i.e. for  $\text{dist}(\mathbf{r}, \partial\Omega) \lesssim \varepsilon |\log \varepsilon|$ . At larger distance from the boundary both functions are indeed exponentially small in  $\varepsilon$  and their mass consequently very small. Note also that, if the condition (1.19) is satisfied,

$$\left\| f_\star^2(\text{dist}(\mathbf{r}, \partial\Omega)/\varepsilon) \right\|_{L^2(\Omega)} \geq c\sqrt{\varepsilon},$$

for some  $c > 0$ , as it immediately follows by observing that  $f_\star(t)$  is independent of  $\varepsilon$  and non-identically zero.

**Remark 1.2 (Limiting Regimes).** We explicitly chose not to address the limiting cases  $b \rightarrow 1^+$  or  $b \rightarrow \Theta_0^{-1}$ . In the former case an adaptation of the method might work (see also [11, Remark 2.1]), while in the latter the analysis is made much more complicate because of the interplay between corner and boundary confinements. In particular a much more detailed knowledge of the behavior of the linear problem, i.e. the ground state energy of the magnetic Schrödinger operator in a sector of angle  $\alpha$ , is needed, e.g., a proof of the conjecture discussed in [6].

The rest of the paper is devoted to the proof of the above result. Before proceeding however, we briefly comment on possible future perspectives. Inspired by the analysis contained in [12] one can indeed aim at capturing higher order corrections to the energy asymptotics (1.20) and isolate the curvature corrections to the energy. At this level, the presence of corners might give rise to an additional contribution to the energy of the same order of the curvature corrections. Compared to the smooth boundary case studied in [12] however, a proof of an  $L^\infty$  estimate of  $|\psi^{\text{GL}}|$  might be harder to obtain, due to the presence of boundary singularities. We plan to address this questions in a future work anyway.

## 2. Proofs

The strategy of the proof is very similar to the arguments contained in [11]. We sketch here the main steps.

The preliminary step, i.e. the *restriction to the boundary layer* (Sec. 2.2), is standard and described in details, e.g., in [17, Sec. 14.4]. The final outcome of this step is a functional restricted to a layer of width  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  along the boundary. The main ingredients are as usual Agmon estimates.

Another common step to both the upper and lower bound proofs, although applied to different magnetic potentials, is the *replacement of the magnetic field* (Sec. 2.4), as, e.g., in [17, Appendix F]. The presence of corners however calls

for suitable modifications, since this step is usually done by exploiting tubular coordinates, which are not defined closed to the boundary singularities. As we are going to see however, such a replacement is needed only in the smooth part of the boundary layer, where it can be done in a rather standard way by making a special choice of the gauge. In fact, the only required modification is an adapted definition of the gauge phase close to the corners.

The energy *upper bound* (Sec. 2.5) is then trivially obtained by testing the energy on a trial configuration with magnetic field equal to the external one and with order parameter  $\psi$  reproducing (1.18). The *lower bound* proof (completed in Sec. 2.6) is more involved and requires few more steps.

In order to extract the 1D effective energy from the GL asymptotics, *boundary coordinates* (Sec. 2.3) are needed, since, according to the expected behavior (1.18), the 1D energy is associated to the variation of  $|\psi^{\text{GL}}|$  along the normal to the boundary, while  $|\psi^{\text{GL}}|$  is approximately constant in the transversal direction. At this stage the non-smoothness of the boundary really affects the proof, because the use of tubular coordinates is clearly prevented near the corners. By a simple *a priori* estimate, however, we show that one can *drop the energy around corners*.

We are thus left with the energy contributions of the smooth pieces of the boundary layer. There we can pass to boundary coordinates and use the same trick, i.e. a suitable *integration by parts*, involved in the proofs of our earlier results [11, 12] and inspired by other works on rotating condensates (see, e.g., [10, 14, 7–9]). Since the region where we perform the integration by parts is not connected however, a naive application of the trick would generate unwanted boundary terms and therefore we will slightly modify the order parameter by introducing a partition of unity around the corners.

Finally, the key estimate to complete the lower bound proof is the *positivity of the cost function*, i.e. a pointwise estimate of an explicit 1D function depending on the effective 1D profile. This step is precisely the same as in [11] and is the only point in the proof where the condition  $1 < b < \Theta_0^{-1}$  comes explicitly into play. However, the assumption  $b > 1$  is required to apply Agmon estimates in the preliminaries, while the condition  $b < \Theta_0^{-1}$  is needed in order to ensure that the 1D minimizing profile is non-trivial.

We start the proof discussion by first recalling some properties of the 1D effective functional (1.16), which is going to provide the effective energy in the surface superconductivity regime.

## 2.1. Analysis of the effective 1D model

Given the functional (1.16), we denote by  $f_\alpha$  any minimizer for  $\alpha \in \mathbb{R}$  and by  $E_\alpha^{\text{1D}}$  the corresponding ground state energy,

$$E_\alpha^{\text{1D}} := \inf_{f \in H^1(\mathbb{R}^+)} \mathcal{E}_\alpha^{\text{1D}}[f],$$

with the convention that  $E_\star^{\text{1D}} = E_{\alpha_\star}^{\text{1D}} = \inf_{\alpha \in \mathbb{R}} E_\alpha^{\text{1D}}$  and  $f_\star = f_{\alpha_\star}$ .

We recall that the minimizer  $f_\star$  is non-trivial if and only if  $b < \Theta_0^{-1}$  [11, Proposition 3.2] and, in this case, it is a unique positive function, which is also monotonically decreasing and solves the variational equation [11, Proposition 3.1]

$$-f_\star'' + (t + \alpha_\star)^2 f_\star = \frac{1}{b}(1 - f_\star^2) f_\star, \quad (2.1)$$

with boundary condition  $f_\star'(0) = 0$ . The decay of  $f_\star$  can be estimated [11, Proposition 3.3]: for any  $b < \Theta_0^{-1}$ , there exist two constants  $0 < c, C < \infty$  such that

$$c \exp \left\{ -\frac{1}{2}(t + \sqrt{2})^2 \right\} \leq f_\star(t) \leq C \exp \left\{ -\frac{1}{2}(t + \alpha)^2 \right\} \quad (2.2)$$

for any  $t \in \mathbb{R}^+$ . As a direct consequence

$$f_\star(t) = \mathcal{O}(\varepsilon^\infty), \quad \text{for } t \geq c_0 |\log \varepsilon|, \quad (2.3)$$

for any constant  $c_0 > 0$ .

The *potential function* associated to  $f_\star$  is

$$F(t) := 2 \int_0^t d\xi f_\star^2(\xi)(\xi + \alpha_\star). \quad (2.4)$$

It can be shown that it is negative (in fact  $\alpha_\star < 0$ ) and vanishes both at  $t = 0$  and  $t = \infty$ . In this second case, the vanishing of  $F$  is implied by the optimality of  $\alpha_\star$ . The *cost function* that will naturally appear in our investigation is

$$K(t) := f_\star^2(t) + F(t). \quad (2.5)$$

In [11, Proposition 3.4] it is proven that

$$K(t) \geq 0, \quad \text{for any } t \in \mathbb{R}^+. \quad (2.6)$$

## 2.2. Restriction to the boundary layer

One of the key features of surface superconductivity is that the critical behavior survives only close to the boundary of the sample, due to the penetration of the external magnetic field. Mathematically this can be formulated by showing that the order parameter decays exponentially far from the boundary, which is the content of Agmon estimates. The presence of corners does not influence such a behavior.

We thus have that, if  $b > 1$ , any configuration  $(\psi, \mathbf{A})$  solving the GL equations satisfies the estimate

$$\begin{aligned} & \int_{\Omega} d\mathbf{r} \exp \left\{ \frac{C_A \text{dist}(\mathbf{r}, \partial\Omega)}{\varepsilon} \right\} \left\{ |\psi|^2 + \varepsilon^2 \left| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 \right\} \\ & \leq \int_{\{\text{dist}(\mathbf{r}, \partial\Omega) \leq \varepsilon\}} d\mathbf{r} |\psi|^2 = \mathcal{O}(\varepsilon), \end{aligned} \quad (2.7)$$

where  $C_A$  is a fixed constant and in the last estimate we have used the bound  $\|\psi\|_\infty \leq 1$ , which remains true also in the case of corners [17, Sec. 15.3.1]. For the

proof of normal Agmon estimates we refer to [6, Theorem 4.4]. Note that because of the exploding exponential factor, (2.7) implies that

$$\int_{\text{dist}(\mathbf{r}, \partial\Omega) \geq c_0 \varepsilon |\log \varepsilon|} d\mathbf{r} \left\{ |\psi|^2 + \varepsilon^2 \left| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 \right\} = \mathcal{O}(\varepsilon^{c_0 C_A + 1}), \quad (2.8)$$

which can be made smaller than any power of  $\varepsilon$  by taking the constant  $c_0$  arbitrarily large. Thanks to this fact and the analogous estimate on the gradient of  $\psi$ , one can easily drop from the energy the contribution of the region further from  $\partial\Omega$  than  $c_0 \varepsilon |\log \varepsilon|$  and, if we set

$$\mathcal{A}_\varepsilon := \{ \mathbf{r} \in \Omega \mid \text{dist}(\mathbf{r}, \partial\Omega) \leq c_0 \varepsilon |\log \varepsilon| \}, \quad (2.9)$$

it holds

$$E^{\text{GL}} = \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}} [\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] + \mathcal{O}(\varepsilon^\infty), \quad (2.10)$$

where  $\mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}}$  stands for

$$\begin{aligned} \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}} [\psi, \mathbf{A}] &= \int_{\mathcal{A}_\varepsilon} d\mathbf{r} \left\{ \left| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi|^2 - |\psi|^4) \right\} \\ &\quad + \frac{1}{\varepsilon^4} \int_{\Omega} d\mathbf{r} |\text{curl} \mathbf{A} - 1|^2. \end{aligned} \quad (2.11)$$

Before proceeding further we state some inequalities which will prove to be helpful in the rest of the paper (for a proof see, e.g., [17, Sec. 15.3] and reference therein): for any weak solution to the GL equations (1.3)

$$\|\psi\|_{L^\infty(\Omega)} \leq 1, \quad (2.12)$$

$$\left\| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon^{-1/2}), \quad (2.13)$$

$$\|\text{curl} \mathbf{A} - 1\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon \|\psi\|_{L^2(\Omega)} \|\psi\|_{L^4(\Omega)}. \quad (2.14)$$

Note that Agmon estimates (2.7) with (2.12) also imply

$$\|\psi\|_{L^p(\Omega)}^p \leq \|\psi\|_{L^2(\Omega)}^2 = \mathcal{O}(\varepsilon), \quad (2.15)$$

which can be used in (2.14) to get

$$\|\text{curl} \mathbf{A} - 1\|_{L^2(\mathbb{R}^2)} = \mathcal{O}(\varepsilon^{7/4}). \quad (2.16)$$

### 2.3. Boundary coordinates

The rest of the proof requires the definition of suitable boundary coordinates  $(\sigma, \tau)$ , e.g., to isolate the singular regions around the corners. As in [17, Appendix F] we introduce a parametrization of the boundary  $\partial\Omega$  denoted by  $\gamma(\sigma)$ ,  $\sigma \in [0, \partial\Omega)$ , which is piecewise smooth. At any point along the boundary, with the exception of corners  $\Sigma$ , the inward normal to the boundary  $\boldsymbol{\nu}(\sigma)$  is well defined and smooth.

The following map however

$$\mathbf{r}(\sigma, \tau) = \boldsymbol{\gamma}(\sigma) + \tau \boldsymbol{\nu}(\sigma), \quad (2.17)$$

with  $\tau = \text{dist}(\mathbf{r}, \partial\Omega)$ , defines a diffeomorphism only in a thin enough strip along  $\partial\Omega$  and far enough from  $\Sigma$ , e.g., in

$$\{\mathbf{r} \mid \text{dist}(\mathbf{r}, \partial\Omega) \leq \tau_0, \text{dist}(\mathbf{r}, \Sigma) \geq \tau'_0\},$$

with  $\tau_0$  small enough and  $\tau'_0$  suitably chosen (of the same order). In that region, we can also define the curvature  $\varsigma(\sigma)$  as

$$\boldsymbol{\gamma}''(\sigma) = \varsigma(\sigma) \boldsymbol{\nu}(\sigma).$$

In order to handle the singularities at corners, we need to define cells covering the region where tubular coordinates are ill defined. This occurs inside a cell of the form described in Fig. 1, where the lengths of the segments  $AC$  and  $BC$  gets smaller of the quantity  $\tau'_0$  mentioned above. Since we need to use boundary coordinates in a tubular neighborhood of  $\partial\Omega$  of width  $c_0\varepsilon|\log\varepsilon|$ , all the segments appearing in Fig. 1 can be taken to be of order  $\mathcal{O}(\varepsilon|\log\varepsilon|)$ , thanks to the independence of  $\partial\Omega$  of  $\varepsilon$ .

For the sake of concreteness, we pick a different boundary cell  $\mathcal{C}_j$ , as shown in Fig. 2, which covers the singular region and meet the following requirements: denoting by  $C$  the vertex of the corner, with coordinate  $\sigma_j$  along  $\partial\Omega$ , we pick two points  $A$  and  $B$  on  $\partial\Omega$ , in such a way that the lengths of the curves  $AC$  and  $BC$  along the boundary are equal and are given by

$$\sigma_j - \sigma_A = \sigma_B - \sigma_j = c_1\varepsilon|\log\varepsilon|. \quad (2.18)$$

where the constant  $c_1$  is chosen large enough (independent of  $\varepsilon$ ) in order to make the whole construction possible. The points  $D$  and  $E$  are then identified on the

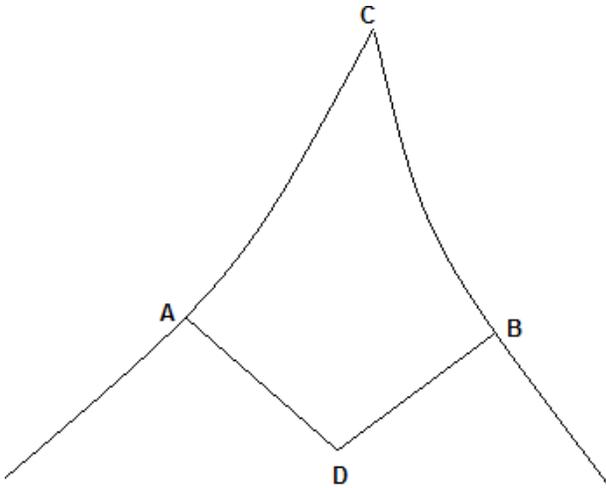


Fig. 1. Region where boundary coordinates might be ill defined.

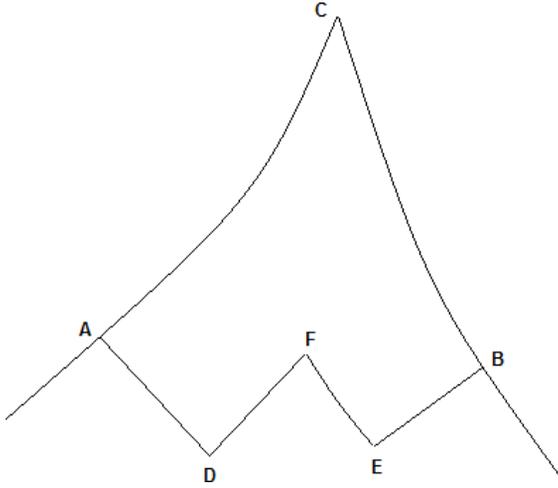


Fig. 2. Cell  $C_j$ .

straight lines orthogonal to  $\partial\Omega$  at  $A$  and  $B$ , respectively, by requiring that the lengths of the segments  $\overline{AD}$  and  $\overline{BE}$  equal  $c_0\varepsilon|\log\varepsilon|$ . Finally the point  $F$  is the unique one (for  $\varepsilon$  small enough) at the same distance  $c_0\varepsilon|\log\varepsilon|$  from both the left and right portion of  $\partial\Omega$ . In tubular coordinates we have

$$\begin{aligned}\overline{AD} &= \{\mathbf{r}(\sigma, \tau) \mid \sigma = \sigma_j - c_1\varepsilon|\log\varepsilon|, t \in [0, c_0\varepsilon|\log\varepsilon|]\}, \\ \overline{BE} &= \{\mathbf{r}(\sigma, \tau) \mid \sigma = \sigma_j + c_1\varepsilon|\log\varepsilon|, t \in [0, c_0\varepsilon|\log\varepsilon|]\},\end{aligned}\quad (2.19)$$

while

$$\begin{aligned}\overline{DF} &= \{\mathbf{r}(\sigma, \tau) \mid \sigma \in [\sigma_j - c_1\varepsilon|\log\varepsilon|, \sigma_j - \delta_-\varepsilon|\log\varepsilon|], t = c_0\varepsilon|\log\varepsilon|\}, \\ \overline{EF} &= \{\mathbf{r}(\sigma, \tau) \mid \sigma \in [\sigma_j + \delta_+\varepsilon|\log\varepsilon|, \sigma_j + c_1\varepsilon|\log\varepsilon|], t = c_0\varepsilon|\log\varepsilon|\},\end{aligned}\quad (2.20)$$

for some  $0 < \delta_{\pm} < c_1$ . The final shape of the cell is described in Fig. 2 for an acute angle. The definition requires no adaptation however for obtuse angles.

The most important property of the corner cells is that they carry a little amount of energy, which allow us to discard them in the estimate of both the upper and lower bounds to the GL energy. This is directly implied by the smallness of those cells, whose area is  $\mathcal{O}(\varepsilon^2|\log\varepsilon|^2)$ . At a heuristic level indeed the energy density is of order  $\varepsilon^{-2}$ , at least close to  $\partial\Omega$ , and therefore the energy contained in  $C_j$  is expected to be  $\mathcal{O}(|\log\varepsilon|^2)$ , i.e. of the same order to the error term in (1.20).

In the rest of the proof, we will also use the rescaled boundary coordinates defined in terms of  $(\sigma, \tau)$  as

$$s = \frac{\sigma}{\varepsilon}, \quad t = \frac{\tau}{\varepsilon},\quad (2.21)$$

so that, with some abuse of notation,<sup>b</sup>

$$\mathcal{A}_\varepsilon = \left\{ (s, t) \mid s \in \left[ 0, \frac{|\partial\Omega|}{\varepsilon} \right], t \in [0, c_0 |\log \varepsilon|] \right\}. \quad (2.22)$$

The boundary layer without the corner cells is called  $\mathcal{A}_{\text{cut}}$ , i.e.

$$\mathcal{A}_{\text{cut}} := \mathcal{A}_\varepsilon \setminus \bigcup_{j=1}^N \mathcal{C}_j. \quad (2.23)$$

We also denote by  $\mathbf{n}(s) := \boldsymbol{\nu}(\varepsilon s)$  and  $k(s) := \zeta(\varepsilon s)$  the normal to  $\partial\Omega$  and the curvature in the new coordinates, respectively.

#### 2.4. Replacement of the magnetic field

In the surface superconductivity regime, the induced magnetic field is actually very close to the applied one. This is typical of this regime and is also the reason why the last term in the GL functional (1.12) is never taken into account, since its contribution is always subleading. Hence the induced field is almost uniform and its strength is approximately 1. There are, however, many magnetic potentials  $\mathbf{A}$  generating such a field and it is useful to exploit gauge invariance to select the most convenient one. Here we discuss how it can be done.

We pick a magnetic potential  $\mathbf{A}$  such that

$$\nabla \cdot \mathbf{A} = 0, \quad \text{in } \Omega, \quad (2.24)$$

i.e. in the Coulomb gauge, and the following inequality holds true (see [17, Eqs. (15.18) and (15.19)]):

$$\|\mathbf{A} - \mathbf{F}\|_{W^{1,2}(\Omega)} \leq C \|\text{curl} \mathbf{A} - 1\|_{L^2(\mathbb{R}^2)}, \quad (2.25)$$

where  $\mathbf{F}(\mathbf{r}) := \frac{1}{2}(-y, x)$ . Since  $\mathbf{A}$  is defined up to an additive constant, this last inequality holds true only if the value of the constant is suitably chosen. We assume that all the magnetic potentials in the sequel are taken in such a way that (2.25) applies.

In smooth domains (2.24) is accompanied by the boundary condition  $\mathbf{A} \cdot \mathbf{n} = 0$  on  $\partial\Omega$ . This is clearly not possible in presence of corners, because of the jumps of  $\mathbf{n}$ : let us stress that at corners  $\mathbf{n}$  is discontinuous but it remains uniformly bounded all over the boundary, so that, for instance,

$$\int_0^{\frac{|\partial\Omega|}{\varepsilon}} ds \mathbf{A}(\mathbf{r}(\varepsilon s, 0)) \cdot \mathbf{n}(s) = 0,$$

i.e. the function  $\mathbf{A} \cdot \mathbf{n}$  is integrable. Moreover Stokes formula and elliptic regularity (see, e.g., [19]) implies that the boundary condition is in fact satisfied in trace sense,

<sup>b</sup>For the sake of simplicity in the notation, we do not change symbol to denote a given set in the rescaled variables, e.g.,  $\mathcal{A}_\varepsilon$  will stand for the boundary layer, either in the  $\mathbf{r}$  or in the  $(s, t)$  variables.

i.e. almost everywhere. In particular it must hold true far from corners, where the normal to the boundary is well defined, e.g.,

$$\mathbf{A}(\mathbf{r}(\varepsilon s, 0)) \cdot \mathbf{n}(s) = 0, \quad \text{for any } |s - s_j| \geq c_1 |\log \varepsilon|. \quad (2.26)$$

Now the key idea for the replacement, described, e.g., in [17, Appendix F] (see also [11, Sec. 4.1] and [12, Proof of Lemma 4]), is that, since at the boundary the normal component of  $\mathbf{A}$  vanishes, close enough to  $\partial\Omega$ , it is possible to find a differentiable function  $\phi_{\mathbf{A}}(s, t)$  such that  $\mathbf{A} - \nabla\phi_{\mathbf{A}}(s, t) = f(s, t)\mathbf{e}_s$ , i.e. the field remains purely tangential close enough to  $\partial\Omega$ . In addition since  $\text{curl}\mathbf{A}$  is approximately 1, the function  $f(s, t)$  is close to  $-t + o(1)$ . Using a gauge transformation built on  $\phi_{\mathbf{A}}$ , one can then replace  $\mathbf{A}$  with a magnetic potential which is of the form described above.

We do not repeat all the details of the procedure but we only comment on the modifications induced by the presence of corners. The first step, i.e. the gauge transformation is essentially the same as for smooth domains: thanks to the integrability of both  $\gamma'(s)$  and  $\mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))$  along  $\partial\Omega$ , which is guaranteed in the first case by the boundedness of  $\gamma'(s)$  and in the second one by Sobolev trace theorem, which gives  $\mathbf{A} \in H^{3/2}(\partial\Omega; \mathbb{R}^2)$ , we can set for any  $(s, t) \in \mathcal{A}_{\text{cut}}$

$$\begin{aligned} \phi_{\mathbf{A}}(s, t) := & -\frac{1}{\varepsilon} \int_0^t d\eta \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon \eta)) \cdot \mathbf{n}(s) \\ & + \frac{1}{\varepsilon} \int_0^s d\xi \mathbf{A}(\mathbf{r}(\varepsilon \xi, 0)) \cdot \gamma'(\varepsilon s) - \varepsilon \delta_\varepsilon s, \end{aligned} \quad (2.27)$$

with<sup>c</sup>

$$\delta_\varepsilon(s, t) = \frac{1}{\varepsilon^2 |\partial\Omega|} \int_{\Omega} d\mathbf{r} \text{curl}\mathbf{A} - \frac{2\pi}{|\partial\Omega|} \left[ \frac{1}{2\pi\varepsilon^2} \int_{\Omega} d\mathbf{r} \text{curl}\mathbf{A} \right]. \quad (2.28)$$

Note that the second term in the expression above is well defined even if  $s$  gets close to the corner points  $\Sigma$ , because  $\gamma'$  is by assumption an integrable function. On the other hand, for any  $(s, t) \in \mathcal{A}_{\text{cut}}^c$ , we can set

$$\begin{aligned} \phi_{\mathbf{A}}(s, t) := & -\frac{1}{\varepsilon} \int_0^t d\eta \chi(s) \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon \eta)) \cdot \mathbf{n}(s) \\ & + \frac{1}{\varepsilon} \int_0^s d\xi \mathbf{A}(\mathbf{r}(\varepsilon \xi, 0)) \cdot \gamma'(\varepsilon s) - \varepsilon \delta_\varepsilon s, \end{aligned} \quad (2.29)$$

where the support of the smooth cut-off function  $\chi$  is contained in the intervals  $[s_j - c_1 |\log \varepsilon|, s - c_1 |\log \varepsilon| + \delta]$  and  $[s_j + c_1 |\log \varepsilon| - \delta, s_j + c_1 |\log \varepsilon|]$ , for some  $\delta$  of order 1, and

$$\begin{aligned} \chi(s_j \pm c_1 |\log \varepsilon| \mp \delta) &= 0, \\ \chi(s_j \pm c_1 |\log \varepsilon|) &= 1. \end{aligned}$$

<sup>c</sup>We denote by  $[\cdot]$  the integer part. Note the missing factors  $2\pi$  in the definitions [11, Eq. (4.8)] and [12, Eq. (5.4)].

It is easy to verify that for any  $(s, t) \in \mathcal{A}_\varepsilon$ , one has

$$\phi_{\mathbf{A}}(s + n|\partial\Omega|, t) = \phi_{\mathbf{A}}(s, t) + 2\pi n, \quad \text{for any } n \in \mathbb{Z}.$$

The change of coordinates  $\mathbf{r} \rightarrow (\varepsilon s, \varepsilon t)$  in  $\mathcal{A}_{\text{cut}}$  and the simultaneous gauge transformation then yields for any  $\Psi \in H^1(\mathcal{A}_\varepsilon)$

$$\begin{aligned} \mathcal{G}_{\varepsilon, \mathcal{A}_{\text{cut}}}^{\text{GL}}[\Psi, \mathbf{A}] &= (1 + \mathcal{O}(\varepsilon |\log \varepsilon|)) \int_{\mathcal{A}_{\text{cut}}} ds dt \left\{ |\partial_t \psi|^2 + |(\partial_s + ia(s, t))\psi|^2 \right. \\ &\quad \left. - \frac{1}{2b} [2|\psi|^2 - |\psi|^4] \right\} + \frac{1}{\varepsilon^4} \int_{\Omega} d\mathbf{r} |\text{curl} \mathbf{A} - 1|^2, \end{aligned} \quad (2.30)$$

where the prefactor  $1 + \mathcal{O}(\varepsilon |\log \varepsilon|)$  is due to an estimate of the jacobian  $1 - \varepsilon k(s)t$  of the change of coordinates  $\mathbf{r} \rightarrow (s, t)$  induced by the diffeomorphism  $\mathbf{r}(\varepsilon s, \varepsilon t)$ , where  $k(s)$  is the curvature of the boundary in the rescaled coordinates. Of course,  $k(s)$  is not defined at corners but admits left and right values and

$$\|k\|_{L^\infty(\partial\Omega)} \leq C. \quad (2.31)$$

Moreover  $\psi$  in the expression above is

$$\psi(s, t) := \Psi(\mathbf{r}(\varepsilon s, \varepsilon t)) e^{-i\phi_{\mathbf{A}}(s, t)}. \quad (2.32)$$

Finally,

$$a_{\mathbf{A}}(s, t) = (1 - \varepsilon k(s)t) \frac{\boldsymbol{\gamma}'(\varepsilon s) \cdot \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon t))}{\varepsilon} - \partial_s \phi_{\mathbf{A}}. \quad (2.33)$$

Note that we have left untouched the last term of the GL functional, because it will be treated in different ways in the upper and lower bound proofs.

**Lemma 2.1.** *Let  $\mathbf{A}$  be such that (2.16) and (2.24) are satisfied, then for any  $c_0 > 0$ ,*

$$\|a_{\mathbf{A}}(s, t) + t\|_{L^2(\mathcal{A}_{\text{cut}})}^2 = \mathcal{O}(\varepsilon |\log \varepsilon|^3). \quad (2.34)$$

**Proof.** Let  $(s, t) \in \mathcal{A}_{\text{cut}}$ , we first observe that

$$a_{\mathbf{A}}(s, 0) = \varepsilon \delta_\varepsilon = \mathcal{O}(\varepsilon),$$

since  $|\delta_\varepsilon| \leq 1$ . The definition of  $a_{\mathbf{A}}$  and the vanishing of the normal component also implies that

$$\partial_t a_{\mathbf{A}}(s, t) = -(1 - \varepsilon k(s)t)(\text{curl} \mathbf{A})(\mathbf{r}(\varepsilon s, \varepsilon t))$$

and therefore

$$\begin{aligned} a_{\mathbf{A}}(s, t) &= \varepsilon \delta_\varepsilon - (1 + \mathcal{O}(\varepsilon |\log \varepsilon|)) \int_0^t d\eta \text{curl} \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon \eta)) \\ &= -t - \int_0^t d\eta [\text{curl} \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon \eta)) - 1] + \mathcal{O}(\varepsilon |\log \varepsilon|). \end{aligned} \quad (2.35)$$

On the other hand, we can estimate

$$\int_0^t d\eta |\text{curl} \mathbf{A}(\mathbf{r}(\varepsilon s, \varepsilon \eta)) - 1| \leq C |\log \varepsilon|^{\frac{1}{2}} \left[ \int_0^{c_0 |\log \varepsilon|} dt |\text{curl} \mathbf{A} - 1|^2 \right]^{1/2}$$

which yields

$$\|a_{\mathbf{A}} + t\|_{L^2(\mathcal{A}_{\text{cut}})}^2 \leq C|\log \varepsilon|^2 \|\text{curl} \mathbf{A} - 1\|_{L^2(\mathcal{A}_{\text{cut}})}^2 + \mathcal{O}(\varepsilon|\log \varepsilon|^3),$$

and therefore the result, via (2.16).  $\square$

## 2.5. Upper bound

We are now able to prove the upper bound to the GL energy.

**Proposition 2.1 (Energy Upper Bound).** *Let  $1 < b < \Theta_0^{-1}$  and  $\varepsilon$  be small enough. Then it holds*

$$E^{\text{GL}} \leq \frac{|\partial\Omega|E_{\star}^{1\text{D}}}{\varepsilon} + \mathcal{O}(|\log \varepsilon|). \quad (2.36)$$

**Proof.** As usual, we prove the result by evaluating the GL energy on a trial state having the expected physical features. This trial state has to be concentrated near the boundary of the sample and its modulus must be approximately constant in the transversal direction. However, since we can not use boundary coordinates at corners, we impose that the function vanishes in a suitable neighborhood of  $\Sigma$ . We will label corners in  $\Sigma$  with their coordinate along the boundary, i.e.

$$\Sigma = \{(s, 0) \mid s = s_j, j = 1, \dots, N\}. \quad (2.37)$$

We thus introduce a cut-off function  $\chi \in C_0^\infty$ , such that  $0 \leq \chi \leq 1$ . Its role is to cut the region close to  $\Sigma$ . For any  $(s, t) \in \mathcal{A}_\varepsilon$ , we require

$$\chi(s) := \begin{cases} 0, & \text{if } |s - s_j| \leq c_1|\log \varepsilon|, \quad \text{for some } j = 1, \dots, N, \\ 1, & \text{if } |s - s_j| \geq 2c_1|\log \varepsilon|, \quad \text{for all } j = 1, \dots, N. \end{cases} \quad (2.38)$$

The transition from 0 to 1 occurs in a one-dimensional region of length  $c_1|\log \varepsilon|$  and therefore we can always assume that

$$|\chi'| = \mathcal{O}(|\log \varepsilon|^{-1}). \quad (2.39)$$

We also define (note that we can not define  $\mathcal{D}_j$  using boundary coordinates in the interior because there they are ill defined)

$$\mathcal{D}_j := \{(s, t) \in \mathcal{A}_\varepsilon \mid |s - s_j| \geq 2c_1|\log \varepsilon|\}^c, \quad (2.40)$$

i.e.  $(\cup_j \mathcal{D}_j)^c$  is the region where  $\chi = 1$ .

It remains to choose the magnetic potential to complete the test configuration: we thus denote by  $\mathbf{F}$  any magnetic potential such that  $\nabla \cdot \mathbf{F} = 0$  and  $\text{curl} \mathbf{F} = 1$  in  $\Omega$ . Our trial state is then

$$(\psi_{\text{trial}}, \mathbf{F}), \quad (2.41)$$

where

$$\psi_{\text{trial}}(s, t) = \chi(s)f_{\star}(t)e^{-i\alpha_{\star}s}e^{i\phi_{\mathbf{F}}(s,t)}, \quad (2.42)$$

with  $\phi_{\mathbf{A}}$  the gauge phase (2.27). Note that  $\alpha_*$  is not necessarily an integer and therefore  $\psi_{\text{trial}}$  might be a multi-valued function, but this does not harm the result since we are here interested in proving an upper bound.

The order parameter decays exponentially as  $t \rightarrow \infty$ , thanks to the pointwise estimate (2.2), and therefore

$$\psi_{\text{trial}}(\mathbf{r}) = \mathcal{O}(\varepsilon^\infty), \quad \text{for } \mathbf{r} \in \mathcal{A}_\varepsilon^c.$$

Hence, as for (2.10), we have

$$\mathcal{G}_\varepsilon^{\text{GL}}[\psi_{\text{trial}}, \mathbf{F}] = \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}}[\psi_{\text{trial}}, \mathbf{F}] + \mathcal{O}(\varepsilon^\infty). \quad (2.43)$$

Also, since  $\text{curl} \mathbf{F} = 1$ , the quantity we have to estimate is actually

$$\begin{aligned} \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}}[\psi_{\text{trial}}, \mathbf{F}] &= \int_{\mathcal{A}_\varepsilon} d\mathbf{r} \left\{ \left| \left( \nabla + i \frac{\mathbf{F}}{\varepsilon^2} \right) \psi_{\text{trial}} \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi_{\text{trial}}|^2 - |\psi_{\text{trial}}|^4) \right\} \\ &= (1 + \mathcal{O}(\varepsilon |\log \varepsilon|)) \int_{\mathcal{A}_\varepsilon} ds dt \left\{ \chi^2 |\partial_t f_\star|^2 + f_\star^2 |\partial_s \chi|^2 \right. \\ &\quad \left. + |a_{\mathbf{F}}(s, t) - \alpha_\star|^2 f_\star^2 \chi^2 - \frac{1}{2b} [2\chi^2 f_\star^2 - \chi^4 f_\star^4] \right\} \end{aligned} \quad (2.44)$$

with  $a_{\mathbf{F}}$  defined in (2.33). For an upper bound we can replace  $\chi$  with 1 wherever  $\chi \neq 0$ , i.e.

$$\begin{aligned} \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}}[\psi_{\text{trial}}, \mathbf{F}] &\leq \int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} ds dt \left\{ f_\star'^2 + (t + \alpha_\star)^2 f_\star^2 - \frac{1}{2b} [2f_\star^2 - f_\star^4] \right\} \\ &\quad + \mathcal{O}(|\log \varepsilon|), \end{aligned} \quad (2.45)$$

where the remainder is due to various factors, we are going to explain. We first notice that

$$\begin{aligned} &\int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} ds dt \left\{ f_\star'^2 + (t + \alpha_\star)^2 f_\star^2 - \frac{1}{2b} [2f_\star^2 - f_\star^4] \right\} \\ &= \int_{\frac{\partial \Omega_{\text{cut}}}{\varepsilon}} ds \int_0^{c_0 |\log \varepsilon|} dt \left\{ f_\star'^2 + (t + \alpha_\star)^2 f_\star^2 - \frac{1}{2b} [2f_\star^2 - f_\star^4] \right\} \\ &= \frac{|\partial \Omega_{\text{cut}}| E_\star^{1\text{D}}}{\varepsilon} \leq \frac{|\partial \Omega| E_\star^{1\text{D}}}{\varepsilon} + \mathcal{O}(|\log \varepsilon|), \end{aligned} \quad (2.46)$$

where  $\partial \Omega_{\text{cut}} = \partial \Omega \cap \partial \mathcal{A}_{\text{cut}}$ , so that the prefactor  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  in (2.44) generates an error of order  $|\log \varepsilon|$ . Moreover

$$\begin{aligned} &-\frac{1}{2b} \int_{\mathcal{A}_\varepsilon} ds dt [2\chi^2 f_\star^2 - \chi^4 f_\star^4] \\ &\leq -\frac{1}{2b} \int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} ds dt [2f_\star^2 - f_\star^4] + C \int_{\mathcal{A}_\varepsilon} ds dt (1 - \chi^2) f_\star^2 \\ &= -\frac{1}{2b} \int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} ds dt [2f_\star^2 - f_\star^4] + \mathcal{O}(|\log \varepsilon|), \end{aligned}$$

and using that the exponential decay of  $f_\star$  given by (2.2),

$$\begin{aligned} \int_{\mathcal{A}_\varepsilon} \text{dsdt} |a_{\mathbf{F}}(s, t) - \alpha_\star|^2 f_\star^2 \chi^2 &\leq \int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} \text{dsdt} (t + \alpha_\star)^2 f_\star^2 \\ &+ C \|a_{\mathbf{F}}(s, t) + t\|_{L^2(\mathcal{A}_{\text{cut}})} \|(t + \alpha_\star) f_\star\|_{L^2(\mathcal{A}_\varepsilon)} + C \|a_{\mathbf{F}}(s, t) + t\|_{L^2(\mathcal{A}_{\text{cut}})}^2 \\ &= \int_{\mathcal{A}_\varepsilon \setminus \cup_j \mathcal{C}_j} \text{dsdt} (t + \alpha_\star)^2 f_\star^2 + \mathcal{O}(1). \end{aligned}$$

Finally the kinetic energy of the cut-off is bounded as

$$\int_{\mathcal{A}_\varepsilon} \text{dsdt} f_\star^2 |\partial_s \chi|^2 \leq C |\log \varepsilon|^{-2} |\cup_j (\mathcal{D}_j \setminus \mathcal{C}_j)| = \mathcal{O}(1),$$

thanks to the assumption (2.39).

Combining (2.45) and (2.46) the energy upper bound is proven.  $\square$

## 2.6. Lower bound and completion of the proof

The first step towards a proof of a suitable lower bound is the control of the energy contributions of corners. This is however rather easy to obtain since

$$\begin{aligned} \mathcal{G}_{\varepsilon, \mathcal{A}_\varepsilon}^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] &\geq \mathcal{G}_{\varepsilon, \mathcal{A}_{\text{cut}}}^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] + \mathcal{O}(|\log \varepsilon|^2) \\ &\geq \mathcal{F}_\varepsilon^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] + \mathcal{O}(|\log \varepsilon|^2), \end{aligned} \quad (2.47)$$

where  $\mathcal{A}_{\text{cut}}$  is given in (2.23),

$$\mathcal{F}_\varepsilon^{\text{GL}}[\psi, \mathbf{A}] := \int_{\mathcal{A}_{\text{cut}}} \text{dr} \left\{ \left| \left( \nabla + i \frac{\mathbf{A}}{\varepsilon^2} \right) \psi \right|^2 - \frac{1}{2b\varepsilon^2} (2|\psi|^2 - |\psi|^4) \right\} \quad (2.48)$$

and the remainder is produced by the only non-positive term of the GL functional i.e.

$$-\frac{1}{b\varepsilon^2} \int_{\mathcal{A}_\varepsilon} \text{dr} |\psi^{\text{GL}}|^2 \geq -\frac{1}{b\varepsilon^2} \int_{\mathcal{A}_{\text{cut}}} \text{dr} |\psi^{\text{GL}}|^2 - C |\log \varepsilon|^2,$$

by (2.12) and the area estimate  $|\mathcal{C}_j| = \mathcal{O}(\varepsilon^2 |\log \varepsilon|^2)$ .

The main result concerning the energy lower bound is the following

**Proposition 2.2 (Energy Lower Bound).** *If  $1 < b < \Theta_0^{-1}$  as  $\varepsilon \rightarrow 0$  then*

$$E^{\text{GL}} \geq \frac{|\partial \Omega| E_\star^{1\text{D}}}{\varepsilon} + \mathcal{O}(|\log \varepsilon|^2). \quad (2.49)$$

The core of the proof is the same argument used in the proof of [11, Proposition 4.2], but in order to get to the spot where one can apply the estimate of the cost function, few adjustments are in order. First of all, the functional  $\mathcal{F}_\varepsilon^{\text{GL}}$  is given on the right domain  $\mathcal{A}_{\text{cut}}$ , where we can pass to tubular coordinates and replace the vector potential  $\mathbf{A}^{\text{GL}}$ , but because  $\mathcal{A}_{\text{cut}}$  is made of several connected components, we need to suitably modify  $\psi^{\text{GL}}$  and impose its vanishing at the normal and inner boundaries of those sets. The reason of this will become clear only

at a later stage of the proof: thanks to so-imposed Dirichlet boundary conditions, several unwanted boundary terms will vanish when integrating by parts the current term in the functional.

We sum up this preliminary steps in the following:

**Lemma 2.2.** *As  $\varepsilon \rightarrow 0$*

$$\mathcal{F}_\varepsilon^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] \geq \mathcal{F}[\psi] + \mathcal{O}(|\log \varepsilon|^2), \quad (2.50)$$

where

$$\mathcal{F}[\psi] := \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t \left\{ |\partial_t \psi|^2 + |(\partial_s - it) \psi|^2 - \frac{1}{2b} [2|\psi|^2 - |\psi|^4] \right\}, \quad (2.51)$$

and, denoting  $\tilde{\mathcal{A}}_\varepsilon := \{(s, t) \in \mathcal{A}_\varepsilon \mid t \leq c_0 |\log \varepsilon| - \varepsilon\}$ ,

$$\psi(s, t) := \begin{cases} \psi^{\text{GL}}(\mathbf{r}(\varepsilon s, \varepsilon t)) \exp \{-i\phi_{\mathbf{A}^{\text{GL}}}(s, t)\}, & \text{in } \tilde{\mathcal{A}}_\varepsilon \setminus \cup_j \mathcal{D}_j, \\ 0, & \text{for } s = s_j \pm c_1 |\log \varepsilon|, \\ 0, & \text{for } t = c_0 |\log \varepsilon|, \end{cases} \quad (2.52)$$

and  $|\psi| \leq |\psi^{\text{GL}}|$  everywhere.

**Proof.** We first pass to boundary coordinates and simultaneously replace the magnetic potential  $\mathbf{A}^{\text{GL}}$  as described in Sec. 2.4: this leads to the lower bound

$$\begin{aligned} \mathcal{F}_\varepsilon^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] &\geq \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t \left\{ |\partial_t \tilde{\psi}|^2 + |(\partial_s + ia_{\mathbf{A}^{\text{GL}}}(s, t)) \tilde{\psi}|^2 \right. \\ &\quad \left. - \frac{1}{2b} [2|\tilde{\psi}|^2 - |\tilde{\psi}|^4] \right\} + \mathcal{O}(|\log \varepsilon|), \end{aligned} \quad (2.53)$$

where  $\tilde{\psi}(s, t) = \psi^{\text{GL}}(\mathbf{r}(\varepsilon s, \varepsilon t)) \exp \{-i\phi_{\mathbf{A}^{\text{GL}}}(s, t)\}$  and  $a_{\mathbf{A}^{\text{GL}}}$  is given in (2.33). The remainder  $\mathcal{O}(|\log \varepsilon|)$  is the product of the prefactor  $\mathcal{O}(\varepsilon |\log \varepsilon|)$  due to the jacobian of the coordinate transformation times the negative term proportional to the  $L^2$  norm of  $\tilde{\psi}$ .

Next, acting as in [11, Eq. (4.26)] and using Lemma 2.1, we can estimate for any  $\delta > 0$ ,

$$\begin{aligned} &\int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t \left[ |(\partial_s + ia_{\mathbf{A}^{\text{GL}}}(s, t)) \tilde{\psi}|^2 - |(\partial_s - it) \tilde{\psi}|^2 \right] \\ &\geq -\delta \left\| \left( \nabla + i \frac{\mathbf{A}^{\text{GL}}}{\varepsilon^2} \right) \psi^{\text{GL}} \right\|_{L^2(\Omega)}^2 - \left( \frac{1}{\delta} + 1 \right) \|a_{\mathbf{A}^{\text{GL}}}(s, t) + t\|_{L^2(\mathcal{A}_{\text{cut}})}^2 \\ &\geq -C [\delta \varepsilon^{-1} + \delta^{-1} \varepsilon^2 |\log \varepsilon|^2] \geq -C \sqrt{\varepsilon} |\log \varepsilon|, \end{aligned}$$

after an optimization over  $\delta$ . Hence we get from (2.53)

$$\mathcal{F}_\varepsilon^{\text{GL}}[\psi^{\text{GL}}, \mathbf{A}^{\text{GL}}] \geq \mathcal{F}[\tilde{\psi}] + \mathcal{O}(|\log \varepsilon|). \quad (2.54)$$

To impose the boundary conditions at the normal and inner boundaries of  $\mathcal{A}_{\text{cut}}$ , we use two different partition of unity, i.e. two pairs of smooth functions  $0 \leq \chi_i, \eta_i \leq 1$ ,  $i = 1, 2$ , such that  $\chi_i^2 + \eta_i^2 = 1$  and

$$\chi_1 = \chi_1(s) = \begin{cases} 1, & \text{in } \mathcal{A}_\varepsilon \setminus \cup_j \mathcal{D}_j, \\ 0, & \text{in } \cup_j \mathcal{C}_j, \end{cases} \quad (2.55)$$

$$\chi_2 = \chi_2(t) = \begin{cases} 1, & \text{for } t \in [0, c_0 |\log \varepsilon| - \varepsilon], \\ 0, & \text{for } t = c_0 |\log \varepsilon|. \end{cases} \quad (2.56)$$

Given the size where  $\chi_i, \eta_i$  are not constant, we can assume the following estimates to hold true

$$|\nabla \chi_1| = \mathcal{O}(|\log \varepsilon|^{-1}), \quad |\nabla \eta_1| = \mathcal{O}(|\log \varepsilon|^{-1}), \quad (2.57)$$

$$|\nabla \chi_2| = \mathcal{O}(\varepsilon^{-1}), \quad |\nabla \eta_2| = \mathcal{O}(\varepsilon^{-1}). \quad (2.58)$$

The IMS formula then yields

$$\begin{aligned} \mathcal{F}[\tilde{\psi}] &\geq \mathcal{F}[\chi_1 \chi_2 \tilde{\psi}] - \int_{\mathcal{A}_\varepsilon} \text{d}s \text{d}t \left[ \chi_1'^2 + \eta_1'^2 \right] |\tilde{\psi}|^2 \\ &\quad - \int_{\mathcal{A}_\varepsilon} \text{d}s \text{d}t \left[ \chi_2'^2 + \eta_2'^2 \right] |\tilde{\psi}|^2 + \mathcal{O}(|\log \varepsilon|^2), \end{aligned} \quad (2.59)$$

where we have estimated

$$\begin{aligned} &\mathcal{F}[\eta_1 \chi_2 \tilde{\psi}] + \mathcal{F}[\eta_2 \chi_1 \tilde{\psi}] + \mathcal{F}[\eta_1 \eta_2 \tilde{\psi}] \\ &\geq -\frac{1}{b} \int_{\mathcal{A}_\varepsilon} \text{d}s \text{d}t \left[ \eta_1^2 + \eta_2^2 + \eta_1^2 \eta_2^2 \right] |\tilde{\psi}|^2 \geq -C |\log \varepsilon|^2. \end{aligned}$$

Using (2.57), it is easy to show that the second term in (2.59) can be absorbed in the remainder, while, thanks to Agmon estimates,

$$\int_{c_0 |\log \varepsilon| - \varepsilon}^{\bar{t}} \text{d}t \int_0^{\frac{|\partial \Omega|}{\varepsilon}} \text{d}s |\tilde{\psi}|^2 \leq \int_{\text{dist}(\mathbf{r}, \partial \Omega) \geq c_0 \varepsilon |\log \varepsilon| - \varepsilon^2} \text{d}\mathbf{r} |\psi^{\text{GL}}|^2 = \mathcal{O}(\varepsilon^{c_0 C_A + 1}),$$

i.e.  $\tilde{\psi}$  is still smaller than any power of  $\varepsilon$  in the support of  $\chi_2'$  and  $\eta_2'$ , which implies that the third term in (2.59) can be discarded as well.

In conclusion we obtained

$$\mathcal{F}[\tilde{\psi}] \geq \mathcal{F}[\chi_1 \chi_2 \tilde{\psi}] + \mathcal{O}(|\log \varepsilon|^2), \quad (2.60)$$

and, setting  $\psi := \chi_1 \chi_2 \tilde{\psi}$ , the claim is proven.  $\square$

The rest of the lower bound proof is very close to the proof of [11, Proposition 4.2]. We sum up the main steps below.

**Proof of Proposition 2.2.** Combining (2.10) with (2.47) and the result of Lemma 2.2, we have

$$E_\varepsilon^{\text{GL}} \geq \mathcal{F}[\psi] + \mathcal{O}(|\log \varepsilon|^2). \quad (2.61)$$

The next step is thus a lower bound to  $\mathcal{F}[\psi]$ .

First of all we extract from  $\mathcal{F}[\psi]$  the desired leading term in the energy asymptotics: by a standard splitting trick, we set

$$\psi(s, t) =: f_\star(t)u(s, t)e^{-i\alpha_\star s}, \quad (2.62)$$

which defines a suitable  $u \in H_{\text{loc}}^1(\mathcal{A}_{\text{cut}})$ . Note that, since  $\alpha_\star$  is in general not an integer,  $u$  is not periodic and therefore a multi-valued function, but  $|u|$  is periodic and this will suffice. Plugging the above ansatz in the functional  $\mathcal{F}$ , we get

$$\mathcal{F}[\psi] = \frac{|\partial\Omega_{\text{cut}}|E_\star^{1\text{D}}}{\varepsilon} + \mathcal{E}[u], \quad (2.63)$$

where

$$\mathcal{E}[u] := \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t f_\star^2 \left\{ |\partial_t u|^2 + |\partial_s u|^2 - 2(t + \alpha_\star)\mathbf{e}_s \cdot \mathbf{j}[u] + \frac{1}{2b}f_\star^2(1 - |u|^2)^2 \right\}, \quad (2.64)$$

and the superconducting current is given by

$$\mathbf{j}[u] := \frac{i}{2}(u\nabla u^* - u^*\nabla u). \quad (2.65)$$

Since

$$|\partial\Omega_{\text{cut}}| = |\partial\Omega| + \mathcal{O}(\varepsilon |\log \varepsilon|),$$

the lower bound is proven if we can show that  $\mathcal{E}[u] \geq 0$ . The rest of the proof is focused on this claim.

In order to investigate the positivity of  $\mathcal{E}[u]$  we use the potential function trick, i.e. we observe that the function  $F$  defined in (2.4) satisfies

$$F'(t) = 2(t + \alpha_\star)f_\star^2(t), \quad (2.66)$$

and therefore

$$-2 \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t (t + \alpha_\star)j_s(u) = - \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t \partial_t F(t) j_s(u) = \int_{\mathcal{A}_{\text{cut}}} \text{d}s \text{d}t F(t) \partial_t j_s[u]$$

where we have denote by  $j_s[u] = \mathbf{e}_s \cdot \mathbf{j}[u]$  the  $s$ -component of the current. Here the boundary terms vanish because  $F(0) = 0$  and

$$u(s, c_0|\log \varepsilon|) = 0, \quad u(s_j \pm c_1|\log \varepsilon|, t) = 0, \quad (2.67)$$

thanks to the boundary conditions inherited from  $\psi$  and the strict positivity of  $f_\star$ .

We now integrate by parts in the  $s$  variable the last two terms:

$$\begin{aligned} & \int_{\mathcal{A}_{\text{cut}}} ds dt F(t) \partial_t j_s[u] \\ &= \frac{i}{2} \int_{\mathcal{A}_{\text{cut}}} ds dt F(t) [\partial_t u \partial_s u^* - \partial_t u^* \partial_s u + u \partial_{s,t}^2 u^* - u^* \partial_{s,t}^2 u] \\ &= i \int_{\mathcal{A}_{\text{cut}}} ds dt F(t) [\partial_t u \partial_s u^* - \partial_t u^* \partial_s u], \end{aligned}$$

where again boundary terms are absent thanks to the vanishing of  $u$  stated in (2.67). At this stage, the non-periodicity of  $u$  could affect the result but this is not the case because  $u^* \partial_t u$  and its complex conjugate are always periodic. The simple estimate

$$\begin{aligned} i \int_{\mathcal{A}_{\text{cut}}} ds dt F(t) [\partial_t u \partial_s u^* - \partial_t u^* \partial_s u] &\geq -2 \int_{\mathcal{A}_{\text{cut}}} ds dt |F(t)| |\partial_t u| |\partial_s u| \\ &\geq \int_{\mathcal{A}_{\text{cut}}} ds dt F(t) [|\partial_t u|^2 + |\partial_s u|^2] \end{aligned}$$

which uses the negativity of  $F$ , then leads us to the lower bound for  $\mathcal{E}[u]$ :

$$\mathcal{E}[u] \geq \int_{\mathcal{A}_{\text{cut}}} ds dt \left\{ K(t) (|\partial_t u|^2 + |\partial_s u|^2) + \frac{1}{2b} f_*^4 (1 - |u|^2)^2 \right\}. \quad (2.68)$$

The pointwise positivity of  $K(t)$  for  $1 < b < \Theta_0^{-1}$  given in (2.6) and the manifest positivity of the second term in the expression above yields the final lower bound

$$\mathcal{E}[u] \geq \frac{1}{2b} \int_{\mathcal{A}_{\text{cut}}} ds dt f_*^4 (1 - |u|^2)^2 \geq 0. \quad (2.69) \quad \square$$

We finalize now the proof of the main result:

**Proof of Theorem 1.1.** The combination of the energy upper (Proposition 2.1) and lower (Proposition 2.2) bounds yields the energy asymptotics (1.20). It only remains to prove the estimate on the  $L^2$  norm of the difference  $|\psi^{\text{GL}}|^2 - f_*^2$ . This is however trivially implied by the lower bound (2.69): if one keeps the positive term appearing on the right-hand side of the inequality and put it together with (2.61), the splitting (2.63) and the upper bound (2.36), the outcome is

$$\int_{\mathcal{A}_{\text{cut}}} ds dt f_*^4 (1 - |u|^2)^2 = \mathcal{O}(|\log \varepsilon|^2). \quad (2.70)$$

By reconstructing first  $\psi(s, t)$  and then using (2.52), one can easily realize that the regions where  $|\psi|$  differs from  $|\psi^{\text{GL}}|$  can be discarded and their contribution be included in the remainder. The same holds true for the corner cells and therefore the final result is (1.21). Note the factor  $\varepsilon^2$  appearing on the right-hand side due to the rescaling  $(s, t) \rightarrow (\sigma, \tau) = (\varepsilon s, \varepsilon t)$ .  $\square$

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