NONLINEAR VIBRATIONS OF VARIABLE-STIFFNESS PLATES USING DIFFERENT SEMI-ANALYTICAL MODELS

A.Y. Cheng[1] , R. Vescovini[1] and E.L. Jansen[2]

[1]Dipartimento di Scienze e Tecnologie Aerospaziali
Politecnico di Milano, Milano, ITALY
Email: chengangelo.yan@mail.polimi.it, riccardo.vescovini@polimi.it

[2]Institute of Structural Analysis
Leibniz University Hannover, Hannover, GERMANY
Email: e.jansen@isd.uni-hannover.de

ABSTRACT

Semi-analytical strategies are presented for the nonlinear vibration analysis of variable-stiffness plates. The formulations are developed within a mixed variational framework, where the out-of-plane displacements and the Airy stress functions are the unknowns of the problem, approximated using the Ritz method. Three different strategies are presented for solving the discrete set of equations originating from the Ritz expansion: direct time integration, an iterative procedure based on the Harmonic Balance Method (HBM) and the Method of Averaging. In addition, a single-mode perturbation approach is developed, and is proposed as a suitable way for efficiently assessing the nonlinear vibration problem of variable-stiffness plates. Comparison against results from literature confirms the validity of the proposed models.

1 INTRODUCTION

Considerable attention has been devoted in the past decades to vibration problems for plates and shells involving moderately large displacements [1]. The motivations can be found in the theoretical relevance of the phenomenon along with several practical applications, in many cases related with structural stability problems. Several studies can be found in the literature for classical isotropic or composite configurations [1, 2]. However, relatively few works cover this topic in relation with variable-stiffness configurations [3, 4]. This work aims at presenting semi-analytical formulations as a viable alternative to finite element techniques for analyzing with improved efficiency the non-linear vibrations of variable-stiffness plates. Different solution strategies are implemented and comparison against reference results is shown.

2 THEORETICAL FRAMEWORK

The approach is developed for the analysis of thin composite plates, obtained by the stacking of plies with non-uniform properties, i.e. variable-stiffness plates (VSP). The orientation of the fibers of a generic ply is specified according to the notation $\langle \alpha | \beta \rangle$, where α and β are the fiber's orientations at the center and the edges of the plate. The formulation is based on thin-plate theory, where a mixed variational functional [5, 6] is introduced for expressing the equilibrium and the compatibility requirements. The functional, expressed in terms of the out-plane deflections and the Airy stress function and including for the contribution due to inertial forces, reads:

$$\begin{aligned}\Pi^* = &-\frac{1}{2}\int_{\mathrm{A}} \left(\mathbf{F}^{\mathrm{T}}\mathbf{a}(x,y)\mathbf{F} - \mathbf{k}^{\mathrm{T}}\mathbf{D}(x,y)\mathbf{k} + F_{,yy}w_{,x}^2 - 2F_{,xy}w_{,x}w_{,y} + F_{,xx}w_{,y}^2\right)\,\mathrm{dA}\, + \\ &-\int_{\mathrm{A}} \left(F_{,yy}w_0 w_{,xx} - 2F_{,xy}w_0 w_{,xy} + F_{,xx}w_0 w_{,yy}\right)\,\mathrm{dA} + \int_{\mathrm{A}} I_0 w_{,tt} w\,\mathrm{dA} + \Pi^*_{load}\end{aligned} \tag{1}$$

where $\mathbf{F}$ and $\mathbf{k}$ are the vectors collecting the second derivatives of the Airy stress function and out-of-plane displacements; the matrices $\mathbf{a}$ and $\mathbf{D}$ are the in-plane compliance and the bending stiffness, both function of the position x, y due to the variability of elastic properties over the plane x-y; the out-of-plane deflections and the initial imperfections are denoted with w and w_0, respectively. In-plane inertia effects are neglected, and just the out-of-plane moment of inertia I_0 is considered. Following Wu et al. [7] the expansion of the unknowns is performed referring to Legendre polynomials and boundary functions, where the membrane stresses are split into a part related to the distribution at the boundaries and another contribution specifying the behaviour inside the plate domain.

Direct time integration

A first strategy for solving the nonlinear dynamic problem consists in integrating the equations of motion. The Ritz approximation is substituted into Eq. (1) and, after imposing the functional to be stationary, the set of nonlinear equations expressing membrane compatibility and dynamic equilibrium along the out-of-plane direction are retrieved. Whenever the loads are directed in the out-of-plane direction, as it is considered here, the compatibility equations are time-independent. Therefore, the problem can be statically condensed into one single set of equations expressing the equilibrium requirements. Referring to a base of n modes, these equations are obtained as:

$$\ddot{q}_i + c_i\dot{q}_i + k_i q_i + \sum_{jk=1}^{nn} a_{ijk}q_j q_k + \sum_{jks=1}^{nnn} b_{ijks}q_j q_k q_s = p_i \qquad \text{for} \quad i = 1, 2, \ldots n \tag{2}$$

where the coefficients k_i, a_{ijk} and b_{ijks} are obtained by numerical integration of the stiffness-related contributions of the functional Π^* and successive projection onto the modal basis. Viscous damping can be added through the term c_i. The equations are integrated using the Matlab ode23s scheme, based on a modified Rosenbrock formula of order 2.

Iterative procedure

A second strategy relies upon an iterative procedure based on the Harmonic Balance Method (HBM), which can be employed for evaluating the forced steady-state response and the nonlinear free vibrations. In the former case, the response of the nonlinear system is approximated as:

$$\mathbf{q} \cong \mathbf{Q_0} + \sum_{k=1}^{5} \left[\mathbf{Q_{c,k}} \cos(k\omega t) + \mathbf{Q_{s,k}} \sin(k\omega t) \right] \tag{3}$$

where ω is the frequency of the forcing term, so super-harmonic contributions are taken as multiples of the fundamental frequency up to the fifth order. Substitution of Eq. (3) into the equation of motion, and successive balancing of the harmonics allows to derive the system of $11n$ nonlinear equations, which are solved using a Newton-Raphson technique.

Method of averaging

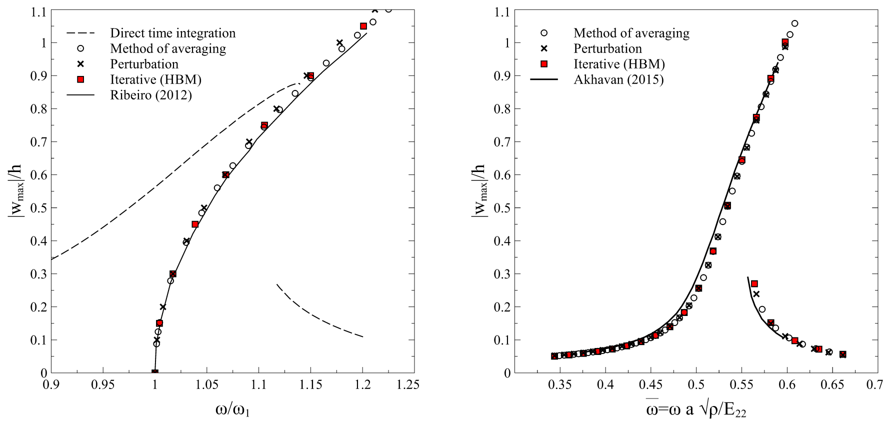
The method of averaging is based on an initial description of the problem as per Eq. (2), where a two-mode approximation is performed, q_1 and q_2 being the modal coordinates. By applying the averaging procedure to eliminate time-dependence, i.e. by replacing the vibration amplitudes and phases with their average values over the time period, the governing equations are found as a set of four algebraic equations. Clearly, the single-mode approximation can be retrieved as a special case, reducing the number of unknowns from four to two.

Perturbation approach

An effective strategy for analysing the nonlinear vibration response relies on a perturbation approach [8], which is essentially an extension of Koiterś approach to the case of vibrations. Main advantage is that the nonlinear problem is transformed into a sequence of linear problems. The unknowns, after introducing the Ritz approximation, are approximated using a perturbation expansion: the out-of-plane deflections are approximated as $w = \xi w^{(1)} + \xi^2 w^{(2)} + ...$, where ξ is the perturbation parameter, and similarly is done for the Airy function.

3 RESULTS

Exemplary results are presented to illustrate the comparison between the different methods presented above and reference numerical simulations from the literature [4, 9]. The full plate properties are not reported here for the sake of conciseness, but can be found in the referenced works. In all the examples, the boundary conditions are those of a fully clamped panel with immovable edges. The first example deals with a variable-stiffness plate with stacking sequence given by $[\langle 135|90 \rangle, \langle -90|-45 \rangle, \langle 90|45 \rangle, \langle 45|0 \rangle]_s$. The nonlinear free vibration response is assessed in Figure 1(a), where the comparison is presented against the backbone curve obtained in Ref. [4]. Note, due to the close matching between the results, they are presented using markers and not continous curves. In the case of the direct integration approach, the procedure is quite cumbersome, as integration needs to be carried-out for a sufficiently large number of periods. Furthermore, several frequency response curves need to the traced to derive the final backbone plot. In this example, a 11-dof model is considered, with a concentrated force applied at the center of the plate (magnitude $f = 0.7N$) and accounting for a slight damping factor ($\xi_{1,1} = 0.0325$).
A second example regards the forced nonlinear response of the variable-stiffness plate analyzed by Akhavan [9], whose lay-up is $[\langle 90|45 \rangle, 90, \langle 90|45 \rangle]_s$. The load is introduced in the form of a uniform pressure with magnitude equal to $2 \times 10^4\ N/m^2$. No damping is accounted for. The results are reported in Figure 1(b) in terms of maximum nondimensional deflection versus nondimensional frequency $\overline{\omega} = \omega a \sqrt{\rho / E_{22}}$. Even in this case, close matching can be observed between the present results and those of Ref. [9], where the p-version finite element and third-order plate theory are used. In both cases, computations are carried out very effectively, with times ranging from 0.5 s to few minutes (on a laptop with Intel i7 4.00 GHz, 32 GB of RAM). The perturbation technique is particularly effective, as fractions of a seconds are needed to trace the plots with a two-term expansion. Iterative procedure and method of averaging require, in general, few seconds, while



(a) (b)

Figure 1: Comparison between different methods and reference results: (a) backbone curves, (b) frequency response curves.

numerical integration is the most computationally intensive approach, requiring a total time of the order of a few minutes.

4 CONCLUDING REMARKS

A semi-analytical strategy has been presented for analyzing the nonlinear vibration response of variable-stiffness plates. The formulation is general enough to allow any set of flexural boundary conditions to be studied, while free, movable and immovable edges can be considered with respect to the in-plane motion. Four different strategies were proposed for solving the nonlinear equations available from the Ritz discretization: direct time integration can be relatively time-consuming, but the three other strategies are extremely attractive in terms of computational time. This is particularly true for the single-mode perturbation approach: the solution is obtained as a sequence of linear problems, and fractions of a second are required for the problem to be solved. These features render the proposed approaches particularly suited for gathering insight into the nonlinear vibration response of variable-stiffness configurations, a relatively new field demanding further investigation.

REFERENCES

[1] M. Amabili. *Nonlinear Vibrations and Stability of Shells and Plates*. Cambridge University Press, 2008.

[2] F. Farbod and M. Amabili. Non-linear vibrations of shells: A literature review from 2003 to 2013. *International Journal of Non-Linear Mechanics*, 58:233–257, 2014.

[3] P. Ribeiro and H. Akhavan. Non-linear vibrations of variable stiffness composite laminated plates. *Composite Structures*, 94(8):2424–2432, 2012.

[4] P. Ribeiro. Non-linear free periodic vibrations of variable stiffness composite laminated plates. *Nonlinear Dynamics*, 70(2):1535–1548, 2012.

[5] V. Giavotto. Sulla meccanica dei pannelli anisotropi ed eterogenei. *Memorie dell' Istituto Lombardo - Accademia di Scienze e Lettere*, XXV(5), 1969. [Italian].

[6] C. Bisagni and R. Vescovini. Analytical formulation for local buckling and post-buckling analysis of stiffened laminated panels. *Thin-Walled Structures*, 47(3):318–334, 2009.

[7] Z. Wu, P.M. Weaver, G. Raju, and B.C. Kim. Buckling analysis and optimisation of variable angle tow composite plates. *Thin-Walled Structures*, 60:163–172, 2012.

[8] L.W. Rehfield. Nonlinear free vibrations of elastic structures. *International Journal of Solids and Structures*, 9(5):581–590, 1973.

[9] H. Akhavan. Non-linear vibrations of tow placed variable stiffness composite laminates. Ph.D. Thesis, Universidade do Porto, 2015.