Vortex Phases of Rotating Superfluids

Michele Correggi[1], Florian Pinsker[2], Nicolas Rougerie[3] and Jakob Yngvason[4]

[1]Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy,
[2]DAMTP, University of Cambridge, Wilbertforce Road, Cambridge CB3 0WA, UK,
[3]Université Grenoble 1 & CNRS, LPMMC (UMR 5493), B.P. 166, 38 042 Grenoble, France,
[4]Faculty of Physics, University of Vienna, Boltzmanngasse 5, and Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria

E-mail: [1]michele.correggi@gmail.com, [2]florian.pinsker@gmail.com, [3]nicolas.rougerie@grenoble.cnrs.fr, [4]jakob.yngvason@univie.ac.at

Abstract. We report on the first mathematically rigorous proofs of a transition to a giant vortex state of a superfluid in rotating anharmonic traps. The analysis is carried out within two-dimensional Gross-Pitaevskii theory at large coupling constant and large rotational velocity and is based on precise asymptotic estimates on the ground state energy. An interesting aspect is a significant difference between 'soft' anharmonic traps (like a quartic plus quadratic trapping potential) and traps with a fixed boundary. In the former case vortices persist in the bulk until the width of the annulus becomes comparable to the size of the vortex cores. In the second case the transition already takes place in a parameter regime where the size of vortices is very small relative to the width of the annulus. Moreover, the density profiles in the annulus are different in the two cases. In both cases rotational symmetry of the density in a true ground state is broken, even though a symmetric variational ansatz gives an excellent approximation to the energy.

1. Introduction

A superfluid confined in a rotating anharmonic trap, where the rotation speed can in principle be arbitrarily large, undergoes several phase transitions as the speed increases. At first the fluid is vortex free [22, 3] but then quantized vortices emerge, eventually forming a vortex lattice [6, 16, 2, 23, 1, 7, 17] that may persist even when the speed is so large that the centrifugal force creates a 'hole' with strongly depleted density in the middle of the trap [19, 8, 9, 10]. Above a certain rotation speed a transition to a *giant vortex state* takes place. In this state the vortices disappear from the annulus where the bulk of the superfluid is concentrated while a macroscopic phase circulation remains. This phenomenon has been studied theoretically by variational and numerical methods in the past [17, 18, 19, 20, 24, 26, 15] but mathematically rigorous proofs of the giant vortex transition have been obtained only very recently [31, 11, 12, 13, 14, 32]. An experimental realization of this transition appears to be still out of reach although anharmonic traps have been available already for some time [4, 34, 21, 37]. In the following we report on the main findings of this analysis with emphasis on [13, 14].

2. Setting the Stage

2.1. The basic many-body Hamiltonian

The quantum mechanical Hamiltonian for N spinless bosons with an external potential, V, and a pair interaction potential, v, in a rotating frame with angular velocity $\mathbf{\Omega}_{\mathrm{rot}}$ is

$$H = \sum_{j=1}^{N} \left(-\tfrac{1}{2}\nabla_j^2 + V(\mathbf{x}_j) - \mathbf{L}_j \cdot \mathbf{\Omega}_{\mathrm{rot}} \right) + \sum_{1 \le i < j \le N} v(|\mathbf{x}_i - \mathbf{x}_j|).$$

Here $\mathbf{x}_j \in \mathbb{R}^3$ and $\mathbf{L}_j = -\mathrm{i}\mathbf{x}_j \times \nabla_j$ is the angular momentum of the jth particle. The Hamiltonian can alternatively be written in the 'magnetic' form

$$H = \sum_{j=1}^{N} \left\{ \tfrac{1}{2}(\mathrm{i}\nabla_j + \mathbf{A}(\mathbf{x}_j))^2 + V(\mathbf{x}_j) - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r_j^2 \right\} + \sum_{1 \le i < j \le N} v(|\mathbf{x}_i - \mathbf{x}_j|) \tag{1}$$

with the vector potential

$$\mathbf{A}(\mathbf{x}) = \mathbf{\Omega}_{\mathrm{rot}} \times \mathbf{x} = \Omega_{\mathrm{rot}} r\, \mathbf{e}_\vartheta$$

where r denotes the distance from the rotation axis and $\mathbf{e}_\vartheta$ the unit vector in the angular direction. This way of writing the Hamiltonian corresponds to the splitting of the rotational effects into Coriolis and centrifugal forces.

2.2. Harmonic vs. anharmonic traps

If V is a harmonic oscillator potential in the direction $\perp$ to $\mathbf{\Omega}_{\mathrm{rot}}$, i.e.,

$$V(\mathbf{x}) = \tfrac{1}{2}\Omega_{\mathrm{trap}} r^2 + V^\|(z)$$

then stability requires $\Omega_{\mathrm{rot}} < \Omega_{\mathrm{trap}}$. Rapid rotation means here that

$$\Omega_{\mathrm{rot}} \to \Omega_{\mathrm{trap}}$$

from below. On the other hand, if V is anharmonic and increases faster than quadratically in the direction $\perp$ to $\mathbf{\Omega}_{\mathrm{rot}}$, e.g. $V(\mathbf{x}) \sim r^s + V^\|(z)$ with $s > 2$, then Ω_{rot} can in principle be as large as one pleases and rapid rotation means simply $\Omega_{\mathrm{rot}} \to \infty$.

These two cases are quite different both physically and mathematically. The former leads to an effective many-body Hamiltonian in the lowest Landau level of the magnetic kinetic energy term in (1) and bosonic analogues of the Fractional Quantum Hall Effect (see [39, 38, 7, 29]). In the case of rapid rotation in an anharmonic trap, as considered here, it is usually sufficient to employ Gross-Pitaevskii (GP) theory for an effective description. We remark, however, that a small anharmonic term appropriately tuned can also lead to interesting modifications of the Quantum Hall states of harmonic traps [33].

2.3. The Gross-Pitaevskii limit theorem

The following basic fact about the many-body Hamiltonian (1) for $N \to \infty$ with Na and Ω_{rot} *fixed*, where a is the scattering length of the (repulsive, short range) interaction potential v was proved in [27]:

There is (possibly fractionated) Bose-Einstein condensation in the ground state as $N \to \infty$, and the wave function of the condensate ("superfluid order parameter") is a minimizer (in general not unique) of the GP energy functional

$$\mathcal{E}_{\mathrm{3D}}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^3} \left\{ \tfrac{1}{2} |(\mathrm{i}\nabla + \mathbf{A})\,\Psi|^2 + (V - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2)|\Psi|^2 + 2\pi Na\,|\Psi|^4 \right\} \tag{2}$$

with $\int |\Psi|^2 = 1$. The Gross-Pitaevskii partial differential equation

$$\left\{ (\mathrm{i}\nabla + \mathbf{A})^2 + (V - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2) + 4\pi N a |\Psi|^2 \right\}\Psi = \mu\Psi$$

with the chemical potential μ is the variational equation corresponding to this minimization problem.

The rigorous proof of this theorem is far from simple, as can be seen from the fact that a Hartree variational ansatz for the Hamiltonian (1) (that would anyhow only lead to an upper bound) is meaningless if the interaction potential has a hard core. Even for 'soft' potentials a naive computation would not lead to (2) with the scattering length as parameter, but rather the integral $\int v$ (that gives only the lowest Born approximation to the scattering length). For the mathematical background of this and related results [28] may be consulted. A limit theorem that holds uniformly the parameters Ω_{rot} and Na as $N \to \infty$, but is restricted to the leading order, was proved in [5].

2.4. 2D Gross-Pitaevskii theory in anharmonic traps

The GP minimization problem has two parameters, Ω_{rot} and Na. We shall be concerned with phenomena that occur in anharmonic traps in the asymptotic regime where both Ω_{rot} and Na are large. For convenience introduce

$$\varepsilon \equiv (2\pi Na)^{-1/2}$$

which is small if Na is large. (In appropriate units ε is the 'healing length'.)

For traps that are sufficiently elongated along the rotational axis (z-direction) the properties of the condensate are to a good approximation independent of z and we may consider a 2D[1] energy functional

$$\mathcal{E}_{\mathrm{2D}}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}\left|(\mathrm{i}\nabla + \mathbf{A})\,\Psi\right|^2 + (V - \tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2)|\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

with a trap potential of the form (for simplicity)

$$V(r) = kr^s \tag{3}$$

with $s > 2$, $k > 0$. Here Ω_{rot} can be arbitrary large.

The limiting case $s \to \infty$ corresponds to a 'flat' trap with fixed boundary at $r = 1$. The effective potential is then simply $-\tfrac{1}{2}\Omega_{\mathrm{rot}}^2 r^2$ and the integration is limited to the unit disc in $\mathbb{R}^2$. A word of caution: The limit $s \to \infty$ can not be interchanged with the limits $\varepsilon \to 0$, $\Omega_{\mathrm{rot}} \to \infty$ as discussed in Section 3.6 below.

The analysis of the GP minimizer is guided by the following heuristics:

- A vortex, i.e., a zero of the wave function $\Psi(\mathbf{x}) = |\Psi(\mathbf{x})| \exp(\mathrm{i}\theta(\mathbf{x}))$ with an accompanying nonzero winding number of the phase factor, reduces the kinetic energy because the associated current $\sim \nabla\theta(\mathbf{x})$ compensates partly the velocity field generated by $\mathbf{A}(\mathbf{x}) = \mathbf{\Omega}_{\mathrm{rot}} \times \mathbf{x}$.

- A vortex causes also a change in the density, however, (mass is moved from the vortex core to the bulk) and this increases the interaction energy that depends on the density at the potential location of the vortex. The energy balance decides whether or not a vortex is favorable, and if that is the case, the size of the vortex core.

- A vortex is the more costly the higher the density. At sufficiently high rotational velocities the compression due to centrifugal forces creates a 'hole' and the density in the bulk increases until, at some point, vortices become too costly.

---

[1] A 2D description is, of course, also appropriate in thin traps where the motion along the z-axis is 'frozen' [36].

2.5. Scaling of the energy functional

The effective potential $(kr^s - \frac{1}{2}\Omega_{\mathrm{rot}}^2 r^2)$ has a unique minimum at $r = (\Omega_{\mathrm{rot}}^2/(sk))^{1/(s-2)}$. Taking this as a length unit we obtain the scaled energy functional

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|(\mathrm{i}\nabla + \Omega x \mathbf{e}_\vartheta)\psi|^2 + \Omega^2 W(x)|\psi|^2 + \varepsilon^{-2}|\psi|^4 \right\}$$

where $x = |\mathbf{x}|$, $\Omega \sim \Omega_{\mathrm{rot}}^{(s+2)/(s-2)}$, and

$$W(x) = \left(\tfrac{1}{s}x^s - \tfrac{1}{2}x^2 \right).$$

The scaled potential has a minimum at $x = 1$, independent of Ω.

3. Analysis of the GP Minimizers

3.1. Critical velocities

The basic facts for traps of the form (3) with $2 < s < \infty$ can be summarized as follows.

As Ω increases there are **three critical velocities:**

- $\Omega_{\mathrm{c1}}' \sim |\log \varepsilon|$ marking the appearance of the first vortex.[2]
- $\Omega_{\mathrm{c2}} \sim \varepsilon^{-1}$ marking the appearance of a 'hole' due to the centrifugal forces.
- $\Omega_{\mathrm{c3}} \sim \varepsilon^{-4}$ marking the transition to a 'giant vortex'.

For the first transition we refer to [2, 22, 1, 30]. For $\Omega_{\mathrm{c1}} \ll \Omega \ll \Omega_{\mathrm{c3}}$ the ground state energy is well approximated by assuming a triangular vortex lattice in the bulk.[3] In the limit $\varepsilon \to 0$ the vorticity becomes uniformly distributed with density Ω [14]. For $\Omega > \Omega_{\mathrm{c3}}$ the bulk is free of vortices but a macroscopic circulation around the origin remains [13, 14].

3.2. The vortex lattice regime

The ground state energy for $\Omega_{\mathrm{c1}} \ll \Omega \ll \Omega_{\mathrm{c3}}$ can be computed exactly to subleading order [14]:

Theorem 1 (Energy between Ω_{c2} and Ω_{c3}) *If $\varepsilon^{-1} \lesssim \Omega \ll \varepsilon^{-4}$ as $\varepsilon \to 0$, then*

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \tfrac{1}{6}\Omega|\log(\varepsilon^4\Omega)|(1 + o(1)).$$

Here E^{TF} denotes the energy without the kinetic term. Below Ω_{2c} a similar formula holds (with a different scaling, $\varepsilon^{-2/(s+2)}$ as length unit):

Theorem 2 (Energy between Ω_{c1} and Ω_{c2}) *If $|\log \varepsilon| \ll \Omega' \lesssim \varepsilon^{-1}$ as $\varepsilon \to 0$, then*

$$E^{\mathrm{GP}'} = E^{\mathrm{TF}'} + \tfrac{1}{2}\Omega'|\log(\varepsilon^2\Omega')|(1 + o(1)).$$

3.3. Vortices reduce kinetic energy

The potential term $\sim \Omega^2$ and the interaction term $\sim \varepsilon^{-2}$ become comparable when

$$\Omega \sim \varepsilon^{-1}.$$

This is the order of the second critical speed Ω_{c2} above which the centrifugal force creates a 'hole'. In the sequel we shall focus on rotation speeds around and above Ω_{c2} which means that $\Omega \gtrsim \varepsilon^{-1}$. In this regime the kinetic energy term $\frac{1}{2}|(\mathrm{i}\nabla + \mathbf{A})\Psi|^2$ is formally also of order $1/\varepsilon^2$ if $\Omega \sim 1/\varepsilon$. Its contribution to the energy is, however, of lower order, namely $\sim \Omega|\log \varepsilon|$, because a lattice of vortices emerges as $\varepsilon \to 0$ and reduces the kinetic energy as remarked in 2.4.

---

[2] Here $\Omega' \sim \varepsilon^{-4/(s+2)}\Omega_{\mathrm{rot}}$. This scaling is more convenient than $\Omega \sim \Omega_{\mathrm{rot}}^{(s+2)/(s-2)}$ for $\Omega \ll 1/\varepsilon$.

[3] The reason why a triangular arrangement with hexagonal unit cells is optimal amongst regular lattices can be made plausible by appealing to an electrostatic analogy and Newton's theorem [10]: Hexagonal cells are as close to being circular as possible and thus have smaller multipole moments and lower interaction energy than other cells.

3.4. The giant vortex regime

Consider a variational ansatz for the wave function of the form[4]

$$\psi(\mathbf{x}) = g(\mathbf{x}) \exp(\mathrm{i}\Omega\vartheta)$$

with a real valued function g, normalized such that $\int g^2 = 1$. This gives

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^2} \left\{ \tfrac{1}{2}|\nabla g|^2 + \tfrac{1}{2}\Omega^2(x - x^{-1})^2 g^2 + \Omega^2\left(\tfrac{1}{s}x^s - \tfrac{1}{2}x^2\right)g^2 + \varepsilon^{-2}g^4 \right\} \equiv \mathcal{E}^{\mathrm{gv}}[g].$$

The unique positive minimizer g_{gv} of $\mathcal{E}^{\mathrm{gv}}$ is rotationally symmetric and we denote the corresponding energy by E^{gv}.

The following results are proved in [13, 14].

Theorem 3 (Energy in the giant vortex regime) *There is a constant $0 < \bar\Omega_0 < \infty$ such that for $\Omega = \Omega_0\,\varepsilon^{-4}$ with $\Omega_0 > \bar\Omega_0$ the ground state energy is*

$$E^{\mathrm{GP}} = E^{\mathrm{gv}} + O(|\log\varepsilon|^{9/2}).$$

Theorem 4 (Absence of vortices in the bulk) *There is a constant $c > 0$ such that for $\Omega = \Omega_0\,\varepsilon^{-4}$ with $\Omega_0 > \bar\Omega_0$ and ε sufficiently small the minimizer ψ^{GP} is free of zeros in the annulus*

$$\mathcal{A} = \{\mathbf{x} : |1 - x| \le c\Omega^{-1/2}|\log\varepsilon|^{1/2}\}.$$

3.5. On the proof of the GV transition

The main issue is a precise lower bound to the energy. We restrict $\mathcal{E}^{\mathrm{gv}}$ to the annulus $\mathcal{A}$, obtaining a positive minimizer g_0. Define $u(\mathbf{x})$ on the annulus by writing

$$\psi^{\mathrm{GP}}(\mathbf{x}) = g_0(x)u(\mathbf{x})\exp(\mathrm{i}\Omega\vartheta).$$

The function u contains all possible zeros of ψ^{GP} in the annulus.

The variational equation for g leads to the lower bound

$$E^{\mathrm{GP}} \ge E^{\mathrm{gv}}_{\mathcal{A}} + \mathcal{E}_{\mathcal{A}}[u]$$

with a functional of Ginzburg-Landau type with g_0^2 as weight:

$$\mathcal{E}_{\mathcal{A}}[u] = \int_{\mathcal{A}} g_0^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) + \varepsilon^{-2}g_0^2(1 - |u|^2)^2 \right\}$$

where $\mathbf{B} = \Omega\left(x - x^{-1}\right)\mathbf{e}_\vartheta$ and $\mathbf{J}(u) = \tfrac{\mathrm{i}}{2}(u\nabla u^* - u^*\nabla u)$. The main task is to estimate the negative term $-\int g_0^2\,\mathbf{B}\cdot\mathbf{J}(u)$.

For this purpose one writes $g^2\mathbf{B} = \nabla^\perp F$ with $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and a potential function F. Integration by parts and estimates of F (this is the key point!) give

$$\int_{\mathcal{A}} g^2 \left\{ \tfrac{1}{2}|\nabla u|^2 - \mathbf{B}\cdot\mathbf{J}(u) \right\} \ge -C\Omega_0^2|\log\varepsilon|^{3/2}$$

leading to the lower energy bound.

A consequence of this bound, combined with the variational upper bound $E^{\mathrm{gv}}_{\mathcal{A}} \le 0$ is an upper bound on the interaction term for large Ω_0:

$$\int_{\mathcal{A}} \varepsilon^{-2}g_0^4(1 - |u|^2)^2 \le C\Omega_0^2|\log\varepsilon|^{3/2}.$$

Together with the upper bound to the kinetic energy and standard inequalities this implies that u must be close to 1, in particular free of zeros.

---

[4] For simplicity of notation we assume that Ω is an integer which is justified since $\Omega \to \infty$.

3.6. Comparison with the 'flat' case

The flat case, $s = \infty$, that is treated in detail in [11, 12], differs from the case $s < \infty$ in several respects:

- The GV transition takes place at $\Omega \sim \varepsilon^{-2} |\log \varepsilon|^{-1}$ rather than $\Omega \sim \varepsilon^{-4}$.
- The density profile in the GV regime is of 'Thomas-Fermi' type in the 'flat' case, but for $s < \infty$ it is gaussian around $x = 1$.
- The 'last' vortices before the GV transition have size $\sim \varepsilon^{3/2}$ that is much smaller than the thickness of the annulus $\sim \varepsilon |\log \varepsilon|$. For $s < \infty$ the size of vortices, $\sim \varepsilon^2$ and the size of the annulus, $\sim \varepsilon^2 |\log \varepsilon|^{1/2}$, are almost comparable.

The techniques of proof in the two cases are also by necessity different: While *vortex ball constructions* and subsequent *jacobian estimates* (see [35]) for the potential function are applicable for the 'small' vortices in a 'flat' trap they are useless for $s < \infty$ and new ideas are required.

3.7. Circulation and symmetry breaking

At low rotation speeds below the onset of the second vortex the GP minimizer has rotationally symmetric density, but a vortex lattice clearly breaks the symmetry. On the other hand, the giant vortex variational ansatz, that gives an excellent approximation to the energy and circulation for $\Omega_0 > \bar{\Omega}_0$, is an eigenfunction of angular momentum. A true minimizer does not have this property, however:

Theorem 5 (Circulation and rotational symmetry breaking) *In the giant vortex regime* $\Omega_0 > \bar{\Omega}_0$ *the circulation of any GP minimizer is* $2\pi\,\Omega + O(\Omega_0\,|\log \varepsilon|^{9/4})$, *but no minimizer is an eigenfunction of angular momentum.*

Theis result holds both for $s < \infty$ and $s = \infty$ [11]–[14].

4. Summary

The study of the GP equation for dilute Bose gases in rotating, anharmonic reveals a surprising rich landscape, both from the mathematical and physical point of view. Detailed analysis can be carried out in an asymptotic regime where both the coupling constant and the rotational speed are large. Among the results found are:

- Energy asymptotics corresponding to a distribution of vorticity in a lattice of vortices for $\Omega_{c1} \ll \Omega \ll \Omega_{c3}$.
- Emergence of a 'hole' with strongly depleted density above a critical rotation speed $\Omega_{c2} \sim \varepsilon^{-1}$.
- Transition to a 'giant vortex' state above $\Omega_{c3} \sim \varepsilon^{-4}$ where the vortex lattice disappears from the bulk and all vorticity resides in the 'hole', creating a macroscopic circulation in the bulk.
- Breaking of rotational symmetry, also in the giant vortex regime.

References

[1] Aftalion A 2006 *Vortices in Bose-Einstein Condensates* (Basel: Birkhäuser)
[2] Aftalion A and Du Q 2001 *Phys. Rev.* A **64** 063603a
[3] Aftalion A, Jerrard R L and Royo-Letelier J 2011 *J. Func. Anal.* **260** 2387–2460
[4] Bretin V, Stock S, Seurin Y and Dalibard J 2004 *Phys. Rev. Lett.* **92** 050403
[5] Bru J B, Correggi M, Pickl P and Yngvason J 2008 *Commun. Math. Phys.* **280**, 517–544
[6] Castin Y and Dum R 1999 *Eur. Phys. J.* D **7** 399
[7] Cooper N R 2008 *Adv. Phys.* **57** 539–616
[8] Correggi M, Rindler-Daller J and Yngvason J 2007 *J. Math. Phys.* **48** 042104
[9] Correggi M, Rindler-Daller J and Yngvason J 2007 *J. Math. Phys.* **48** 102103.
[10] Correggi M and Yngvason J 2008 *J. Phys. A: Math. Theor.* **41** 445002
[11] Correggi M, Rougerie N and Yngvason J 2011 *Commun. Math. Phys.* **303** 451–508
[12] Correggi M, Pinsker F, Rougerie N and Yngvason J 2011 *J. Stat. Phys.* **143** 261–305
[13] Correggi M, Pinsker F, Rougerie N and Yngvason J 2011 *Phys. Rev.* A **84** 053614
[14] Correggi M, Pinsker F, Rougerie N and Yngvason J 2012 *J. Math. Phys.* **53** 095203
[15] Danaila I and P Kazemi P 2010 *SIAM J. Sci. Comp.* **32** 2447
[16] Fetter A L 2001 *Phys. Rev.* A **64** 063608
[17] Fetter A L 2009 *Rev. Mod. Phys.* **81** 647–691
[18] Fetter A L, Jackson B and Stringari S 2005 *Phys. Rev.* A **71** 013605
[19] Fischer U R and Baym G 2003 *Phys. Rev. Lett.* **90** 140402
[20] Fu H and Zaremba E 2006 *Phys. Rev.* A **73** 013614
[21] Henderson K, Ryu C, Mac Cormick C and Boshier M G 2009 *New J. Phys.* **11** 043030
[22] Ignat R and Millot V 2006 *J. Funct. Anal.* **233** 260–306
[23] Ignat R and Millot V 2006 *Rev. Math. Phys.* **18** 119–162
[24] Kasamatsu K, Tsubota M and Ueda M 2002 *Phys. Rev.* A **66** 053606
[25] Kavoulakis G M and Baym G 2003 *New J. Phys.* **5** 51.1-11
[26] Kim J K and Fetter A L 2005 *Phys. Rev.* A **72** 023619
[27] Lieb E H and Seiringer R 2006 *Comm. Math. Phys.* **264** 505–537
[28] Lieb E H, Seiringer R, Solovej J P and Yngvason J 2005 *The Mathematics of the Bose Gas and its Condensation* (Basel: Birkhäuser)
[29] Lewin M and Seiringer R 2009 *J. Stat. Phys.* **137** 1040–1062
[30] Rindler-Daller T 2008 *Physica* A **387** 1851–1874
[31] Rougerie N 2011 *J. math. pures et appl.* **95** 296
[32] Rougerie N 2012 *Arch. Rat. Mech. Anal.* **203** 69
[33] Rougerie N, Serfaty S and Yngvason J 2012 Quantum Hall states of bosons in rotating anharmonic traps *Preprint* arXiv:1212.1085
[34] Ryu C, Andersen M F , Clade P, Natarajan V, Helmerson K and Phillips W D 2007 *Phys. Rev. Lett.* **99** 260401
[35] Sandier E and Serfaty S 2007 *Vortices in the Magnetic Ginzburg-Landau Model* (Basel: Birkhäuser)
[36] Schnee S and Yngvason J 2007 *Commun. Math. Phys.* **269** 659–691
[37] Sherlock B E, Gildemeister M, Owen E, Nugent E and Foot C J 2011 *Phys. Rev.* A **83** 043408
[38] Viefers S 2008 *J.Phys.* C **12** 123202
[39] Wilkin N K and Gunn J M F 2000 *Phys. Rev. Lett.* **84** 6-9