A NOTE ON OPTIMAL SPECTRAL BOUNDS FOR NONOVERLAPPING DOMAIN DECOMPOSITION PRECONDITIONERS FOR hp–VERSION DISCONTINUOUS GALERKIN METHODS

PAOLA F. ANTONIETTI, PAUL HOUSTON, AND IAIN SMEARS

Abstract. In this article, we consider the derivation of hp–optimal spectral bounds for a class of domain decomposition preconditioners based on the Schwarz framework for discontinuous Galerkin finite element approximations of second–order elliptic partial differential equations. In particular, we improve the bounds derived in our earlier article [P.F. Antonietti and P. Houston, *J. Sci. Comput.*, 46(1):124–149, 2011] in the sense that the resulting bound on the condition number of the preconditioned system is not only explicit with respect to the coarse and fine mesh sizes H and h, respectively, and the fine mesh polynomial degree p, but now also explicit with respect to the polynomial degree q employed for the coarse grid solver. More precisely, we show that the resulting spectral bounds are of order $p^2H/(qh)$ for the hp–version of the discontinuous Galerkin method.

Key words. Schwarz preconditioners, hp–discontinuous Galerkin methods.

AMS Subject Classification: 65N30, 65N55, 65F08.

1. Introduction

In this article, we study a class of nonoverlapping Schwarz preconditioners employed for the hp–version discontinuous Galerkin finite element (DGFEM) approximation of second–order elliptic partial differential equations. We stress that Schwarz-type preconditioners are particularly suited to DGFEMs, in the sense that uniform scalability of the underlying iterative method may be established without the need to overlap the subdomain partition of the computational mesh. In a parallel setting, this is a particularly attractive property, since the absence of overlapping subdomains reduces communication between processors.

In the h–version setting, spectral bounds of order H/h for the underlying preconditioned system may be established, where H and h denote the granularity of the coarse and fine meshes, respectively, cf., for example, [12, 1, 2, 3, 4, 9, 10]. We note that h-version results generally do not specify the dependence of the spectral bounds on the polynomial degree of the finite element space, as they are left implicit in the constants carried through the analysis. The extension of the above results to the hp–version setting has been undertaken in our previous articles [5, 6]; in particular, we showed that the condition number of the preconditioned system is of order p^2H/h, where p denotes the polynomial degree employed on the fine finite element mesh (of granularity h). While this bound is indeed optimal with respect to H, h, and p, when the polynomial degree q employed for the coarse grid solver is kept fixed, the dependence on q may not be explicitly determined from this analysis. Indeed, on the basis of the computations presented in [5], we conjectured a spectral bound on the preconditioned system to be of order $p^2H/(qh)$; in the present article, we now provide a proof of this conjecture. The key aspect of this analysis is the derivation of an hp–optimal approximation property between the coarse and fine finite element spaces. With this in mind, we follow the recent analysis presented in [20] for problems posed within the H^2–context to deduce analogous results in the present setting.

This article is organised as follows. In Section 2 we introduce the model problem, together with its hp–version DGFEM discretization. Section 3 derives a crucial result concerning the approximation of discontinuous functions by a conforming H^1-approximant. In Section 4 we recall the additive and multiplicative Schwarz preconditioners analyzed in [5]. Finally, hp–optimal spectral bounds are deduced in Section 5 which are explicit with respect to both the fine and coarse mesh sizes h and H, respectively, as well as the polynomial degrees p and q exploited within the fine and coarse mesh solvers, respectively. Throughout this article, we use the notation $x \lesssim y$ to signify that there exists a positive constant C, independent of the discretization parameters, such that $x \leq C\, y$.

2. Discontinuous Galerkin methods

Given a bounded, convex polygonal/polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and a function $f \in L^2(\Omega)$, we consider the following model problem: find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \ \mathrm{d}x = \int_\Omega f v \ \mathrm{d}x \quad \forall\, v \in H_0^1(\Omega). \tag{1}$$

Let $\mathcal{T}_h = \{\mathcal{K}\}$ be a *shape-regular*, quasi-uniform, conforming decomposition of Ω with granularity $h = \max_{\mathcal{K}\in\mathcal{T}_h} h_{\mathcal{K}}$, where $h_{\mathcal{K}}$ denotes the diameter of element $\mathcal{K}$, $\mathcal{K} \in \mathcal{T}_h$. We assume that every element $\mathcal{K} \in \mathcal{T}_h$ is the image of a fixed master element $\widehat{\mathcal{K}}$, i.e., $\mathcal{K} = F_{\mathcal{K}}(\widehat{\mathcal{K}})$, where $\widehat{\mathcal{K}}$ is either the open unit d-simplex or the open unit hypercube in $\mathbb{R}^d$, $d = 2, 3$. We collect all the interior and boundary faces of $\mathcal{T}_h$ in the sets $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$, respectively, and set $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$.

Next we introduce standard jump and average trace operators, cf. [8]. To this end, given an interior face $F \in \mathcal{F}_h^I$, shared by two neighboring elements $\mathcal{K}^\pm \in \mathcal{T}_h$, we write $v^\pm$ to denote the trace of a (sufficiently regular) function v on the face F, taken within the interior of $\mathcal{K}^\pm$, respectively. Similarly, given a (sufficiently regular) vector-valued function $\boldsymbol{q}$, $\boldsymbol{q}^\pm$ is defined in an analogous (componentwise) manner. With this notation, we define

$$[\![\boldsymbol{q}]\!] = \boldsymbol{q}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{q}^- \cdot \boldsymbol{n}^-, \qquad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-,$$
$$\{\!\{\boldsymbol{q}\}\!\} = \frac{1}{2}(\boldsymbol{q}^+ + \boldsymbol{q}^-), \qquad \{\!\{v\}\!\} = \frac{1}{2}(v^+ + v^-),$$

where $\boldsymbol{n}^\pm$ denotes the unit outward normal vector on the boundary of $\mathcal{K}^\pm$, respectively. On a boundary face $F \in \mathcal{F}_h^B$, we set $[\![\boldsymbol{q}]\!] = \boldsymbol{q} \cdot \boldsymbol{n}$, $[\![v]\!] = v\,\boldsymbol{n}$, $\{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}$, and $\{\!\{v\}\!\} = v$, where $\boldsymbol{n}$ denotes the outward unit normal vector on the boundary $\partial\Omega$ of the computational domain Ω.

Given an integer $p \geq 1$, the polynomial degree, the corresponding hp–DGFEM finite element space is defined by

$$\mathcal{V}_{hp} = \{u \in L^2(\Omega) \ : \ u \circ F_{\mathcal{K}} \in \mathbb{M}^p(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{K} \in \mathcal{T}_h\}, \tag{2}$$

where $\mathbb{M}^p(\widehat{\mathcal{K}})$ is either the space $\mathbb{P}_p(\widehat{\mathcal{K}})$ of polynomials of degree at most p on $\widehat{\mathcal{K}}$, if $\widehat{\mathcal{K}}$ is the reference d-simplex, or the space $\mathbb{Q}_p(\widehat{\mathcal{K}})$ of all tensor–product polynomials on $\widehat{\mathcal{K}}$ of degree p in each coordinate direction, if $\widehat{\mathcal{K}}$ is the unit reference hypercube in $\mathbb{R}^d$, $d = 2, 3$.

Using the convention that

$$\int_{\mathcal{F}_h} \varphi \, \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F \varphi \, \mathrm{d}s,$$

for a sufficiently regular function φ, we introduce the following lifting operators:

$$\begin{aligned} &\mathcal{R} : [L^1(\mathcal{F}_h)]^d \to [\mathcal{V}_{hp}]^d, \quad && \int_\Omega \mathcal{R}(\boldsymbol{q}) \cdot \boldsymbol{\eta} \, \mathrm{d}x = - \int_{\mathcal{F}_h} \boldsymbol{q} \cdot \{\!\{\boldsymbol{\eta}\}\!\} \, \mathrm{d}s \quad && \forall \boldsymbol{\eta} \in [\mathcal{V}_{hp}]^d, \\ &\mathcal{L} : L^1(\mathcal{F}_h^I) \to [\mathcal{V}_{hp}]^d, && \int_\Omega \mathcal{L}(z) \cdot \boldsymbol{\eta} \, \mathrm{d}x = - \int_{\mathcal{F}_h^I} z \, [\![\boldsymbol{\eta}]\!] \, \mathrm{d}s && \forall \boldsymbol{\eta} \in [\mathcal{V}_{hp}]^d. \end{aligned} \tag{3}$$

Remark 2.1. *For the sake of simplicity we have assumed that the underlying computational mesh $\mathcal{T}_h$ is both conforming and quasi-uniform and that the polynomial degree does not vary elementwise. However, we stress that these assumptions can be relaxed; indeed, our forthcoming analysis naturally extends to the case when non-uniform, non-matching grids are employed, together with a variable polynomial degree vector, provided that both the mesh size and the polynomial degree distribution satisfy a local-bounded variation property, cf.* [13, 17]*, for example.*

Writing ∇_h to denote the elementwise application of the operator ∇, we introduce the bilinear form $\mathcal{A} : \mathcal{V}_{hp} \times \mathcal{V}_{hp} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{A}(w, v) = &\int_\Omega (\nabla_h w + \mathcal{R}([\![w]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![w]\!]) \cdot (\nabla_h v + \mathcal{R}([\![v]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v]\!]) \, \mathrm{d}x \\ &- \theta \int_\Omega \mathcal{R}([\![w]\!]) \cdot \mathcal{R}([\![w]\!]) \, \mathrm{d}x + \int_{\mathcal{F}_h} \sigma \, [\![w]\!] \cdot [\![v]\!] \, \mathrm{d}s. \end{aligned} \tag{4}$$

Here, $\mathcal{R}(\cdot)$ and $\mathcal{L}(\cdot)$ are the lifting operators defined as in (3), $\theta, \boldsymbol{\beta}$ are parameters that will be specified later on, and the penalty stabilization function σ is defined as

$$\sigma = C_\sigma \, p^2 h^{-1}, \tag{5}$$

where $C_\sigma \geq 1$ (at our disposal) is independent of the meshsize and the approximation order. Then, the hp–DGFEM approximation of (1) is given by: find $u_h \in \mathcal{V}_{hp}$ such that

$$\mathcal{A}(u_h, v_h) = \int_\Omega f v_h \, \mathrm{d}x \qquad \forall \, v_h \in \mathcal{V}_{hp}. \tag{6}$$

For $\theta = 1$ and $\boldsymbol{\beta} = \mathbf{0}$, the bilinear form (4) corresponds to the symmetric interior penalty (SIP) DGFEM formulation [7], whereas for $\theta = 0$ and $\boldsymbol{\beta}$ a uniformly bounded (and possibly null) vector in $\mathbb{R}^d$ we obtain the local discontinuous Galerkin (LDG) bilinear form [11].

For a sufficiently regular function v, we adopt the convention that

$$\|v\|^2_{L^2(\mathcal{F}_h)} = \sum_{F \in \mathcal{F}_h} \|v\|^2_{L^2(F)};$$

with this notation we define the following DGFEM (mesh-dependent) norm

$$\|v\|^2_{\mathrm{DG}} = \|\nabla_h v\|^2_{L^2(\Omega)} + \|\sigma^{1/2} \, [\![v]\!] \, \|^2_{L^2(\mathcal{F}_h)}. \tag{7}$$

Equipped with $\|\cdot\|_{\mathrm{DG}}$, it can be shown that the bilinear form $\mathcal{A}(\cdot,\cdot)$ is continuous and coercive on $\mathcal{V}_{hp}\times\mathcal{V}_{hp}$, i.e.,

$$\mathcal{A}(u,v) \lesssim \|u\|_{\mathrm{DG}}\|v\|_{\mathrm{DG}} \qquad \forall\, u,v\in\mathcal{V}_{hp}, \tag{8}$$

$$\mathcal{A}(u,u) \gtrsim \|u\|_{\mathrm{DG}}^2 \qquad \forall\, u\in\mathcal{V}_{hp}, \tag{9}$$

respectively, cf. [5, 15, 17] and the references cited therein. We point out that coercivity of the SIP formulation requires that the constant C_σ appearing in the definition (5) of the penalty stabilization function σ must be chosen sufficiently large.

Given a particular (fixed) basis for the discrete space $\mathcal{V}_{hp}$, problem (6) can be recast as the following linear system of equations: find $\mathbf{U}\in\mathbb{R}^m$, $m=\dim(\mathcal{V}_{hp})$, such that

$$\mathbf{A}\mathbf{U}=\mathbf{f}, \tag{10}$$

where $\mathbf{A}$ is an $m\times m$ symmetric, positive definite matrix. We recall from [5, Corollary 2.9] that for a given set of basis functions, which are orthonormal on the reference element $\widehat{\mathcal{K}}$, the spectral condition number $\kappa(\mathbf{A})$ of the stiffness matrix $\mathbf{A}$ can be bounded by

$$\kappa(\mathbf{A})\lesssim C_\sigma\, p^4 h^{-2}. \tag{11}$$

In the next section, we discuss the efficient preconditioning of the underlying DGFEM matrix problem (10); first, however, we recall the following standard results.

Given a face $F\in\mathcal{F}_h$ of an element $\mathcal{K}\in\mathcal{T}_h$, i.e., $F\subset\partial\mathcal{K}$, the following inverse inequality holds:

$$\|v\|_{L^2(F)}^2 \lesssim \frac{p^2|F|}{|\mathcal{K}|}\|v\|_{L^2(\mathcal{K})}^2 \quad \forall v\in\mathbb{M}^p(\mathcal{K}), \tag{12}$$

cf. [18, 19]. Given that the elements $\mathcal{K}$, $\mathcal{K}\in\mathcal{T}_h$, are *shape-regular* and that the mesh $\mathcal{T}_h$ is conforming, the diameter of the faces of each element $\mathcal{K}\in\mathcal{T}_h$ are of comparable size to the diameter of the corresponding element. In particular, we have that

$$h_{\mathcal{K}}^{d-1}\lesssim|F|\lesssim h_{\mathcal{K}}^{d-1};$$

thereby, exploiting the quasi-uniformity of the mesh, the inverse inequality (12) can be rewritten in the following manner:

$$\|v\|_{L^2(F)}^2 \lesssim \frac{p^2}{h}\|v\|_{L^2(\mathcal{K})}^2 \quad \forall v\in\mathbb{M}^p(\mathcal{K}).$$

Employing the above inverse estimate, together with the arguments presented in [5], we deduce the following stability bounds for the lifting operators $\mathcal{R}$ and $\mathcal{L}$.

Lemma 2.1. *For any $v\in\mathcal{V}_{hp}$ we have that*

$$\begin{aligned}\|\mathcal{R}([\![v]\!])\|_{L^2(\Omega)} &\lesssim \|\sigma^{1/2}\,[\![v]\!]\,\|_{L^2(\mathcal{F}_h)},\\ \|\mathcal{L}(\boldsymbol{\beta}\cdot[\![v]\!])\|_{L^2(\Omega)} &\lesssim \|\sigma^{1/2}\,[\![v]\!]\,\|_{L^2(\mathcal{F}_h)}.\end{aligned}$$

Finally, we recall the following interpolation estimates. To this end, for any real number $s\geq 0$, we write $H^s(\mathcal{T}_h)$ to denote the broken Sobolev space of piecewise H^s functions with norm and semi-norm denoted by $\|\cdot\|_{H^s(\mathcal{T}_h)}$ and $|\cdot|_{H^s(\mathcal{T}_h)}$, respectively.

Then, for any function $v \in H^s(\mathcal{T}_h)$, there exists $\Pi_h v \in \mathcal{V}_{hp}$ such that, for any element $\mathcal{K} \in \mathcal{T}_h$, we have

$$\begin{aligned} \|v - \Pi_h v\|_{H^r(\mathcal{K})} &\lesssim \frac{h^{\min(s,p+1)-r}}{p^{s-r}} \|v\|_{H^s(\mathcal{K})} && \forall r,\ 0 \le r \le s, \\ \|D^\alpha (v - \Pi_h v)\|_{L^2(\partial\mathcal{K})} &\lesssim \frac{h^{\min(s,p+1)-|\alpha|-1/2}}{p^{s-|\alpha|-1/2}} \|v\|_{H^s(\mathcal{K})} && \forall \alpha,\ 0 \le |\alpha| \le k, \end{aligned} \tag{13}$$

where $\alpha \in \mathbb{N}_0^d$ is a multi-index of length $|\alpha|$. Here, the second inequality holds provided $s > 1/2$ and k is the greatest non-negative integer strictly less than $s-1/2$.

3. Approximation of $\mathcal{V}_{hp}$ functions by H^1-functions

The main result of this section is an approximation result which demonstrates that any $v_h \in \mathcal{V}_{hp}$ can be approximated by an H^1-function. We remark that results of this type have been exploited within the context of *a posteriori* error estimation of DGFEMs; see, in particular, [14] and [16]. In this section, we present a bound for the $L^2(\Omega)$-norm of the error between $v_h \in \mathcal{V}_{hp}$ and its conforming approximant based on exploiting the analysis presented in [20]. To this end, we introduce the discrete operator

$$\mathcal{G}_h : \mathcal{V}_{hp} \longrightarrow [\mathcal{V}_{hp}]^d, \quad \mathcal{G}_h(v_h) = \nabla_h v_h + \mathcal{R}([\![v_h]\!]) + \mathcal{L}(\boldsymbol{\beta} \cdot [\![v_h]\!]), \tag{14}$$

where the lifting operators $\mathcal{R}$ and $\mathcal{L}$ are as defined in (3). With this notation, we consider the following problem: for a given $v_h \in \mathcal{V}_{hp}$, find $\mathcal{H}(v_h) \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla \mathcal{H}(v_h) \cdot \nabla w \ \mathrm{d}x = \int_\Omega \mathcal{G}_h(v_h) \cdot \nabla w \ \mathrm{d}x \quad \forall\, w \in H_0^1(\Omega). \tag{15}$$

Note that in general $\mathcal{H}(v_h)$ is not an element of $\mathcal{V}_{hp}$; however, we shall demonstrate that the function $\mathcal{H}(v_h)$ possesses good approximation properties in terms of providing an H^1-conforming approximant of the discontinuous function v_h.

Theorem 3.1. *Let Ω be a bounded convex polygonal/polyhedral domain in $\mathbb{R}^d$, $d = 2, 3$. Given $v_h \in \mathcal{V}_{hp}$, we write $\mathcal{H}(v_h) \in H_0^1(\Omega)$ to be the approximation defined in* (15)*. Then, the following approximation and stability results hold:*

$$\|v_h - \mathcal{H}(v_h)\|_{L^2(\Omega)} \lesssim \frac{h}{p} \|\sigma^{1/2} [\![v_h]\!] \|_{L^2(\mathcal{F}_h)}, \tag{16}$$

$$|\mathcal{H}(v_h)|_{H^1(\Omega)} \lesssim \|v_h\|_{DG}, \tag{17}$$

where the constant is independent of v_h.

Proof. First we present the proof of (16); to this end, since Ω is convex, there exists $z \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta z &= v_h - \mathcal{H}(v_h) && \text{in } \Omega, \\ z &= 0 && \text{on } \partial\Omega; \end{aligned} \tag{18}$$

morover, z satisfies the bound $\|z\|_{H^2(\Omega)} \lesssim \|v_h - \mathcal{H}(v_h)\|_{L^2(\Omega)}$. Employing integration by parts, we deduce that

$$\begin{aligned}
\|v_h - \mathcal{H}(v_h)\|^2_{L^2(\Omega)} &= -\int_\Omega (v_h - \mathcal{H}(v_h))\,\Delta z \;\mathrm{d}x \\
&= \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (\nabla v_h - \nabla\mathcal{H}(v_h)) \cdot \nabla z \;\mathrm{d}x - \int_{\mathcal{F}_h} \nabla z \cdot [\![v_h]\!] \;\mathrm{d}s \\
&= \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} (\mathcal{G}_h(v_h) - \nabla\mathcal{H}(v_h)) \cdot \nabla z \;\mathrm{d}x + \int_{\mathcal{F}_h} \nabla z \cdot [\![v_h]\!] \;\mathrm{d}s \\
&\quad - \int_\Omega (\mathcal{R}([\![v_h]\!]) + \mathcal{L}(\boldsymbol{\beta}\cdot[\![v_h]\!])) \cdot \nabla z \;\mathrm{d}x.
\end{aligned}$$

Using the definition of $\mathcal{H}(v_h)$ in (15) and the definition of the lifting operators $\mathcal{R}$ and $\mathcal{L}$ given in (3), for any $z_h \in \mathcal{V}_{hp}$, we get

$$\begin{aligned}
\|v_h - \mathcal{H}(v_h)\|^2_{L^2(\Omega)} &= -\int_\Omega (\mathcal{R}([\![v_h]\!]) + \mathcal{L}(\boldsymbol{\beta}\cdot[\![v_h]\!])) \cdot \nabla z \;\mathrm{d}x - \int_{\mathcal{F}_h} \nabla z \cdot [\![v_h]\!] \;\mathrm{d}s \\
&= -\int_\Omega (\mathcal{R}([\![v_h]\!]) + \mathcal{L}(\boldsymbol{\beta}\cdot[\![v_h]\!])) \cdot (\nabla z - \nabla z_h) \;\mathrm{d}x \\
&\quad + \int_{\mathcal{F}_h} [\![v_h]\!] \cdot \{\!\{\nabla z_h\}\!\} \;\mathrm{d}s + \int_{\mathcal{F}_h^I} \boldsymbol{\beta}\cdot[\![v_h]\!]\,[\![\nabla z_h]\!] \;\mathrm{d}s \\
&\quad - \int_{\mathcal{F}_h} \nabla z \cdot [\![v_h]\!] \;\mathrm{d}s.
\end{aligned} \tag{19}$$

Given that $z \in H^2(\Omega)$, we deduce that the jump $[\![\nabla z]\!] = 0$ for all faces $F \in \mathcal{F}_h^I$. Thereby, (19) may be written in the following equivalent form

$$\begin{aligned}
\|v_h - \mathcal{H}(v_h)\|^2_{L^2(\Omega)} &= -\int_\Omega (\mathcal{R}([\![v_h]\!]) + \mathcal{L}(\boldsymbol{\beta}\cdot[\![v_h]\!])) \cdot (\nabla z - \nabla z_h) \;\mathrm{d}x \\
&\quad - \int_{\mathcal{F}_h} [\![v_h]\!] \cdot \{\!\{\nabla z - \nabla z_h\}\!\} \;\mathrm{d}s \\
&\quad - \int_{\mathcal{F}_h^I} \boldsymbol{\beta}\cdot[\![v_h]\!]\,[\![\nabla z - \nabla z_h]\!] \;\mathrm{d}s,
\end{aligned}$$

for any $z_h \in \mathcal{V}_{hp}$. We now select z_h to be the projection of z, i.e., $z_h = \Pi_h z$. Then, exploiting the interpolation bounds given in (13), together with the stability of the lifting operators $\mathcal{R}$ and $\mathcal{L}$, cf. Lemma 2.1, we deduce that

$$\|v_h - \mathcal{H}(v_h)\|^2_{L^2(\Omega)} \lesssim \frac{h}{p} \|z\|_{H^2(\Omega)} \|\sigma^{1/2}\,[\![v_h]\!]\,\|_{L^2(\mathcal{F}_h)}. \tag{20}$$

The approximation result stated in (16) now immediately follows from (20), based on employing the inequality $\|z\|_{H^2(\Omega)} \lesssim \|v_h - \mathcal{H}(v_h)\|_{L^2(\Omega)}$. Finally, to prove the stability estimate given in (17), we first note that by selecting $w = \mathcal{H}(v_h)$ in (15), upon application of the Cauchy–Schwarz inequality, we get

$$|\mathcal{H}(v_h)|_{H^1(\Omega)} \leq \|\mathcal{G}_h(v_h)\|_{L^2(\Omega)}.$$

Employing the definition of $\mathcal{G}_h(v_h)$ given in (14), together with Lemma 2.1, and the definition of DGFEM norm, cf. (7), we immediately deduce the desired result. $\square$

4. Nonoverlapping domain decomposition preconditioners

In this section we define the nonoverlapping domain decomposition preconditioners analyzed in [5]. To this end, we assume that the fine partition $\mathcal{T}_h$ has been obtained after successive uniform refinements of a given *shape regular* and quasi-uniform coarse mesh $\mathcal{T}_H$ with granularity H. We then agglomerate the coarse elements to obtain a subdomain partition $\mathcal{T}_{\mathcal{S}} = \{\Omega_i\}_{i=1}^N$ consisting of N nonoverlapping *star-shaped* subdomains. Next we introduce the *local* and *coarse* solvers, which represent the key ingredients in the definition of the underlying preconditioners.

Local solvers. The local spaces are defined as the restriction of the DGFEM finite element space $\mathcal{V}_{hp}$, cf. (2), to the subdomain Ω_i, i.e.,

$$\mathcal{V}_{hp}^i = \{v \in L^2(\Omega_i) \ : \ v \circ F_{\mathcal{K}} \in \mathbb{M}^p(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{K} \in \mathcal{T}_h, \mathcal{K} \subset \Omega_i\},$$

$i = 1, \dots, N$. The local bilinear forms are defined by

$$\mathcal{A}_i : \mathcal{V}_{hp}^i \times \mathcal{V}_{hp}^i \longrightarrow \mathbb{R}, \quad \mathcal{A}_i(u_i, v_i) = \mathcal{A}(R_i^\top u_i, R_i^\top v_i) \quad \forall\, u_i, v_i \in \mathcal{V}_{hp}^i,$$

$i = 1, \dots, N$, where $R_i^\top : \mathcal{V}_{hp}^i \longrightarrow \mathcal{V}_{hp}$, $i = 1, \dots, N$, denotes the classical injection operator from $\mathcal{V}_{hp}^i$ to $\mathcal{V}_{hp}$.

Coarse solver. The DGFEM finite element space associated to the coarse partition $\mathcal{T}_H$ is defined by

$$\mathcal{V}_{Hq} = \{v \in L^2(\Omega) \ : \ v \circ F_{\mathcal{P}} \in \mathbb{M}^q(\widehat{\mathcal{K}}) \quad \forall\, \mathcal{P} \in \mathcal{T}_H\},$$

where the integer q is chosen so that $1 \le q \le p$. Notice that with the above choice of $\mathcal{T}_H$ and q, the following inclusion holds: $\mathcal{V}_{Hq} \subseteq \mathcal{V}_{hp}$. The *coarse* bilinear form is defined by

$$\mathcal{A}_0 : \mathcal{V}_{Hq} \times \mathcal{V}_{Hq} \longrightarrow \mathbb{R}, \quad \mathcal{A}_0(u_0, v_0) = \mathcal{A}(R_0^\top u_0, R_0^\top v_0) \quad \forall\, u_0, v_0 \in \mathcal{V}_{Hq}, \tag{21}$$

where $R_0^\top : \mathcal{V}_{Hq} \longrightarrow \mathcal{V}_{hp}$ is the classical injection operator from $\mathcal{V}_{Hq}$ to $\mathcal{V}_{hp}$.

Introducing the projection operators $P_i = R_i^\top \widetilde{P}_i : \mathcal{V}_{hp} \longrightarrow \mathcal{V}_{hp}$, $i = 0, 1, \dots N$, where

$$\begin{aligned} &\widetilde{P}_i : \mathcal{V}_{hp} \longrightarrow \mathcal{V}_{hp}^i, \quad \mathcal{A}_i(\widetilde{P}_i v_h, w_i) = \mathcal{A}(v_h, R_i^\top w_i) \quad \forall w_i \in \mathcal{V}_{hp}^i, \ i = 1, \dots, N, \\ &\widetilde{P}_0 : \mathcal{V}_{hp} \longrightarrow \mathcal{V}_{Hq}, \quad \mathcal{A}_0(\widetilde{P}_0 v_h, w_0) = \mathcal{A}(v_h, R_0^\top w_0) \quad \forall w_0 \in \mathcal{V}_{Hq}, \end{aligned}$$

the additive and multiplicative Schwarz operators are defined, respectively, by

$$P_{\mathrm{ad}} = \sum_{i=0}^N P_i, \qquad P_{\mathrm{mu}} = I - E_{\mathrm{mu}}, \tag{22}$$

where the error propagation operator E_{mu} is given by

$$E_{\mathrm{mu}} = (I - P_N)(I - P_{N-1}) \cdots (I - P_0).$$

Then, the Schwarz method consists of solving, by a suitable Krylov iterative solver, the system of equations

$$P u_h = g,$$

for a suitable right hand side g, where P is either P_{ad} or P_{mu}.

Algebraically, by fixing a given basis for the discrete space $\mathcal{V}_{hp}$, the Schwarz operators P_{ad} and P_{mu} can be written as products of suitable preconditioners, namely $\mathbf{B}_{\mathrm{ad}}$ or $\mathbf{B}_{\mathrm{mu}}$, respectively, with the matrix $\boldsymbol{A}$. Thereby, the Schwarz method consists of solving the preconditioned system of equations: find $\mathbf{U} \in \mathbb{R}^m$, $m = \dim(\mathcal{V}_{hp})$, such that

$$\mathbf{BAU} = \mathbf{Bf},$$

where $\mathbf{B}$ is either $\mathbf{B}_{\mathrm{ad}}$ or $\mathbf{B}_{\mathrm{mu}}$.

5. Analysis of nonoverlapping preconditioners

The main result of this section is to establish spectral bounds for the Schwarz operators introduced in the previous section. To this end, it is easy to see that the additive Schwarz operator is self-adjoint (with respect to the inner-product induced by $\mathcal{A}(\cdot,\cdot)$) and positive definite. Therefore, we can define its spectral condition number $\kappa(P_{\mathrm{ad}})$ by

$$\kappa(P_{\mathrm{ad}}) = \frac{\lambda_{\max}(P_{\mathrm{ad}})}{\lambda_{\min}(P_{\mathrm{ad}})},$$

where

$$\lambda_{\max}(P_{\mathrm{ad}}) = \sup_{\substack{v_h \in \mathcal{V}_{hp} \\ v_h \neq 0}} \frac{\mathcal{A}(P_{\mathrm{ad}} v_h, v_h)}{\mathcal{A}(v_h, v_h)}, \qquad \lambda_{\min}(P_{\mathrm{ad}}) = \inf_{\substack{v_h \in \mathcal{V}_{hp} \\ v_h \neq 0}} \frac{\mathcal{A}(P_{\mathrm{ad}} v_h, v_h)}{\mathcal{A}(v_h, v_h)}.$$

On the other hand, in general, the multiplicative Schwarz operator P_{mu} is not self-adjoint. Thereby, we consider a Richardson iteration applied to the multiplicative preconditioned system and show that it converges. This, will be undertaken by proving that the $\mathcal{A}$–norm of the error propagation operator E_{mu} defined by

$$\|E_{\mathrm{mu}}\|_{\mathcal{A}} = \sup_{\substack{v_h \in \mathcal{V}_{hp} \\ v_h \neq 0}} \frac{\mathcal{A}(E_{\mathrm{mu}} v_h, E_{\mathrm{mu}} v_h)}{\mathcal{A}(v_h, v_h)}$$

is strictly less than one, which indeed represents a sufficient condition for the convergence of the Richardson scheme.

Before proceeding, we first derive a result concerning the approximation of any function $v_h \in \mathcal{V}_{hp}$, by an approximant $v_H \in \mathcal{V}_{Hq}$; the proof is an extension of [20, Theorem 6].

Lemma 5.1. *For any $v_h \in \mathcal{V}_{hp}$, there exists $v_H \in \mathcal{V}_{Hq}$ such that*

$$\|v_h - v_H\|_{L^2(\Omega)} \lesssim \frac{H}{q} \|v_h\|_{DG}, \tag{23}$$

$$|v_h - v_H|_{H^1(\mathcal{T}_h)} \lesssim \|v_h\|_{DG}. \tag{24}$$

Proof. For any $v_h \in \mathcal{V}_{hp}$, let $\mathcal{H}(v_h) \in H^1_0(\Omega)$ be defined as in (15), and let $v_H = \Pi_H \mathcal{H}(v_h)$, where Π_H is the projection operator onto the coarse space $\mathcal{V}_{Hq}$ satisfying (13). Firstly, we establish the approximation result stated in (23); to this end, employing the triangle inequality we get

$$\|v_h - v_H\|_{L^2(\Omega)} \leq \|v_h - \mathcal{H}(v_h)\|_{L^2(\Omega)} + \|\mathcal{H}(v_h) - \Pi_H \mathcal{H}(v_h)\|_{L^2(\Omega)}.$$

Exploiting the interpolation estimates stated in (13), the Poincaré–Friedrichs inequality, together with Theorem 3.1, we obtain

$$\|v_h - v_H\|_{L^2(\Omega)} \lesssim \frac{h}{p} \|\sigma^{1/2} [\![v_h]\!] \|_{L^2(\mathcal{F}_h)} + \frac{H}{q} \|\mathcal{H}(v_h)\|_{H^1(\Omega)} \lesssim \frac{h}{p} \|v_h\|_{\mathrm{DG}} + \frac{H}{q} \|v_h\|_{\mathrm{DG}}.$$

Using the fact that $q \le p$ and $h \le H$, we deduce that

$$\|v_h - v_H\|_{L^2(\Omega)} \lesssim \frac{H}{q}\|v_h\|_{\mathrm{DG}},$$

as required. Finally, we consider the proof of (24); as before, employing the triangle inequality and the definition of the DGFEM norm, gives

$$|v_h - v_H|_{H^1(\mathcal{T}_h)} \le |v_h|_{H^1(\mathcal{T}_h)} + |v_H|_{H^1(\mathcal{T}_h)} \le \|v_h\|_{\mathrm{DG}} + |v_H|_{H^1(\mathcal{T}_h)}\,. \tag{25}$$

Hence, to prove (24), it is sufficient to show that $|v_H|_{H^1(\mathcal{T}_h)} \lesssim \|v_h\|_{\mathrm{DG}}$. To this end, exploiting the triangle inequality, the interpolation estimate (13), the Poincaré–Friedrichs inequality, together with (17), we get

$$\begin{aligned}
|v_H|_{H^1(\mathcal{T}_h)} &\le |v_H - \mathcal{H}(v_h)|_{H^1(\mathcal{T}_h)} + |\mathcal{H}(v_h)|_{H^1(\Omega)}\\
&\lesssim \|\mathcal{H}(v_h)\|_{H^1(\Omega)} + |\mathcal{H}(v_h)|_{H^1(\Omega)}\\
&\lesssim |\mathcal{H}(v_h)|_{H^1(\Omega)} \lesssim \|v_h\|_{\mathrm{DG}}. \qquad (26)
\end{aligned}$$

The approximation result stated in equation (24) immediately follows by inserting (26) into (25).

□

Next we recall the following preliminary result which will be utilized in the forthcoming analysis, cf. [5].

Lemma 5.2. [5, Lemma 4.2]. *Any $v_h \in \mathcal{V}_{hp}$ can be decomposed (uniquely) as $v_h = \sum_{i=1}^N R_i^\top v_i$, with $v_i \in \mathcal{V}_{hp}^i$, $i = 1, \ldots, N$, and the following identity holds:*

$$\mathcal{A}(v_h, v_h) = \sum_{i=1}^N \mathcal{A}_i(v_i, v_i) + \sum_{\substack{i,j=1\\ i\neq j}}^N \mathcal{A}(R_i^\top v_i, R_j^\top v_j). \tag{27}$$

Moreover, the second term on the right hand side can be bounded as

$$\left|\sum_{\substack{i,j=1\\ i\neq j}}^N \mathcal{A}(R_i^\top v_i, R_j^\top v_j)\right| \lesssim \|v_h\|_{DG}^2 + \sigma \sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h\|_{L^2(\partial\mathcal{P})}^2,$$

where $\sigma = C_\sigma p^2 h^{-1}$ is the penalty stabilization function defined in (5).

We also require the following trace inequality derived in [20]; the result is an extension of the trace inequality proved in [12].

Lemma 5.3. *For any $v_h \in \mathcal{V}_{hp}$, the following trace inequality holds*

$$\begin{aligned}
\sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h\|_{L^2(\partial\mathcal{P})}^2 \quad \lesssim \quad & |v_h|_{H^1(\mathcal{T}_h)}\, \|v_h\|_{L^2(\Omega)} + \frac{1}{H}\|v_h\|_{L^2(\Omega)}^2\\
& + \left(\sum_{\mathcal{P}\in\mathcal{T}_H} \sum_{\substack{F\in\mathcal{F}_h^I\\ F\subset\mathcal{P}}} \|\sigma^{1/2}\, [\![v_h]\!]\,\|_{L^2(F)}^2\right)^{1/2} \|v_h\|_{L^2(\Omega)}.
\end{aligned}$$

Equipped with Lemma 5.2 and Lemma 5.3, we now prove the following key result.

Theorem 5.1 (Stable decomposition). *Every $v_h \in \mathcal{V}_{hp}$ admits a decomposition of the form $v_h = \sum_{i=0}^N R_i^\top v_i$, with $v_0 \in \mathcal{V}_{Hq}$ and $v_i \in \mathcal{V}_{hp}^i$, $i = 1, \ldots, N$, which satisfies the bound*

$$\sum_{i=0}^{N} \mathcal{A}_i(v_i, v_i) \le C_\natural^2 \; \mathcal{A}(v_h, v_h),$$

where

$$C_\natural^2 = C_\sigma \frac{H}{h} \frac{p^2}{q}.$$

Proof. Given $v_h \in \mathcal{V}_{hp}$, let $v_0 \in \mathcal{V}_{Hq}$ be defined as in Lemma 5.1. Then, we uniquely decompose $v_h - R_0^\top v_0$ as follows

$$v_h - R_0^\top v_0 = \sum_{i=1}^{N} R_i^\top v_i.$$

Thereby, from (27) we can write

$$\mathcal{A}(v_h - R_0^\top v_0, v_h - R_0^\top v_0) = \sum_{i=1}^{N} \mathcal{A}_i(v_i, v_i) + \sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathcal{A}(R_i^\top v_i, R_j^\top v_j).$$

Adding $\mathcal{A}_0(v_0, v_0)$ ($\equiv \mathcal{A}(R_0^\top v_0, R_0^\top v_0)$) to both sides and exploiting the triangle inequality gives

$$\begin{aligned} \left|\sum_{i=0}^{N} \mathcal{A}_i(v_i, v_i)\right| \;\le\;& \left|\mathcal{A}(v_h - R_0^\top v_0, v_h - R_0^\top v_0)\right| + \left|\mathcal{A}(R_0^\top v_0, R_0^\top v_0)\right| \\ &+ \left|\sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathcal{A}(R_i^\top v_i, R_j^\top v_j)\right|. \end{aligned} \tag{28}$$

Exploiting the continuity and coercivity of the DGFEM bilinear form $\mathcal{A}(\cdot,\cdot)$, cf. (8) and (9), respectively, the first two terms on the right hand side of (28) can be bounded by

$$\left|\mathcal{A}(v_h - R_0^\top v_0, v_h - R_0^\top v_0)\right| + \left|\mathcal{A}(R_0^\top v_0, R_0^\top v_0)\right| \lesssim |\mathcal{A}(v_h, v_h)| + \|v_h - R_0^\top v_0\|_{\mathrm{DG}}^2. \tag{29}$$

The third term on the right hand side of (28) can be bounded using Lemma 5.2, with v_h replaced by $v_h - R_0^\top v_0$, i.e.,

$$\left|\sum_{\substack{i,j=1\\ i\neq j}}^{N} \mathcal{A}(R_i^\top v_i, R_j^\top v_j)\right| \lesssim \|v_h - R_0^\top v_0\|_{\mathrm{DG}}^2 + \sigma \sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h - R_0^\top v_0\|_{L^2(\partial\mathcal{P})}^2. \tag{30}$$

Inserting the bounds (29) and (30) into (28) gives

$$\left|\sum_{i=0}^{N} \mathcal{A}_i(v_i, v_i)\right| \lesssim |\mathcal{A}(v_h, v_h)| + \|v_h - R_0^\top v_0\|_{\mathrm{DG}}^2 + \sigma \sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h - R_0^\top v_0\|_{L^2(\partial\mathcal{P})}^2.$$

Using the fact that $[\![R_0^\top v_0]\!] = 0$ on each face $F \in \mathcal{F}_h^I$, such that $F \not\subseteq \partial\mathcal{P}$, for any $\mathcal{P} \in \mathcal{T}_H$, we deduce that

$$\left|\sum_{i=0}^{N} \mathcal{A}_i(v_i, v_i)\right| \lesssim \mathcal{A}(v_h, v_h) + \left|v_h - R_0^\top v_0\right|^2_{H^1(\mathcal{T}_h)} + \sigma \sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h - R_0^\top v_0\|^2_{L^2(\partial\mathcal{P})}. \tag{31}$$

Employing Lemma 5.1, together with (9), the second term on the right-hand side of (31) may be bounded as follows:

$$\left|v_h - R_0^\top v_0\right|^2_{H^1(\mathcal{T}_h)} \lesssim \|v_h\|^2_{\mathrm{DG}} \lesssim \mathcal{A}(v_h, v_h). \tag{32}$$

To bound the third term on the right-hand side of (31), we employ the trace inequality stated in Lemma 5.3, together with Lemma 5.1 and (9); thereby, we deduce that

$$\sigma \sum_{\mathcal{P}\in\mathcal{T}_H} \|v_h - R_0^\top v_0\|^2_{L^2(\partial\mathcal{P})} \lesssim \sigma \frac{H}{q}\left(1 + \frac{1}{q}\right)\mathcal{A}(v_h, v_h). \tag{33}$$

The statement of the theorem now immediately follows upon inserting (32) and (33) into (31) and employing the definition of the penalty stabilization function given (5). $\square$

Writing N_S to denote the maximum number of adjacent partitions that any given subdomain in the partition $\mathcal{T}_S$ might possess, utilizing the stable splitting given in Theorem 5.1, together with the abstract framework for the analysis of Schwarz methods [21, 22], we now state the main result of this article.

Theorem 5.2. *Given $C_\natural$ is defined as in Theorem 5.1, i.e., $C_\natural^2 = C_\sigma H p^2/(hq)$, then, the condition number of the additive Schwarz operator satisfies*

$$\kappa(P_{ad}) \leq C_\natural^2 \left(N_S + 2\right).$$

Moreover, the error propagation operator $E_{mu} = (I - P_N)\cdots(I - P_0)$ of the multiplicative Schwarz operator satisfies

$$\|E_{mu}\|^2_{\mathcal{A}} \leq 1 - \frac{1}{\left(2(N_S+1)^2 + 1\right) C_\natural^2}.$$

Remark 5.1. *We point out that the expression derived in Theorem 5.1 for $C_\natural$ leads to the optimal spectral bounds stated in Theorem 5.2; these bounds are in agreement with the numerical experiments presented in our previous article* [5]. *However, the analogous spectral bounds presented in* [5] *were not explicit with respect to the polynomial degree q employed in the coarse grid solver; indeed, Proposition 4.3 in* [5] *only provided the estimate $C_\natural^2 = C_\sigma H p^2/h$. As a final remark, we note that while the case $q = 0$ is not directly covered in this article, this has already been treated in* [5].

6. Acknowledgements

Paola F. Antonietti has been partially funded by SIR Project n. RBSI14VT0S "PolyPDEs: Non-conforming polyhedral finite element methods for the approximation of partial differential equations" funded by MIUR and by Indam-GNCS Project 2015: "Non-standard numerical methods for Geophysics".

References

[1] P. F. Antonietti and B. Ayuso. Schwarz domain decomposition preconditioners for discontinuous Galerkin approximations of elliptic problems: Non-overlapping case. *Math. Model. Numer. Anal.*, 41(1):21–54, 2007.

[2] P. F. Antonietti and B. Ayuso. Multiplicative Schwarz methods for discontinuous Galerkin approximations of elliptic problems. *M2AN Math. Model. Numer. Anal.*, 42(3):443–469, 2008.

[3] P. F. Antonietti and B. Ayuso. Two-level Schwarz preconditioners for super penalty discontinuous Galerkin methods. *Commun. Comput. Phys.*, 5(2–4):398–412, 2009.

[4] P. F. Antonietti, B. Ayuso de Dios, S. C. Brenner, and L.-Y. Sung. Schwarz methods for a preconditioned WOPSIP method for elliptic problems. *Comp. Meth. Appl. Math.*, 12(3):241–272, 2012.

[5] P. F. Antonietti and P. Houston. A class of domain decomposition preconditioners for *hp*-discontinuous Galerkin finite element methods. *J. Sci. Comput.*, 46(1):124–149, 2011.

[6] P. F. Antonietti and P. Houston. Preconditioning high-order discontinuous Galerkin discretizations of elliptic problems. *Lecture Notes in Computational Science and Engineering*, 91:231–238, 2013.

[7] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.

[8] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39(5):1749–1779 (electronic), 2001/02.

[9] A. T. Barker, S. C. Brenner, E.-H. Park, and L.-Y. Sung. Two-level additive Schwarz preconditioners for a weakly over-penalized symmetric interior penalty method. *J. Sci. Comp.*, 47:27–49, 2011.

[10] S. C. Brenner and K. Wang. Two-level additive Schwarz preconditioners for C^0 interior penalty methods. *Numer. Math.*, 102(2):231–255, 2005.

[11] B. Cockburn and C.-W. Shu. The local discontinuous Galerkin method for time-dependent convection-diffusion systems. *SIAM J. Numer. Anal.*, 35(6):2440–2463 (electronic), 1998.

[12] X. Feng and O. A. Karakashian. Two-level additive Schwarz methods for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 39(4):1343–1365 (electronic), 2001.

[13] E. H. Georgoulis, E. Hall, and P. Houston. Discontinuous Galerkin methods on *hp*-anisotropic meshes. I. A priori error analysis. *Int. J. Comput. Sci. Math.*, 1(2-4):221–244, 2007.

[14] P. Houston, D. Schötzau, and T. P. Wihler. Energy norm a posteriori error estimation of *hp*-adaptive discontinuous Galerkin methods for elliptic problems. *Math. Models Methods Appl. Sci.*, 17(1):33–62, 2007.

[15] P. Houston, C. Schwab, and E. Süli. Discontinuous *hp*-finite element methods for advection-diffusion-reaction problems. *SIAM J. Numer. Anal.*, 39(6):2133–2163 (electronic), 2002.

[16] O. Karakashian and F. Pascal. A posteriori error estimation for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.*, 41:2374–2399, 2003.

[17] I. Perugia and D. Schötzau. An *hp*-analysis of the local discontinuous Galerkin method for diffusion problems. *J. Sci. Comp.*, 17:561–571, 2002.

[18] B. Rivière. *Discontinuous Galerkin methods for solving elliptic and parabolic equations*, volume 35 of *Frontiers in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008. Theory and implementation.

[19] C. Schwab. *p- and hp-finite element methods*. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 1998. Theory and applications in solid and fluid mechanics.

[20] I. Smears. Nonoverlapping domain decomposition preconditioners for discontinuous Galerkin finite element methods in H^2-type norms. Technical report, arXiv:1409.4202, 2014.

[21] B. F. Smith, P. E. Bjørstad, and W. D. Gropp. *Domain decomposition. Parallel multilevel methods for elliptic partial differential equations*. Cambridge University Press, Cambridge, 1996.

[22] A. Toselli and O. Widlund. *Domain decomposition methods—algorithms and theory*, volume 34 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2005.

MOX–*Modeling and Scientific Computing*, Dipartimento di Matematica, Politecnico di Milano, Piazza Leondardo da Vinci 32, 20133 Milano, Italy.
E-mail: `paola.antonietti@polimi.it`

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, UK.
E-mail: `Paul.Houston@nottingham.ac.uk`

Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK.
E-mail: `Iain.Smears@maths.ox.ac.uk`