FREE BOUNDARY REGULARITY FOR FULLY NONLINEAR NON-HOMOGENEOUS TWO-PHASE PROBLEMS

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ABSTRACT. We prove that flat or Lipschitz free boundaries of two-phase free boundary problems governed by fully nonlinear uniformly elliptic operators and with non-zero right hand side are $C^{1,\gamma}$.

1. Introduction and main results

In this paper we continue the development of the regularity theory for free boundary problems with forcing term, started in [D] and [DFS]. We will focus on the following problem

$$(1.1)\qquad \begin{cases} \mathcal{F}(D^2u) = f, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1, & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain and

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) := \{x \in \Omega : u(x) \leq 0\}^\circ,$$

while u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively. Also, $f \in L^\infty(\Omega)$ is continuous in $\Omega^+(u) \cup \Omega^-(u)$. $F(u)$ is called the *free boundary*.

$\mathcal{F}$ is a fully nonlinear uniformly elliptic operator, that is there exist $0 < \lambda \leq \Lambda$ positive constants such that for every $M, N \in \mathcal{S}^{n\times n}$, with $N \geq 0$,

$$\lambda\|N\| \leq \mathcal{F}(M+N) - \mathcal{F}(M) \leq \Lambda\|N\|,$$

where $\mathcal{S}^{n\times n}$ denotes the set of real $n \times n$ symmetric matrices. We write $N \geq 0$, whenever N is non-negative definite. Also, $\|M\|$ denotes the (L^2, L^2)-norm of M, that is $\|M\| = \sup_{|x|=1} |Mx|$. Finally, we assume that $\mathcal{F}(0) = 0$.

When $f \equiv 0$ and $\mathcal{F}$ is homogeneous of degree one, several authors extended the results of the seminal works of Caffarelli ([C1, C2]) to various kind of nonlinear operators. Wang ([W1, W2]) considered $\mathcal{F} = \mathcal{F}\left(D^2u\right)$ concave, Feldman ([F1]) enlarged the class of operators to $\mathcal{F} = \mathcal{F}\left(D^2u, Du\right)$ without concavity assumptions, Ferrari and Argiolas ([Fe1, AF]) added x-dependence in $\mathcal{F}$, with $\mathcal{F}(0, 0, x) \equiv 0$.

All these papers follows the general strategy developed in [C1, C2], that however seems not so suitable when distributed sources are present.

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In [D], De Silva introduced a new technique to prove the smoothness of the free boundary for one-phase problems governed by non-homogeneous linear elliptic equations. As we show in [DFS], her method is flexible enough to deal with general two-phase problems for linear operators. Here we enforce the same technique to prove regularity of flat free boundaries for problem (1.1). Our main result is the following, where we denote with B_r the ball of radius r centered at 0 (for the definition of viscosity solution see Section 2).

Theorem 1.1 (Flatness implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to* (1.1) *in B_1. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a constant $\bar{\delta} > 0$, depending only on $n, \lambda, \Lambda, \|f\|_\infty$ and $Lip(u)$ such that, if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \tag{1.2}$$

with $0 \leq \delta \leq \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Expressely note that we assume for $\mathcal{F}$ neither concavity nor homogeneity of degree one.

When $\mathcal{F}$ is homogeneous of degree one (or when $\mathcal{F}_r(M)$ has a limit $\mathcal{F}^*(M)$, as $r \to 0$, which is homogeneous of degree one) we can also prove the following *Lipschitz implies smoothness* result.

Theorem 1.2 (Lipschitz implies $C^{1,\gamma}$). *Let $\mathcal{F}$ be homogeneous of degree one and u be a Lipschitz viscosity solution to* (1.1) *in B_1, with $0 \in F(u)$. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

Theorem 1.2 follows from Theorem 1.1 and the main result in [F1], via a blow-up argument.

As we have already mentioned, to prove Theorem 1.1 we will use the technique introduced in [D, DFS]. In particular, the structure of our paper parallel the one in [DFS]. Thus, the proof of Theorem 1.1 is obtained through an iterative improvement of flatness via a suitable compactness and linearization argument. A crucial tool is the $C^{1,\alpha}$ regularity of the solution of the linearized problem, that turns out to be a transmission problem in the unit ball, governed by two different fully nonlinear operators in two half-balls. Section 3 is devoted to prove this regularity result, that, we believe, could be interesting in itself.

As it is common in two-phase free boundary problems, the main difficulty in the analysis comes from the case when u^- is degenerate, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of u to an "optimal" (two-plane) configuration. Thus one needs to work only with the positive phase u^+ to balance the situation in which u^+ highly predominates over u^- and the case in which u^- is not too small with respect to u^+. For this reason, throughout the paper we distinguish two cases, which we refer to as the non-degenerate and the degenerate case.

The paper is organized as follows. In Section 2, we provide basic definitions and reduce our main flatness theorem to a proper "normalized" situation (i.e. closeness to a two-plane solution). As already mentioned above, Section 3 is devoted to the linearized problem. In Section 4 we obtain the necessary Harnack inequalities which rigorously allow the linearization of the problem. Section 5 provides the proof of the improvement of flatness lemmas. Finally, the main theorems are proved in the last section.

A remark on further generalization is in order. We have choosen the particular free boundary condition in problem (1.1) in order to better emphasize the ideas involved in our proofs. Also, to avoid the machinery of L^p-viscosity solution, we assume that f is bounded in Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$ but everything works with f merely bounded, measurable.

Following the lines of Sections 7-9 in [DFS], our results can be extended to a more general class of operators $\mathcal{F} = \mathcal{F}(M, p)$, uniformly Lipschitz with respect to p, $\mathcal{F}(0,0) = 0$, (homogeneous of degree one in both arguments for Theorem 1.2) with free boundary conditions given by

$$u_\nu^+ = G(u_\nu^-, x),$$

where

$$G : [0, \infty) \times \Omega \to (0, \infty)$$

satisfies the following assumptions:

(1) $G(\eta, \cdot) \in C^{0,\bar{\gamma}}(\Omega)$ uniformly in η; $G(\cdot, x) \in C^{1,\bar{\gamma}}([0, L])$ for every $x \in \Omega$.
(2) $G'(\cdot, x) > 0$ with $G(0, x) \geq \gamma_0 > 0$ uniformly in x.
(3) There exists $N > 0$ such that $\eta^{-N} G(\eta, x)$ is strictly decreasing in η, uniformly in x.

A last remark concerns existence. In our generality, the existence of Lipschitz viscosity solutions with proper measure theoretical properties of the free boundary is an open problem and it will be object of future investigations. When $f = 0$, and $\mathcal{F} = \mathcal{F}\left(D^2 u\right)$ is concave, homogeneous of degree one, the existence issue has been settled by Wang in [W3].

Other two recent papers, namely [AT], [RT], deal with well posedness and regularity for free boundary problems governed by fully nonlinear operators. In [AT] the authors perform a complete analysis of singular perturbation problems and their limiting free boundary problems. Of particular interest is the limiting free boundary condition, obtained through a homogenization of the governing operator, under suitable hypotheses such as rotational invariance and e.g. concavity. In [RT], a free boundary problem with power type singular absorption term is considered. In this interesting paper the authors establish existence, optimal regularity and non degeneracy of a minimal solution, together with fine measure theoretical properties of the free boundary. Further regularity of the free boundary seems to be a challenging problem.

2. Preliminaries

In this section, we state basic definitions and we show that our flatness Theorem 1.1 follows from Theorem 2.1 below. From now on, U_β denotes the one-dimensional function,

$$U_\beta(t) = \alpha t^+ - \beta t^-, \quad \beta \geq 0, \quad \alpha = \sqrt{1 + \beta^2},$$

where

$$t^+ = \max\{t, 0\}, \quad t^- = -\min\{t, 0\}.$$

Here and henceforth, all constants depending only on $n, \lambda, \Lambda, \|f\|_\infty$ and $Lip(u)$ will be called universal.

Theorem 2.1. *Let u be a (Lipschitz) solution to* (1.1) *in B_1 with $Lip(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \leq \bar{\varepsilon} \quad \text{for some } 0 \leq \beta \leq L, \tag{2.1}$$

and

$$\{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\},$$

and

$$\|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

The rest of the paper is devoted to the proof of Theorem 2.1, following the strategy developed in [DFS].

We recall some standard fact about fully nonlinear uniformly elliptic operators. For a comprehensive treatment of fully nonlinear elliptic equations, we refer the reader to [CC].

From now on, the class of all uniformly elliptic operators with ellipticity constants λ, Λ and such that $\mathcal{F}(0) = 0$ will be denoted by $\mathcal{E}(\lambda, \Lambda)$.

We start with the definition of the extremal Pucci operators, $\mathcal{M}^-_{\lambda,\Lambda}$ and $\mathcal{M}^+_{\lambda,\Lambda}$. Given $0 < \lambda \leq \Lambda$, we set

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,$$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,$$

with the $e_i = e_i(M)$ the eigenvalues of M.

In the rest of the paper, whenever it is obvious, the dependance of the extremal operators from λ, Λ will be omitted.

We recall that if $\mathcal{F} \in \mathcal{E}(\lambda, \Lambda)$ then

$$\mathcal{M}^-_{\frac{\lambda}{n},\Lambda}(M) \leq \mathcal{F}(M) \leq \mathcal{M}^+_{\frac{\lambda}{n},\Lambda}(M),$$

a fact which will be used very often throughout the paper.

Finally, it is readily verified that if $\mathcal{F} \in \mathcal{E}(\lambda, \Lambda)$ is the rescaling operator defined by

$$\mathcal{F}_r(M) = \frac{1}{r}\mathcal{F}(rM), \quad r > 0$$

then $\mathcal{F}_r$ is still an operator in our class $\mathcal{E}(\lambda, \Lambda)$.

We now introduce the definition of viscosity solution to our free boundary problem (1.1). First we recall some standard notion.

Given $u, \varphi \in C(\Omega)$, we say that φ touches u by below (resp. above) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly by below (resp. above).

Let $\mathcal{F} \in \mathcal{E}(\lambda, \Lambda)$. If $v \in C^2(O)$, O open subset in $\mathbb{R}^n$, satisfies

$$\mathcal{F}(D^2 v) > f \quad (\text{resp.} < f) \quad \text{in } O,$$

with $f \in C(O)$, we call v a (strict) classical subsolution (resp. supersolution) to the equation $\mathcal{F}(D^2v) = f$ in O.

We say that $u \in C(O)$ is a viscosity solution to

$$\mathcal{F}(D^2v) = f \quad \text{in } O,$$

if u cannot be touched by above (resp. below) by a strict classical subsolution (resp. supersolution) at an interior point $x_0 \in O$.

We now turn to the definition of viscosity solution to our free boundary problem (1.1).

Definition 2.2. We say that $v \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1.1) in Ω, if and only if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

(i) $\mathcal{F}(D^2v) > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;

(ii) If $x_0 \in F(v)$, then

$$(v_\nu^+)^2 - (v_\nu^-)^2 > 1 \quad (\text{resp. } (v_\nu^+)^2 - (v_\nu^-)^2 < 1,\, v_\nu^+(x_0) \neq 0).$$

Notice that by the implicit function theorem, according to our definition the free boundary of a comparison subsolution/supersolution is C^2.

Definition 2.3. Let u be a continuous function in Ω. We say that u is a viscosity solution to (1.1) in Ω, if the following conditions are satisfied:

(i) $\mathcal{F}(D^2u) = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense;

(ii) Any (strict) comparison subsolution v (resp. supersolution) cannot touch u by below (resp. by above) at a point $x_0 \in F(v)$ (resp. $F(u)$.)

The next lemma shows that "$\delta-$flat" viscosity solutions (in the sense of our main Theorem 1.1) enjoy non-degeneracy of the positive part δ-away from the free boundary. Precisely,

Lemma 2.4. *Let u be a solution to* (1.1) *in B_2 with $Lip(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. If*

$$\{x_n \leq g(x') - \delta\} \subset \{u^+ = 0\} \subset \{x_n \leq g(x') + \delta\},$$

with g a Lipschitz function, $Lip(g) \leq L, g(0) = 0$, then

$$u(x) \geq c_0(x_n - g(x')), \quad x \in \{x_n \geq g(x') + 2\delta\} \cap B_{\rho_0},$$

for some $c_0, \rho_0 > 0$ depending on n, λ, Λ, L as long as $\delta \leq c_0$.

Proof. The proof follows the lines of the analogous result in [DFS]. For completeness we present the details. All constants in this proof depend on n, λ, Λ, L.

It suffices to show that our statement holds for $\{x_n \geq g(x') + C\delta\}$ for a possibly large constant C. Then one can apply Harnack inequality to obtain the full statement.

We prove the statement above at $x = de_n$ (recall that $g(0) = 0$). Precisely, we want to show that

$$u(de_n) \geq c_0 d, \quad d \geq C\delta.$$

After rescaling (for simplicity we drop all subindices in the rescalings and remark that the rescaled operator preserves the same ellipticity constants as $\mathcal{F}$), we reduce to proving that

$$u(e_n) \geq c_0$$

as long as $\delta \leq 1/C$, and $\|f\|_\infty$ is sufficiently small. Let $\gamma > \max\{0, \frac{\Lambda}{\lambda}n(n-1)-1\}$ and

$$w(x) = \frac{1}{2\gamma}(1 - |x|^{-\gamma})$$

be defined on the closure of the annulus $B_2 \setminus \overline{B}_1$. Since $w(x) = w(|x|)$ is a radial function ($r = |x|$), we easily compute that in the appropriate system of coordinates,

$$D^2 w = \frac{1}{2} r^{-\gamma-2} diag\{-(\gamma+1), 1, 1, \dots, 1\}.$$

Thus,

$$\mathcal{M}^+_{\frac{\lambda}{n},\Lambda}(D^2 w) = \frac{1}{2}|x|^{-\gamma-2}((n-1)\Lambda - \frac{\lambda}{n}(\gamma+1)).$$

Hence, for $\|f\|_\infty$ small enough

$$\mathcal{F}(D^2 w) \leq \mathcal{M}^+_{\frac{\lambda}{n},\Lambda}(D^2 w) < -\|f\|_\infty \quad \text{on } B_2 \setminus \overline{B}_1.$$

Let

$$w_t(x) = w(x + te_n).$$

Notice that

$$|\nabla w_0| < 1 \quad \text{on } \partial B_1.$$

From our flatness assumption for $t >, 0$ sufficiently large (depending on the Lipschitz constant of g), w_t is strictly above u. We decrease t and let $\bar{t}$ be the first t such that w_t touches u by above. Since $w_{\bar{t}}$ is a strict supersolution to $\mathcal{F}(D^2 u) = f$ in $B_2 \setminus \bar{B}_1$ the touching point z can occur only on the $\eta := \frac{1}{2\gamma}(1 - 2^{-\gamma})$ level set in the positive phase of u, and $|z| \leq C = C(L)$.

Since u is Lipschitz continuous, $0 < u(z) = \eta \leq Ld(z, F(u))$, that is a full ball around z of radius η/L is contained in the positive phase of u. Thus, for $\bar{\delta}$ small depending on η, L we have that $B_{\eta/2L}(z) \subset \{x_n \geq g(x') + 2\bar{\delta}\}$.

Since $x_n = g(x') + 2\bar{\delta}$ is Lipschitz we can connect e_n and z with a chain of intersecting balls included in the positive side of u with radii comparable to $\eta/2L$. The number of balls depends on L . Then we can apply Harnack inequality and obtain

$$u(e_n) \geq cu(z) = c_0,$$

as desired.

Next, we state a compactness lemma. Since its proof is standard (see Lemma 2.5 in [DFS] and Proposition 2.9 in [CC]), we omit the details.

Lemma 2.5. *Let u_k be a sequence of viscosity solutions to* (1.1) *with operators $\mathcal{F}_k \in \mathcal{E}(\lambda, \Lambda)$ and right-hand-sides f_k satisfying $\|f_k\|_{L^\infty} \leq L$. Assume $\mathcal{F}^k \to \mathcal{F}^*$ uniformly on compact sets of matrices, $u_k \to u^*$ uniformly on compact sets, and $\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance. Then*

$$-L \leq \mathcal{F}^*(D^2 u^*) \leq L, \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

in the viscosity sense and u^* satisfies the free boundary condition

$$(u_\nu^{*+})^2 - (u_\nu^{*-})^2 = 1 \quad \text{on } F(u^*)$$

in the viscosity sense of Definition 2.3.

We are now ready to re-formulate our main Theorem 1.1. To do so, we prove the following Lemma 2.6. Then, using this lemma, our main Theorem 1.1 follows by rescaling from Theorem 2.1, as desired.

Lemma 2.6. *Let u be a solution to* (1.1) *in* B_1 *with* $Lip(u) \leq L$ *and* $\|f\|_{L^\infty} \leq L$. *For any* $\varepsilon > 0$ *there exist* $\bar{\delta}, \bar{r} > 0$ *depending on* $\varepsilon, n, \lambda, \Lambda$ *and* L *such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, *then*

$$\|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r} \tag{2.2}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and $\bar{r}$ depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \to 0$ and a sequence of solutions u_k to the problem (1.1) with operators $\mathcal{F}_k \in \mathcal{E}(\lambda, \Lambda)$, and right-hand-sides f_k such that $Lip(u_k), \|f_k\| \leq L$ and

$$\{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\}, \tag{2.3}$$

but the u_k do not satisfy the conclusion (2.2).

Then, up to a subsequence, the u_k converge uniformly on compacts to a function u^*, and by the uniform ellipticity, $\mathcal{F}_k$ converges uniformly (up to a subsequence) on compact sets of matrices. In view of (2.3) and the non-degeneracy of u_k^+ $2\delta_k$-away from the free boundary (Lemma 2.4), we can apply our compactness lemma and conclude that

$$-L \leq \mathcal{F}^*(D^2 u^*) \leq L, \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

in the viscosity sense and also

$$({u_n^*}^+)^2 - ({u_n^*}^-)^2 = 1 \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\}, \tag{2.4}$$

with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus, by the Remark 3.9 in Section 3,

$$u^* \in C^{1,\gamma}(B_{1/2} \cap \{x_n \geq 0\}) \cap C^{1,\gamma}(B_{1/2} \cap \{x_n \leq 0\})$$

for some $\gamma = \gamma(n, \lambda, \Lambda)$ and in view of (2.4) we have that (for any $\bar{r}$ small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq C(n, L)\bar{r}^{1+\gamma}$$

with $\alpha^2 = 1 + \beta^2$. If $\bar{r}$ is chosen depending on ε so that

$$C(n, L)\bar{r}^{1+\gamma} \leq \frac{\varepsilon}{2}\bar{r},$$

since the u_k converge uniformly to u^* on $B_{1/2}$ we obtain that for all k large

$$\|u_k - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r},$$

a contradiction.

3. The linearized problem

Theorem 2.1 follows from the regularity properties of viscosity solutions to the following transmission problem,

$$\begin{cases} \mathcal{F}^+(D^2\tilde{u}(x)) = 0 & \text{in } B^+_\rho, \\ \\ \mathcal{F}^-(D^2\tilde{u}(x)) = 0 & \text{in } B^-_\rho, \\ \\ a(\tilde{u}_n)^+ - b(\tilde{u}_n)^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}, \end{cases} \tag{3.1}$$

where $(\tilde{u}_n)^+$ (resp. $(\tilde{u}_n)^-$) denotes the derivative in the e_n direction of $\tilde{u}$ restricted to $\{x_n > 0\}$ (resp. $\{x_n < 0\}$.) Here $\mathcal{F}^\pm \in \mathcal{E}(\lambda, \Lambda)$, $a > 0$ and $b \geq 0$. Finally, if B_ρ is the ball of radius ρ centered at zero, we denote

$$B^+_\rho := B_\rho \cap \{x_n > 0\}, \quad B^-_\rho = B_\rho \cap \{x_n < 0\}.$$

In what follows, we sometimes write $\mathcal{F}^\pm(D^2\tilde{u}(x)) = 0$ in $B^\pm_\rho$, to denote both the interior equations in (3.1).

Definition 3.1. We say that $\tilde{u} \in C(B_1)$ is a viscosity subsolution (resp. supersolution) to (3.1) if

(i) $\mathcal{F}^\pm(D^2\tilde{u}(x)) \geq 0$ (resp. ≤ 0) in $B^\pm_\rho$, in the viscosity sense;

(ii) If ($\delta > 0$ small)

$$\varphi \in C^2(\overline{B}^+_\delta) \cap C^2(\overline{B}^-_\delta)$$

touches $\tilde{u}$ by above (resp. by below) at $x_0 \in \{x_n = 0\}$, then

$$a\varphi^+_n(x_0) - b\varphi^-_n(x_0) \geq 0 \quad (\text{resp.} \leq 0).$$

If $\tilde{u}$ is both a viscosity subsolution and supersolution to (3.1), we say that $\tilde{u}$ is a viscosity solution to (3.1).

Equivalently, the condition (ii) above can be replaced by the following one:

(ii') If $P(x')$ denotes a quadratic polynomial in x' and

$$P(x') + px^+_n - qx^-_n$$

touches $\tilde{u}$ by above (resp. by below) at $x_0 \in \{x_n = 0\}$, then

$$ap - bq \geq 0 \quad (\text{resp.} \leq 0).$$

Indeed, let $\varphi \in C^2(\overline{B}^+_\delta) \cap C^2(\overline{B}^-_\delta)$ touch $\tilde{u}$ say by above at $0 \in \{x_n = 0\}$. Then, by Taylor's theorem we obtain that $\varphi(0) + D' \cdot x' + (px^+_n - qx^-_n) + C|x|^2$ also touches u by above at 0 with $p = \varphi^+_n(0), q = \varphi^-_n(0)$. Then, for all $\varepsilon > 0$ small, in a sufficiently small neighborhood of 0 we get that

$$\varphi(0) + D' \cdot x' + C|x'|^2 + (p + \varepsilon)x^+_n - (q - \varepsilon)x^-_n$$

also touches u by above at 0 and hence by (ii')

$$a(p + \varepsilon) - b(q - \varepsilon) \geq 0.$$

The desired inequality follows by letting $\varepsilon \to 0$.

The objective of this section is to prove the following regularity result for viscosity solutions to the linearized problem (3.1). Constants depending on n, λ, Λ are called universal.

Theorem 3.2. *Let $\tilde{u}$ be a solution to* (3.1) *in B_1 such that $\|\tilde{u}\|_\infty \leq 1$. Then $u \in C^{1,\gamma}(\overline{B^+_{1/2}}) \cap C^{1,\gamma}(\overline{B^-_{1/2}})$ with a universal bound on the $C^{1,\gamma}$ norm. In particular, there exists a universal constant $\tilde{C}$ such that*

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq \tilde{C}r^{1+\gamma}, \quad \text{in } B_r \tag{3.2}$$

for all $r \leq 1/4$ and with

$$a\tilde{p} - b\tilde{q} = 0. \tag{3.3}$$

Towards proving the theorem above, we introduce the following special classes of functions, in the spirit of [CC]. From now on, since the parameters a, b in the transmission condition are defined up to a multiplicative constant, and the problem is invariant under reflection with respect to $\{x_n = 0\}$, we can assume without loss of generality that $a = 1, 0 \leq b \leq 1$.

For $0 < \lambda \leq \Lambda$, and $0 \leq b \leq 1$, we denote by $\underline{\mathcal{S}}_{\lambda,\Lambda}$ the class of continuous functions u in B_1 such that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq 0 \quad \text{in } B_1^+ \cup B_1^-,$$

and u satisfies the condition

$$(u_n)^+ - b(u_n)^- \geq 0, \quad \text{on } B_1 \cap \{x_n = 0\},$$

in the viscosity sense of Definition 3.1 (with comparison with test function touching u by above).

Analogously, we denote by $\overline{\mathcal{S}}_{\lambda,\Lambda}$ the class of continuous functions u in B_1 such that

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq 0 \quad \text{in } B_1^+ \cup B_1^-,$$

and u satisfies the condition

$$(u_n)^+ - b(u_n)^- \leq 0, \quad \text{on } B_1 \cap \{x_n = 0\},$$

in the viscosity sense of Definition 3.1 (with comparison with test functions touching u by below).

Finally we denote by

$$\mathcal{S}_{\lambda,\Lambda} := \underline{\mathcal{S}}_{\lambda,\Lambda} \cap \overline{\mathcal{S}}_{\lambda,\Lambda}.$$

First we prove the following Hölder regularity result.

Theorem 3.3. *Let $u \in \mathcal{S}_{\lambda,\Lambda}$ with $\|u\|_\infty \leq 1$. Then $u \in C^\alpha(B_{1/2})$ for some α universal, and with a universal bound on the C^α norm.*

The Theorem above immediately follows from the next Lemma.

Lemma 3.4. *Let $u \in \mathcal{S}_{\lambda,\Lambda}$ with $\|u\|_\infty \leq 1$. Assume that*

$$u(\frac{1}{5}e_n) > 0. \tag{3.4}$$

Then, there exists a universal constant $c > 0$ such that

$$u \geq -1 + c \quad \text{in } B_{1/3}.$$

Proof. By Harnack inequality (see Theorem 4.3 in [CC]) and assumption (3.4) we have that ($\bar{x} = \frac{1}{5}e_n$)

$$u + 1 > \tilde{c} \quad \text{in } B_{1/20}(\bar{x}).$$

Let

$$w = \eta(\Gamma^\gamma(|x - \bar{x}|) + \delta x_n^+), \quad \Gamma^\gamma(|x - \bar{x}|) = |x - \bar{x}|^{-\gamma} - (2/3)^{-\gamma}$$

be defined in the closure of the annulus

$$D := B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$$

with $\gamma > \max\{0, \frac{(n-1)\Lambda}{\lambda} - 1\}$, and η, δ to be made precise later. Since $\Gamma^\gamma(|x-\bar{x}|)$ is a radial function ($r = |x-\bar{x}|$), we find that in the appropriate system of coordinates,

$$D^2 w = \eta\gamma r^{-\gamma-2} diag\{(\gamma+1), -1, \dots, -1\}$$

then, in D

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2 w) = \eta\gamma|x-\bar{x}|^{-\gamma-2}(\lambda(\gamma+1) - \Lambda(n-1)) > 0.$$

Since

$$\partial_n \Gamma^\gamma|_{\{x_n=0\}} > 0,$$

the transmission condition

$$(w_n)^+ - b(w_n)^- > 0 \quad \text{on } x_n = 0,$$

is satisfied.

Finally, notice that on $\partial B_{3/4}(\bar{x})$ we have that $w \leq 0$ as long as δ is chosen sufficiently small (universal). Also, we choose η so that

$$w \leq \tilde{c} \quad \text{on } \partial B_{1/20}(\bar{x}),$$

that is

$$\eta(\Gamma^\gamma(1/20) + \delta(\frac{1}{20} + \frac{1}{5})) \leq \tilde{c}.$$

Combining all the facts above we obtain that

$$w \leq u + 1 \quad \text{on } \partial D$$

and w is a strict (classical) subsolution to the transmission problem in D. By the the definition of viscosity solution, we conclude that

$$w \leq u + 1 \quad \text{in } D.$$

Our desired statement now follows from the fact that

$$w \geq c \quad \text{on } B_{1/3}$$

for c universal.

Now, we wish to prove the following main result.

Proposition 3.5. *Let u be a subsolution to* (3.1) *in B_1 and let v be a supersolution to* (3.1) *in B_1. Then*

$$u - v \in \underline{\mathcal{S}}_{\frac{\lambda}{n},\Lambda}.$$

Corollary 3.6. *Let u be a viscosity solution to* (3.1) *then for any unit vector e' in the x' direction,*

$$\frac{u(x+\varepsilon e') - u(x)}{\varepsilon} \in \mathcal{S}_{\frac{\lambda}{n},\Lambda}.$$

In view of Theorem 3.3 and the Corollary above we obtain by standard arguments (see Chapter 5 in [CC]) the following result.

Theorem 3.7. *Let u be a viscosity solution to* (3.1) *with $\|u\|_\infty \leq 1$. Then, for some α universal, $u \in C^{1,\alpha}$ in the x'-direction in $B_{3/4}$ with $C^{1,\alpha}$ norm bounded by a universal constant.*

The desired estimate (3.2) in Theorem 3.2 now follows from the corollary above and the (boundary) regularity theory for fully nonlinear uniformly elliptic equations (for the regularity result up to the boundary see [MW] or the Appendix in [MS].)

We show below how to obtain (3.3), which concludes the proof of Theorem 3.2.

Proof of (3.3). Let us prove that $a\tilde{p} - b\tilde{q} \leq 0$. (The other inequality follows similarly.) Without loss of generality (after subtracting a linear function), we can assume that (r small)

$$|\tilde{u} - (\tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq \tilde{C}|x|^{1+\gamma}, \quad |x| \leq r. \tag{3.5}$$

For any $\delta > 0$ small, we define,

$$w_\delta(x) = C(-|x|^2 + Kx_n^2) - \delta|x_n| + \tilde{p}x_n^+ - \tilde{q}x_n^-,$$

where

$$r^\gamma = \frac{1}{2}\frac{\delta}{\tilde{C}K}, \quad C = \tilde{C}r^{\gamma-1},$$

and $K = K(n, \lambda, \Lambda)$ is chosen large enough so that

$$\mathcal{M}^-(D^2(-|x|^2 + Kx_n^2)) = \frac{\lambda}{n}2(K-1) - 2\Lambda n > 0. \tag{3.6}$$

Then, using (3.5) it is easy to verify that

$$w_\delta < \tilde{u}, \quad \text{on } \partial B_r.$$

Let

$$m = \min_{\overline{B}_r}(\tilde{u} - w_\delta) = (\tilde{u} - w_\delta)(x_0), \quad x_0 \in \overline{B}_r.$$

In view of (3.6), the minimum cannot occur in the interior $B_r \cap \{x_n \neq 0.\}$ Also, since $(\tilde{u} - w_\delta)(0) = 0$, we have $m \leq 0$ and hence the minimum cannot occur on ∂B_r. Thus, x_0 occurs on $\{x_n = 0\}$, then $w_\delta + m$ touches $\tilde{u}$ by below at x_0 and by definition

$$a(\tilde{p} - \delta) - b(\tilde{q} + \delta) \leq 0. \tag{3.7}$$

The conclusion follows by letting $\delta \to 0$ in (3.7).

We are now left with the proof of our main Proposition 3.5. First, we remark that in the proof of this proposition we will need a pointwise boundary regularity result of the following type (see [MW]).

Proposition 3.8. *Let u satisfy ($\mathcal{F} \in \mathcal{E}(\lambda, \Lambda)$)*

$$\mathcal{F}(D^2u) = f \quad \text{in } B_1^+, \quad u(x', 0) = \varphi(x') \quad \text{on } B_1 \cap \{x_n = 0\}$$

with φ pointwise $C^{1,\alpha}$ at 0 and $f \in L^\infty(\overline{B_1^+})$. Then u is pointwise $C^{1,\alpha}$ at 0, that is there exists a linear function L_u such that for all r small

$$|u - L_u| \leq Cr^{1+\alpha}, \quad \text{in } \overline{B_r^+(0)}$$

with C depending on $n, \lambda, \Lambda, \|f\|_\infty$ and the pointwise $C^{1,\alpha}$ bound on φ.

Remark 3.9. Clearly, from the proposition above and the interior regularity estimates it follows that regularity up to the boundary holds also for a problem with right-hand side as used in Lemma 2.6.

Towards the proof of Proposition 3.5, we now introduce the following regularizations. Given a continuous function u in B_1 and an arbitrary ball B_ρ with $\overline{B}_\rho \subset B_1$ we define for $\varepsilon > 0$ the upper ε-envelope of u in the x'-direction,

$$u^\varepsilon(y', y_n) = \sup_{x \in \overline{B}_\rho \cap \{x_n = y_n\}} \{u(x', y_n) - \frac{1}{\varepsilon}|x' - y'|^2\}, \quad y = (y', y_n) \in B_\rho.$$

The proof of the following facts is standard (see [CC]):

(1) $u^\varepsilon \in C(B_\rho)$ and $u_\varepsilon \to u$ uniformly in B_ρ as $\varepsilon \to 0$.

(2) u^ε is $C^{1,1}$ in the x'-direction by below in B_ρ. Thus, u^ε is pointwise second order differentiable in the x'-direction at almost every point in B_ρ.

(3) If u is a viscosity subsolution to (3.1) in B_1 and $\overline{B}_r \subset B_\rho$, then for $\varepsilon \leq \varepsilon_0$ (ε_0 depending on u, ρ, r) u^ε is a viscosity subsolution to (3.1) in B_r. This fact follows from the obvious remark that the maximum of solutions of (3.1) is a viscosity subsolution.

Analogously we can define u_ε, the lower ε-envelope of u in the x'-direction which enjoys the corresponding properties.

We are now ready to prove our main proposition.

Proof of Proposition 3.5. In what follows, for notational simplicity, we omit the dependence of the Pucci operators from $\lambda/n, \Lambda$.

By Theorem 5.3 in [CC] we only need to show that the free boundary condition is satisfied in the viscosity sense. Let

$$\varphi(x) = P(x') + px_n^+ - qx_n^-$$

touch $u - v$ by above at a point $x_0 \in \{x_n = 0\}$, with P a quadratic polynomial. Assume by contradiction that

$$p - bq < 0.$$

Without loss of generality we can assume that φ touches $w := u - v$ strictly and also that $\mathcal{M}^+(D^2P) < 0$ (by modifying p, q, allowing quadratic dependence on x_n and possibly restricting the neighborhood around x_0). Let us say that on the annulus $\overline{B}_{2\delta}(x_0) \setminus B_{\delta/2}(x_0)$, $\varphi - w \geq \eta > 0$. Now, since $w_\varepsilon := u^\varepsilon - v_\varepsilon$ converges to $u - v$ uniformly, and φ touches w strictly by above, for ε small enough we have that (up to adding a small constant) φ touches w_ε at some x_ε by above and say $\varphi - w_\varepsilon \geq \eta/2$ on $\partial B_\delta(x_\varepsilon)$. By property (3) above and the fact that $\mathcal{M}^+(D^2P) < 0$ we get from the comparison principle that $x_\varepsilon \in \{x_n = 0\}$.

Now, call

$$\psi = \varphi - w_\varepsilon - \eta/2$$

Since $\psi \geq 0$ on $\partial B_\delta(x_\varepsilon)$ and $\psi(x_\varepsilon) < 0$ we obtain by ABP estimates (see Lemma 3.5 in [CC]) that the set of points in $B_\delta(x_\varepsilon) \cap \{x_n = 0\}$ where ψ admits a touching plane $l(x')$, of slope less than some arbitrarily small number, by below in the x'-direction is a set of positive measure. We choose the slope of l small enough so that $\bar{\varphi} = \varphi - l - \eta/2$ is above w_ε on $\partial B_\delta(x_\varepsilon)$ and hence in the interior. By property (2) above we can then conclude that $\bar{\varphi}$ touches w_ε by above at some point $y_\varepsilon \in \{x_n = 0\}$ where u^ε and v_ε are twice pointwise differentiable in the x'-direction.

Now call $\bar{u}^\varepsilon$ (resp. $\bar{v}_\varepsilon$) the solution to $\mathcal{F}^\pm(D^2w) = 0$ in $B^\pm_\delta(x_\varepsilon)$ with $w = u^\varepsilon$ (resp. v_ε) on $\partial B^\pm_\delta(x_\varepsilon)$. Also call $\bar{w}_\varepsilon = \bar{u}^\varepsilon - \bar{v}_\varepsilon$. Since (by Theorem 5.3 in [CC]) $\mathcal{M}^+(D^2\bar{w}_\varepsilon) \geq 0$ in $B^\pm_\delta(x_\varepsilon)$ and $\mathcal{M}^+(D^2\bar{\varphi}) < 0$ in $B^\pm_\delta(x_\varepsilon)$ with $\bar{\varphi} \geq \bar{w}_\varepsilon = w_\varepsilon$ on the boundary (recall that l is below ψ on $\{x_n = 0\}$), we conclude that $\bar{\varphi}$ is above $\bar{w}_\varepsilon$ also in the interior and therefore it touches it by above at y_ε.

Since the boundary data is twice pointwise differentiable at y_ε, thus in particular it is pointwise $C^{1,\alpha}$ we conclude by pointwise $C^{1,\alpha}$ regularity that $\bar{u}^\varepsilon$ is $C^{1,\alpha}$ up to y_ε that is, there exist linear functions L_u, L_v such that for all r small

$$|\bar{u}^\varepsilon - L_u| \leq Cr^{1+\alpha}, \quad \text{in } B_r^+(y_\varepsilon),$$

$$|\bar{v}_\varepsilon - L_v| \leq Cr^{1+\alpha}, \quad \text{in } B_r^+(y_\varepsilon).$$

Since $\bar{\varphi}$ touches $\bar{w}_\varepsilon$ by above at y_ε we get that:

$$p \geq p_u^+ - p_v^+$$

where

$$p_u^+ = (\bar{u}^\varepsilon)_n^+(y_\varepsilon), \quad p_v^+ = (\bar{v}_\varepsilon)_n^+(y_\varepsilon).$$

Arguing similarly in $B_r^-(y_\varepsilon)$ we also get,

$$q \leq q_u^- - q_v^-,$$

where

$$q_u^- = (\bar{u}^\varepsilon)_n^-(y_\varepsilon), \quad q_v^- = (\bar{v}_\varepsilon)_n^-(y_\varepsilon).$$

We therefore contradict the fact that $p - bq < 0$ if we show that

$$p_u^+ - bq_u^- \geq 0, \quad p_v^+ - bq_v^- \leq 0.$$

Since the replacement $\bar{u}^\varepsilon$ (resp. $\bar{v}_\varepsilon$) is still a viscosity subsolution (resp. supersolution) to the boundary condition, the inequalities above follow from the next Lemma. Thus our proof is concluded. □

Lemma 3.10. *Let u be a viscosity solution to ($0 \leq b \leq 1$)*

$$\begin{cases} \mathcal{F}^\pm(D^2u) = 0, & \text{in } B_1^\pm \\ \\ u_n^+ - bu_n^- \geq 0, & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{3.8}$$

Assume that u is twice differentiable at zero in the x'-direction. Then, u is differentiable at 0 and

$$u_n^+(0) - bu_n^-(0) \geq 0.$$

Proof. From pointwise boundary regularity we know that there exist linear functions L_u such that for all r small

$$|u - L_u| \leq Cr^{1+\alpha}, \quad \text{in } \overline{B_r^+(0)}.$$

Let us assume (by subtracting a linear function) that $L_u = d^+x_n$.

Let w solve

$$\mathcal{F}^+(D^2w) = 0 \quad \text{in } B_r^+, \quad w = \phi_r \quad \text{on } \partial B_r^+,$$

where

$$\phi_r := \begin{cases} C_1|x|^{1+\alpha} & \text{if } x \in \partial B_r^+ \cap \{x_n > 0\} \\ C_2|x'|^2 & \text{if } x \in B_r \cap \{x_n = 0\} \end{cases}$$

with $C_1 = 2C$ and $C_2 = C_1 r^{\alpha-1}$. Therefore, by the assumption that u is twice pointwise differentiable at 0 we get (for r small enough)

$$u - d^+x_n \leq \phi_r \quad \text{on } \partial B_r^+.$$

Then, by the comparison principle,

$$u - d^+ x_n \leq w \quad \text{in } B_r^+. \tag{3.9}$$

Let $\tilde{w}$ be the rescaling:

$$\tilde{w}(x) = \frac{1}{r^{1+\alpha}} w(rx), \quad x \in B_1^+.$$

Then $\tilde{w}$ solves

$$\mathcal{G}(D^2 \tilde{w}) = 0 \quad \text{in } B_1^+, \quad \tilde{w} = C_1 |x'|^2 \quad \text{on } B_1 \cap \{x_n = 0\}$$

where $\mathcal{G}(M) = r^{1-\alpha} \mathcal{F}^+(r^{\alpha-1} M)$ depend on r but has the same ellipticity constants as $\mathcal{F}^+$.

By boundary $C^{1,\alpha}$ estimates we obtain that

$$\|\tilde{w}\|_{1,\alpha} \leq C_3 \quad \text{in } \overline{B}^{\pm}_{1/2}$$

with C_3 universal. Then, in particular

$$\tilde{w} \leq C_1 |x'|^2 + C_3 x_n \quad \text{in } \overline{B}^+_{1/2}$$

and by rescaling we conclude that

$$w \leq C_1 r^{\alpha-1} |x'|^2 + C_3 r^\alpha x_n, \quad \text{in } \overline{B}^+_{r/2}.$$

Thus, by (3.9)

$$u \leq C_1 r^{\alpha-1} |x'|^2 + C_3 r^\alpha x_n + d^+ x_n^+, \quad \text{in } \overline{B}^+_{r/2},$$

where we recall that $d^+ = (u_n)^+(0)$. Arguing similarly in B_r^- we also obtain that

$$u \leq C_1 r^{\alpha-1} |x'|^2 + C_3 r^\alpha x_n - d^- x_n^-, \quad \text{in } \overline{B}^-_{r/2}.$$

where $d^- = (u_n)^-(0)$. Thus,

$$\varphi = C_1 r^{\alpha-1} |x'|^2 + C_3 r^\alpha x_n + d^+ x_n^+ - d^- x_n^-$$

touches u by above at zero with $p = Mr^\alpha + d^+$ and $q = -Mr^\alpha + d^-$. We conclude that

$$Mr^\alpha + bMr^\alpha + d^+ - bd^- \geq 0$$

for all r small, from which our desired claim follows.

4. Harnack inequality

In this section we prove a Harnack-type inequality for “flat” solutions to our free boundary problem (1.1). Our strategy follows closely the arguments in [DFS]. We especially point out the main technical differences in the proofs, which mostly consist of the choice of the barriers.

Throughout this section we consider a Lipschitz solution u to (1.1) with $Lip(u) \leq L$. As pointed out in the introduction, we distinguish two cases, the non-degenerate and the degenerate one.

4.1. **Non-degenerate case.** In this case u is trapped between two translations of a "true" two-plane solution U_β that is with $\beta \neq 0$.

Theorem 4.1 (Harnack inequality)**.** *There exists a universal constant $\bar{\varepsilon}$, such that if u is a solution of* (1.1) *that satisfies at some point $x_0 \in B_2$*

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_2, \tag{4.1}$$

with

$$\|f\|_{L^\infty} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_\beta(x_n + a_1) \leq u(x) \leq U_\beta(x_n + b_1) \quad \text{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

Let

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon} & \text{in } B_2^+(u) \cup F(u) \\ \\ \dfrac{u(x) - \beta x_n}{\beta \varepsilon} & \text{in } B_2^-(u). \end{cases}$$

From a standard iterative argument (see [DFS]), we obtain the following corollary.

Corollary 4.2. *Let u be as in Theorem* 4.1 *satisfying* (4.1) *for $r = 1$. Then in $B_1(x_0)$ $\tilde{u}_\varepsilon$ has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e. for all $x \in B_1(x_0)$ with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$*

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The main tool in the proof of the Harnack inequality is the following lemma.

Lemma 4.3. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u(x) \geq U_\beta(x), \quad \text{in } B_1 \tag{4.2}$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L, \tag{4.3}$$

and at $\bar{x} = \dfrac{1}{5} e_n$

$$u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon), \tag{4.4}$$

then

$$u(x) \geq U_\beta(x_n + c\varepsilon) \quad \text{in } \overline{B}_{1/2}, \tag{4.5}$$

for some $0 < c < 1$ universal. Analogously, if

$$u(x) \leq U_\beta(x) \quad \text{in } B_1$$

and

$$u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$$

then

$$u(x) \le U_\beta(x_n - c\varepsilon) \quad \textit{in } \overline{B}_{1/2}.$$

Proof. We prove the first statement. For notational simplicity we drop the subindex β from U_β and the dependence of the Pucci operators from $\lambda/n, \Lambda$.

Let

$$w = c(|x-\bar{x}|^{-\gamma} - (3/4)^{-\gamma})$$

be defined in the closure of the annulus

$$A := B_{3/4}(\bar{x}) \setminus \overline{B}_{1/20}(\bar{x})$$

where $\gamma > 0$ will be fixed later on. The constant c is such that w satisfies the boundary conditions

$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$

Extend w to be equal to 1 on $B_{1/20}(\bar{x})$. We claim that,

$$\mathcal{F}(D^2 w) \ge k(n, \lambda, \Lambda) > 0, \quad \text{in } A.$$

Indeed, since $w(x) = w(|x-\bar{x}|)$ is a radial function ($r = |x-\bar{x}|$), we find that in the appropriate system of coordinates,

$$D^2 w = c\gamma r^{-\gamma-2} diag\{(\gamma+1), -1, \dots, -1\}.$$

Then, in A

$$\mathcal{M}^-(D^2 w) = c\gamma|x-\bar{x}|^{-\gamma-2}(\frac{\lambda}{n}(\gamma+1) - \Lambda(n-1)) \ge k(n,\lambda,\Lambda) > 0,$$

as long as $\gamma > \max\{0, \frac{n(n-1)\Lambda}{\lambda} - 1\}$.

Thus

$$\mathcal{F}(D^2 w) \ge \mathcal{M}^-(D^2 w) \ge k(n,\lambda,\Lambda) > 0, \quad 0 \le w \le 1 \quad \text{in } A.$$

Having provided the appropriate barrier, the proof now proceeds as in Lemma 4.3 in [DFS]. For the reader's convenience we provide the details.

Notice that since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u \ge U$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - U \ge 0$ and solves $\mathcal{F}(D^2(u-U)) = \mathcal{F}(D^2 u) = f$ in $B_{1/10}(\bar{x})$. We can apply Harnack inequality to obtain

$$u(x) - U(x) \ge c(u(\bar{x}) - U(\bar{x})) - C\|f\|_{L^\infty} \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{4.6}$$

From the assumptions (4.3) and (4.4) we conclude that (for ε small enough)

$$u - U \ge \alpha c\varepsilon - C\varepsilon^2\beta \ge \alpha c_0\varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{4.7}$$

Now set $\psi = 1 - w$ and

$$v(x) = U(x_n - \varepsilon c_0\psi(x)), \quad x \in \overline{B}_{3/4}(\bar{x}), \tag{4.8}$$

and for $t \ge 0$,

$$v_t(x) = U(x_n - \varepsilon c_0\psi(x) + t\varepsilon), \quad x \in \overline{B}_{3/4}(\bar{x}).$$

Then,

$$v_0(x) = U(x_n - \varepsilon c_0\psi(x)) \le U(x) \le u(x) \quad x \in \overline{B}_{3/4}(\bar{x}).$$

Let $\bar t$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar x).$$

We want to show that $\bar t \geq c_0$. Then we get the desired statement. Indeed,

$$u(x) \geq v_{\bar t}(x) = U(x_n - \varepsilon c_0 \psi + \bar t \varepsilon) \geq U(x_n + c\varepsilon) \quad \text{in } B_{1/2} \subset\subset B_{3/4}(\bar x)$$

with c universal. In the last inequality we used that $\|\psi\|_{L^\infty(B_{1/2})} < 1$.

Suppose $\bar t < c_0$. Then at some $\tilde x \in \overline{B}_{3/4}(\bar x)$ we have

$$v_{\bar t}(\tilde x) = u(\tilde x).$$

We show that such touching point can only occur on $\overline{B}_{1/20}(\bar x)$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar x)$ from the definition of v_t we get that for $\bar t < c_0$

$$v_{\bar t}(x) = U(x_n - \varepsilon c_0 \psi(x) + \bar t \varepsilon) < U(x) \leq u(x) \quad \text{on } \partial B_{3/4}(\bar x).$$

We now show that $\tilde x$ cannot belong to the annulus A. Indeed, in $A^+(v_{\bar t})$

$$\mathcal{F}(D^2 v_{\bar t}(x)) = \mathcal{F}(\alpha\varepsilon c_0 D^2 w) \geq \mathcal{M}^-(\alpha\varepsilon c_0 D^2 w) \geq \alpha\varepsilon c_0 k \geq \beta\varepsilon^2 \geq \|f\|_\infty$$

for ε small enough. An analogous computation holds in $A^-(v_{\bar t})$.

Finally,

$$(v_{\bar t}^+)^2_\nu - (v_{\bar t}^-)^2_\nu = 1 + \varepsilon^2 c_0^2 |\nabla \psi|^2 - 2\varepsilon c_0 \psi_n \quad \text{on } F(v_{\bar t}) \cap A.$$

Thus,

$$(v_{\bar t}^+)^2_\nu - (v_{\bar t}^-)^2_\nu > 1 \quad \text{on } F(v_{\bar t}) \cap A$$

as long as

$$\psi_n < 0 \quad \text{on } F(v_{\bar t}) \cap A.$$

This can be easily verified from the formula for ψ (for ε small enough.)

Thus, $v_{\bar t}$ is a strict subsolution to (1.1) in A which lies below u, hence by the definition of viscosity solutions $\tilde x$ cannot belong to A.

Therefore, $\tilde x \in \overline{B}_{1/20}(\bar x)$ and

$$u(\tilde x) = v_{\bar t}(\tilde x) = U(\tilde x_n + \bar t \varepsilon) \leq U(\tilde x) + \alpha \bar t \varepsilon < U(\tilde x) + \alpha c_0 \varepsilon$$

contradicting (4.7).

We can now prove our Theorem 4.1.

Proof of Theorem 4.1. Assume without loss of generality that $x_0 = 0, r = 1$. We distinguish three cases.

Case 1. $a_0 < -1/5$. In this case it follows from 4.1 that $B_{1/10} \subset \{u < 0\}$ and

$$0 \leq v(x) := \frac{u(x) - \beta(x_n + a_0)}{\beta\varepsilon} \leq 1.$$

We recall that the operator $\mathcal{F}_{\epsilon\beta}(M) = \frac{1}{\epsilon\beta}\mathcal{F}(\epsilon\beta M) \in \mathcal{E}(\lambda, \Lambda)$, and

$$\mathcal{F}_{\epsilon\beta}(D^2 v) = \frac{f}{\beta\epsilon}.$$

Moreover

$$|\mathcal{F}_{\varepsilon\beta}(D^2 v)| \leq \varepsilon \quad \text{in } B_{1/10}$$

since $\|f\| < \beta\epsilon^2$. The desired claim follows from standard Harnack inequality applied to the function v.

Case 2. $a_0 > 1/5$. In this case it follows from (4.1) that $B_{1/5} \subset \{u > 0\}$ and

$$0 \leq v(x) := \frac{u(x) - \alpha(x_n + a_0)}{\alpha\varepsilon} \leq 1.$$

As before,

$$\mathcal{F}_{\epsilon\alpha}(D^2 v) = \frac{1}{\epsilon\alpha}\mathcal{F}(\epsilon\alpha D^2 v) = \frac{f}{\alpha\epsilon},$$

hence

$$|\mathcal{F}_{\varepsilon\alpha}(D^2 v)| \leq \varepsilon \quad \text{in } B_{1/5}$$

since $\|f\| \leq \beta\epsilon^2 \leq \alpha\varepsilon^2$. Again, the desired claim follows from standard Harnack inequality for v.

Case 3. $|a_0| \leq 1/5$. In this case we argue exactly as in the Laplacian case (see Theorem 4.1 in [DFS]) using the key Lemma 4.3.

4.2. **Degenerate case.** In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0$).

Theorem 4.4 (Harnack inequality)**.** *There exists a universal constant $\bar{\varepsilon}$, such that if u satisfies at some point $x_0 \in B_2$*

$$U_0(x_n + a_0) \leq u^+(x) \leq U_0(x_n + b_0) \quad \textit{in } B_r(x_0) \subset B_2, \tag{4.9}$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon, \quad \|f\|_{L^\infty} \leq \varepsilon^3$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_0(x_n + a_1) \leq u^+(x) \leq U_0(x_n + b_1) \quad \textit{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1-c)\varepsilon r,$$

and $0 < c < 1$ universal.

From the theorem above we conclude the following.

Corollary 4.5. *Let u be as in Theorem 4.4 satisfying (4.1) for $r = 1$. Then in $B_1(x_0)$*

$$\tilde{u}_\varepsilon := \frac{u^+(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0, outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

Again, the proof of the Harnack inequality relies on the following lemma.

Lemma 4.6. *There exists a universal constant* $\bar{\varepsilon} > 0$ *such that if* u *satisfies*

$$u^+(x) \geq U_0(x), \quad \textit{in } B_1$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4, \tag{4.10}$$

and at $\bar{x} = \dfrac{1}{5} e_n$

$$u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon), \tag{4.11}$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon), \quad \textit{in } \overline{B}_{1/2}, \tag{4.12}$$

for some $0 < c < 1$ *universal. Analogously, if*

$$u^+(x) \leq U_0(x), \quad \textit{in } B_1$$

and

$$u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon),$$

then

$$u^+(x) \leq U_0(x_n - c\varepsilon), \quad \textit{in } \overline{B}_{1/2}.$$

Proof. We prove the first statement. The proof follows the same line as in the non-degenerate case. The dependence of the Pucci operators on $\lambda/n, \Lambda$ is omitted.

Since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u^+ \geq U_0$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - x_n \geq 0$ and solves $\mathcal{F}(D^2(u - x_n)) = f$ in $B_{1/10}(\bar{x})$. We can apply Harnack inequality and the assumptions (4.10) and (4.11) to obtain that (for ε small enough)

$$u - x_n \geq c_0 \varepsilon \quad \text{in } \overline{B}_{1/20}(\bar{x}). \tag{4.13}$$

Let w be as in the proof of Lemma 4.3 and $\psi = 1 - w$. Set

$$v(x) = (x_n - \varepsilon c_0 \psi(x))^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x))^-, \quad x \in \overline{B}_{3/4}(\bar{x}),$$

and for $t \geq 0$,

$$v_t(x) = (x_n - \varepsilon c_0 \psi + t\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x) + t\varepsilon)^-, \quad x \in \overline{B}_{3/4}(\bar{x}).$$

Here C_1 is a universal constant to be made precise later. We claim that

$$v_0(x) = v(x) \leq u(x) \quad x \in \overline{B}_{3/4}(\bar{x}).$$

This is readily verified in the set where u is non-negative using that $u \geq x_n^+$. To prove our claim in the set where u is negative we wish to use the following fact:

$$u^- \leq C x_n^- \varepsilon^2, \quad \text{in } B_{\frac{19}{20}},\ C \text{ universal}. \tag{4.14}$$

This estimate is obtained remarking that in the set $\{u < 0\}$, u^- satisfies

$$\mathcal{M}^+(D^2 u^-) = -\mathcal{M}^-(D^2 u) \geq -\mathcal{F}(D^2 u) = -f > -\varepsilon^4.$$

Hence, the inequality follows using that $\{u < 0\} \subset \{x_n < 0\}$, $\|u^-\|_\infty < \varepsilon^2$ and the comparison principle with the function w satisfying

$$\mathcal{M}^+(D^2 w) = -\varepsilon^2 \leq -\varepsilon^4 \quad \text{in } B_1 \cap \{x_n < 0\},$$

$$w = \varepsilon^2 \quad \text{on } \partial B_1 \cap \{x_n < 0\}, \quad w = 0 \quad \text{on } x_n = 0.$$

Notice that $\mathcal{M}^+$ is a convex operator, thus w/ε^2 is an explicit barrier which has $C^{2,\alpha}$ estimates up to $\{x_n = 0\}$. Hence $u^- \leq w \leq C x_n^- \varepsilon^2$ in $B_{19/20} \cap \{x_n \leq 0\}$.

Thus our claim immediately follows from the fact that for $x_n < 0$ and $C_1 \geq C$,

$$\varepsilon^2 C_1(x_n - \varepsilon c_0 \psi(x)) \leq C x_n \varepsilon^2.$$

Let $\bar{t}$ be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}).$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed, it is easy to check that if

$$u(x) \geq v_{\bar{t}}(x) = (x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon)^- \quad \text{in } B_{3/4}(\bar{x})$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } B_{1/2} \subset\subset B_{3/4}(\bar{x})$$

with c universal, $c < c_0 \inf_{B_1/2} w$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \overline{B}_{3/4}(\bar{x})$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\overline{B}_{1/20}(\bar{x})$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$

$$v_{\bar{t}}(x) = (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0 + \bar{t}\varepsilon)^- < u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

In the set where $u \geq 0$ this can be seen using that $u \geq x_n^+$ while in the set where $u < 0$ again we can use the estimate (4.14).

We now show that $\tilde{x}$ cannot belong to the annulus A. Indeed,

$$\mathcal{F}(D^2 v_{\bar{t}}) \geq \mathcal{M}^-(D^2 v_{\bar{t}}) \geq \varepsilon^3 c_0 k(n) > \varepsilon^4 \geq \|f\|_\infty, \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough.

Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = (1 - \varepsilon^4 C_1^2)(1 + \varepsilon^2 c_0^2 |\nabla\psi|^2 - 2\varepsilon c_0 \psi_n) \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A$$

as long as ε is small enough (as in the non-degenerate case one can check that $\inf_{F(v_{\bar{t}}) \cap A}(-\psi_n) > c > 0$, c universal). Thus, $v_{\bar{t}}$ is a strict subsolution to (1.1) in A which lies below u, hence by definition $\tilde{x}$ cannot belong to A.

Therefore, $\tilde{x} \in \overline{B}_{1/20}(\bar{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = (\tilde{x}_n + \bar{t}\varepsilon) < \tilde{x}_n + c_0\varepsilon$$

contradicting (4.13).

5. Improvement of flatness

In this section we prove our key "improvement of flatness" lemmas. As in Section 4, we need to distinguish two cases. Recall that $\mathcal{E}(\lambda, \Lambda)$ is the class of all uniformly elliptic operators $\mathcal{F}(M)$ with ellipticity constants λ, Λ and such that $\mathcal{F}(0) = 0$.

5.1. **Non-degenerate case.** In this case u is trapped between two translations of a two-plane solution U_β with $\beta \neq 0$. We show that when we restrict to smaller balls, u is trapped between closer translations of another two-plane solution (in a different system of coordinates).

Lemma 5.1 (Improvement of flatness)**.** *Let u satisfy*

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \tag{5.1}$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta.$$

If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then

$$U_{\beta'}(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \leq u(x) \leq U_{\beta'}(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r, \tag{5.2}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\varepsilon$, and $|\beta - \beta'| \leq \tilde{C}\beta\varepsilon$ for a universal constant $\tilde{C}$.

Proof. We divide the proof of this Lemma into 3 steps.

Step 1 – Compactness. Fix $r \leq r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1.1) in B_1 with for a sequence of operators $\mathcal{F}_k \in \mathcal{E}(\lambda, \Lambda)$ and right hand sides f_k with L^∞ norm bounded by $\varepsilon_k^2 \beta_k$, such that

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{for } x \in B_1,\ 0 \in F(u_k), \tag{5.3}$$

with $L \geq \beta_k > 0$, but u_k does not satisfy the conclusion (5.2) of the lemma.

Set ($\alpha_k^2 = 1 + \beta_k^2$),

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\[2ex] \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Then (5.3) gives,

$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1. \tag{5.4}$$

From Corollary 4.2, it follows that the function $\tilde{u}_k$ satisfies

$$|\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C|x - y|^\gamma, \tag{5.5}$$

for C universal and

$$|x - y| \geq \varepsilon_k/\bar{\varepsilon}, \quad x, y \in B_{1/2}.$$

From (5.3) it clearly follows that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. This fact and (5.5) together with Ascoli-Arzela give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$. Also, up to a subsequence

$$\beta_k \to \tilde{\beta} \geq 0$$

and hence

$$\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}.$$

Step 2 – Limiting Solution. We now show that $\tilde{u}$ solves

$$\begin{cases} \mathcal{F}^{\pm}(D^2\tilde{u}(x)) = 0 & \text{in } B^{\pm}_{1/2} \cap \{x_n \neq 0\}, \\ \\ a(\tilde{u}_n)^+ - b(\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}, \end{cases} \tag{5.6}$$

with $\mathcal{F}^{\pm} \in \mathcal{E}(\lambda, \Lambda)$, and $a = \tilde{\alpha}^2 > 0, b = \tilde{\beta}^2 \geq 0$.

Set,

$$\mathcal{F}^{+}_k(M) = \frac{1}{\alpha_k \epsilon_k} \mathcal{F}_k(\alpha_k \epsilon_k M)$$
$$\mathcal{F}^{-}_k(M) = \frac{1}{\beta_k \epsilon_k} \mathcal{F}_k(\beta_k \epsilon_k M).$$

Then $\mathcal{F}^{\pm}_k \in \mathcal{E}(\lambda, \Lambda)$. Thus, up to extracting a subsequence,

$$\mathcal{F}^{\pm}_k \to \mathcal{F}^{\pm}, \quad \text{uniformly on compact subsets of matrices.}$$

Moreover,

$$\mathcal{F}^{+}_k(D^2\tilde{u}_k(x)) = \frac{f_k}{\epsilon_k \alpha_k}, \quad B^{+}_1(u_k)$$

and

$$\mathcal{F}^{-}_k(D^2\tilde{u}_k(x)) = \frac{f_k}{\epsilon_k \beta_k}, \quad B^{-}_1(u_k),$$

with

$$|\mathcal{F}^{\pm}_k(D^2 u_k(x))| \leq \varepsilon_k$$

in $B^{\pm}_1(u_k)$, since $\|f\|_\infty \leq \varepsilon_k^2 \beta_k$.

Then, by standard arguments (see Proposition 2.9 in [CC]), we conclude that $\tilde{u}$ solves in the viscosity sense

$$\mathcal{F}^{\pm}(D^2\tilde{u}) = 0 \quad \text{in } B^{\pm}_{1/2}(\tilde{u}).$$

Next, we prove that $\tilde{u}$ satisfies the boundary condition in (5.6) in the viscosity sense. By a slight modification of the argument after the Definition 3.1 (ii'), it is enough to test that if $\tilde{\phi}$ is a function of the form (γ a specific constant to be made precise later)

$$\tilde{\phi}(x) = A + p x_n^+ - q x_n^- + B Q^{\gamma}(x - y)$$

with

$$Q^{\gamma}(x) = \frac{1}{2}[\gamma x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0,$$

and

$$ap - bq > 0,$$

then $\tilde{\phi}$ cannot touch u strictly by below at a point $x_0 = (x_0', 0) \in B_{1/2}$.

The analogous statement by above follows with a similar argument.

Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point. Let

$$\Gamma^{\gamma}(x) = \frac{1}{2\gamma}[(|x'|^2 + |x_n - 1|^2)^{-\gamma} - 1], \tag{5.7}$$

where γ is sufficiently large (to be made precise later), and let

$$\Gamma^{\gamma}_k(x) = \frac{1}{B\varepsilon_k} \Gamma^{\gamma}(B\varepsilon_k(x - y) + AB\varepsilon_k^2 e_n). \tag{5.8}$$

Now, call

$$\phi_k(x) = a_k\Gamma_k^{\gamma+}(x) - b_k\Gamma_k^{\gamma-}(x) + \alpha_k(d_k^+(x))^2\varepsilon_k^{3/2} + \beta_k(d_k^-(x))^2\varepsilon_k^{3/2}$$

where

$$a_k = \alpha_k(1+\varepsilon_k p), \quad b_k = \beta_k(1+\varepsilon_k q)$$

and $d_k(x)$ is the signed distance from x to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k))$.

Finally, let

$$\tilde{\phi}_k(x) = \begin{cases} \dfrac{\phi_k(x) - \alpha_k x_n}{\alpha_k\varepsilon_k}, & x \in B_1^+(\phi_k) \cup F(\phi_k) \\ \\ \dfrac{\phi_k(x) - \beta_k x_n}{\beta_k\varepsilon_k}, & x \in B_1^-(\phi_k). \end{cases}$$

By Taylor's theorem

$$\Gamma(x) = x_n + Q^\gamma(x) + O(|x|^3) \quad x \in B_1,$$

thus it is easy to verify that

$$\Gamma_k^\gamma(x) = A\varepsilon_k + x_n + B\varepsilon_k Q^\gamma(x-y) + O(\varepsilon_k^2) \quad x \in B_1,$$

with the constant in $O(\varepsilon_k^2)$ depending on A, B, and $|y|$ (later this constant will depend also on p, q).

It follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ $(Q^{\gamma,y}(x) = Q^\gamma(x-y))$

$$\tilde{\phi}_k(x) = A + BQ^{\gamma,y} + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^{\gamma,y} + \varepsilon_k^{1/2}d_k^2 + O(\varepsilon_k)$$

and analogously in $B_1^-(\phi_k)$

$$\tilde{\phi}_k(x) = A + BQ^{\gamma,y} + qx_n + A\varepsilon_k p + Bq\varepsilon_k Q^{\gamma,y} + \varepsilon_k^{1/2}d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly by below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k by below at x_k. We thus get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is

$$\begin{cases} \mathcal{F}_k(D^2\psi_k) > \varepsilon_k^2\beta_k \geq \|f_k\|_\infty, & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ \\ (\psi_k^+)_\nu^2 - (\psi_k^-)_\nu^2 > 1, & \text{on } F(\psi_k). \end{cases}$$

For k large enough, say, in the positive phase of ψ_k (denoting $\bar{x} = x + \varepsilon_k c_k e_n$ and dropping the dependance of the Pucci operator from $\lambda/n, \Lambda$), we have that

$$\mathcal{M}^-(D^2\psi_k(x)) \geq a_k\mathcal{M}^-(D^2\Gamma_k^\gamma(\bar{x})) + \alpha_k\varepsilon_k^{3/2}\mathcal{M}^-(d_k^2(\bar{x})).$$

As computed several times throughout the paper (see for example Lemma 4.3), for γ large enough depending on n, λ, Λ we have that $\mathcal{M}^-(D^2\Gamma_k^\gamma(\bar{x})) > 0$. Moreover, in the appropriate system of coordinates,

$$D^2 d_k^2(\bar{x}) = diag\{-d_k(\bar{x})\kappa_1(\bar{x}), \ldots, -d_k(\bar{x})\kappa_{n-1}(\bar{x}), 1\}$$

where the $\kappa_i(\bar{x})$ denote the curvature of the surface parallel to $\partial B_{\frac{1}{B\varepsilon_k}}(y+e_n(\frac{1}{B\varepsilon_k}-A\varepsilon_k))$ which passes through $\bar{x}$. Thus,

$$\kappa_i(\bar{x}) = \frac{B\varepsilon_k}{1-B\varepsilon_k d_k(\bar{x})}.$$

For k large enough we conclude that $\mathcal{M}^-(d_k^2(\bar{x})) > \lambda/2n$ and hence,

$$\mathcal{F}_k(D^2\psi_k) \geq \mathcal{M}^-(D^2\psi_k(x)) \geq \alpha_k\varepsilon_k^{3/2}\frac{\lambda}{2n} > \beta_k\varepsilon_k^2 \geq \|f_k\|_\infty,$$

as desired.

An analogous estimate holds in the negative phase.

Finally, since on the zero level set $|\nabla\Gamma_k| = 1$ and $|\nabla d_k^2| = 0$, the free boundary condition reduces to show that

$$a_k^2 - b_k^2 > 1.$$

Recalling the definition of a_k, b_k we need to check that

$$(a_k^2p^2 - \beta_k^2q^2)\varepsilon + 2(\alpha_k^2p - \beta_k^2q) > 0.$$

This inequality holds for k large since

$$\tilde{\alpha}^2p - \tilde{\beta}^2q = ap - bq > 0.$$

Thus $\tilde{u}$ is a solution to the linearized problem.

Step 3 – Contradiction. The conclusion now follows exactly as in the case of [DFS], using the regularity estimates for the solution of the transmission problem from Theorem 3.2. □

5.2. **Degenerate case.** In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0$). We prove below that in this setting only u^+ enjoys an improvement of flatness.

Lemma 5.2 (Improvement of flatness)**.** *Let u satisfy*

(5.9) $$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n+\varepsilon) \quad \text{in } B_1, \quad 0 \in F(u),$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4,$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2.$$

If $0 < r \leq r_1$ for r_1 universal, and $0 < \varepsilon \leq \varepsilon_1$ for some ε_1 depending on r, then

(5.10) $$U_0(x\cdot\nu_1 - r\frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x\cdot\nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r,$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ for a universal constant C.

Proof. We argue similarly as in the non-degenerate case.

Step 1 – Compactness. Fix $r \leq r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1.1) in B_1 for operators $\mathcal{F}_k \in \mathcal{E}(\lambda,\Lambda)$ and right hand sides f_k with L^∞ norm bounded by ε_k^4, such that

(5.11) $$U_0(x_n-\varepsilon_k) \leq u_k(x) \leq U_0(x_n+\varepsilon_k) \quad \text{for } x \in B_1,\ 0 \in F(u_k),$$

with
$$\|u_k^-\|_\infty \leq \varepsilon_k^2$$
but u_k does not satisfy the conclusion (5.10) of the lemma.

Set
$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$
Then (5.3) gives,
$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1^+(u_k) \cup F(u_k).$$

As in the previous case, it follows from Corollary 4.5 that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2} \cap \{x_n \geq 0\}$.

Step 2 – Limiting Solution. We now show that $\tilde{u}$ solves the following Neumann problem

$$\begin{cases} \mathcal{F}_*^+(D^2\tilde{u}) = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}, \end{cases} \tag{5.12}$$

with $\mathcal{F}_*^+ \in \mathcal{E}(\lambda, \Lambda)$. As before, the interior condition follows easily, thus we focus on the boundary condition. It is enough to test that if $\tilde{\phi}$ is a function of the form (γ a precise constant to be specified later)
$$\tilde{\phi}(x) = A + px_n + BQ(x - y)$$
with
$$Q^\gamma(x) = \frac{1}{2}[\gamma x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0, \eta > 0$$
and
$$p > 0,$$
then $\tilde{\phi}$ cannot touch $\tilde{u}$ strictly by below at a point $x_0 = (x_0', 0) \in B_{1/2}$. Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let Γ_k^γ be as in the proof of the non-degenerate case (see (5.8)). Call
$$\phi_k(x) = a_k \Gamma_k^{\gamma+}(x) + (d_k^+(x))^2 \varepsilon_k^2, \quad a_k = (1 + \varepsilon_k p)$$
where $d_k(x)$ is the signed distance from x to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k))$.

Let
$$\tilde{\phi}_k(x) = \frac{\phi_k(x) - x_n}{\varepsilon_k}.$$

As in the previous case, it follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ ($Q^{\gamma,y}(x) = Q^\gamma(x - y)$)
$$\tilde{\phi}_k(x) = A + BQ^{\gamma,y} + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^{\gamma,y} + \varepsilon_k d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2} \cap \{x_n \geq 0\}$. Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ and $\tilde{\phi}$ touches $\tilde{u}$ strictly by below at x_0, we conclude that there exist a sequence of constants $c_k \to 0$ and of points $x_k \to x_0$ such that the function
$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k by below at $x_k \in B_1^+(u_k) \cup F(u_k)$. We claim that x_k cannot belong to $B_1^+(u_k)$. Otherwise, in a small neighborhood N of x_k we would have that (with a similar computation as in the non-degenerate case, and γ large enough universal)

$$\mathcal{M}^-(D^2\psi_k) > \varepsilon_k^4 \geq \|f_k\|_\infty \geq \mathcal{F}(D^2u_k) \geq \mathcal{M}^-(D^2u_k),$$

$\psi_k < u_k$ in $N \setminus \{x_k\}, \psi_k(x_k) = u_k(x_k)$, a contradiction.

Thus $x_k \in F(u_k) \cap \partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k))$. For simplicity we call

$$\mathcal{B} := B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k)).$$

Let $\mathcal{N}$ be a neighborhood of x_k. In the set $\{u_k < 0\}$,

$$\mathcal{M}^+(D^2u_k^-) = -\mathcal{M}^-(D^2u_k) \geq F(D^2u_k) = f \geq -\varepsilon^4.$$

Hence, since $\|u_k^-\|_\infty \leq \varepsilon_k^2$, we can compare u_k^- with the function $\varepsilon_k^2 w$ where w solves the following problem:

$$\mathcal{M}^+(D^2w) = -1 \quad \text{in } \mathcal{N} \setminus \overline{\mathcal{B}},$$

$$w = 1 \quad \text{on } \partial N \setminus \mathcal{B}, \quad w = 0 \quad \text{on } \mathcal{N} \cap \partial\mathcal{B}.$$

Let

$$\Psi_k(x) = \begin{cases} \psi_k & \text{in } \mathcal{N} \cap \mathcal{B} \\ \\ -\varepsilon_k^2 w & \text{in } \mathcal{N} \setminus \mathcal{B}. \end{cases} \tag{5.13}$$

Then Ψ_k touches u_k strictly by below at $x_k \in F(u_k) \cap F(\Psi_k)$. We reach a contradiction if we show that

$$(\Psi_k^+)_\nu^2 - (\Psi_k^-)_\nu^2 > 1, \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that (for c small universal)

$$a_k^2 - c\varepsilon_k^4 > 1$$

or

$$(1 + \varepsilon_k p)^2 - c\varepsilon_k^4 > 1.$$

This holds for k large enough since $p > 0$, and our proof is concluded.

Step 3 – Contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [D]. $\square$

6. Proof of the main Theorem.

In this section we exhibit the proofs of our main results, Theorem 1.1 and Theorem 1.2. We recall the following elementary lemma from [DFS] which holds for any continuous function u.

Lemma 6.1. *Let u be a continuous function. If for $\eta > 0$ small,*

$$\|u - U_\beta\|_{L^\infty(B_2)} \leq \eta, \quad 0 \leq \beta \leq L,$$

and

$$\{x_n \leq -\eta\} \subset B_2 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\},$$

then

- *If* $\beta \geq \eta^{1/3}$, *then*
$$U_\beta(x_n - \eta^{1/3}) \leq u(x) \leq U_\beta(x_n + \eta^{1/3}), \quad \text{in } B_1$$

- *If* $\beta < \eta^{1/3}$, *then*
$$U_0(x_n - \eta^{1/3}) \leq u^+(x) \leq U_0(x_n + \eta^{1/3}), \quad \text{in } B_1.$$

6.1. **Proof of Theorem 1.1.** To complete the analysis of the degenerate case, we need to deal with the situation when u is close to a one-plane solution and however the size of u^- is not negligible. Precisely, we prove the following lemma.

Lemma 6.2. *Let* u *solve* (1.1) *in* B_2 *with*
$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4$$
and satisfy
$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \tag{6.1}$$
$$\|u^-\|_{L^\infty(B_2)} \leq \bar{C}\varepsilon^2, \quad \|u^-\|_{L^\infty(B_1)} > \varepsilon^2,$$
for a universal constant $\bar{C}$. *If* $\varepsilon \leq \varepsilon'$ *universal, then the rescaling*
$$u_\varepsilon(x) = \varepsilon^{-1/2} u(\varepsilon^{1/2} x)$$
satisfies in B_1
$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \leq u_\varepsilon(x) \leq U_{\beta'}(x_n + C'\varepsilon^{1/2})$$
with $\beta' \sim \varepsilon^2$ *and* $C' > 0$ *depending on* $\bar{C}$.

Proof. As usually, we omit the dependence of the Pucci operators from $\lambda/n, \Lambda$. For notational simplicity we set
$$v = \frac{u^-}{\varepsilon^2}.$$

From our assumptions we can deduce that
$$F(v) \subset \{-\varepsilon \leq x_n \leq \varepsilon\}$$
$$v \geq 0 \quad \text{in } B_2 \cap \{x_n \leq -\varepsilon\}, \quad v \equiv 0 \quad \text{in } B_2 \cap \{x_n > \varepsilon\}. \tag{6.2}$$

Also,
$$\mathcal{M}^+(D^2 v) = \frac{1}{\varepsilon^2}\mathcal{M}^+(-D^2 u) = -\frac{1}{\varepsilon^2}\mathcal{M}^-(D^2 u) \geq \frac{1}{\varepsilon^2}\mathcal{F}(D^2 u) \geq -\epsilon^2$$
in $B_2 \cap \{x_n < -\varepsilon\}$, and
$$0 \leq v \leq C \quad \text{on } \partial B_2, \tag{6.3}$$
$$v(\bar{x}) > 1 \quad \text{at some point } \bar{x} \text{ in } B_1. \tag{6.4}$$

Hence, using comparison with the function w such that
$$\mathcal{M}^+(D^2 w) = -1 \quad \text{in } D := B_2 \cap \{x_n < \varepsilon\},$$
$$w = C \quad \text{on } \partial B_2 \cap \{x_n < \varepsilon\}, \quad w = 0 \quad \text{on } \{x_n = \varepsilon\}$$
($\mathcal{M}^+$ is a convex operator hence w is an explicit barrier which has $C^{1,1}$ estimates up to $\{x_n = \varepsilon\}$,) we get that for some $k > 0$ universal
$$v \leq k|x_n - \varepsilon|, \quad \text{in } B_1. \tag{6.5}$$

This fact forces the point $\bar{x}$ in (6.4) to belong to $B_1 \cap \{x_n < -\varepsilon\}$ at a fixed distance δ from $x_n = -\varepsilon$.

Analogously,

$$\mathcal{M}^-(D^2 v) = \frac{1}{\varepsilon^2}\mathcal{M}^-(-D^2 u) = -\frac{1}{\varepsilon^2}\mathcal{M}^+(D^2 u) \leq \frac{-1}{\varepsilon^2}\mathcal{F}(D^2 u) \leq \epsilon^2$$

in $B_2 \cap \{x_n < -\varepsilon\}$. Thus if w is such that $\mathcal{M}^-(D^2 w) = 0$ in $B_1 \cap \{x_n < -\varepsilon\}$ such that

$$w = 0 \quad \text{on } B_1 \cap \{x_n = -\varepsilon\}, \quad w = v \quad \text{on } \partial B_1 \cap \{x_n \leq -\varepsilon\},$$

then

$$\mathcal{M}^-(D^2(w + \frac{1}{2\lambda}\varepsilon^2(|x|^2 - 3)) \geq \varepsilon^2.$$

By the comparison principle we conclude that

$$w + \frac{1}{2\lambda}\varepsilon^2(|x|^2 - 3) \leq v \quad \text{on} \quad B_1 \cap \{x_n < -\varepsilon\}. \tag{6.6}$$

Also, for ε small, in view of (6.5) we obtain that

$$w - k\varepsilon(|x|^2 - 3) \geq v \quad \text{on} \quad \partial(B_1 \cap \{x_n < -\varepsilon\}) \tag{6.7}$$

and hence also in the interior. Thus we conclude that

$$|w - v| \leq c\varepsilon \quad \text{in } B_1 \cap \{x_n < -\varepsilon\}. \tag{6.8}$$

In particular this is true at $\bar{x}$ which forces

$$w(\bar{x}) \geq 1/2. \tag{6.9}$$

By expanding w around $(0, -\varepsilon)$ we then obtain, say in $B_{1/2} \cap \{x_n \leq -\varepsilon\}$

$$|w - a|x_n + \varepsilon|| \leq C|x|^2 + C\varepsilon.$$

This combined with (6.8) gives that

$$|v - a|x_n + \varepsilon|| \leq C\varepsilon, \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \leq -\varepsilon\}. \tag{6.10}$$

Moreover, in view of (6.9) and the fact that $\bar{x}$ occurs at a fixed distance from $\{x_n = -\varepsilon\}$ we deduce from Hopf lemma that

$$a \geq c > 0$$

with c universal. In conclusion (see (6.5))

$$|u^- - b\varepsilon|x_n + \varepsilon|| \leq C\varepsilon^3, \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \leq -\varepsilon\}, \quad u^- \leq b\varepsilon^2|x_n - \varepsilon|, \quad \text{in } B_1 \tag{6.11}$$

with b comparable to a universal constant.

Combining (6.11) and the assumption (6.1) we conclude that in $B_{\varepsilon^{1/2}}$

$$(x_n - \varepsilon)^+ - b\varepsilon(x_n - C\varepsilon)^- \leq u(x) \leq (x_n + \varepsilon)^+ - b\varepsilon(x_n + C\varepsilon)^- \tag{6.12}$$

with $C > 0$ universal and b larger than a universal constant. Rescaling, we obtain that in B_1

$$(x_n - \varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq (x_n + \varepsilon^{1/2})^+ - \beta'(x_n + C\varepsilon^{1/2})^- \tag{6.13}$$

with $\beta' \sim \varepsilon^2$. We finally need to check that this implies the desired conclusion in B_1

$$\alpha'(x_n - C\varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq \alpha'(x_n + C\varepsilon^{1/2})^+ - \beta'(x_n + C\varepsilon^{1/2})^-$$

with $\alpha'^2 = 1 + \beta^2 \sim 1 + \varepsilon^4$. This clearly holds in B_1 for ε small, say by possibly enlarging C so that $C \geq 2$.

We are finally ready to exhibit the proof of our main Theorem 2.1. Having provided all the necessary ingredients, the proof now follows as in [DFS]. For the reader's convenience we present the details.

Proof of Theorem 2.1. Let us fix $\bar{r} > 0$ to be a universal constant such that

$$\bar{r} \leq r_0, r_1, 1/8,$$

with r_0, r_1 the universal constants in the improvement of flatness Lemmas 5.1-5.2. Also, let us fix a universal constant $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon} \leq \varepsilon_0(\bar{r}), \frac{\varepsilon_1(\bar{r})}{2}, \frac{1}{2\tilde{C}}, \frac{\varepsilon_0(\bar{r})^2}{2C'}, \frac{\varepsilon'}{4}, C''$$

with $\varepsilon_0, \varepsilon_1, \varepsilon', \tilde{C}, C', \bar{C}$, the constants in the Lemmas 5.1-5.2-6.2 and C'' universal to be specified later.

Now, let

$$\bar{\varepsilon} = \tilde{\varepsilon}^3.$$

We distinguish two cases. For notational simplicity we assume that u satisfies our assumptions in the ball B_2 and $0 \in F(u)$.

Case 1. $\beta \geq \tilde{\varepsilon}$.

In this case, in view of Lemma 6.1and our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the assumptions of Lemma 5.1,

$$U_\beta(x_n - \tilde{\varepsilon}) \leq u(x) \leq U_\beta(x_n + \tilde{\varepsilon}) \quad \text{in } B_1, \quad 0 \in F(u),$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \tilde{\varepsilon}^3 \leq \tilde{\varepsilon}^2 \beta.$$

Thus we can conclude that, $(\beta_1 = \beta')$

$$U_{\beta_1}(x \cdot \nu_1 - \bar{r}\frac{\tilde{\varepsilon}}{2}) \leq u(x) \leq U_{\beta_1}(x \cdot \nu_1 + \bar{r}\frac{\tilde{\varepsilon}}{2}) \quad \text{in } B_{\bar{r}}, \tag{6.14}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\tilde{\varepsilon}$, and $|\beta - \beta_1| \leq \tilde{C}\beta\tilde{\varepsilon}$. In particular, by our choice of $\tilde{\varepsilon}$ we have

$$\beta_1 \geq \tilde{\varepsilon}/2.$$

We can therefore rescale and iterate the argument above. Precisely, set $(k = 0, 1, 2....)$

$$\rho_k = \bar{r}^k, \quad \varepsilon_k = 2^{-k}\tilde{\varepsilon}$$

and

$$\mathcal{F}_k(M) = \rho_k \mathcal{F}(\frac{1}{\rho_k}M), \quad u_k(x) = \frac{1}{\rho_k}u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x).$$

Notice that $F_k \in \mathcal{E}(\lambda, \Lambda)$ hence our flatness theorem holds.

Also, let β_k be the constants generates at each k-iteration, hence satisfying $(\beta_0 = \beta)$

$$|\beta_k - \beta_{k+1}| \leq \tilde{C}\beta_k\varepsilon_k.$$

Then we obtain by induction that each u_k satisfies

$$U_{\beta_k}(x \cdot \nu_k - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1, \tag{6.15}$$

with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq \tilde{C}\tilde{\varepsilon}_k$ $(\nu_0 = e_n.)$

Case 2. $\beta < \tilde{\varepsilon}$.

In view of Lemma 6.1 we conclude that

$$U_0(x_n - \tilde{\varepsilon}) \leq u^+(x) \leq U_0(x_n + \tilde{\varepsilon}) \quad \text{in } B_1. \tag{6.16}$$

Moreover, from the assumption (2.1) and the fact that $\beta < \tilde{\varepsilon}$ we also obtain that

$$\|u^-\|_{L^\infty(B_1)} < 2\tilde{\varepsilon}.$$

Call $\varepsilon' = 2\tilde{\varepsilon}$. Then u satisfies the assumptions of the (degenerate) improvement of flatness Lemma 5.2.

$$U_0(x_n - \varepsilon') \leq u^+(x) \leq U_0(x_n + \varepsilon') \quad \text{in } B_1, \tag{6.17}$$

with

$$\|f\|_{L^\infty(B_1)} \leq (\varepsilon')^3, \quad \|u^-\|_{L^\infty(B_1)} < \varepsilon'.$$

We conclude that

$$U_0(x \cdot \nu_1 - \bar{r}\frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + \bar{r}\frac{\varepsilon}{2}) \quad \text{in } B_{\bar{r}}, \tag{6.18}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon'$ for a universal constant C. We now rescale as in the previous case and set $(k = 0, 1, 2....)$

$$\rho_k = \bar{r}^k, \quad \varepsilon_k = 2^{-k}\varepsilon'$$

and

$$\mathcal{F}_k(M) = \rho_k \mathcal{F}(\frac{1}{\rho_k}M), \quad u_k(x) = \frac{1}{\rho_k}u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x).$$

We can iterate our argument and obtain that (with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq C\varepsilon_k$)

$$U_0(x \cdot \nu_k - \varepsilon_k) \leq u_k^+(x) \leq U_0(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1, \tag{6.19}$$

as long as we can verify that

$$\|u_k^-\|_{L^\infty(B_1)} < \varepsilon_k^2.$$

Let $\bar{k}$ be the first integer $\bar{k} \geq 1$ for which this fails, that is

$$\|u_{\bar{k}}^-\|_{L^\infty(B_1)} \geq \varepsilon_{\bar{k}}^2,$$

and

$$\|u_{\bar{k}-1}\|_{L^\infty(B_1)} < \varepsilon_{\bar{k}-1}^2.$$

Also,

$$U_0(x \cdot \nu_{\bar{k}-1} - \varepsilon_{\bar{k}-1}) \leq u_{\bar{k}-1}^+(x) \leq U_0(x \cdot \nu_{\bar{k}-1} + \varepsilon_{\bar{k}_-1}) \quad \text{in } B_1.$$

As argued several times, we can then conclude from the comparison principle that

$$u_{\bar{k}-1}^- \leq M|x_n - \varepsilon_{\bar{k}-1}|\varepsilon_{\bar{k}-1}^2 \quad \text{in } B_{19/20},$$

for a universal constant $M > 0$. Thus, by rescaling we get that

$$\|u_{\bar{k}}^-\|_{L^\infty(B_2)} < \bar{C}\varepsilon_{\bar{k}}^2$$

with $\bar{C}$ universal depending on the fixed $\bar{r}$. We obtain that $u_{\bar{k}}$ satisfies all the assumptions of that Lemma and hence the rescaling

$$v(x) = \varepsilon_{\bar{k}}^{-1/2} u_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2}x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon_{\bar{k}}^{1/2}) \leq v(x) \leq U_{\beta'}(x_n + \bar{C}'\varepsilon_{\bar{k}}^{1/2})$$

with $\beta' \sim \varepsilon_{\bar{k}}^2$. Call $\eta = \bar{C}\varepsilon_{\bar{k}}^{1/2}$. Then v satisfies our free boundary problem in B_1 for the operator

$$\mathcal{G}(M) = \varepsilon_{\bar{k}}^{1/2}\mathcal{F}_{\bar{k}}(\frac{1}{\varepsilon_{\bar{k}}^{1/2}}M) \in \mathcal{E}(\lambda, \Lambda)$$

with right hand side

$$g(x) = \varepsilon_{\bar{k}}^{1/2} f_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2}x)$$

and the flatness assumption

$$U_{\beta'}(x_n - \eta) \leq v(x) \leq U_{\beta'}(x_n + \eta)$$

Since $\beta' \sim \varepsilon_{\bar{k}}^2$ with a universal constant,

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon_{\bar{k}}^{1/2}\varepsilon_{\bar{k}}^4 \leq \eta^2\beta'$$

as long as $\tilde{\varepsilon} \leq C''$ universal depending on $\bar{C}$. In conclusion v falls under the assumptions of the (non-degenerate) improvement of flatness Lemma 5.1 and we can use an iteration argument as in Case 1.

6.2. **Proof of Theorem 1.2.** To provide the proof of Theorem 1.2, we use the following Liouville type result for global viscosity solutions to a two-phase homogeneous free boundary problem, that could be of independent interest.

Lemma 6.3. *Let U be a global viscosity solution to*

$$\begin{cases} \mathcal{G}(D^2U) = 0, & \text{in } \{U > 0\} \cup \{U \leq 0\}^0, \\ \\ (U_\nu^+)^2 - (U_\nu^-)^2 = 1, & \text{on } F(U) := \partial\{U > 0\}. \end{cases} \tag{6.20}$$

Assume that $\mathcal{G} \in \mathcal{E}(\lambda, \Lambda)$ and $\mathcal{G}$ is homogeneous of degree 1. Also, $F(U) = \{x_n = g(x'), x' \in \mathbb{R}^{n-1}\}$ with $Lip(g) \leq M$. Then g is linear and $U(x) = U_\beta(x)$ for some $\beta \geq 0$.

Proof. Assume for simplicity, $0 \in F(U)$. Also, balls (of radius ρ and centered at 0) in $\mathbb{R}^{n-1}$ are denoted by $\mathcal{B}_\rho$.

By the regularity theory in [F1] , since U is a solution in B_2, the free boundary $F(U)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n, λ, Λ and M. Thus,

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1$$

with C depending only on n, λ, Λ, M. Moreover, since U is a global solution, the rescaling

$$g_R(x') = \frac{1}{R}g(Rx'), \quad x' \in \mathcal{B}_2$$

which preserves the same Lipschitz constant as g, satisfies the same inequality as above i.e.

$$|g_R(x') - g_R(0) - \nabla g_R(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

This reads,

$$|g(Rx') - g(0) - \nabla g(0) \cdot Rx'| \leq CR|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

Thus,

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \leq C\frac{1}{R^\alpha}|y'|^{1+\alpha}, \quad y' \in \mathcal{B}_R.$$

Passing to the limit as $R \to \infty$ we obtain the desired claim.

Now the proof of Theorem 1.2, follows exactly as in the Laplacian case [DFS].

Proof of Theorem 1.2. Let $\bar{\varepsilon}$ be the universal constant in Theorem 2.1. Consider the blow-up sequence

$$u_k(x) = \frac{u(\delta_k x)}{\delta_k}$$

with $\delta_k \to 0$ as $k \to \infty$. Each u_k solves (1.1) with operator $\mathcal{F}_k$ and right hand side f_k given by

$$\mathcal{F}_k(M) = \delta_k F_k(\frac{1}{\delta_k} M) \in \mathcal{E}(\lambda, \Lambda), \quad f_k(x) = \delta_k f(\delta_k x)$$

and

$$\|f_k(x)\| \leq \delta_k \|f\|_{L^\infty} \leq \bar{\varepsilon}$$

for k large enough. Standard arguments (see for example [ACF]) using the uniform Lischitz continuity of the u_k's and the nondegeneracy of their positive part u_k^+ (see Lemma 2.4) imply that (up to a subsequence)

$$u_k \to \tilde{u} \quad \text{uniformly on compacts}$$

and

$$\{u_k^+ = 0\} \to \{\tilde{u} = 0\} \quad \text{in the Hausdorff distance.}$$

Moreover, up to a subsequence, the $\mathcal{F}_k$ converge uniformly on compact subsets of matrices to an operator $\tilde{\mathcal{F}} \in \mathcal{E}(\lambda, \Lambda)$. Since all the $\mathcal{F}_k$'s are homogeneous of degree 1, also $\tilde{\mathcal{F}}$ is homogeneous of degree 1. The blow-up limit $\tilde{u}$ solves the global two-phase free boundary problem

$$\begin{cases} \tilde{\mathcal{F}}(D^2\tilde{u}) = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \leq 0\}^0 \\ \\ (\tilde{u}_\nu^+)^2 - (\tilde{u}_\nu^-)^2 = 1, & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases} \tag{6.21}$$

Since $F(u)$ is a Lipschitz graph in a neighborhood of 0, it follows from Lemma 6.3 that $\tilde{u}$ is a two-plane solutions, $\tilde{u} = U_\beta$ for some $\beta \geq 0$. Thus, for k large enough

$$\|u_k - U_\beta\|_{L^\infty} \leq \bar{\varepsilon}$$

and

$$\{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\}.$$

Therefore, we can apply our flatness Theorem 2.1 and conclude that $F(u_k)$ and hence $F(u)$ is smooth.

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