NONEXISTENCE OF SOLUTIONS TO PARABOLIC DIFFERENTIAL INEQUALITIES WITH A POTENTIAL ON RIEMANNIAN MANIFOLDS

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Abstract. We are concerned with nonexistence results of nonnegative weak solutions for a class of quasilinear parabolic problems with a potential on complete noncompact Riemannian manifolds. In particular, we highlight the interplay between the geometry of the underlying manifold, the power nonlinearity and the behavior of the potential at infinity.

1. Introduction

In this paper we investigate the nonexistence of nonnegative, nontrivial weak solutions (in the sense of Definition 2.1 below) to parabolic differential inequalities of the type

$$\begin{cases} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq V(x,t)u^q & \text{in } M\times(0,\infty) \\ u = u_0 & \text{in } M\times\{0\}, \end{cases} \tag{1.1}$$

where M is a complete, m–dimensional, noncompact Riemannian manifold with metric g, div and ∇ are respectively the divergence and the gradient with respect to g, $p>1, q>\max\{p-1,1\}$, the potential satisfies $V=V(x,t)>0$ a.e. in $M\times(0,\infty)$ and the initial condition u_0 is nonnegative.

Local existence, finite time blow-up and global existence of solutions to parabolic Cauchy problems have attracted much attention in the literature. In particular, the following semilinear parabolic Cauchy problem

$$\begin{cases} \partial_t u - \Delta u \,=\, u^q & \text{in } \mathbb{R}^m\times(0,\infty) \\ \\ u = u_0 & \text{in } \mathbb{R}^m\times\{0\}\,, \end{cases} \tag{1.2}$$

where $q>1, u_0\geq 0, u_0\in L^\infty(\mathbb{R}^m)$, has been largely investigated. Indeed (see [5], [6] and [15]), problem (1.2) does not admit global bounded solutions for $1<q\leq 1+\frac{2}{m}$. On the contrary, for $q>1+\frac{2}{m}$ global bounded solutions exist, provided that u_0 is sufficiently small. For initial conditions $u_0\in L^p(\mathbb{R}^m)$ similar results have been obtained in the framework of mild solutions in the space $C\big([0,T);L^p(\mathbb{R}^m)\big)$ in [29], [30].

Problem (1.1) with $(M,g)=(\mathbb{R}^m, g_{\text{flat}})$, where g_{flat} is the standard flat metric in the Euclidean space, together with its generalization to a wider class of operators of p–Laplace type or related to the porous medium equation, has also been largely studied; without claim of completeness we refer the reader to [7], [8], [9], [20], [21], [23], [26], and references therein. In particular, in [20] it is shown that problem (1.1) with $M=\mathbb{R}^m$ and $V\equiv 1$ does not admit nontrivial nonnegative weak solutions, provided that

$$p>\frac{2m}{m+1}\,,\quad q\leq p-1+\frac{p}{m}\,.$$

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Moreover, the blow-up result given in [5] has been extended to the setting of Riemannian manifolds. To further describe such results, let us introduce some notation. Let (M, g) be a complete noncompact Riemannian manifold, endowed with a smooth Riemannian metric g. Fix any point $x_0 \in M$, and for any $x \in M$ denote by $r(x) = \mathrm{dist}(x_0, x)$ the Riemannian distance between x_0 and x. Moreover, let $B(x_0, r)$ be the geodesics ball with center $x_0 \in M$ and radius $r > 0$, and let μ be the Riemannian volume on M with volume density $\sqrt{g}$.

In [31] it is proved that no nonnegative nontrivial weak solutions to problem (1.1) with $p = 2$ exist, provided there exist $C > 0, \alpha > 2, \beta > -2$ such that, for all $r > 0$ large enough:

(*a*) $\mu(B(x, r)) \leq C r^\alpha$ for all $x \in M$;
(*b*) $\frac{\partial \log \sqrt{g}}{\partial r} \leq \frac{C}{r}$;
(*c*) $V = V(x)$, $V \in L^\infty_{loc}(M)$ and $C^{-1} r(x)^\beta \leq V(x) \leq C r(x)^\beta$;

Observe that if the Ricci curvature of M is nonnnegative, then $(a) - (b)$ are satisfied, see e.g. [**?**]. On the other hand (see Theorem 5.2.10 in [4], or Section 10.1 of [10]), hypotheses $(a) - (b)$ imply that $\lambda_1(M) = 0$, where $\lambda_1(M)$ is the infimum of the L^2- spectrum of the operator $-\Delta$ on M.

The semilinear Cauchy problem

$$\begin{cases} \partial_t u = \Delta u + h(t) u^\nu & \text{in } \mathbb{H}^m \times (0, T) \\ u = u_0 & \text{in } \mathbb{H}^m \times \{0\} \end{cases} \tag{1.3}$$

has been studied in [1], where $\mathbb{H}^m$ is the $m-$dimensional hyperbolic space, u_0 is nonnegative and bounded on M and h is a positive continuous function defined in $[0, \infty)$; note that in this case we have $\lambda_1(\mathbb{H}^N) = \frac{(N-1)^2}{4}$.

To be specific, it has been shown that if $h(t) \equiv 1$ $(t \geq 0)$, or if

$$\alpha_1 t^q \leq h(t) \leq \alpha_2 t^q \quad \text{for any } t > t_0, \tag{1.4}$$

for some $\alpha_1 > 0, \alpha_2 > 0, t_0 > 0$ and $q > -1$, then there exist global bounded solutions for sufficiently small initial data u_0. Moreover, when $h(t) = e^{\alpha t}$ $(t \geq 0)$ for some $\alpha > 0$, the authors showed that:

(*i*) if $1 < q < 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^m)}$, then every nontrivial bounded solution of problem (1.3) blows up in finite time;
(*ii*) if $q > 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^m)}$, then problem (1.3) posses global bounded solutions for small initial data ;
(*iii*) if $q = 1 + \frac{\alpha}{\lambda_1(\mathbb{H}^m)}$ and $\alpha > \frac{2}{3}\lambda_1(\mathbb{H}^m)$, then there exist global bounded solutions of problem (1.3) for small initial data.

Analogous results to those established in [1] have been obtained in [24], for the problem

$$\begin{cases} \partial_t u = \Delta u + h(t) u^q & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{1.5}$$

where M is a Cartan-Hadamard Riemannian manifold with sectional curvature bounded above by a negative constant, and $u_0 \in L^\infty(M)$. Moreover, for initial conditions $u_0 \in L^p(M)$ similar results have been established for mild solutions belonging to $C\big([0, T); L^p(M)\big)$ in [25].

Let us mention that nonexistence results of nonnegative nontrivial solutions have been also much investigated for solutions to elliptic equations and inequalities both on $\mathbb{R}^m$ (see, e.g., [2], [19], [18], [21], [22], [3]) and on Riemannian manifolds (see [11], [12], [14] [16], [17], [27], [28]). In particular, the present paper is the natural continuation of [16], where some ideas and methods introduced in [12], [11] and [14] have been developed. Indeed, our results can be regarded as the parabolic counterpart of those shown in [16], concerning nonnegative weak solutions to the inequality

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) \geq V(x) u^q \quad \text{in } M.$$

In [16], as well as in [11], [12], [27] and [28], the key assumptions are concerned with the parameters p, q and the behavior of a suitable weighted volume of geodesic balls, with density a negative power of the potential $V(x)$.

As for the case of $\mathbb{R}^m$, also on Riemannian manifolds the parabolic case presents substantial differences with respect to the elliptic one. In fact, new test functions have to be used, and suitable estimates of new integral terms are necessary. On the other hand, as in the case of elliptic inequalities on Riemannian manifolds, a simple adaptation of the methods used in $\mathbb{R}^m$ does not allow to obtain results as accurate as those we prove in the present work. In the next two subsections we describe our main results and some of their consequences; furthermore, we compare them with results in the literature.

1.1. **Main results.** In order to formulate our main results, we shall introduce some further notation and hypotheses. For each $R > 0$, $\theta_1 \geq 1$, $\theta_2 \geq 1$ let $S := M \times [0, \infty)$ and

$$E_R := \{(x,t) \in S \,:\, r(x)^{\theta_2} + t^{\theta_1} \leq R^{\theta_2}\}\,.$$

Let

$$\begin{aligned} \bar{s}_1 &:= \frac{q}{q-1}\theta_2\,, & \bar{s}_2 &:= \frac{1}{q-1}\,, \\ \bar{s}_3 &:= \frac{pq}{q-p+1}\theta_2\,, & \bar{s}_4 &:= \frac{p-1}{q-p+1}\,. \end{aligned}$$

The following conditions, that we call **HP1** and **HP2**, are the main hypotheses under which we will derive our nonexistence results for nonnegative nontrivial weak solutions of problem (1.1).

HP1. Assume that: (i) there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 > 0$, $C > 0$, $R_0 > 0$, $\varepsilon_0 > 0$ such that for every $R > R_0$ and for every $0 < \varepsilon < \varepsilon_0$ one has

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon}\, d\mu dt \leq C R^{\bar{s}_1 + C_0\varepsilon} (\log R)^{s_2}\,, \tag{1.6}$$

for some $0 \leq s_2 < \bar{s}_2$;

(ii) for the same constants as above, for every $R > R_0$ and for every $0 < \varepsilon < \varepsilon_0$ one has

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\, d\mu dt \leq C R^{\bar{s}_3 + C_0\varepsilon} (\log R)^{s_4}\,, \tag{1.7}$$

for some $0 \leq s_4 < \bar{s}_4$.

HP2. Assume that: (i) there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 > 0$, $C > 0$, $R_0 > 0$, $\varepsilon_0 > 0$ such that for every $R > R_0$ and for every $0 < \varepsilon < \varepsilon_0$ one has

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon}\, d\mu dt \leq C R^{\bar{s}_1 + C_0\varepsilon} (\log R)^{\bar{s}_2}\,, \tag{1.8}$$

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon}\, d\mu dt \leq C R^{\bar{s}_1 + C_0\varepsilon} (\log R)^{\bar{s}_2}\,; \tag{1.9}$$

(ii) for the same constants as above, for every $R > R_0$ and for every $0 < \varepsilon < \varepsilon_0$ one has

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\, d\mu dt \leq C R^{\bar{s}_3 + C_0\varepsilon} (\log R)^{\bar{s}_4}\,, \tag{1.10}$$

$$\int\!\!\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon}\, d\mu dt \leq C R^{\bar{s}_3 + C_0\varepsilon} (\log R)^{\bar{s}_4}\,. \tag{1.11}$$

Remark 1.1. Passing to the limit as $\varepsilon \to 0$ we see that, if **HP1** holds, then for the same constants as above conditions (1.6) and (1.7) hold also for $\varepsilon = 0$. Similarly, if **HP2** holds then (1.8) and (1.10) (or equivalently (1.9) and (1.11)) are satisfied also with $\varepsilon = 0$.

We prove the following theorems (for the definition of weak solution see Definition 2.1 below).

Theorem 1.2. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V > 0$ a.e. in $M \times (0, \infty)$, $V \in L^1_{loc}(M \times [0, \infty))$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$ a.e. in M. Let u be a nonnegative weak solution of problem* (1.1). *Assume condition* **HP1**. *Then $u = 0$ a.e. in S.*

Theorem 1.3. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V > 0$ a.e. in $M \times (0, \infty)$, $V \in L^1_{loc}(M \times [0, \infty))$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$ a.e. in M. Let u be a nonnegative weak solution of problem* (1.1). *Assume condition* **HP2**. *Then $u = 0$ a.e. in S.*

We should note that, to the best of our knowledge, no nonexistence results for linear or nonlinear parabolic equations on complete, noncompact Riemannian manifolds have been obtained in the literature under conditions similar to **HP1** and **HP2**, nor using the techniques that we exploit to prove Theorems 1.2 and 1.3. Even if Theorems 1.2 and 1.3 can be regarded as the natural parabolic counterparts of the results in [16] for elliptic equations, their proofs are substantially different from those in the elliptic case. Moreover, we should also observe that in [16] a nonexistence result for the stationary problem was obtained under a different assumption than the stationary counterparts of the conditions **HP1** and **HP2** introduced in the present work (see [16, condition **HP3**]). An analogous result which could give rise to nontrivial applications cannot be deduced using our methods for parabolic equations, and the question whether a hypothesis corresponding to [16, condition **HP3**] can be introduced also in the parabolic setting in order to prove nonexistence results still remains to be understood.

1.2. **Applications.** This subsection is devoted to the discussion of some consequences of Theorems 1.2 and 1.3 and to comparison with existing results in the literature.

Corollary 1.4. *Let $(M, g) = (\mathbb{R}^m, g_{flat})$, $V \equiv 1$, $p > 1$. Suppose that*

$$\max\{1, p-1\} < q \leq \frac{p}{m} + p - 1 \,. \tag{1.12}$$

Let u be a nonnegative weak solution of problem (1.1). *Then $u = 0$ a.e. in S.*

Note that condition (1.12) in particular requires that $p > \frac{2m}{m+1}$. Note also that Corollary 1.4 agrees with results in [20]. Furthermore, for $p = 2$ we recover the results on the Laplace operator in [5, 13].

Corollary 1.5. *Let M be a complete noncompact Riemannian manifold, $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$ a.e. in M. Suppose the potential $V \in L^1_{loc}(M \times [0, \infty))$ satisfies*

$$V(x,t) \geq f(t)h(x) \quad \text{for a.e. } (x,t) \in S, \tag{1.13}$$

where $f : (0, \infty) \to \mathbb{R}$, $h : M \to \mathbb{R}$ are two functions satisfying

$$0 < f(t) \leq C(1+t)^{\alpha} \ \text{ for a.e. } t \in (0, \infty) \qquad \text{and} \qquad 0 < h(x) \leq C(1 + r(x))^{\beta} \ \text{ for a.e. } x \in M \tag{1.14}$$

and

$$\int_0^T f(t)^{-\frac{1}{q-1}}\, dt \leq C T^{\sigma_2} (\log T)^{\delta_2} \,, \qquad \int_0^T f(t)^{-\frac{p-1}{q-p+1}}\, dt \leq C T^{\sigma_4} (\log T)^{\delta_4} \,, \tag{1.15}$$

$$\int_{B_R} h(x)^{-\frac{1}{q-1}}\, d\mu \leq C R^{\sigma_1} (\log R)^{\delta_1} \,, \qquad \int_{B_R} h(x)^{-\frac{p-1}{q-p+1}}\, d\mu \leq C R^{\sigma_3} (\log R)^{\delta_3} \tag{1.16}$$

for T, R large enough, with $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \delta_1, \delta_2, \delta_3, \delta_4 \geq 0$ and $C > 0$. Assume that

i) $\delta_1 + \delta_2 < \frac{1}{q-1}$, $\quad \delta_3 + \delta_4 < \frac{p-1}{q-p+1}$;
ii) $0 \leq \sigma_2 \leq \frac{q}{q-1}$, $\quad 0 \leq \sigma_3 \leq \frac{pq}{q-p+1}$;
iii) *if* $\sigma_2 = \frac{q}{q-1}$ *then* $\sigma_1 = 0$, *if* $\sigma_3 = \frac{pq}{p-q+1}$ *then* $\sigma_4 = 0$;
iv) $\sigma_1 \sigma_4 \leq \left(\frac{q}{q-1} - \sigma_2\right)\left(\frac{pq}{q-p+1} - \sigma_3\right)$.

Then problem (1.1) *does not admit any nontrivial nonnegative weak solution.*

Corollary 1.6. *Let M be a complete noncompact Riemannian manifold, $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$ a.e. in M. Assume that $V \in L^1_{loc}(M \times [0, \infty))$ satisfies condition* (1.13) *with*

$f:(0,\infty)\to\mathbb{R}$, $h:M\to\mathbb{R}$ *such that*

$$
\begin{aligned}
&C^{-1}(1+t)^{-\alpha}\le f(t)\le C(1+t)^{\alpha} && \text{for a.e. } t\in(0,\infty)\\
&C^{-1}(1+r(x))^{-\beta}\le h(x)\le C(1+r(x))^{\beta} && \text{for a.e. } x\in M
\end{aligned}
\tag{1.17}
$$

and (1.15), (1.16) *hold for* T,R *sufficiently large,* $\alpha,\beta,\sigma_1,\sigma_2,\sigma_3,\sigma_4,\delta_1,\delta_2,\delta_3,\delta_4\ge 0$ *and* $C>0$. *Suppose that*

i) $\delta_1+\delta_2\le\frac{1}{q-1}\,,\quad \delta_3+\delta_4\le\frac{p-1}{q-p+1}\,;$
ii) $0\le\sigma_2\le\frac{q}{q-1}\,,\quad 0\le\sigma_3\le\frac{pq}{q-p+1}\,;$
iii) *if* $\sigma_2=\frac{q}{q-1}$ *then* $\sigma_1=0$, *if* $\sigma_3=\frac{pq}{p-q+1}$ *then* $\sigma_4=0$;
iv) $\sigma_1\sigma_4\le\left(\frac{q}{q-1}-\sigma_2\right)\left(\frac{pq}{q-p+1}-\sigma_3\right).$

Then problem (1.1) *does not admit any nontrivial nonnegative weak solution.*

Remark 1.7. i) We explicitly note that the hypotheses in Corollaries 1.5 and 1.6 allow for a potential V that can also be independent of $x\in M$ or of $t\in[0,\infty)$.

ii) In the particular case of the Laplace–Beltrami operator, i.e. for $p=2$, from Corollaries 1.5, 1.6 we have the following results:

Let V *satisfy condition* (1.13), *with* $f:(0,\infty)\to\mathbb{R}$, $hM\to\mathbb{R}$ *such that* (1.14) *holds and*

$$
\int_{B_R} h(x)^{-\frac{1}{q-1}}\,d\mu\le CR^{\sigma_1}(\log R)^{\delta_1}\,,\qquad \int_0^T f(t)^{-\frac{1}{q-1}}\,dt\le CT^{\sigma_2}(\log T)^{\delta_2}
\tag{1.18}
$$

for T,R *large enough, with* $\alpha,\beta,\sigma_1,\sigma_2,\delta_1,\delta_2\ge 0$, $C>0$ *and*

$$
\delta_1+\delta_2<\frac{1}{q-1}\,,\qquad \sigma_1+2\sigma_2\le\frac{2q}{q-1}.
$$

Then there exists no nonnegative, nontrivial weak solution of problem (1.1) *with* $p=2$.

Similarly, if condition (1.13) *on* V *holds with* f,h *satisfying* (1.17) *and* (1.18) *for* T,R *sufficiently large,* $\alpha,\beta,\sigma_1,\sigma_2,\delta_1,\delta_2\ge 0$, $C>0$ *and if*

$$
\delta_1+\delta_2\le\frac{1}{q-1}\,,\qquad \sigma_1+2\sigma_2\le\frac{2q}{q-1}\,,
$$

then there exists no nonnegative, nontrivial weak solution of problem (1.1) *with* $p=2$.

We should note that, even if in view of Remark 1.7-i) problem (1.3) on the hyperbolic space could in principle be addressed, we cannot actually obtain nonexistence results for it using our results. In fact, condition (1.16) is not satisfied if $M=\mathbb{H}^m$ and $h\equiv 1$, due to the exponential volume growth of geodesic balls in the hyperbolic space. Therefore, we do not recover the results given in [1] (see also [24]). This is essentially due to the fact that in [1] spectral analysis and heat kernel estimates on $\mathbb{H}^m$ have been used. Similar methods have also been used on Cartan-Hadamard manifolds in [24]. Clearly, such tools are not at disposal on general Riemannian manifolds, that are the object of our investigation. On the other hand, our hypotheses **HP1** and **HP2** include a large class of Riemannian manifolds for which results in [1] or in [24] cannot be applied. In particular, this includes the case of Riemannian manifolds that satisfy $(a),(b),(c)$ above, also treated in [31].

In [31] quite different methods from ours have been employed, but also porous medium type nonlinear operators have been considered. However, we remark that in this work we introduce new techniques in the setting of parabolic equations on Riemannian manifolds. We obtain completely new results in the case of the p-Laplace operator, which improve on those already present in the literature even in the particular case of semililinear equations involving the Laplacian. Indeed, we obtain more general nonexistence results than those in [31] (see Example 4.1 below).

The paper is organized as follows: in Section 2 we prove some preliminary results, that will be used in the proof of the theorems and corollaries stated in the Introduction; Section 3 contains the proof of Theorems 1.2 and 1.3, while Section 4 is devoted to the proof of the Corollaries.

2. Auxiliary results

We begin with

Definition 2.1. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V > 0$ a.e. in $M \times (0, \infty)$, $V \in L^1_{loc}(M \times [0, \infty))$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$ a.e. in M. We say that $u \in W^{1,p}_{loc}(M \times [0, \infty)) \cap L^q_{loc}(M \times [0, \infty); V d\mu dt)$ is a* weak solution *of problem (1.1) if $u \geq 0$ a.e. in $M \times (0, \infty)$ and for every $\psi \in W^{1,p}(M \times [0, \infty))$, with $\psi \geq 0$ a.e. in $M \times [0, \infty)$ and compact support, one has*

$$\int_0^\infty \int_M \psi u^q V \, d\mu dt \leq \int_0^\infty \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle \, d\mu dt - \int_0^\infty \int_M u \, \partial_t \psi \, d\mu dt - \int_M u_0 \psi(x, 0) \, d\mu. \tag{2.1}$$

The next lemmas will be the crucial tools we will use in the proof of Theorems 1.2 and 1.3.

Lemma 2.2. *Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in \frac{1}{2}(-\min\{1, p-1\}, 0)$, every nonnegative weak solution u of problem (1.1) and every $\varphi \in \mathrm{Lip}(M \times [0, \infty))$ with compact support and $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned} &\frac{1}{2} \int_0^\infty \int_M V u^{q+\alpha} \varphi^s \, d\mu dt + \frac{3}{4} |\alpha| \int_0^\infty \int_M |\nabla u|^p u^{\alpha-1} \varphi^s \, d\mu dt \\ &\quad \leq C \left\{ |\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_M |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} \, d\mu dt + \int_0^\infty \int_M |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} \, d\mu dt \right\}. \end{aligned} \tag{2.2}$$

Proof. For any $\varepsilon > 0$ let $u_\varepsilon := u + \varepsilon$. Define $\psi = u_\varepsilon^\alpha \varphi^s$; then ψ is an admissible test function for problem (1.1), with

$$\nabla \psi = \alpha u_\varepsilon^{\alpha-1} \varphi^s \nabla u + s \varphi^{s-1} u_\varepsilon^\alpha \nabla \varphi, \quad \partial_t \psi = \alpha u_\varepsilon^{\alpha-1} \varphi^s \partial_t u + s \varphi^{s-1} u_\varepsilon^\alpha \partial_t \varphi.$$

Inequality (2.1) gives

$$\int_0^\infty \int_M u^q u_\varepsilon^\alpha \varphi^s V \, d\mu dt \leq \alpha \int_0^\infty \int_M |\nabla u|^p u_\varepsilon^{\alpha-1} \varphi^s \, d\mu dt + s \int_0^\infty \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle u_\varepsilon^\alpha \varphi^{s-1} \, d\mu dt + I, \tag{2.3}$$

where

$$I = -\alpha \int_0^\infty \int_M u_\varepsilon^{\alpha-1} \varphi^s u \partial_t u \, d\mu dt - s \int_0^\infty \int_M u u_\varepsilon^\alpha \varphi^{s-1} \partial_t \varphi \, d\mu dt - \int_M u_0 (u_0 + \varepsilon)^\alpha \varphi^s(x, 0) \, d\mu. \tag{2.4}$$

Now we have

$$\begin{aligned} -\alpha \int_0^\infty \int_M u_\varepsilon^{\alpha-1} \varphi^s u \partial_t u \, d\mu dt &= -\alpha \int_0^\infty \int_M u_\varepsilon^\alpha \varphi^s \partial_t u \, d\mu dt - \alpha\varepsilon \int_0^\infty \int_M u_\varepsilon^{\alpha-1} \varphi^s \partial_t u \, d\mu dt \\ &= -\frac{\alpha}{\alpha+1} \int_0^\infty \int_M \partial_t \left(u_\varepsilon^{\alpha+1}\right) \varphi^s \, d\mu dt + \varepsilon \int_0^\infty \int_M \partial_t \left(u_\varepsilon^\alpha\right) \varphi^s \, d\mu dt \, . \end{aligned}$$

Since $u_\varepsilon^\alpha, u_\varepsilon^{\alpha+1} \in W^{1,p}_{loc}(M \times [0, \infty))$ with $p > 1$ and since $\varphi^s \in W^{1,p'}(M \times [0, \infty))$ and has compact support, integrating by parts we obtain

$$\begin{aligned} -\alpha \int_0^\infty \int_M u_\varepsilon^{\alpha-1} \varphi^s u \partial_t u \, d\mu dt =& \frac{\alpha s}{\alpha+1} \int_0^\infty \int_M \varphi^{s-1} u_\varepsilon^{\alpha+1} \partial_t \varphi \, d\mu dt - \varepsilon s \int_0^\infty \int_M u_\varepsilon^\alpha \varphi^{s-1} \partial_t \varphi \, d\mu dt \\ &+ \frac{\alpha}{\alpha+1} \int_M \varphi^s(x, 0) (u_0 + \varepsilon)^{\alpha+1} \, d\mu - \varepsilon \int_M \varphi^s(x, 0) (u_0 + \varepsilon)^\alpha \, d\mu, \end{aligned}$$

thus, recalling that $u_\varepsilon = u + \varepsilon$, we have

$$I = -\frac{s}{\alpha+1} \int_0^\infty \int_M u_\varepsilon^{\alpha+1} \varphi^{s-1} \partial_t \varphi \, d\mu dt - \frac{1}{\alpha+1} \int_M (u_0 + \varepsilon)^{\alpha+1} \varphi^s(x, 0) \, d\mu. \tag{2.5}$$

This, combined with (2.3), yields

(2.6)
$$\int_0^\infty\int_M u^q u_\varepsilon^\alpha \varphi^s V\,d\mu dt \le \alpha\int_0^\infty\int_M |\nabla u|^p u_\varepsilon^{\alpha-1}\varphi^s\,d\mu dt + s\int_0^\infty\int_M |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle u_\varepsilon^\alpha\varphi^{s-1}\,d\mu dt - \frac{s}{\alpha+1}\int_0^\infty\int_M u_\varepsilon^{\alpha+1}\varphi^{s-1}\partial_t\varphi\,d\mu dt - \frac{1}{\alpha+1}\int_M (u_0+\varepsilon)^{\alpha+1}\varphi^s(x,0)\,d\mu$$

and then

$$|\alpha|\int_0^\infty\int_M |\nabla u|^p u_\varepsilon^{\alpha-1}\varphi^s\,d\mu dt + \int_0^\infty\int_M u^q u_\varepsilon^\alpha\varphi^s V\,d\mu dt + \frac{1}{\alpha+1}\int_M (u_0+\varepsilon)^{\alpha+1}\varphi^s(x,0)\,d\mu \le s\int_0^\infty\int_M |\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle u_\varepsilon^\alpha\varphi^{s-1}\,d\mu dt - \frac{s}{\alpha+1}\int_0^\infty\int_M u_\varepsilon^{\alpha+1}\varphi^{s-1}\partial_t\varphi\,d\mu dt. \tag{2.7}$$

Now we estimate the first integral in the right-hand side of (2.7) using Young's inequality, obtaining

$$\begin{aligned} s&\int_0^\infty\int_M \varphi^{s-1}u_\varepsilon^\alpha|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,d\mu dt\\ &\le s\int_0^\infty\int_M \varphi^{s-1}u_\varepsilon^\alpha|\nabla u|^{p-1}|\nabla\varphi|\,d\mu dt\\ &= \int_0^\infty\int_M \left(|\alpha|^{\frac{p-1}{p}}\varphi^{s\frac{p-1}{p}}u_\varepsilon^{-(|\alpha|+1)\frac{p-1}{p}}|\nabla u|^{p-1}\right)\left(s|\alpha|^{-\frac{p-1}{p}}\varphi^{\frac{s}{p}-1}u_\varepsilon^{1-\frac{|\alpha|+1}{p}}|\nabla\varphi|\right)d\mu dt\\ &\le \frac{|\alpha|}{4}\int_0^\infty\int_M \varphi^s u_\varepsilon^{\alpha-1}|\nabla u|^p\,d\mu dt + \frac{s}{p}\left[\frac{4s(p-1)}{|\alpha|p}\right]^{p-1}\int_0^\infty\int_M \varphi^{s-p}u_\varepsilon^{p-(|\alpha|+1)}|\nabla\varphi|^p\,d\mu dt. \end{aligned}$$

From (2.7) we deduce

$$\frac{3}{4}|\alpha|\int_0^\infty\int_M |\nabla u|^p u_\varepsilon^{\alpha-1}\varphi^s\,d\mu dt + \int_0^\infty\int_M u^q u_\varepsilon^\alpha\varphi^s V\,d\mu dt + \frac{1}{\alpha+1}\int_M (u_0+\varepsilon)^{\alpha+1}\varphi^s(x,0)\,d\mu \le \frac{s}{p}\left[\frac{4s(p-1)}{|\alpha|p}\right]^{p-1}\int_0^\infty\int_M \varphi^{s-p}u_\varepsilon^{p-(|\alpha|+1)}|\nabla\varphi|^p\,d\mu dt + \frac{s}{\alpha+1}\int_0^\infty\int_M u_\varepsilon^{\alpha+1}\varphi^{s-1}|\partial_t\varphi|\,d\mu dt. \tag{2.8}$$

Note that, by Young's inequality,

$$\begin{aligned} \frac{s}{p}&\left[\frac{4s(p-1)}{|\alpha|p}\right]^{p-1}\int_0^\infty\int_M \varphi^{s-p}u_\varepsilon^{p-(|\alpha|+1)}|\nabla\varphi|^p\,d\mu dt\\ &= \frac{s}{p}\left[\frac{4s(p-1)}{|\alpha|p}\right]^{p-1}\int_0^\infty\int_M \left(u_\varepsilon^{p+\alpha-1}\varphi^{s\left(\frac{p+\alpha-1}{q+\alpha}\right)}V^{\frac{p+\alpha-1}{q+\alpha}}\right)\left(|\nabla\varphi|^p\varphi^{s-p-s\left(\frac{p+\alpha-1}{q+\alpha}\right)}V^{-\frac{p+\alpha-1}{q+\alpha}}\right)d\mu dt\\ &\le \frac{1}{4}\int_0^\infty\int_M u_\varepsilon^{q+\alpha}\varphi^s V\,d\mu dt + C(\alpha,s)\int_0^\infty\int_M |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\varphi^{s-p\left(\frac{q+\alpha}{q-p+1}\right)}V^{-\frac{p+\alpha-1}{q+p-1}}\,d\mu dt \end{aligned}$$

and

$$\begin{aligned} &\frac{s}{\alpha+1}\int_0^\infty\int_M u_\varepsilon^{\alpha+1}\varphi^{s-1}|\partial_t\varphi|\,d\mu dt\\ &\quad= \frac{s}{\alpha+1}\int_0^\infty\int_M \left(u_\varepsilon^{\alpha+1}\varphi^{s\left(\frac{\alpha+1}{q+\alpha}\right)}V^{\frac{\alpha+1}{q+\alpha}}\right)\left(\varphi^{-s\left(\frac{\alpha+1}{q+\alpha}\right)+s-1}|\partial_t\varphi|V^{-\frac{\alpha+1}{q+\alpha}}\right)d\mu dt\\ &\quad\le \frac{1}{4}\int_0^\infty\int_M u_\varepsilon^{q+\alpha}\varphi^s V\,d\mu dt + D(\alpha,s)\int_0^\infty\int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\varphi^{s-\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt \end{aligned}$$

where

$$C(\alpha,s) = \frac{s}{p}\left[\frac{4s(p-1)}{|\alpha|p}\right]^{p-1}\frac{q-p+1}{q+\alpha}\left[\frac{(q+\alpha)p}{4s(p+\alpha-1)}\left(\frac{4s(p-1)}{p|\alpha|}\right)^{1-p}\right]^{-\frac{p+\alpha-1}{q-p+1}}$$

and

$$D(\alpha,s) = \frac{s}{\alpha+1}\frac{q-1}{q+\alpha}\left(\frac{4s}{q+\alpha}\right)^{\frac{\alpha+1}{q-1}}.$$

Substituting in (2.8) we have

$$\frac{3}{4}|\alpha|\int_0^\infty\int_M |\nabla u|^p u_\varepsilon^{\alpha-1}\varphi^s\,d\mu dt+\int_0^\infty\int_M u^q u_\varepsilon^\alpha\varphi^s V\,d\mu dt$$
$$-\frac{1}{2}\int_0^\infty\int_M u_\varepsilon^{q+\alpha}\varphi^s V\,d\mu dt+\frac{1}{\alpha+1}\int_M (u_0+\varepsilon)^{\alpha+1}\varphi^s(x,0)\,d\mu$$
$$\leq C(\alpha,s)\int_0^\infty\int_M |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\varphi^{s-p\left(\frac{q+\alpha}{q-p+1}\right)}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt+D(\alpha,s)\int_0^\infty\int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\varphi^{s-\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt.$$

Now letting $\varepsilon\to 0$ and applying Fatou's lemma, we get

$$\frac{3}{4}|\alpha|\int_0^\infty\int_M |\nabla u|^p u^{\alpha-1}\varphi^s\,d\mu dt+\frac{1}{2}\int_0^\infty\int_M V u^{q+\alpha}\varphi^s\,d\mu dt$$
$$\leq C(\alpha,s)\int_0^\infty\int_M |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\varphi^{s-p\left(\frac{q+\alpha}{q-p+1}\right)}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt$$
$$+D(\alpha,s)\int_0^\infty\int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\varphi^{s-\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\,, \tag{2.9}$$

where we use the convention $|\nabla u|^p u^{\alpha-1}\equiv 0$ on the set where $u=0$, since $\nabla u=0$ a.e. on level sets of u. Now since there exists a positive constant C, depending on s,p,q, such that

$$C(\alpha,s)\leq C|\alpha|^{-\frac{(p-1)q}{q-p+1}},\quad D(\alpha,s)\leq C,$$

and since $0\leq\varphi\leq 1$ on $M\times[0,\infty)$, by our assumptions on s the conclusion follows from (2.9).

Lemma 2.3. *Let $s\geq\max\left\{1,\frac{q+1}{q-1},\frac{2pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C>0$ such that for every nonnegative weak solution u of equation (1.1), every function $\varphi\in\mathrm{Lip}(S)$ with compact support and $0\leq\varphi\leq 1$ and every $\alpha\in\left(-\frac{1}{2}\min\left\{1,p-1,q-1,\frac{q-p+1}{p-1}\right\},0\right)$ one has*

(2.10)

$$\int_0^\infty\int_M \varphi^s u^q V\,d\mu dt$$
$$\leq C\left(|\alpha|^{-1-\frac{(p-1)q}{(q-p+1)}}\int_0^\infty\int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt+|\alpha|^{-1}\int_0^\infty\int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right)^{\frac{p-1}{p}}$$
$$\times\left(\int\int_{S\setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\,d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{pq}}\left(\int\int_{S\setminus K}\varphi^s u^q V\,d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{pq}}$$
$$+C\left(\int\int_{S\setminus K}\varphi^s u^{q+\alpha}V\,d\mu dt\right)^{\frac{1}{q+\alpha}}\left(\int_0^\infty\int_M V^{-\frac{1}{q+\alpha-1}}|\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.$$

with $K=\{(x,t)\in S:\varphi(x,t)=1\}$.

Proof. Under our assumptions $\psi=\varphi^s$ is a feasible test function in equation (2.1). Thus we obtain

(2.11)

$$\int_0^\infty\int_M \varphi^s u^q V\,d\mu dt\leq s\int_0^\infty\int_M \varphi^{s-1}|\nabla u|^{p-2}\langle\nabla u,\nabla\varphi\rangle\,d\mu dt-s\int_0^\infty\int_M u\varphi^{s-1}\partial_t\varphi\,d\mu dt-\int_M u_0(x)\varphi^s(x,0)\,d\mu\,.$$

Through an application of Hölder's inequality we obtain

(2.12)

$$\int_0^\infty\int_M u\varphi^{s-1}|\partial_t\varphi|\,d\mu dt\leq\left(\int\int_{S\setminus K}u^{q+\alpha}V\varphi^s\,d\mu dt\right)^{\frac{1}{q+\alpha}}\left(\int_0^\infty\int_M V^{-\frac{1}{q+\alpha-1}}\varphi^{\frac{(s-1)(q+\alpha)-s}{q+\alpha-1}}|\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.$$

On the other hand, using again Hölder's inequality we obtain

$$
\begin{aligned}
(2.13)\qquad & \int_0^\infty \int_M s\varphi^{s-1}|\nabla u|^{p-1}|\nabla\varphi|\,d\mu dt \\
&= s\int_0^\infty \int_M \left(\varphi^{\frac{p-1}{p}s}|\nabla u|^{p-1}u^{-\frac{p-1}{p}(1-\alpha)}\right)\left(\varphi^{\frac{s}{p}-1}u^{\frac{p-1}{p}(1-\alpha)}|\nabla\varphi|\right)d\mu dt \\
&\leq s\left(\int_0^\infty \int_M \varphi^s|\nabla u|^p u^{\alpha-1}\,d\mu dt\right)^{\frac{p-1}{p}}\left(\int_0^\infty \int_M \varphi^{s-p}u^{(p-1)(1-\alpha)}|\nabla\varphi|^p\,d\mu dt\right)^{\frac{1}{p}}.
\end{aligned}
$$

Moreover from equation (2.2) we deduce

$$
\begin{aligned}
(2.14)\qquad \int_0^\infty \int_M \varphi^s|\nabla u|^p u^{\alpha-1}\,d\mu dt &\leq C|\alpha|^{-1-\frac{(p-1)q}{q-p+1}}\int_0^\infty \int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt \\
&\quad + C|\alpha|^{-1}\int_0^\infty \int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt,
\end{aligned}
$$

with $C>0$ depending on s. Thus from (2.11), (2.12), (2.13) and (2.14) we obtain

$$
\begin{aligned}
(2.15)\quad & \int_0^\infty \int_M \varphi^s u^q V\,d\mu dt \\
&\leq C\left\{|\alpha|^{-1-\frac{(p-1)q}{q-p+1}}\int_0^\infty \int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt + |\alpha|^{-1}\int_0^\infty \int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right\}^{\frac{p-1}{p}} \\
&\times\left(\int_0^\infty \int_M \varphi^{s-p}u^{(p-1)(1-\alpha)}|\nabla\varphi|^p\,d\mu dt\right)^{\frac{1}{p}} \\
&+ C\left(\int\int_{S\setminus K} u^{q+\alpha}V\varphi^s\,d\mu dt\right)^{\frac{1}{q+\alpha}}\left(\int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}}\varphi^{\frac{(s-1)(q+\alpha)-s}{q+\alpha-1}}|\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned}
$$

We use again Hölder's inequality with exponents

$$
a=\frac{q}{(1-\alpha)(p-1)},\qquad b=\frac{a}{a-1}=\frac{q}{q-(1-\alpha)(p-1)}
$$

to obtain

$$
\begin{aligned}
& \int_0^\infty \int_M \varphi^{s-p}u^{(p-1)(1-\alpha)}|\nabla\varphi|^p\,d\mu dt \\
&\leq \left(\int\int_{S\setminus K} \varphi^s u^q V\,d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{q}}\left(\int\int_{S\setminus K} \varphi^{s-\frac{pq}{q-(1-\alpha)(p-1)}}V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\,d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{q}}.
\end{aligned}
$$

Substituting into (2.15) we have

(2.16)

$$
\begin{aligned}
& \int_0^\infty \int_M \varphi^s u^q V\,d\mu dt \\
&\leq C\left\{|\alpha|^{-1-\frac{(p-1)q}{q-p+1}}\int_0^\infty \int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt + |\alpha|^{-1}\int_0^\infty \int_M |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right\}^{\frac{p-1}{p}} \\
&\times\left(\int\int_{S\setminus K} \varphi^s u^q V\,d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{qp}}\left(\int\int_{S\setminus K} \varphi^{s-\frac{pq}{q-(1-\alpha)(p-1)}}V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\,d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{qp}} \\
&+ C\left(\int\int_{S\setminus K} u^{q+\alpha}V\varphi^s\,d\mu dt\right)^{\frac{1}{q+\alpha}}\left(\int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}}\varphi^{\frac{(s-1)(q+\alpha)-s}{q+\alpha-1}}|\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned}
$$

Now inequality (2.10) immediately follows from the previous relation, by our assumptions on s,α and since $0\leq\varphi\leq 1$.

Corollary 2.4. *Under the hypotheses of Lemma 2.3 one has*

$$\begin{aligned}(2.17)\quad &\int_0^\infty\int_M \varphi^s u^q V\,d\mu dt\\ &\le C\left\{|\alpha|^{-1-\frac{(p-1)q}{q-p+1}}\int_0^\infty\int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt+|\alpha|^{-1}\int_0^\infty\int_M|\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right\}^{\frac{p-1}{p}}\\ &\times\left(\int\int_{S\setminus K}\varphi^s u^q V\,d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{qp}}\left(\int\int_{S\setminus K}V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\,d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{qp}}\\ &+C\left(|\alpha|^{-\frac{(p-1)q}{q-p+1}}\int_0^\infty\int_M|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt+\int_0^\infty\int_M|\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right)^{\frac{1}{q+\alpha}}\\ &\times\left(\int_0^\infty\int_M V^{-\frac{1}{q+\alpha-1}}|\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.\end{aligned}$$

Proof. The conclusion immediately follows combining (2.10) and (2.2).

Lemma 2.5. *Let* $s\ge\max\left\{1,\frac{q+1}{q-1},\frac{2pq}{q-p+1}\right\}$ *be fixed. Then there exists a constant* $C>0$ *such that for every nonnegative weak solution* u *of equation* (1.1)*, every function* $\varphi\in\mathrm{Lip}(S)$ *with compact support and* $0\le\varphi\le1$ *and every* $\alpha\in\left(-\frac12\min\left\{1,p-1,q-1,\frac{q-p+1}{p-1}\right\},0\right)$ *one has*

(2.18)

$$\begin{aligned}&\int_0^\infty\int_M \varphi^s u^q V\,d\mu dt\\ &\le C\left(|\alpha|^{-1-\frac{(p-1)q}{(q-p+1)}}\int_0^\infty\int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\,d\mu dt+|\alpha|^{-1}\int_0^\infty\int_M|\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\right)^{\frac{p-1}{p}}\\ &\times\left(\int\int_{S\setminus K}V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\,d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{pq}}\left(\int\int_{S\setminus K}\varphi^s u^q V\,d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{pq}}\\ &+C\left(\int\int_{S\setminus K}\varphi^s u^q V\,d\mu dt\right)^{\frac1q}\left(\int_0^\infty\int_M V^{-\frac{1}{q-1}}|\partial_t\varphi|^{\frac{q}{q-1}}\,d\mu dt\right)^{\frac{q-1}{q}}.\end{aligned}$$

with $S=M\times[0,\infty)$ *and* $K=\{(x,t)\in S:\varphi(x,t)=1\}$.

Proof. Inequality (2.18) can be proved in the same way as (2.10), where the only difference with respect to the above argument is that in this case one has to use inequality (2.12) with $\alpha=0$. □

3. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. For any fixed $R>0$ sufficiently large, let $\alpha:=-\frac{1}{\log R}$. Fix any $C_1>\frac{C_0+\theta_2+1}{\theta_2}$ with C_0 and θ_2 as in **HP1**. Define for all $(x,t)\in S$

$$(3.1)\qquad \varphi(x,t):=\begin{cases}1 & \text{if } (x,t)\in E_R\,,\\ \left(\frac{r(x)^{\theta_2}+t^{\theta_1}}{R^{\theta_2}}\right)^{C_1\alpha} & \text{if } (x,t)\in E_R^c\,,\end{cases}$$

and for all $n\in\mathbb{N}$

$$(3.2)\qquad \eta_n(x,t):=\begin{cases}1 & \text{if } (x,t)\in E_{nR}\,,\\ 2-\frac{r(x)^{\theta_2}+t^{\theta_1}}{(nR)^{\theta_2}} & \text{if } (x,t)\in E_{2^{1/\theta_2}nR}\setminus E_{nR}\,,\\ 0 & \text{if } (x,t)\in E^c_{2^{1/\theta_2}nR}\,.\end{cases}$$

Let

$$\varphi_n(x,t) := \eta_n(x,t)\varphi(x,t) \quad \text{for all } (x,t) \in S\,. \tag{3.3}$$

We have $\varphi_n \in \mathrm{Lip}(S)$ with $0 \le \varphi_n \le 1$; furthermore,

$$\partial_t\varphi_n = \eta_n\partial_t\varphi + \varphi\partial_t\eta_n, \qquad \nabla\varphi_n = \eta_n\nabla\varphi + \varphi\nabla\eta_n$$

a.e. in S, and for every $a \ge 1$

$$|\partial_t\varphi_n|^a \le 2^{a-1}(|\partial_t\varphi|^a + \varphi^a|\partial_t\eta_n|^a), \qquad |\nabla\varphi_n|^a \le 2^{a-1}(|\nabla\varphi|^a + \varphi^a|\nabla\eta_n|^a)$$

a.e. in S. Now we use φ_n in formula (2.2), with any fixed $s \ge \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$, and we see that for some positive constant C and for every $n \in \mathbb{N}$ and every small enough $|\alpha| > 0$, we have

$$\begin{aligned}
&\int_0^\infty\int_M Vu^{q+\alpha}\varphi_n^s\,d\mu dt \\
&\le C\Big\{|\alpha|^{-\frac{(p-1)q}{q-p+1}}\int_0^\infty\int_M|\nabla\varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt + \int_0^\infty\int_M|\partial_t\varphi_n|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\Big\} \\
&\le C\Big\{|\alpha|^{-\frac{(p-1)q}{q-p+1}}\int_0^\infty\int_M|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt \\
&\quad + |\alpha|^{-\frac{(p-1)q}{q-p+1}}\int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}\varphi^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\eta_n|^{\frac{p(q+\alpha)}{q-p+1}}d\mu dt \\
&\quad + \int_0^\infty\int_M|\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt + \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}\varphi^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}|\partial_t\eta_n|^{\frac{q+\alpha}{q-1}}d\mu dt\Big\} \\
&\le C\Big\{|\alpha|^{-\frac{(p-1)q}{q-p+1}}(I_1+I_2)+I_3+I_4\Big\},
\end{aligned} \tag{3.4}$$

where

$$I_1 := \int_0^\infty\int_M|\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}\,d\mu dt\,, \tag{3.5}$$

$$I_2 := \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}\varphi^{\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\eta_n|^{\frac{p(q+\alpha)}{q-p+1}}d\mu dt\,, \tag{3.6}$$

$$I_3 := \int_0^\infty\int_M|\partial_t\varphi|^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,d\mu dt\,, \tag{3.7}$$

$$I_4 := \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}\varphi^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}|\partial_t\eta_n|^{\frac{q+\alpha}{q-1}}d\mu dt\,. \tag{3.8}$$

In view of (3.1) and (3.2) and assumption **HP1**-(ii) (see (1.7)) with $\varepsilon = -\frac{\alpha}{q-p+1} > 0$, for every $n \in \mathbb{N}$ and every small enough $|\alpha| > 0$ we get

$$\begin{aligned}
I_2 &\le \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}\left[\theta_2\left(\frac{1}{nR}\right)^{\theta_2}r(x)^{\theta_2-1}|\nabla r(x)|\right]^{\frac{p(q+\alpha)}{q-p+1}}n^{C_1\theta_2\alpha\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}d\mu dt \\
&\le C(nR)^{-\frac{\theta_2p(q+\alpha)}{q-p+1}}n^{C_1\theta_2\alpha\frac{p(q+\alpha)}{q-p+1}}\int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}}r(x)^{(\theta_2-1)\frac{p(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}d\mu dt \\
&\le C(nR)^{-\frac{\theta_2p(q+\alpha)}{q-p+1}}n^{C_1\theta_2\alpha\frac{p(q+\alpha)}{q-p+1}}(nR)^{\frac{\theta_2pq}{q-p+1}+C_0\frac{|\alpha|}{q-p+1}}(\log(nR))^{s_4}\,.
\end{aligned} \tag{3.9}$$

Now note that for any constant $\bar{C} \in \mathbb{R}$ and for $R > 0$ and $\alpha = -\frac{1}{\log R}$ we have

$$R^{|\alpha|\bar{C}} = e^{|\alpha|\bar{C}\log R} = e^{\bar{C}} \le C\,. \tag{3.10}$$

Thus, also using the fact that

$$\frac{|\alpha|[\theta_2p - C_1\theta_2p(q+\alpha)+C_0]}{q-p+1} \le -\frac{|\alpha|}{q-p+1} \,<\, 0\,,$$

from (3.9) we deduce

$$I_2 \le C n^{-\frac{|\alpha|}{q-p+1}}[\log(nR)]^{s_4}\,. \tag{3.11}$$

In a similar way we can estimate I_4, using **HP1**-(i) (see (1.6)). Indeed, for $R>0$ large enough,

$$\begin{aligned} I_4 &\le \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} \left(\frac{\theta_1}{(nR)^{\theta_2}}t^{\theta_1-1}\right)^{\frac{q+\alpha}{q-1}} n^{C_1\theta_2\alpha\frac{q+\alpha}{q-1}}V^{\frac{-1+|\alpha|}{q-1}}d\mu dt \\ &\le C(nR)^{-\theta_2\frac{q+\alpha}{q-1}}n^{C_1\theta_2\alpha\frac{q+\alpha}{q-1}}(nR)^{\frac{\theta_2 q+C_0|\alpha|}{q-1}}[\log(nR)]^{s_2} \\ &\le C n^{\frac{\alpha}{q-1}(-\theta_2+C_1\theta_2 q-C_0+C_1\theta_2\alpha)}[\log(nR)]^{s_2} \;\le\; C n^{-\frac{|\alpha|}{q-1}}[\log(nR)]^{s_2}\,. \end{aligned} \tag{3.12}$$

In order to estimate I_1 we observe that if $f:[0,\infty)\to[0,\infty)$ is a nonincreasing function and if **HP1**-(ii) holds (see (1.7)), then

$$\int\int_{E_R^c} f\big([r(x)^{\theta_2}+t^{\theta_1}]^{\frac{1}{\theta_2}}\big)r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}V^{-\frac{p-1}{q-p+1}+\varepsilon}d\mu dt \le C\int_{R/2^{1/\theta_2}}^{\infty} f(z)z^{\bar{s}_3+C_0\varepsilon-1}(\log z)^{s_4}dz\,, \tag{3.13}$$

for every $0<\varepsilon<\varepsilon_0$ and $R>0$ large enough. This can shown by minor variations in the proof of [12, formula (2.19)].

Now, since for a.e. $x\in M$ we have $|\nabla r(x)|\le 1$, we obtain for a.e. $(x,t)\in S$

$$|\nabla\varphi(x,t)| \le C_1|\alpha|\theta_2\left(\frac{r(x)^{\theta_2}+t^{\theta_1}}{R^{\theta_2}}\right)^{C_1\alpha-1}\frac{r(x)^{\theta_2-1}}{R^{\theta_2}}\,. \tag{3.14}$$

Thus, using (3.10) for every sufficiently large $R>0$ we get

$$\begin{aligned} |\alpha|^{-\frac{(p-1)q}{q-p+1}}I_1 &\le C|\alpha|^{-\frac{(p-1)q}{q-p+1}}\int\int_{E_R^c}V^{-\frac{p+\alpha-1}{q-p+1}}\left[C_1|\alpha|\theta_2\left(\frac{r(x)^{\theta_2}+t^{\theta_1}}{R^{\theta_2}}\right)^{C_1\alpha-1}\frac{r(x)^{\theta_2-1}}{R^{\theta_2}}\right]^{\frac{p(q+\alpha)}{q-p+1}}d\mu dt \\ &\le C|\alpha|^{\frac{p(q+\alpha)-(p-1)q}{q-p+1}}\int\int_{E_R^c}\left\{\left[r(x)^{\theta_2}+t^{\theta_1}\right]^{\frac{1}{\theta_2}}\right\}^{\theta_2\frac{(C_1\alpha-1)p(q+\alpha)}{q-p+1}} r(x)^{(\theta_2-1)p\frac{(q+\alpha)}{q-p+1}}V^{-\frac{p+\alpha-1}{q-p+1}}d\mu dt\,. \end{aligned}$$

Now, using (3.13) with $\varepsilon=\frac{|\alpha|}{q-p+1}$,

$$|\alpha|^{-\frac{(p-1)q}{q-p+1}}I_1 \le C|\alpha|^{\frac{p(q+\alpha)-(p-1)q}{q-p+1}}\int_{R/2^{1/\theta_2}}^{\infty} z^{\theta_2(C_1\alpha-1)\frac{p(q+\alpha)}{q-p+1}+\bar{s}_3+C_0\frac{|\alpha|}{q-p+1}-1}(\log z)^{s_4}dz\,. \tag{3.15}$$

By our choice of C_1 and by the very definition of $\bar{s}_3$ we have

$$b := \theta_2(C_1\alpha-1)\frac{p(q+\alpha)}{q-p+1}+\bar{s}_3+C_0\frac{|\alpha|}{q-p+1} \le -\frac{|\alpha|}{q-p+1}\,.$$

Then using the change of variable $y:=|b|\log z$ in the right hand side of (3.15) we obtain for $\alpha>0$ small enough

$$\begin{aligned} |\alpha|^{-\frac{(p-1)q}{q-p+1}}I_1 &\le C|\alpha|^{\frac{p(q+\alpha)-(p-1)q}{q-p+1}}\int_0^\infty e^{-y}\left(\frac{y}{|b|}\right)^{s_4}\frac{dy}{|b|} \\ &\le C|\alpha|^{\frac{p(q+\alpha)-(p-1)q}{q-p+1}-s_4-1} \;\le C|\alpha|^{\frac{p-1}{q-p+1}-s_4}\,. \end{aligned} \tag{3.16}$$

The term I_3 can be estimated similarly. Indeed, we start noting that if $f:[0,\infty)\to[0,\infty)$ is nonincreasing function and if **HP1**-(i) holds (see (1.6)), then for any sufficiently small $\varepsilon>0$ and every large enough $R>0$ we get

$$\int\int_{E_R^c} f\big([r(x)^{\theta_2}+t^{\theta_1}]^{\frac{1}{\theta_2}}\big)t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}V^{-\frac{1}{q-1}+\varepsilon}d\mu dt \le C\int_{R/2^{1/\theta_2}}^{\infty} f(z)z^{\bar{s}_1+C_0\varepsilon-1}(\log z)^{s_2}dz\,; \tag{3.17}$$

this can be shown by minor changes in the proof of [12, formula (2.19)]. Since for a.e. $(x,t)\in S$

$$|\partial_t\varphi(x,t)| \le C_1|\alpha|\theta_1\left(\frac{r(x)^{\theta_2}+t^{\theta_1}}{R^{\theta_2}}\right)^{C_1\alpha-1}\frac{t^{\theta_1-1}}{R^{\theta_2}}, \tag{3.18}$$

also using (3.10), we have for every $R>0$ large enough

$$I_3 \leq C|\alpha|^{\frac{q+\alpha}{q-1}} \int\int_{E_R^c} \left[(r(x)^{\theta_2}+t^{\theta_1})^{\frac{1}{\theta_2}}\right]^{\theta_2(C_1\alpha-1)\frac{q+\alpha}{q-1}} t^{(\theta_1-1)\frac{q+\alpha}{q-1}} V^{-\frac{1+\alpha}{q-1}} d\mu dt\,.$$

Now due to (3.17) with $\varepsilon=\frac{|\alpha|}{q-1}$

$$I_3 \leq C|\alpha|^{\frac{q+\alpha}{q-1}} \int_{R/2^{1/\theta_2}}^{\infty} z^{\theta_2(C_1\alpha-1)\frac{q+\alpha}{q-1}+C_0\frac{|\alpha|}{q-1}+\bar{s}_1-1} (\log z)^{s_2} dz\,. \tag{3.19}$$

By our choice of C_1 and the very definition of $\bar{s}_1$ we have that

$$\beta:=\theta_2(C_1\alpha-1)\frac{q+\alpha}{q-1}+\bar{s}_1+C_0\frac{|\alpha|}{q-1}\leq -\frac{|\alpha|}{q-1}\,.$$

Using the change of variable $y:=|\beta|\log z$ in (3.19) we obtain

$$\begin{aligned} I_3 &\leq C|\alpha|^{\frac{q}{q-1}} \int_0^{\infty} e^{-y}\left(\frac{y}{|\beta|}\right)^{s_2}\frac{dy}{|\beta|} \\ &\leq C|\alpha|^{\frac{1}{q-1}-s_2}\,. \end{aligned} \tag{3.20}$$

Inserting (3.11), (3.12), (3.16) and (3.20) into (3.4) we obtain for every $n\in\mathbb{N}$ and every sufficiently large $R>0$

$$\begin{aligned} \int\int_{E_R} u^{q+\alpha} V d\mu dt &\leq \int_0^{\infty}\int_M u^{q+\alpha}\varphi_n^s V\, d\mu dt \\ &\leq C\left(|\alpha|^{\frac{p-1}{q-p+1}-s_4}+|\alpha|^{-\frac{(p-1)q}{q-p+1}} n^{-\frac{|\alpha|}{p-q+1}}[\log(nR)]^{s_4}+|\alpha|^{\frac{1}{q-1}-s_2}+n^{-\frac{|\alpha|}{q-1}}[\log(nR)]^{s_2}\right) \end{aligned}$$

with C independent of n and R. Passing to the lim inf as $n\to\infty$ we deduce that

$$\int\int_{E_R} u^{q+\alpha} V d\mu dt \leq C\left(|\alpha|^{\frac{p-1}{q-p+1}-s_4}+|\alpha|^{\frac{1}{q-1}-s_2}\right)\,.$$

Therefore, letting $R\to\infty$ (and thus $\alpha\to 0$), by Fatou's lemma, we have

$$\int_0^{\infty}\int_M u^q V\, d\mu dt = 0$$

in view of our assumptions on s_2, s_4, which concludes the proof.

Proof of Theorem 1.3. We claim that $u^q\in L^1(S, V d\mu dt)$. To see this, we will show that

$$\int_0^{\infty}\int_M u^q V\, d\mu dt \leq A\Big(\int_0^{\infty}\int_M u^q V\, d\mu dt\Big)^{\sigma}+B \tag{3.21}$$

for some constants $A>0$, $B>0$, $0<\sigma<1$. In order to prove (3.21) we consider (2.17) with φ replaced by the family of functions φ_n defined in (3.3), for any fixed $s\geq\max\left\{1,\frac{q+1}{q-1},\frac{2pq}{q-p+1}\right\}$ and $C_1>\max\left\{\frac{1+C_0+\theta_2}{\theta_2},\frac{2(C_0+1)}{\theta_2(q-p+1)},\frac{2(C_0+1)}{\theta_2 q(q-1)}\right\}$ with C_0, θ_2 as in **HP2** and with $R>0$ sufficiently large and

$\alpha = -\frac{1}{\log R}$. Thus we have

$$
\begin{aligned}
(3.22)\qquad & \int_0^\infty \int_M \varphi_n^s u^q V \, d\mu dt \\
& \le C\left(|\alpha|^{-1-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_M V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} \, d\mu dt\right)^{\frac{p-1}{p}} \left(\int\int_{E_R^c} \varphi_n^s u^q V \, d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{qp}} \\
& \quad \times \left(\int\int_{E_R^c} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} \, d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{qp}} \\
& \quad + C\left(|\alpha|^{-1} \int_0^\infty \int_M |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} \, d\mu dt\right)^{\frac{p-1}{p}} \left(\int\int_{E_R^c} \varphi_n^s u^q V \, d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{qp}} \\
& \quad \times \left(\int\int_{E_R^c} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} \, d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{qp}} \\
& \quad + C\Bigg(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_M |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} \, d\mu dt + \int_0^\infty \int_M |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} \, d\mu dt\Bigg)^{\frac{1}{q+\alpha}} \\
& \quad \times \left(\int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}} \, d\mu dt\right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned}
$$

Let us prove that for $R > 0$ large enough, and thus for $|\alpha| = \frac{1}{\log R}$ sufficiently small,

$$
\begin{aligned}
(3.23)\qquad & \limsup_{n\to\infty} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1\right) \le C, \\
(3.24)\qquad & \limsup_{n\to\infty} \left(|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3\right) \le C, \\
(3.25)\qquad & \limsup_{n\to\infty} J_2 \le C, \\
(3.26)\qquad & \limsup_{n\to\infty} J_4 \le C,
\end{aligned}
$$

for some $C > 0$ independent of α, where

$$
\begin{aligned}
(3.27)\qquad J_1 &:= \int_0^\infty \int_M V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} \, d\mu dt, \\
(3.28)\qquad J_2 &:= \int_0^\infty \int_M |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} \, d\mu dt, \\
(3.29)\qquad J_3 &:= \int\int_{E_R^c} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}} \, d\mu dt, \\
(3.30)\qquad J_4 &:= \int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}} \, d\mu dt.
\end{aligned}
$$

Note that

$$
(3.31)\qquad J_1 \le C(I_1 + I_2),
$$

with I_1 and I_2 defined in (3.5) and (3.6), respectively. Due to (1.10) in **HP2**-(ii), by the same arguments used to obtain (3.16) and (3.11) with s_4 replaced by $\bar{s}_4$, for every $n \in \mathbb{N}, R > 0$ large enough and $\alpha = \frac{1}{\log R}$ we have

$$
|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 \le C\left(1 + |\alpha|^{-\frac{(p-1)q}{q-p+1}} n^{-\frac{|\alpha|}{q-p+1}} [\log(nR)]^{\bar{s}_4}\right).
$$

Letting $n \to \infty$ we get (3.23).

Next we observe that

$$
(3.32)\qquad J_2 \le C(I_3 + I_4),
$$

with I_3 and I_4 defined in (3.7) and (3.8), respectively. By the same computations used to obtain (3.20) and (3.12), with s_2 replaced by $\bar{s}_2$, we have for every $n \in \mathbb{N}, R > 0$ large enough and $\alpha = \frac{1}{\log R}$

$$J_2 \leq C\left(1 + n^{-\frac{|\alpha|}{q-1}}[\log(nR)]^{\bar{s}_2}\right).$$

Again, letting $n \to \infty$ we obtain (3.25).

We now proceed to estimate J_4; note that

$$J_4 \leq C(I_5 + I_6), \tag{3.33}$$

where

$$I_5 := \int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi|^{\frac{q+\alpha}{q+\alpha-1}} \, d\mu dt \tag{3.34}$$

$$I_6 := \int_0^\infty \int_M V^{-\frac{1}{q+\alpha-1}} \varphi^{\frac{q+\alpha}{q+\alpha-1}} |\partial_t \eta_n|^{\frac{q+\alpha}{q+\alpha-1}} \, d\mu dt. \tag{3.35}$$

Due to (3.18) and (3.10), we have for every $R > 0$ large enough

$$I_5 \leq C|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int\int_{E_R^c} \left[(r(x)^{\theta_2} + t^{\theta_1})^{\frac{1}{\theta_2}}\right]^{\theta_2(C_1\alpha-1)\left(\frac{q}{q-1}+\frac{|\alpha|}{(q+\alpha-1)(q-1)}\right)} \times t^{(\theta_1-1)\left(\frac{q}{q-1}+\frac{|\alpha|}{(q+\alpha-1)(q-1)}\right)} V^{-\frac{1}{q-1}-\frac{|\alpha|}{(q+\alpha-1)(q-1)}} \, d\mu dt\,. \tag{3.36}$$

Note that if $f : [0, \infty) \to [0, \infty)$ is a nonincreasing function and if (1.9) in **HP2**-(i) holds, then for any sufficiently small $\varepsilon > 0$ and every large enough $R > 0$ we get

$$\int\int_{E_R^c} f\left([r(x)^{\theta_2} + t^{\theta_1}]^{\frac{1}{\theta_2}}\right) t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon} d\mu dt \leq C \int_{R/2^{1/\theta_2}}^\infty f(z) z^{\bar{s}_1 + C_0\varepsilon - 1} (\log z)^{\bar{s}_2} dz\,; \tag{3.37}$$

this can be shown by minor changes in the proof of [12, formula (2.19)]. Now due to (3.36), (3.37) with $\varepsilon = \frac{|\alpha|}{(q+\alpha-1)(q-1)}$

$$I_5 \leq C|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_{R/2^{1/\theta_2}}^\infty z^{\theta_2(C_1\alpha-1)\frac{q+\alpha}{q+\alpha-1} + C_0\frac{|\alpha|}{(q+\alpha-1)(q-1)} + \bar{s}_1 - 1} (\log z)^{\bar{s}_2} dz\,. \tag{3.38}$$

By our choice of C_1 and the very definition of $\bar{s}_1$, we have for sufficiently small $|\alpha| > 0$

$$\gamma := \theta_2(C_1\alpha - 1)\frac{q+\alpha}{q+\alpha-1} + C_0\frac{|\alpha|}{(q+\alpha-1)(q-1)} + \bar{s}_1 < -\frac{|\alpha|}{(q-1)^2}\,.$$

Using the change of variable $y := |\gamma| \log z$ in the right hand side of (3.38), due to the very definition of $\bar{s}_2$ we obtain for every $R > 0$ large enough

$$I_5 \leq C|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_0^\infty e^{-y} \left(\frac{y}{|\gamma|}\right)^{\bar{s}_2} \frac{dy}{|\gamma|} \leq C\,. \tag{3.39}$$

Moreover, using (3.10) and (1.9) in **HP2**, for every $n \in \mathbb{N}$ and $\alpha > 0$ sufficiently small, we have

$$\begin{aligned} I_6 &\leq \int\int_{E_{2^{1/\theta_2}nR} \setminus E_{nR}} \left(\frac{\theta_1}{(nR)^{\theta_2}} t^{\theta_1-1}\right)^{\frac{q+\alpha}{q+\alpha-1}} n^{C_1\theta_2\alpha\frac{q+\alpha}{q+\alpha-1}} V^{-\frac{1}{q-1}-\frac{|\alpha|}{(q+\alpha-1)(q-1)}} d\mu dt \\ &\leq C(nR)^{-\theta_2\frac{q+\alpha}{q+\alpha-1}} n^{C_1\theta_2\alpha\frac{q+\alpha}{q+\alpha-1}} (nR)^{\frac{\theta_2 q}{q-1} + C_0\frac{|\alpha|}{(q+\alpha-1)(q-1)}} [\log(nR)]^{\bar{s}_2} \\ &\leq Cn^{-\frac{|\alpha|}{(q-1)^2}} [\log(nR)]^{\bar{s}_2}\,. \end{aligned} \tag{3.40}$$

In view of (3.33), (3.39), (3.40) we obtain

$$J_4 \leq C\left(1 + n^{-\frac{|\alpha|}{(q-1)^2}} [\log(nR)]^{\bar{s}_2}\right).$$

Letting $n \to \infty$ we get (3.26).

In order to estimate the integral J_3 we start by defining $\Lambda = \frac{(p-1)q|\alpha|}{(q-p+1)[q-(1-\alpha)(p-1)]}$, and we note that

$$\frac{(p-1)q}{(q-p+1)^2}|\alpha| < \Lambda < \frac{2(p-1)q}{(q-p+1)^2}|\alpha| < \varepsilon^* \tag{3.41}$$

for every small enough $|\alpha| > 0$, and that

$$\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)} = \bar{s}_4 + \Lambda \qquad \text{and} \qquad \frac{pq}{q-(1-\alpha)(p-1)} = \frac{\bar{s}_3}{\theta_2} + \Lambda p \,.$$

By our definition of the functions φ_n, for every $n \in \mathbb{N}$ and every small enough $|\alpha| > 0$ we have

$$\begin{aligned} J_3 &= \int_0^\infty \int_M V^{-\bar{s}_4-\Lambda} |\nabla\varphi_n|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt \\ &\leq C\Big[\int_0^\infty \int_M V^{-\bar{s}_4-\Lambda}\eta_n^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}|\nabla\varphi|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt + \int_0^\infty \int_M V^{-\bar{s}_4-\Lambda}\varphi^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}|\nabla\eta_n|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt\Big] \\ &\leq C\Big[\int\int_{E_R^c} V^{-\bar{s}_4-\Lambda}|\nabla\varphi|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt + \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} V^{-\bar{s}_4-\Lambda}\varphi^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}|\nabla\eta_n|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt\Big] \\ &:= C(I_7+I_8). \end{aligned} \tag{3.42}$$

Now we use condition (1.11) in **HP2**-(ii) with $\varepsilon = \Lambda$, and we obtain for every $n \in \mathbb{N}$ and $R > 0$ large enough

$$\begin{aligned} I_8 &= \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} V^{-\bar{s}_4-\Lambda}\varphi^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}|\nabla\eta_n|^{\frac{\bar{s}_3}{\theta_2}+\Lambda p}\, d\mu dt \\ &\leq \Big(\sup_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} \varphi\Big)^{\frac{\bar{s}_3}{\theta_2}+\Lambda p} \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} \Big(\frac{\theta_2 r(x)^{\theta_2-1}}{(nR)^{\theta_2}}\Big)^{\frac{\bar{s}_3}{\theta_2}+\Lambda p} V^{-\bar{s}_4-\Lambda}\, d\mu dt \\ &\leq C n^{\frac{C_1\theta_2\alpha pq}{q-(1-\alpha)(p-1)}}(nR)^{-\frac{\theta_2 pq}{q-(1-\alpha)(p-1)}} \int\int_{E_{2^{1/\theta_2}nR}\setminus E_{nR}} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\Lambda\right)} V^{-\frac{p-1}{q-p+1}-\Lambda} d\mu dt \\ &\leq C n^{\frac{C_1\theta_2\alpha pq}{q-(1-\alpha)(p-1)}}(nR)^{-\frac{\theta_2 pq}{q-(1-\alpha)(p-1)}}(nR)^{\frac{\theta_2 pq}{q-p+1}+C_0\Lambda}(\log(nR))^{\bar{s}_4}. \end{aligned}$$

By our definition of C_1, Λ and by relation (3.41) we easily find

$$\begin{aligned} &\frac{C_1\theta_2\alpha pq}{q-(1-\alpha)(p-1)} - \frac{\theta_2 pq}{q-(1-\alpha)(p-1)} + \frac{\theta_2 pq}{q-p+1} + C_0\Lambda \\ &\qquad < \frac{pq\alpha C_1\theta_2}{q-(1-\alpha)(p-1)} - \frac{\alpha q(p-1)C_0}{[q-(1-\alpha)(p-1)](q-p+1)} \\ &\qquad \leq \frac{q\alpha p}{[q-(1-\alpha)(p-1)](q-p+1)} < \frac{q\alpha p}{(q-p+1)^2} < 0, \end{aligned} \tag{3.43}$$

for any small enough $|\alpha| > 0$. Moreover by (3.10), since $\alpha = -\frac{1}{\log R}$, we have

$$R^{-\frac{\theta_2 pq}{q-(1-\alpha)(p-1)}+\frac{\theta_2 pq}{q-p+1}+C_0\Lambda} \leq C\,.$$

Thus, for any sufficiently large $R > 0$ and every $n \in \mathbb{N}$,

$$I_8 \leq C n^{\frac{q\alpha p}{(q-p+1)^2}}(\log(nR))^{\bar{s}_4}. \tag{3.44}$$

In order to estimate I_7 we observe that if $f : [0,\infty) \to [0,\infty)$ is a nonincreasing function and if **HP2**-(ii) holds (see (1.11)), then

$$\int\int_{E_R^c} f\big([r(x)^{\theta_2}+t^{\theta_1}]^{\frac{1}{\theta_2}}\big) r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon}\, d\mu dt \leq C\int_{R/2^{1/\theta_2}}^\infty f(z) z^{\bar{s}_3+C_0\varepsilon-1}(\log z)^{\bar{s}_4}\, dz\,, \tag{3.45}$$

for every $0 < \varepsilon < \varepsilon_0$ and $R > 0$ large enough. This can againbe shown by minor variations in the proof of [12, formula (2.19)]. Thus, similarly to (3.36) and (3.38), using (3.10), (3.41) and (3.45), we have for

$R>0$ large enough and $\alpha=-\frac{1}{\log R}$

(3.46)

$$I_7 \le C|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int\int_{E_R^c} \left[(r(x)^{\theta_2}+t^{\theta_1})^{\frac{1}{\theta_2}}\right]^{\theta_2(C_1\alpha-1)\frac{pq}{q-(1-\alpha)(p-1)}} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\Lambda\right)} V^{-\frac{p-1}{q-p+1}-\Lambda}\, d\mu dt$$
$$\le C|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int_{R/2^{1/\theta_2}}^{\infty} z^{\theta_2(C_1\alpha-1)\frac{pq}{q-(1-\alpha)(p-1)}+\bar{s}_3+C_0\Lambda-1}(\log z)^{\bar{s}_4}\, dz.$$

By our choice of C_1 and the definition of $\bar{s}_3$ and Λ we have

$$a := \theta_2(C_1\alpha-1)\frac{pq}{q-(1-\alpha)(p-1)} + \bar{s}_3 + C_0\Lambda < -\frac{qp|\alpha|}{(q-p+1)^2} < 0,$$

thus using the change of variable $y=|a|\log z$ in the last integral in (3.46) we obtain

$$I_7 \le\ C|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int_0^{\infty} e^{-y}\left(\frac{y}{|a|}\right)^{\bar{s}_4}\frac{dy}{|a|} \tag{3.47}$$
$$\le\ C|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}-\bar{s}_4-1} \ =\ C|\alpha|^{\frac{(p-1)q}{q-(1-\alpha)(p-1)}+\frac{(p-1)q|\alpha|}{(q-p+1)(q-(1-\alpha)(p-1))}}\,.$$

Thus for any sufficiently large $R>0$ and every $n\in\mathbb{N}$, by (3.42), (3.44) and (3.47)

$$|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3 \le C|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}}\left(|\alpha|^{\frac{(p-1)q}{q-(1-\alpha)(p-1)}+\frac{(p-1)q|\alpha|}{(q-p+1)(q-(1-\alpha)(p-1))}} + n^{\frac{q\alpha p}{(q-p+1)^2}}(\log(nR))^{\bar{s}_4}\right).$$

Letting $n\to\infty$, for every $R>0$ large enough and $\alpha=-\frac{1}{\log R}$ we obtain

$$|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3 \le C|\alpha|^{\frac{(p-1)q|\alpha|}{(q-p+1)(q-(1-\alpha)(p-1))}} \le C,$$

that is (3.24).

Now using (3.23)–(3.26) in (3.22), since $\varphi_n\equiv 1$ on E_R and $0\le\varphi_n\le 1$ on $M\times[0,\infty)$, for every $R>0$ large enough we have

$$\int\int_{E_R} u^q V\, d\mu dt \ \le\ \limsup_{n\to\infty}\left(\int_0^{\infty}\int_M \varphi_n^s u^q V\, d\mu dt\right) \ \le A\left(\int_0^{\infty}\int_M u^q V\, d\mu dt\right)^{\sigma} + B$$

for some positive constants A, B and $\sigma\in(0,1)$. Passing to the limit as $R\to\infty$ we obtain (3.21), and hence we conclude that $u^q\in L^1(S, V d\mu dt)$ as claimed.

Next we want to show that

$$\int_0^{\infty}\int_M u^q V\, d\mu dt = 0,$$

and thus that $u=0$ a.e., since $V>0$ a.e. on $M\times[0,\infty)$. To this aim, we consider (2.18) with φ replaced by the family of functions φ_n. Since $\varphi_n\equiv 1$ on E_R and since $0\le\varphi_n\le 1$ on $M\times[0,\infty)$, for every $n\in\mathbb{N}$, every $R>0$ large enough and $\alpha=-\frac{1}{\log R}$ we have

(3.48)

$$\int\int_{E_R} u^q V\, d\mu dt \ \le\ \int_0^{\infty}\int_M \varphi_n^s u^q V\, d\mu dt$$
$$\le C\left(|\alpha|^{-1-\frac{(p-1)q}{(q-p+1)}}\int_0^{\infty}\int_M V^{-\frac{p+\alpha-1}{q-p+1}}|\nabla\varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}\, d\mu dt + |\alpha|^{-1}\int_0^{\infty}\int_M |\partial_t\varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, d\mu dt\right)^{\frac{p-1}{p}}$$
$$\times\left(\int\int_{E_R^c} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}|\nabla\varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, d\mu dt\right)^{\frac{q-(1-\alpha)(p-1)}{pq}}\left(\int\int_{E_R^c} u^q V\, d\mu dt\right)^{\frac{(1-\alpha)(p-1)}{pq}}$$
$$+ C\left(\int\int_{E_R^c} u^q V\, d\mu dt\right)^{\frac{1}{q}}\left(\int_0^{\infty}\int_M V^{-\frac{1}{q-1}}|\partial_t\varphi_n|^{\frac{q}{q-1}}\, d\mu dt\right)^{\frac{q-1}{q}}.$$

Now we claim that for $R>0$ sufficiently large

$$\limsup_{n\to\infty} J_5 \le C, \tag{3.49}$$

where

$$J_5 := \int_0^\infty \int_M V^{-\frac{1}{q-1}} |\partial_t \varphi_n|^{\frac{q}{q-1}}\, d\mu dt.$$

This can be shown similarly to inequality (3.26). Indeed

$$J_5 \le C(I_9 + I_{10}), \tag{3.50}$$

where

$$I_9 := \int_0^\infty \int_M V^{-\frac{1}{q-1}} |\partial_t \varphi|^{\frac{q}{q-1}}\, d\mu dt\,, \qquad I_{10} := \int_0^\infty \int_M V^{-\frac{1}{q-1}} \varphi^{\frac{q}{q-1}} |\partial_t \eta_n|^{\frac{q}{q-1}}\, d\mu dt.$$

By (3.18) and (3.10), for $R > 0$ sufficiently large

$$I_9 \le C|\alpha|^{\frac{q}{q-1}} \int\int_{E_R^c} \left[(r(x)^{\theta_2} + t^{\theta_1})^{\frac{1}{\theta_2}} \right]^{\theta_2(C_1\alpha - 1)\frac{q}{q-1}} t^{(\theta_1 - 1)\frac{q}{q-1}} V^{-\frac{1}{q-1}} d\mu dt\,. \tag{3.51}$$

Now note that if $f : [0, \infty) \to [0, \infty)$ is a nonincreasing function and if (1.9) in **HP2**-(i) holds, then for every $R > 0$ sufficiently large we get

$$\int\int_{E_R^c} f\big([r(x)^{\theta_2} + t^{\theta_1}]^{\frac{1}{\theta_2}}\big) t^{(\theta_1-1)\frac{q}{q-1}} V^{-\frac{1}{q-1}} d\mu dt \le C \int_{R/2^{1/\theta_2}}^\infty f(z) z^{\bar{s}_1 - 1} (\log z)^{\bar{s}_2} dz\,; \tag{3.52}$$

indeed, the proof of (3.52) is similar to that of [12, formula (2.19)], where here one uses condition (1.9) with $\varepsilon = 0$, see also Remark 1.1. Then

$$\begin{aligned} I_9 &\le C|\alpha|^{\frac{q}{q-1}} \int_{R/2^{1/\theta_2}}^\infty z^{\theta_2(C_1\alpha - 1)\frac{q}{q-1} + \bar{s}_1 - 1} (\log z)^{\bar{s}_2} dz \\ &\le C|\alpha|^{\frac{q}{q-1}} \int_1^\infty z^{\frac{\theta_2 C_1 \alpha q}{q-1}} (\log z)^{\bar{s}_2} \frac{dz}{z} \;\le\; C|\alpha|^{\frac{q}{q-1} - \bar{s}_2 - 1} \;\le\; C\,. \end{aligned} \tag{3.53}$$

Moreover, for every $n \in \mathbb{N}$ by (3.10) and by (1.8) with $\varepsilon = 0$, see also Remark 1.1,

$$\begin{aligned} I_{10} \;&\le\; \int\int_{E_{2^{1/\theta_2} nR} \setminus E_{nR}} \left(\frac{\theta_1}{(nR)^{\theta_2}} t^{\theta_1 - 1} \right)^{\frac{q}{q-1}} n^{C_1 \theta_2 \alpha \frac{q}{q-1}} V^{-\frac{1}{q-1}} d\mu dt \\ &\le\; C(nR)^{-\theta_2 \frac{q}{q-1}} n^{C_1\theta_2\alpha\frac{q}{q-1}} (nR)^{\frac{\theta_2 q}{q-1}} [\log(nR)]^{\bar{s}_2} \;=\; Cn^{-\frac{C_1\theta_2 q}{q-1}|\alpha|} [\log(nR)]^{\bar{s}_2}\,. \end{aligned} \tag{3.54}$$

In view of (3.50), (3.53), (3.54) we have

$$J_5 \le C\left(1 + n^{-\frac{C_1\theta_2 q}{q-1}|\alpha|} [\log(nR)]^{\bar{s}_2}\right)\,.$$

Letting $n \to \infty$ we get our claim, inequality (3.49).

Now consider again (3.48); passing to the limsup as $n \to \infty$ and using (3.23)–(3.25) and (3.49), we obtain for some constant $C > 0$

$$\int\int_{E_R} u^q V\, d\mu dt \le C\left[\left(\int\int_{E_R^c} u^q V\, d\mu dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} + \left(\int\int_{E_R^c} u^q V\, d\mu dt \right)^{\frac{1}{q}} \right]. \tag{3.55}$$

Now we can pass to the limit in (3.55) as $R \to \infty$, and thus as $\alpha \to 0$, and conclude by using Fatou's Lemma and the fact that $u^q \in L^1(S, V d\mu dt)$ that

$$\int_0^\infty \int_M u^q V\, d\mu dt = 0.$$

Thus $u = 0$ a.e. on $M \times [0, \infty)$.

4. Proof of Corollaries 1.4, 1.5 and 1.6

Proof of Corollary 1.4. We now show that under our assumptions hypothesis **HP1** is satisfied (see conditions (1.6) and (1.7)). Observe that for small $\varepsilon > 0$

$$\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}\,dxdt \,\le\, CR^m\int_0^{2^{1/\theta_1}R^{\frac{\theta_2}{\theta_1}}} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}dt \,\le\, CR^m R^{\frac{\theta_2}{\theta_1}\left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+1\right]}.$$

Hence, condition (1.6) is satisfied, if

$$\frac{\theta_2}{\theta_1} \ge (q-1)m\,. \tag{4.1}$$

On the other hand, for small $\varepsilon > 0$,

$$\begin{aligned}\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} |x|^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}\,dxdt &\le CR^{\frac{\theta_2}{\theta_1}}\int_0^{2^{1/\theta_2}R} \varrho^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+m-1}d\varrho\\ &\le CR^{\frac{\theta_2}{\theta_1}+\left[(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+m\right]}.\end{aligned}$$

Therefore condition (1.7) is satisfied, if

$$\frac{\theta_2}{\theta_1} \le \frac{pq}{q-p+1} - m\,. \tag{4.2}$$

Now note that we can find $\theta_1 \ge 1, \theta_2 \ge 1$ such that conditions (4.1) and (4.2) hold simultaneously, if (1.12) holds. Thus, from Theorem 1.2 the conclusion follows. □

Proof of Corollary 1.5. Under our assumptions, for $R > 0$ large and $\varepsilon > 0$ small enough we have

$$\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}V^{-\frac{1}{q-1}+\varepsilon}\,d\mu dt \le CR^{\frac{\theta_2}{\theta_1}(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_2+\sigma_1}(\log R)^{\delta_1+\delta_2}.$$

Hence condition (1.6) in **HP1** is satisfied if we choose $C_0 \ge \max\left\{0, \frac{\theta_2}{\theta_1}(\alpha+1)+\beta-\theta_2\right\}$ and if

$$\frac{\theta_2}{\theta_1}\left(\sigma_2-\frac{q}{q-1}\right)+\sigma_1 \le 0\,, \qquad \delta_1+\delta_2 < \frac{1}{q-1}\,. \tag{4.3}$$

Similarly for sufficiently large $R > 0$ and small $\varepsilon > 0$ we have

$$\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}V^{-\frac{p-1}{q-p+1}+\varepsilon}d\mu dt \le CR^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_4+\sigma_3}(\log R)^{\delta_3+\delta_4}\,.$$

Therefore condition (1.7) in **HP1** is satisfied if $C_0 \ge \max\left\{0, \beta+\frac{\theta_2}{\theta_1}\alpha-(\theta_2-1)p\right\}$ and if

$$\left(-\frac{pq}{q-p+1}+\sigma_3\right)+\frac{\theta_2}{\theta_1}\sigma_4 \le 0\,, \qquad \delta_3+\delta_4 < \frac{p-1}{q-p+1}\,. \tag{4.4}$$

Now for conditions (4.3) and (4.4) to be satisfied, by our assumptions it is sufficient to choose $\theta_1 \ge 1$, $\theta_2 \ge 1$ such that

$$\sigma_1\left(\frac{q}{q-1}-\sigma_2\right)^{-1} \le \frac{\theta_2}{\theta_1} \qquad \text{if } 0 \le \sigma_2 < \frac{q}{q-1}\,, \tag{4.5}$$

$$\frac{\theta_2}{\theta_1} \le \left(\frac{pq}{q-p+1}-\sigma_3\right)\sigma_4^{-1} \qquad \text{if } 0 \le \sigma_3 < \frac{pq}{q-p+1}\,. \tag{4.6}$$

Thus we can apply Theorem 1.2 and conclude.

Proof of Corollary 1.6. By our assumptions for large $R > 0$ and small $\varepsilon > 0$ we have

$$\begin{aligned}\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}V^{-\frac{1}{q-1}+\varepsilon}\,d\mu dt &\le CR^{\frac{\theta_2}{\theta_1}(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_2+\sigma_1}(\log R)^{\delta_1+\delta_2}\,,\\ \int\int_{E_{2^{1/\theta_2}R}\setminus E_R} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)}V^{-\frac{1}{q-1}-\varepsilon}\,d\mu dt &\le CR^{\frac{\theta_2}{\theta_1}(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_2+\sigma_1}(\log R)^{\delta_1+\delta_2}\,.\end{aligned}$$

Thus conditions (1.8)–(1.9) of **HP2** are satisfied if we choose $C_0 \geq \max\left\{0, \frac{\theta_2}{\theta_1}(\alpha-1)+\beta+\theta_2\right\}$ and

$$\frac{\theta_2}{\theta_1}\left(\sigma_2 - \frac{q}{q-1}\right) + \sigma_1 \leq 0\,, \qquad \delta_1+\delta_2 \leq \frac{1}{q-1}\,. \tag{4.7}$$

Similarly if $R>0$ is large and $\varepsilon>0$ is small enough we have

$$\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon} d\mu dt \leq C R^{(\theta_2-1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_4+\sigma_3}(\log R)^{\delta_3+\delta_4}\,,$$

$$\int\int_{E_{2^{1/\theta_2}R}\setminus E_R} r(x)^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon} d\mu dt \leq C R^{(\theta_2-1)p\left(\frac{q}{q-p+1}+\varepsilon\right)+\frac{\theta_2}{\theta_1}\alpha\varepsilon+\beta\varepsilon+\frac{\theta_2}{\theta_1}\sigma_4+\sigma_3}(\log R)^{\delta_3+\delta_4}\,.$$

Thus conditions (1.10)–(1.11) in **HP2** are satisfied if $C_0 \geq \max\left\{0, \beta+\frac{\theta_2}{\theta_1}\alpha+(\theta_2-1)p\right\}$ and

$$\left(-\frac{pq}{q-p+1}+\sigma_3\right)+\frac{\theta_2}{\theta_1}\sigma_4 \leq 0\,, \qquad \delta_3+\delta_4 \leq \frac{p-1}{q-p+1}\,. \tag{4.8}$$

Hence, arguing as in the proof of Corollary 1.5, we have that under our assumptions **HP2** holds, and we can apply Theorem 1.3 to conclude. □

We conclude with the next example, where we show that our results extend those in [31] in the case of the Laplace–Beltrami operator on a complete noncompact manifold M.

Let us start by fixing a point $o \in M$ and denote by $\mathrm{Cut}(o)$ the *cut locus* of o. For any $x \in M \setminus \left[\mathrm{Cut}(o)\cup\{o\}\right]$, one can define the *polar coordinates* with respect to o, see e.g. [10]. Namely, for any point $x \in M\setminus\left[\mathrm{Cut}(o)\cup\{o\}\right]$ there correspond a polar radius $r(x) := dist(x,o)$ and a polar angle $\theta\in\mathbb{S}^{m-1}$ such that the shortest geodesics from o to x starts at o with the direction θ in the tangent space T_oM. Since we can identify T_oM with $\mathbb{R}^m$, θ can be regarded as a point of $\mathbb{S}^{m-1}$.

The Riemannian metric in $M\setminus\left[\mathrm{Cut}(o)\cup\{o\}\right]$ in polar coordinates reads

$$ds^2 = dr^2 + A_{ij}(r,\theta)d\theta^i d\theta^j,$$

where $(\theta^1,\dots,\theta^{m-1})$ are coordinates in $\mathbb{S}^{m-1}$ and (A_{ij}) is a positive definite matrix. It is not difficult to see that the Laplace-Beltrami operator in polar coordinates has the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r,\theta)\frac{\partial}{\partial r} + \Delta_{S_r},$$

where $\mathcal{F}(r,\theta) := \frac{\partial}{\partial r}\left(\log\sqrt{A(r,\theta)}\right)$, $A(r,\theta) := \det(A_{ij}(r,\theta))$, Δ_{S_r} is the Laplace-Beltrami operator on the submanifold $S_r := \partial B(o,r)\setminus \mathrm{Cut}(o)$.

M is a *manifold with a pole*, if it has a point $o\in M$ with $\mathrm{Cut}(o)=\emptyset$. The point o is called *pole* and the polar coordinates (r,θ) are defined in $M\setminus\{o\}$.

A manifold with a pole is a *spherically symmetric manifold* or a *model*, if the Riemannian metric is given by

$$ds^2 = dr^2 + \psi^2(r)d\theta^2, \tag{4.9}$$

where $d\theta^2$ is the standard metric in $\mathbb{S}^{m-1}$, and

$$\psi \in \mathcal{A} := \left\{f\in C^\infty((0,\infty))\cap C^1([0,\infty)) : f'(0)=1,\ f(0)=0,\ f>0 \text{ in } (0,\infty)\right\}. \tag{4.10}$$

In this case, we write $M\equiv M_\psi$; furthermore, we have $\sqrt{A(r,\theta)}=\psi^{m-1}(r)$, so the boundary area of the geodesic sphere ∂S_R is computed by

$$S(R) = \omega_m \psi^{m-1}(R),$$

ω_m being the area of the unit sphere in $\mathbb{R}^m$. Also, the volume of the ball $B_R(o)$ is given by

$$\mu(B_R(o)) = \int_0^R S(\xi)d\xi\,.$$

Observe that for $\psi(r) = r$, $M = \mathbb{R}^m$, while for $\psi(r) = \sinh r$, M is the $m-$dimensional hyperbolic space $\mathbb{H}^m$.

Example 4.1. Let M be an $m-$dimensional model manifold with pole o and metric given by (4.9) with

$$\psi(r) := \begin{cases} r & \text{if } 0 \le r < 1\,, \\ [r^{\alpha-1}(\log r)^{\beta}]^{\frac{1}{m-1}} & \text{if } r > 2\,; \end{cases}$$

where $\alpha > 1$ and $\beta \in \left(0, \frac{1}{q-1}\right]$. We consider problem (1.1) with $V \equiv 1$ and $p = 2$. Note that for $R > 0$ large enough

$$\mu(B_R) \simeq CR^{\alpha}(\log R)^{\beta} \le CR^{\alpha+\sigma}\,,$$

for any $\sigma > 0$, while

$$\lim_{R\to+\infty} \frac{\mu(B_R)}{R^{\alpha}} = +\infty.$$

Furthermore,

$$\frac{d}{dr}\Big(\log\sqrt{A(r)}\Big) = \frac{d}{dr}\big(\log\big([\psi(r)]^{m-1}\big)\big) \le \frac{C}{r} \quad \text{for all } r > 0\,.$$

Thus, for $\alpha > 2$, from [31, Theorem A] we can infer that problem (1.1) does not admit nonnegative nontrivial solutions, provided that $1 < q \le 1 + \frac{2}{\alpha+\sigma}$ for some $\sigma > 0$, that is provided that

$$1 < q < 1 + \frac{2}{\alpha}\,.$$

On the other hand, just assuming $\alpha > 1$, we can apply Corollary 1.6 with $p = 2$ (see also Remark 1.7), where $f(t) \equiv 1$, $g(x) \equiv 1$, $\sigma_1 = \alpha$, $\sigma_2 = 1$, $\delta_1 = \beta$, $\delta_2 = 0$, and thus we can deduce that problem (1.1) does not admit nonnegative nontrivial solutions, provided that

$$1 < q \le 1 + \frac{2}{\alpha}\,.$$

So, we can exclude existence of nontrivial solutions also in the particular case when $q = 1 + \frac{2}{\alpha}$.

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