AN EXTENSION OF HERGLOTZ'S THEOREM TO THE QUATERNIONS

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Abstract. A classical theorem of Herglotz states that a function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ is positive definite if and only there exists a $\mathbb{C}^{s\times s}$-valued positive measure μ on $[0, 2\pi]$ such that $r(n) = \int_0^{2\pi} e^{int} d\mu(t)$for $n \in \mathbb{Z}$. We prove a quaternionic analogue of this result when the function is allowed to have a number of negative squares. A key tool in the argument is the theory of slice hyperholomorphic functions, and the representation of such functions which have a positive real part in the unit ball of the quaternions. We study in great detail the case of positive definite functions.

1. Introduction

The main purpose of this paper is to prove a version of a theorem of Herglotz on positive functions in the quaternionic and indefinite setting. To set the framework we first recall some definitions and results pertaining to the complex numbers setting. A function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ is called positive definite if the associated function (also called kernel) $K(n-m)$ is positive definite on $\mathbb{Z}$. This means that for every choice of $N \in \mathbb{N}$ and $n_1, \ldots, n_N \in \mathbb{Z}$, the $N \times N$ block matrix with (j, ℓ) block entry equal to the matrix $r(n_j - n_\ell)$ is non-negative, that is, all the block Toeplitz matrices

$$\mathbb{T}_N \stackrel{\text{def.}}{=} \begin{pmatrix} r(0) & r(1) & \cdots & r(N) \\ r(-1) & r(0) & \cdots & r(N-1) \\ & & & \\ r(-N) & r(1-N) & \cdots & r(0) \end{pmatrix} \tag{1.1}$$

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are non-negative. We will use the notation $\mathbb{T}_N \succeq 0$. The positivity implies in particular that $r(-n) = r(n)^*$, where $r(n)^*$ denotes the adjoint of $r(n)$.

A result of Herglotz, also known as Bochner's theorem, asserts that:

Theorem 1.1. *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ is positive definite if and only if there exists a unique positive $\mathbb{C}^{s\times s}$-valued measure μ on $[0, 2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{1.2}$$

See for instance [17, p. 38], [13] and the discussion in [18, p. 19]. We note that (1.2) can be rewritten as

$$r(n) = C^* U^n C, \quad n \in \mathbb{Z}, \tag{1.3}$$

where U denotes the unitary operator of multiplication by e^{it} in $\mathbf{L}_2([0, 2\pi], d\mu)$, and C denotes the operator defined by $C\xi = \xi$ from $\mathbb{C}^s$ into $\mathbf{L}_2([0, 2\pi], d\mu)$.

A result of Carathéodory [9], which is related to Theorem 1.1 asserts that:

Theorem 1.2. *If the function $n \mapsto r(n)$ from $-N, \ldots, N$ into $\mathbb{C}^{s\times s}$ is positive definite, i.e., $\mathbb{T}_N \succeq 0$, then exists a function $n \mapsto \tilde{r}(n)$ from $\mathbb{Z}$ to $\mathbb{C}^{s\times s}$ which is positive definite and satisfies*

$$r(n) = \tilde{r}(n), \quad n \in \{-N, \ldots, N\}. \tag{1.4}$$

In commutative harmonic analysis, Theorem 1.1 is a special case of a general result of Weil [23] on the representation of positive definite functions on a group in terms of the characters of the group. See for instance [12, (22.7.10), p 65] or [22, Theorem 5.4.3, p. 65].

A key result in one of the proofs (see for instance [17, pp. 148-149]) of Theorem 1.1 is Herglotz's representation theorem, which states that a $\mathbb{C}^{s\times s}$-valued function φ is analytic and with a real positive part in the open unit disk $\mathbb{D}$ if and only if it can be written as

$$\varphi(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) + ia, \tag{1.5}$$

where $d\mu$ is as in Theorem 1.1 and $a \in \mathbb{C}^{s\times s}$ satisfies $a + a^* = 0$. There are a number of ways to prove (1.5). It can be obtained from Cauchy's formula and from the weak-$*$ compactness of the family of finite variation measures on $[0, 2\pi]$; see for instance the discussion in [1, p. 207].

Krein extended the notion of positive definite functions to the notion of functions having a number of negative squares; see [15]. We first recall the definition of this notion in the present setting:

Definition 1.3. *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ satisfying $r(n) = r(-n)^*$ has a finite number of negative squares, say κ, if by definition the function $K(n, m) = r(n - m)$ has κ negative squares, that is, if all the block Toeplitz matrices $\mathbb{T}_N$ defined in* (1.1) *(which are Hermitian since $r(n) = r(-n)^*$) have at most κ strictly negative eigenvalues and exactly κ strictly negative eigenvalues for some choice of N and $n_1, \ldots n_N$.*

Theorem 1.1 was extended, in the scalar case, by Iohvidov [16] to case where the function $K(n, m)$ has a finite number of negative squares. Formula (1.2) is then replaced by a more involved expression. More precisely, he obtained the following extension of (1.2) (there is a minus sign with respect to [16] and [14] because they work there with positive squares rather than negative squares):

$$\begin{aligned} r(n) = & \int_0^{2\pi} \frac{e^{int} - S_n(t)}{\prod_{k=1}^{u}\left(\sin\left(\frac{t-\varphi_k}{2}\right)^{2\rho_k}\right)} d\mu(t) \\ & - \left(\sum_{j=1}^{r} Q_j(in)\lambda_j^n + \overline{Q_j(in)\lambda_j}^{-n} + \sum_{k=1}^{u} R_k(in) e^{in\varphi_k} \right). \end{aligned} \tag{1.6}$$

In this expression, the λ_j are of modulus strictly bigger than 1, the Q_j and R_j are polynomials and S_n is a regularizing correction. These terms follow from the structure of a contraction in a Pontryagin space, and in particular from the fact that such an operator has always a strictly negative invariant subspace, on which it is one-to-one. See [14, (20.2), p. 319], where Iohvidov and Krein prove that such a representation is unique.

In this paper we shall prove in particular a quaternionic analogue of Theorem 1.1, where $\mathbb{H}$ denote the quaternions:

Theorem 1.4. *Let $(r(n))_{n\in\mathbb{Z}}$ be a sequence of $s\times s$ matrices with quaternionic entries. Then:*
(1) The function $K(n,m) = r(n-m)$ has a finite number of negative squares κ if and only if there exists a right quaternionic Pontryagin space $\mathcal{P}$, a unitary operator $U \in \mathbf{L}(\mathcal{P})$ and a linear operator $C \in \mathbf{L}(\mathbb{H}^s, \mathcal{P})$ such that

$$r(n) = C^*U^nC, \quad n \in \mathbb{Z}. \tag{1.7}$$

(2) Assume that

$$\bigcup_{n\in\mathbb{Z}} \operatorname{ran} U^nC \tag{1.8}$$

is dense in $\mathcal{P}$ where ran U^nC *denotes the range of U^nC. Then, the realization* (1.7) *is unique up to a unitary map.*

Some remarks:
(1) The sufficiency of condition (1.7) follows from the inner product representation

$$c^*r(n-m)d = c^*C^*U^{n-m}Cd = \langle U^{-m}Cd\,,\, U^{-n}Cc\rangle_{\mathcal{P}}, \quad n,m \in \mathbb{Z}, \quad c,d \in \mathbb{H}^s.$$

The proof of the necessity is done using the theory of slice hyperholomorphic functions. We use in particular a representation theorem from [2] for functions φ slice-hyperholomorphic in some open subset of the unit ball and with a certain associated kernel $K_\varphi(p,q)$ (defined by (4.4) below) having a finite number of negative squares there. We also note that the arguments in [2] rely on the theory of linear relations in Pontryagin spaces.
(2) The more precise integral representation of Iohvidov and Krein relies on the theory of unitary operators in Pontryagin spaces. Such results are still lacking in the setting of quaternionic Pontryagin spaces.
(3) We also consider the positive definite case. There, the lack of a properly established spectral theorem for unitary operators in quaternionic Hilbert spaces prevents to get a direct counterpart of the integral representation (1.2).

The outline of the paper is as follows. The paper consists of seven sections, besides the introduction. In Section 2 we review some results from the theory of slice hyperholomorphic functions. Some definitions and results on quaternionic Pontryagin spaces are recalled in Section 3 as well as Herglotz-type theorem for matrix valued functions. The proof of the necessity and uniqueness in Theorem 1.4 is done in Section 4. Section 5 deals with the analogue of Herglotz's theorem in the quaternionic setting. In Section 6 we prove a quaternionic analogue of Theorem 1.2. Section 7 contains the characterization of quaternionic, bounded, Hermitian sequences of matrices with κ negative squares. It uses results proved in Section 5. In Section 8 we prove an Herglotz representation theorem for scalar valued functions slice hyperholomophic in the unit ball of the quaternions, and with a positive real part there.

2. Slice hyperholomorphic functions

The kernels we will use in this paper are slice hyperholomorphic, so we recall their definition. For more details and the proofs of the results in this section see [10].
The imaginary units in $\mathbb{H}$ are denoted by i, j and k, and an element in $\mathbb{H}$ is of the form $p = x_0 + ix_1 + jx_2 + kx_3$, for $x_\ell \in \mathbb{R}$. The real part, the imaginary part and conjugate of p are defined as $\mathrm{Re}(p) = x_0$, $\mathrm{Im}(p) = ix_1 + jx_2 + kx_3$ and by $\bar{p} = x_0 - ix_1 - jx_2 - kx_3$, respectively. The unit sphere of purely imaginary quaternions $\mathbb{S}$ is defined by

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$; for this reason the elements of $\mathbb{S}$ are also called imaginary units. Note that $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4$. Given a nonreal quaternion $p = x_0 + \mathrm{Im}(p) = x_0 + I|\mathrm{Im}(p)|$, $I = \mathrm{Im}(p)/|\mathrm{Im}(p)| \in \mathbb{S}$, we can associate to it the 2-dimensional sphere defined by

$$[p] = \{x_0 + I|\mathrm{Im}(p)| \ : \ I \in \mathbb{S}\}.$$

We will denote an element in the complex plane $\mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ by $x + Iy$.

Definition 2.1 (Slice hyperholomorphic functions)**.** *Let Ω be an open set in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. Denote by f_I the restriction of f to the complex plane $\mathbb{C}_I$.*
We say that f is (left) slice hyperholomorphic (or (left) slice regular) if, on $\Omega \cap \mathbb{C}_I$, f_I satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0,$$

for all $I \in \mathbb{S}$.
We say that f is right slice hyperholomorphic (or right slice regular) if, on $\Omega \cap \mathbb{C}_I$, f_I satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy) I\right) = 0,$$

for all $I \in \mathbb{S}$.

An immediate consequence of the definition of slice regularity is that the monomial $p^n a$, with $a \in \mathbb{H}$, is left slice regular, so power series with quaternionic coefficients written on the right are left slice regular where they converge. As one can easily verify only power series with center at real points are slice regular.
We introduce a class of domains, which includes the balls with center at a real point, on which slice regular functions have good properties.

Definition 2.2 (Axially symmetric domain)**.** *Let $U \subseteq \mathbb{H}$. We say that U is* axially symmetric *if, for all $x + Iy \in U$, the whole 2-sphere $[x + Iy]$ is contained in U.*

Definition 2.3 (Slice domain)**.** *Let $U \subseteq \mathbb{H}$ be a domain in $\mathbb{H}$. We say that U is a* slice domain (s-domain for short) *if $U \cap \mathbb{R}$ is non empty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.*

Lemma 2.4 (Splitting Lemma)**.** *Let Ω be an s-domain in $\mathbb{H}$. If $f : \Omega \to \mathbb{H}$ is left slice hyperholomorphic, then for every $I \in \mathbb{S}$, and every $J \in \mathbb{S}$, perpendicular to I, there are two holomorphic functions $F, G : \Omega \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for any $z = x + Iy$, it is*

$$f_I(z) = F(z) + G(z)J.$$

Note that the decomposition given in the Splitting Lemma is highly non-canonical. In fact, for any $I \in \mathbb{S}$ there is an infinite number of choices of $J \in \mathbb{S}$ orthogonal to it.

Theorem 2.5 (Representation Formula). *Let Ω be an axially symmetric s-domain $\Omega \subseteq \mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice hyperholomorphic function on Ω. Then the following equality holds for all $x + yI, x \pm Jy \in \Omega$:*

$$f(x+Iy) = \frac{1}{2}\Big[f(x+Jy) + f(x-Jy)\Big] + I\frac{1}{2}\Big[J[f(x-Jy) - f(x+Jy)]\Big]. \tag{2.1}$$

3. Quaternionic Pontryagin spaces

A Hermitian form on a right quaternionic vector space $\mathcal{P}$ is an $\mathbb{H}$-valued map $[\cdot,\cdot]$ defined on $\mathcal{P} \times \mathcal{P}$ and such that

$$\begin{aligned}[a,b] &= \overline{[b,a]}\\ [ap,bq] &= \overline{q}[a,b]p, \quad \forall a,b \in \mathcal{P} \text{ and } p,q \in \mathbb{H}.\end{aligned}$$

$\mathcal{P}$ is called a (right quaternionic) Pontryagin space if it can written as

$$\mathcal{P} = \mathcal{P}_+ \, [\dot{+}] \, \mathcal{P}_-, \tag{3.1}$$

where:
(a) The space $\mathcal{P}_+$ endowed with the form $[\cdot,\cdot]$ is a Hilbert space.
(b) The space $\mathcal{P}_-$ endowed with the form $-[\cdot,\cdot]$ is a finite dimensional Hilbert space.
(c) The sum is direct and orthogonal, meaning that $\mathcal{P}_+ \cap \mathcal{P}_- = \{0\}$ and

$$[a,b] = 0, \quad \forall (a,b) \in \mathcal{P}_+ \times \mathcal{P}_-.$$

The decomposition (3.1) is called a fundamental decomposition. It not unique unless one of the components reduces to $\{0\}$. The dimension of $\mathcal{P}_-$ is the same for all fundamental decompositions, and is called the index of the Pontryagin space. The space $\mathcal{P}$ endowed with the form

$$\langle a,b\rangle = [a_+,b_+] - [a_-,b_-] \tag{3.2}$$

with $a_\pm$ and $b_\pm \in \mathcal{P}_\pm$ is a Hilbert space. The inner product (3.2) depends on the given fundamental decomposition, but all the associated norms are equivalent,and hence define the same topology. We refer to [5] for a proof of these facts. We refer to [5, 3] for more details on quaternionic Pontryagin spaces and to[6, 8, 15] for the theory of Pontryagin spaces in the complex case. A reproducing kernel Pontryagin space will be a Pontryagin space of functions for which the point evaluations are bounded. The definition of negative squares makes sense in the quaternionic setting since an Hermitian quaternionic matrix H is diagonalizable: It can be written as $T = UDU^*$, where U is unitary and D is unique and with real entries. The number of strictly negative eigenvalues of T is exactly the number of strictly negative elements of D. See [24]. The one-to-one correspondence between reproducing kernel Pontryagin spaces and functions with a finite number of negative squares, proved in the classical case by [20, 21], extends to the Pontryagin space setting, see [5].

Definition 3.1. *An $\mathbb{H}^{s\times s}$-valued function φ slice hyperholomorphic in a neighborhood $\mathcal{V}$ of the origin is called a generalized Carathéodory function if the kernel*

$$k_\varphi(p,q) = \Sigma_{\ell=0}^{\infty} p^\ell(\varphi(p) + \overline{\varphi(q)})\overline{q}^\ell$$

has a finite number of negative squares in $\mathcal{V}$.

The following result is Theorem 10.2 in [2].

Theorem 3.2. *A $\mathbb{H}^{s\times s}$-valued function φ is a generalized Carathéodory function if and only if it can be written as*

$$\varphi(p) = \frac{1}{2}C \star (I_{\mathcal{P}} + pV) \star (I_{\mathcal{P}} - pV)^{-\star} C^* J + \frac{\varphi(0) - \varphi(0)^*}{2} \tag{3.3}$$

where $\mathcal{P}$ is a right quaternionic Pontryagin space of index κ, V is a coisometry in $\mathcal{P}$, and C is a bounded operator from $\mathcal{P}$ to $\mathbb{H}^N$, and the pair (C, A) is observable.

Remark 3.3. When $\kappa = 0$ the representation (3.3) is the counterpart of Herglotz representation theorem for functions slice hyperholomorphic in the open unit ball and with a positive real part. In the last section we shall discuss a scalar version of this result.

4. Proof of the necessity and uniqueness of the realization

In this section we assume that the function $K(n, m) = r(n - m)$ has a finite number of negative squares for $n, m \in \mathbb{Z}$, and prove that the function $r(n)$ has a representation of the form (1.7). We also prove the uniqueness of this representation under hypothesis (1.8). We begin with a preliminary proposition.

Proposition 4.1. *There exists $C > 0$ and $K > 0$ such that*

$$\|r(n)\| \leq K \cdot C^{|n|}, \quad n \in \mathbb{Z}. \tag{4.1}$$

Proof. The claim is true in the scalar complex-valued case and follows from (1.6); see [14]. The idea is to reduce the problem to this case. We write $r(n) = a(n) + jb(n)$ where $a(n)$ and $b(n)$ are $\mathbb{C}^{s\times s}$-valued. We obtain a bound of the required form for every entry of $a(n)$ and $b(n)$. The coefficients K and C in (4.1) will depend on the given entry. Since there are $2s^2$ entries, we obtain a bound independent of the entry.

STEP 1: *For every choice of $(e, f) \in \mathbb{H}^s \times \mathbb{H}^s$, the function*

$$K_{e,f}(n, m) = e^* a(n - m)e + f^* \overline{a(n - m)} f + e^* b(n - m) f - f^* \overline{b(n - m)} e$$

has at most 2κ negative squares.

Indeed, the $\mathbb{C}^{2s\times 2s}$ function

$$K_1(n, m) = \begin{pmatrix} a(n - m) & b(n - m) \\ -\overline{b(n - m)} & \overline{a(n - m)} \end{pmatrix}$$

has 2κ negative squares (See [5, Proposition 11.4, p. 466]), and so, for every fixed choice of $(e, f) \in \mathbb{H}^s \times \mathbb{H}^s$, the function

$$K_{e,f}(n - m) = \begin{pmatrix} e^* & f^* \end{pmatrix} \begin{pmatrix} a(n - m) & b(n - m) \\ -\overline{b(n - m)} & \overline{a(n - m)} \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix}$$

has at most 2κ negative squares.

STEP 2: *The claim holds for every diagonal entry of $a(n)$.*

Take $e = e_j \in \mathbb{H}^s$ to be the vector with all entries equal to 0, except the j-th one equal to 1 and $f = 0$. We have

$$K_{e,f}(n, m) = a_{jj}(n - m),$$

and the result follows from [14].

STEP 3: *The claim holds for all the entries of $a(n)$.*

Let $\ell \neq j \in \{1, \ldots, s\}$. We now take $e = e_{\ell j}(\varepsilon) \in \mathbb{H}^s$ to be the vector with all entries equal to 0, except the ℓ-th one equal to 1, and the j-th entry equal to ϵ (where ε will be determined) and $f = 0$. We have

$$K_{e,f}(n, m) = a_{\ell\ell}(n-m) + a_{jj}(n-m) + \overline{\varepsilon} a_{j\ell}(n-m) + \varepsilon a_{\ell j}(n-m).$$

This function has at most κ negative squares and so the sequence

$$a_{\ell\ell}(n) + a_{jj}(n) + \overline{\varepsilon} a_{j\ell}(n) + \varepsilon a_{\ell j}(n).$$

has a bound of the form (4.1) (where K and C depend on ℓ, j and ϵ). The choices $\epsilon = 1$ and $\varepsilon = i$ gives that the functions

$$a_{\ell\ell}(n-m) + a_{jj}(n-m) + a_{j\ell}(n-m) + a_{\ell j}(n-m)$$

and

$$a_{\ell\ell}(n-m) + a_{jj}(n-m) + i(-a_{j\ell}(n-m) + a_{\ell j}(n-m))$$

have at most κ negative squares and so the functions

$$|a_{\ell\ell}(n) + a_{jj}(n) + a_{j\ell}(n) + a_{\ell j}(n)| \leq K_1 C_1^{|n|}$$

and

$$|a_{\ell\ell}(n) + a_{jj}(n) + i(-a_{j\ell}(n) + a_{\ell j}(n))| \leq K_2 C_2^{|n|}$$

where the constants depend on (ℓ, j). Since $a_{\ell\ell}(n)$ and $a_{jj}(n)$ admit similar bounds we get that both $a_{\ell j}(n)$ and $a_{j\ell}(n)$ admit bounds of the form (4.1).

STEP 4: *The claim holds for the diagonal entries of* $b(n)$.

We now take $e = e_\ell$ and $f = \varepsilon e_j$, where ε is of modulus 1. We have

$$K_{e,f}(n-m) = A(n-m) + B(n-m)$$

where

$$\begin{aligned} A(n-m) &= a_{\ell\ell}(n-m) + a_{jj}(n-m), \\ B(n-m) &= \overline{\varepsilon} b_{\ell\ell}(n-m) + \varepsilon b_{jj}(n-m). \end{aligned}$$

The choice $\varepsilon = 1$ and $\varepsilon = i$ lead to the conclusion that $b_{\ell\ell}(n)$ admits a bound of the form (4.1) since, as follows from the previous step, $A(n-m)$ admits such a bound.

STEP 5: *The claim holds for all the entries of* $b(n)$.

We now take $e = e_{\ell j}(\varepsilon_1)$ and $f = \varepsilon e_{\ell j}(e_2)$, where ε_1 and ε_2 are of modulus 1. We have now $K_{e,f}(n-m) = A(n-m) + B(n-m)$ with

$$\begin{aligned} A(n-m) &= e^* a(n-m) e + f^* \overline{a(n-m)} f^* \\ B(n-m) &= e^* b(n-m) f - f^* \overline{b(n-m)} e \\ &= b_{\ell\ell}(n-m) + \overline{\varepsilon_1} \varepsilon_2 b_{jj}(n-m) + \overline{\varepsilon_1} b_{\ell j}(n-m) + \varepsilon_2 b_{j\ell}(n-m) - \\ &\quad - \overline{b_{\ell\ell}(n-m)} - \overline{\varepsilon_2} \varepsilon_1 \overline{b_{jj}(n-m)} - \overline{\varepsilon_2} \overline{b_{\ell j}(n-m)} - \varepsilon_1 \overline{b_{\ell j}(n-m)}. \end{aligned}$$

In view of the previous steps the sequence

$$\overline{\varepsilon_1} b_{j\ell}(n) + \varepsilon_2 b_{\ell j}(n) - \overline{\varepsilon_2} \overline{b_{j\ell}(n)} - \varepsilon_1 \overline{b_{\ell j}(n)}$$

admits a bound of the form (4.1). The choices

$$(\varepsilon_1, \varepsilon_2) \in \{(1, 1), (1, -1), (i, i), (i, -i)\}$$

lead to the functions

$$\begin{aligned}
&(\overline{b_{j\ell}(n)} - b_{j\ell}(n)) + (b_{\ell j}(n) - \overline{b_{\ell j}(n)})\\
&(\overline{b_{j\ell}(n)} + b_{j\ell}(n)) + (b_{\ell j}(n) + \overline{b_{\ell j}(n)})\\
&(\overline{b_{j\ell}(n)} - b_{j\ell}(n)) + (b_{\ell j}(n) - \overline{b_{\ell j}(n)})\\
&-(\overline{b_{j\ell}(n)} + b_{j\ell}(n)) + i(b_{\ell j}(n) - \overline{b_{\ell j}(n)})\\
&-i(\overline{b_{j\ell}(n)} + b_{j\ell}(n)) - i(b_{\ell j}(n) + \overline{b_{\ell j}(n)})
\end{aligned}$$

all admit a bound of the form (4.1).

Proof of Theorem 1.4. We proceed in a number of steps to prove the necessity part of the theorem. The first step is a direct computation which is omitted.

STEP 1: *Let V be a coisometry (that is, $VV^* = I$) in the quaternionic Pontryagin space $\mathcal{P}$. Then,*

$$U = \begin{pmatrix} V^* & I - V^*V \\ 0 & V \end{pmatrix}$$

is unitary from $\mathcal{P}^2$ into itself, and is such that

$$V^n = \begin{pmatrix} 0 & I \end{pmatrix} U^n \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad n = 0, 1, 2, \ldots \tag{4.2}$$

STEP 2: *The series*

$$\begin{aligned}
\varphi(p) &= r(0) + 2\sum_{n=1}^{\infty} p^n r(n),\\
K_\varphi(p,q) &= \sum_{n,m\in\mathbb{Z}} p^n r(n-m)\overline{q}^m,
\end{aligned}$$

converge for p and q in a neighborhood Ω of the origin, and it holds that

$$K_\varphi(p,q) - pK_\varphi(p,q)\overline{q} = \frac{\varphi(p) + \varphi(q)^*}{2}, \quad p, q \in \Omega. \tag{4.3}$$

The asserted convergences follow from (4.1), while (4.3) is a direct computation.

STEP 4: *It holds that*

$$K_\varphi(p,q) = \sum_{n=0}^{\infty} p^n \left(\frac{\varphi(p) + \varphi(q)^*}{2}\right) \overline{q}^n \tag{4.4}$$

This is because equation (4.3) has a unique solution, and that the right side of (4.4) solves (4.3).

STEP 5: *$K_\varphi(p,q)$ is has a finite number of negative squares in Ω.*
Note that for every $N \in \mathbb{N}$ the function

$$K_{\varphi,N}(p,q) = \begin{pmatrix} I_s & I_s p & \cdots & I_s p^N \end{pmatrix} \mathbb{T}_N \begin{pmatrix} I_s \\ I_s\overline{q} \\ \vdots \\ I_s\overline{q}^N \end{pmatrix}$$

has a finite number of negative squares, uniformly bounded by κ in Ω. The claim then follows from

$$K_\varphi(p,q)=\lim_{N\to\infty}K_{\varphi,N}(p.q)$$

STEP 6: *There exist a right quaternionic Pontryagin space $\mathcal{P}$, a unitary operator $U\in\mathbf{L}(\mathcal{P})$ and a linear operator $C\in\mathbf{L}(\mathbb{H}^s,\mathcal{P})$ such that*

$$\varphi(p)=\frac{CC^*}{2}+\sum_{n=1}^{\infty}p^nC^*U^nC,\quad p\in\Omega \tag{4.5}$$

Indeed, since the expression in the right side of (4.4) defines a kernel with a finite number of negative squares, we can apply [2, Theorem 10.2] to see that there exists a right quaternionic Pontryagin space $\mathcal{P}_1$, a coisometric operator $V\in\mathbf{L}(\mathcal{P}_1)$ and a bounded operator $C_1\in\mathbf{L}(\mathcal{P},\mathcal{P}_1)$ such that

$$r(n)=C_1^*V^nC_1,\quad n=0,1,\ldots$$

We now apply STEP 1 to write

$$r(n)=C_1^*\begin{pmatrix}0 & I\end{pmatrix}U^n\begin{pmatrix}0\\ I\end{pmatrix}C_1,\quad n=0,1,\ldots,$$

which concludes the proof with $C=\begin{pmatrix}0\\ C_1\end{pmatrix}$ and $n\geq 0$. That the formula still holds for negative n follows from $r(-n)=r(n)^*$ and from the unitarity of U.

To conclude the proof, we turn to the uniqueness of the representation (1.7). Consider two representations (1.7),

$$r(n)=C_1^*U_1^nC_1=C_2^*U_2^nC_2,\quad n\in\mathbb{Z},$$

where U_1 and U_2 are unitary operators in quaternionic Pontryagin spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Consider the space of pairs

$$R=\left\{(U_1^nC_1c\,,\,U_2^nC_2c)\,,\,n\in\mathbb{Z},\ c\in\mathbb{H}^s\right\}.$$

When condition (1.8) is in force for both representations R, defines a linear isometric relation with dense domain and range, and hence, by the quaternionic version of a theorem of Shmulyan (see [2, Theorem 7.2] and see [4, p. 29-30] for the complex version of this theorem and for the definition of a linear relation in Pontryagin spaces), R extends to the graph of a unitary operator, say S, from $\mathcal{H}_1$ into $\mathcal{H}_2$:

$$SU_1^nC_1c=U_2^nC_2c,\quad n\in\mathbb{Z},\ c\in\mathbb{H}^s$$

Setting $n=0$ we get $SC_1=C_2$. Then, taking $n=1$ leads to $(SU_1)C_1=(U_2S)C_1$, and more generally

$$(SU_1)U_1^nC_1=(U_2S)U_1^nC_1,\quad n\in\mathbb{Z},$$

and so $SU_1=U_2S$.

5. Herglotz's theorem in the quaternionic setting

Herglotz's theorem has been already recalled in Section 1, see Theorem 1.1. Here we state a related result which will be useful in the sequel (see [18, Theorem 1.3.6]).

Theorem 5.1. *Let μ and ν be $\mathbb{C}^{s\times s}$-valued measures on $[0,2\pi]$. If*

$$\int_0^{2\pi}e^{int}d\mu(t)=\int_0^{2\pi}e^{int}d\nu(t),\quad n\in\mathbb{Z},$$

then $\mu=\nu$.

Given $P \in \mathbb{H}^{s\times s}$, there exist unique $P_1, P_2 \in \mathbb{C}^{s\times s}$ such that $P = P_1 + P_2 j$. Thus there is a bijective homomorphism $\chi : \mathbb{H}^{s\times s} \to \mathbb{C}^{2s\times 2s}$ given by

$$\chi P = \begin{pmatrix} P_1 & P_2 \\ -\overline{P}_2 & \overline{P}_1 \end{pmatrix} \quad \text{where } P = P_1 + P_2 j, \tag{5.1}$$

Definition 5.2. *Given an $\mathbb{H}^{s\times s}$-valued measure ν, write $\nu = \nu_1 + \nu_2 j$, where ν_1 and ν_2 are uniquely determined $\mathbb{C}^{s\times s}$-valued measures. We call a measure ν on $[0, 2\pi]$ q-positive if the $\mathbb{C}^{2s\times 2s}$-valued measure*

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}, \quad \text{where } d\nu_3(t) = d\bar{\nu}_1(2\pi - t), \;\; t \in [0, 2\pi) \tag{5.2}$$

is positive and, in addition,

$$d\nu_2(t) = -d\nu_2(2\pi - t)^T, \quad t \in [0, 2\pi),$$

Remark 5.3. If ν is q-positive, then $\nu = \nu_1 + \nu_2 j$, where ν_1 is a uniquely determined positive $\mathbb{C}^{s\times s}$-valued measure and ν_2 is a uniquely determined $\mathbb{C}^{s\times s}$-valued measure.

Remark 5.4. If $r = (r(n))_{n\in\mathbb{Z}}$ is a $\mathbb{H}^{s\times s}$-valued sequence on $\mathbb{Z}$ such that

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t),$$

where ν is a q-positive measure, then r is Hermitian, i.e., $r(-n)^* = r(n)$. Indeed, write $\nu = \nu_1 + \nu_2 j$, where ν_1 and ν_2 are as in Definition 5.2. Then

$$\begin{aligned} r(-n)^* &= \int_0^{2\pi} (d\nu_1(t) - j d\nu_2(t)^*) e^{int} \\ &= \int_0^{2\pi} e^{int} d\nu_1(t) - \int_0^{2\pi} e^{-int} (-d\nu_2(t)^T) j \\ &= \int_0^{2\pi} e^{int} d\nu_1(t) + \int_0^{2\pi} e^{int} (-d\nu_2(2\pi - t)^T) j \\ &= \int_0^{2\pi} e^{int} d\nu_1(t) + \int_0^{2\pi} e^{int} d\nu_2(t) \\ &= r(n), \qquad n \in \mathbb{Z} \end{aligned}$$

Theorem 5.5. *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{H}^{s\times s}$ is positive definite if and only if there exists a unique q-positive measure ν on $[0, 2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}. \tag{5.3}$$

Proof. Let $(r(n))_{n\in\mathbb{Z}}$ be a positive definite sequence and write $r(n) = r_1(n) + r_2(n) j$, where $r_1(n), r_2(n) \in \mathbb{C}^{s\times s}$, $n \in \mathbb{Z}$. Put $R(n) = \chi r(n)$, $n \in \mathbb{Z}$. It is easily seen that $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence if and only if $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence. Thus, by Theorem 1.1 there exists a unique positive $\mathbb{C}^{2s\times 2s}$-valued measure μ on $[0, 2\pi]$ such that

$$R(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{5.4}$$

Write

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12}^* & \mu_{22} \end{pmatrix} : \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix} \to \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix}.$$

It follows from

$$R(n)=\begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)}\end{pmatrix}, \quad n\in\mathbb{Z}$$

and (5.4) that

$$r_1(n)=\int_0^{2\pi} e^{int}d\mu_{11}(t)=\int_0^{2\pi} e^{-int}d\bar{\mu}_{22}(t), \quad n\in\mathbb{Z}$$

and hence

$$\int_0^{2\pi} e^{int}d\mu_{11}(t)=\int_0^{2\pi} e^{int}d\bar{\mu}_{22}(2\pi-t), \quad n\in\mathbb{Z}.$$

Thus, Theorem 5.1 yields that $d\mu_{11}(t)=d\bar{\mu}_{22}(2\pi-t)$ for $t\in[0,2\pi)$. Similarly,

$$r_2(n)=\int_0^{2\pi} e^{int}d\mu_{12}(t)=-\int_0^{2\pi} e^{-int}d\mu_{12}(t)^T, \quad n\in\mathbb{Z}$$

and hence

$$\int_0^{2\pi} e^{int}d\mu_{12}(t)=\int_0^{2\pi} e^{int}(-d\mu_{12}(2\pi-t)^T), \quad n\in\mathbb{Z}.$$

Thus, Theorem 5.1 yields that $d\mu_{12}(t)=-d\mu_{12}(2\pi-t)^T$ for $t\in[0,2\pi)$.
It is easy to show that

$$\begin{pmatrix} I_s & -jI_s\end{pmatrix} R(n)\begin{pmatrix} I_s \\ jI_s\end{pmatrix}=2r(n)$$

and hence (5.4) yields

$$\begin{aligned}
2r(n) &= \int_0^{2\pi} \begin{pmatrix} e^{int} & -je^{int}\end{pmatrix}\begin{pmatrix} d\mu_{11}(t)+d\mu_{12}(t)j \\ d\mu_{12}(t)^*+d\mu_{22}(t)j\end{pmatrix} \\
&= \int_0^{2\pi} e^{int}d\mu_{11}(t)+\int_0^{2\pi} e^{int}d\mu_{12}(t)j-\int_0^{2\pi} e^{-int}d\mu_{12}(t)^Tj \\
&\quad +\int_0^{2\pi} e^{-int}d\bar{\mu}_{22}(t) \\
&= \int_0^{2\pi} e^{int}d\mu_{11}(t)+\int_0^{2\pi} e^{int}d\mu_{12}(t)j-\int_0^{2\pi} e^{int}d\mu_{12}(2\pi-t)^Tj \\
&\quad +\int_0^{2\pi} e^{int}d\bar{\mu}_{22}(2\pi-t) \\
&= 2\int_0^{2\pi} e^{int}d\mu_{11}(t)+2\int_0^{2\pi} e^{int}d\mu_{12}(t)j, \quad n\in\mathbb{Z},
\end{aligned}$$

where the last line follows from $d\mu_{11}(t)=d\bar{\mu}_{22}(2\pi-t)$ and $d\mu_{12}(t)=-d\mu_{12}(2\pi-t)^T$. If we put $\nu=\mu_{11}+\mu_{12}j$, then ν is a q-positive measure which satisfies (5.3).
Conversely, suppose $\nu=\nu_1+\nu_2 j$ is a q-positive measure on $[0,2\pi]$ and put

$$r(n)=\int_0^{2\pi} e^{int}d\nu(t), \quad n\in\mathbb{Z}.$$

Since ν is q-positive,

$$\mu=\begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3\end{pmatrix}, \quad \text{where } d\nu_3(t)=d\bar{\nu}_1(2\pi-t), \ t\in[0,2\pi),$$

is a positive $\mathbb{C}^{2s\times 2s}$-valued measure on $[0,2\pi]$ and

$$d\nu_2(t)=-d\nu_2(2\pi-t)^T, \quad t\in[0,2\pi).$$

Since μ is a positive $\mathbb{C}^{2s\times 2s}$-valued measure, $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence, where

$$R(n) := \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z},$$

Moreover, $R(n)$ can be written in form

$$R(n) = \begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

where

$$r_1(n) = \int_0^{2\pi} e^{int} d\nu_1(t), \quad n \in \mathbb{Z};$$
$$r_2(n) = \int_0^{2\pi} e^{int} d\nu_2(t), \quad n \in \mathbb{Z}.$$

Thus, $R(n) = \chi r(n)$, where

$$r(n) = r_1(n) + r_2(n)j = \int_0^{2\pi} e^{int} d\nu(t).$$

Since $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence we get that $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence.
Finally, suppose that the q-positive measure ν were not unique, i.e., there exists $\tilde{\nu}$ so that $\tilde{\nu} \neq \nu$ and

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t) = \int_0^{2\pi} e^{int} d\tilde{\nu}(t), \quad n \in \mathbb{Z}.$$

Write $\nu = \nu_1 + \nu_2 j$ and $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2 j$ as in Remark 5.3. If we consider $R(n) = \chi r(n), n \in \mathbb{Z}$, then it follows from Theorem 1.1 that $\nu_1 = \tilde{\nu}_1$ and $\nu_2 = \tilde{\nu}_2$ and hence that $\nu = \tilde{\nu}$, a contradiction. $\square$

Remark 5.6. The statement and proof of Herglotz's theorem have been written using an exponential involving the imaginary unit i of the quaternions. Analogous statements can be written using the imaginary units j or k in the basis or with respect to new basis elements chosen in $\mathbb{S}$.

6. A theorem of Carathéodory in the quaternionic setting

Definition 6.1. *A function $r : \{-N, \dots, N\} \to \mathbb{H}^{s\times s}$ is called positive definite if $\mathbb{T}_N \succeq 0$, where $\mathbb{T}_N$ is the matrix defined in (1.1).*

Definition 6.2. *Let $r : \{-N, \dots, N\} \to \mathbb{H}^{s\times s}$ be positive definite. We will say that r has a positive definite extension if there exists a positive definite function $\tilde{r} : \mathbb{Z} \to \mathbb{H}^{s\times s}$ such that*

$$\tilde{r}(n) = r(n), \qquad n = -N, \dots, N.$$

Theorem 6.3. *If $r : \{-N, \dots, N\} \to \mathbb{H}^{s\times s}$ is positive definite, then r has a positive definite extension.*

Remark 6.4. The strategy for proving Theorem 6.3 is to establish the existence of $r(N+1), r(N+2), \ldots$ so that the block matrices

$$\mathbb{T}_{N+1} = \begin{pmatrix} r(0) & \cdots & r(N+1) \\ \vdots & \ddots & \vdots \\ r(-N-1) & \cdots & r(0) \end{pmatrix} \succeq 0$$
$$\mathbb{T}_{N+2} = \begin{pmatrix} r(0) & \cdots & r(N+2) \\ \vdots & \ddots & \vdots \\ r(-N-2) & \cdots & r(0) \end{pmatrix} \succeq 0, \quad \ldots$$

Here we let $r(-N-1) = r(N+1)^*$, $r(-N-2) = r(N+2)^*, \ldots$. We must first establish some lemmas before proving Theorem 6.3. The proofs of Lemmas 6.5, 6.6 and 6.8 are adapted from Lemma 2.4.2, Corollary 2.4.3 and Theorem 2.4.5 in Bakonyi and Woerdeman [7], respectively.

Lemma 6.5. *If $A \in \mathbb{H}^{t\times s}$ and $B \in \mathbb{H}^{u\times s}$, then*

$$B^*B \succeq A^*A$$

if and only if there exists a contraction $G : \operatorname{ran} B \to \operatorname{ran} A$ such that $A = GB$. Moreover, G is unique and an isometry if and only if $B^*B = A^*A$.

Proof. If there exists a contraction $G : \operatorname{ran} B \to \operatorname{ran} A$ such that $A = GB$, then it is easy to verify that $B^*B \succeq A^*A$. Conversely, if $B^*B \succeq A^*A$, then let $y \in \operatorname{ran} B$, i.e $y = Bx$ for some $x \in \mathbb{H}^s$. Let $G : \operatorname{ran} B \to \operatorname{ran} A$ be given by

$$Gy = Ax.$$

To check that G is well-defined, suppose that

$$y = Bx = B\tilde{x},$$

where $\tilde{x} \in \mathbb{H}^s$. Using $B^*B \succeq A^*A$ we get that

$$0 \le (x - \tilde{x})^*A^*A(x - \tilde{x}) \le (x - \tilde{x})^*B^*B(x - \tilde{x}) = 0$$

and hence $Ax = A\tilde{x}$. Therefore, G is well-defined.
We will now show that G is a contraction. Let $\{y_n\}_{n=1}^{\infty}$ be a convergent sequence in $\operatorname{ran} B$. If $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{H}^s$ so that

$$Bx_n = y_n,$$

then

$$\begin{aligned}(Gy_n - Gy_m)^*(Gy_n - Gy_m) &= [A(x_n - x_m)]^*[A(x_n - x_m)] \\ &\le [B(x_n - x_m)]^*[B(x_n - x_m)] \\ &= (y_n - y_m)^*(y_n - y_m). \end{aligned} \tag{6.1}$$

Since $\{y_n\}_{n=1}^{\infty}$ is a convergent sequence in $\operatorname{ran} B$, $\{y_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in $\operatorname{ran} B$ and hence $\{Gy_n\}_{n=1}^{\infty}$ is a Cauchy sequence as well. Thus,

$$\lim_{n\uparrow\infty} Gy_n$$

exists. The inequality given in (6.1) readily yields that $y^*G^*Gy \le y^*y$, whence G is a contraction. Note that G is unique by construction, since the equation $A = GB$ requires that whenever $y = Bx$ we get that $Gy = Ax$.
Finally, if G is an isometry then it follows from the equality $A = GB$ that $A^*A = B^*B$. Conversely, if $A^*A = B^*B$, then $y = Bx$ and $Gy = Ax$ yield that

$$y^*G^*Gy = x^*A^*Ax = x^*B^*Bx = y^*y.$$

Thus, $u^*G^*Gy = y^*y$ for $y \in \operatorname{ran} B$.

Lemma 6.6. *If*

$$K = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathbb{H}^{(t+u)\times(s+u)},$$

then $K \succeq 0$ if and only if the following conditions hold:

(i) *$A \succeq 0$ and $C \succeq 0$;*
(ii) *$B = A^{1/2}GC^{1/2}$ for some contraction $G : \operatorname{ran} C \to \operatorname{ran} A$.*

Proof. Suppose conditions (i) and (ii) are in force. It follows from (i) that there exist P and Q such that $A = P^*P$ and $C = Q^*Q$. Thus,

$$K = \begin{pmatrix} P^* & 0 \\ 0 & Q^* \end{pmatrix} \begin{pmatrix} I & G \\ G^* & I \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \succeq 0,$$

since G is a contraction. Conversely, suppose $K \succeq 0$ and let P and Q be given by

$$\begin{pmatrix} P^* \\ Q^* \end{pmatrix} \begin{pmatrix} P & Q \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

Thus, $P^*P = A^{1/2}A^{1/2}$ and $Q^*Q = C^{1/2}C^{1/2}$. Using Lemma 6.5 we arrive at the isometries $G_1 : \operatorname{ran} A \to \operatorname{ran} P$ and $G_2 : \operatorname{ran} C \to \operatorname{ran} Q$ which satisfy $P = G_1A^{1/2}$ and $Q = G_2C^{1/2}$, respectively. Therefore,

$$B = P^*Q = A^{1/2}G_1^*G_2C^{1/2}$$

and thus $B = A^{1/2}GC^{1/2}$, where $G = G_1^*G_2$ is a contraction.

Definition 6.7. *We will call a block matrix, with quaternionic entries,*

$$K = \begin{pmatrix} A & B & ? \\ B^* & C & D \\ ? & D^* & E \end{pmatrix}$$

partially positive semidefinite if all principle specified minors are nonnegative. We will say that K has a positive semidefinite completion if there exists a quaternionic matrix X so that

$$\begin{pmatrix} A & B & X \\ B^* & C & D \\ X^* & D^* & E \end{pmatrix} \succeq 0.$$

Lemma 6.8. *If*

$$K = \begin{pmatrix} A & B & ? \\ B^* & C & D \\ ? & D^* & E \end{pmatrix}$$

is partially positive semidefinite, then K has a positive semidefinite completion given as follows. Let $G_1 : \operatorname{ran} C \to \operatorname{ran} A$ and $G_2 : \operatorname{ran} E \to \operatorname{ran} C$ be contractions so that $B = A^{1/2}G_1C^{1/2}$and $D = C^{1/2}G_2E^{1/2}$. Choosing the $(1,3)$ block entry of K to be $A^{1/2}G_1G_2E^{1/2}$ results in a positive semidefinite completion.

Proof. Since K is partially positive semidefinite,

$$K_1 = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \succeq 0 \qquad \text{and} \qquad K_2 = \begin{pmatrix} C & D \\ D^* & E \end{pmatrix} \succeq 0.$$

Use Lemma 6.6 on K_1 and K_2 to produce contractions G_1 and G_2, resepectively, so that $B = A^{1/2}G_1C^{1/2}$ and $D = C^{1/2}G_2E^{1/2}$. Since G_1 and G_2 are contractions, the factorization

$$\widetilde{K} = \begin{pmatrix} A & B & A^{1/2}G_1G_2E^{1/2} \\ B^* & C & D \\ E^{1/2}(G_2)^*(G_1)^*A^{1/2} & D^* & E \end{pmatrix}$$

$$= \begin{pmatrix} A & A^{1/2}G_1C^{1/2} & A^{1/2}G_1G_2E^{1/2} \\ C^{1/2}(G_1)^*A^{1/2} & C & C^{1/2}G_2E^{1/2} \\ E^{1/2}(G_2)^*(G_1)^*A^{1/2} & E^{1/2}(G_2)^*C^{1/2} & E \end{pmatrix}$$

$$= \begin{pmatrix} A^{1/2} & 0 & 0 \\ 0 & C^{1/2} & 0 \\ 0 & 0 & E^{1/2} \end{pmatrix} \begin{pmatrix} I & G_1 & G_1G_2 \\ (G_1)^* & I & G_2 \\ (G_2)^*(G_1)^* & (G_2)^* & I \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 & 0 \\ 0 & C^{1/2} & 0 \\ 0 & 0 & E^{1/2} \end{pmatrix}$$

$$= \begin{pmatrix} A^{1/2} & 0 & 0 \\ 0 & C^{1/2} & 0 \\ 0 & 0 & E^{1/2} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ (G_1)^* & I & 0 \\ (G_1G_2)^* & (G_2)^* & I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I-(G_1)^*G_1 & 0 \\ 0 & 0 & I-(G_2)^*G_2 \end{pmatrix}$$

$$\times \begin{pmatrix} I & G_1 & G_1G_2 \\ 0 & I & G_2 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 & 0 \\ 0 & C^{1/2} & 0 \\ 0 & 0 & E^{1/2} \end{pmatrix},$$

shows that $\widetilde{K}$ is a positive semidefinite completion of K.

We are now ready to prove Theorem 6.3.

Proof of Theorem 6.3. Let $r : \{-N, \ldots, N\} \to \mathbb{H}^{s\times s}$ be positive definite. It follows from Lemma 6.8 with

$$\begin{aligned} A &= r(0); \\ B &= \begin{pmatrix} r(1) & \cdots & r(N) \end{pmatrix}; \\ C &= \begin{pmatrix} r(0) & \cdots & r(N) \\ \vdots & \ddots & \vdots \\ r(-N) & \cdots & r(0) \end{pmatrix}; \\ D &= \begin{pmatrix} r(N)^T & \cdots & r(1)^T \end{pmatrix}^T; \\ E &= A, \end{aligned}$$

that there exist contractions G_1 and G_2 so that if we put $r(N+1) = A^{1/2}G_1G_2E^{1/2}$ and $r(-N-1) = r(N+1)^*$, then

$$\begin{pmatrix} r(0) & \cdots & r(N+1) \\ \vdots & \ddots & \vdots \\ r(-N-1) & \cdots & r(0) \end{pmatrix} \succeq 0.$$

Continuing in this fashion, we can choose $r(N+2), r(N+3), \ldots$ so that

$$\begin{pmatrix} r(0) & \cdots & r(N+2) \\ \vdots & \ddots & \vdots \\ r(-N-2) & \cdots & r(0) \end{pmatrix} \succeq 0, \quad \begin{pmatrix} r(0) & \cdots & r(N+3) \\ \vdots & \ddots & \vdots \\ r(-N-3) & \cdots & r(0) \end{pmatrix} \succeq 0, \ldots.$$

Thus we have contructed $\tilde{r} : \mathbb{Z} \to \mathbb{H}^{s\times s}$ which is positive definite and satisfies

$$\tilde{r}(n) = r(n), \qquad n = -N, \ldots, N.$$

7. A theorem of Krein and Iohvidov in the quaternionic setting

Sasvári [19] attributes the following theorem to Krein and Iohvidov [14].

Theorem 7.1. *Let $a = (a(n))_{n\in\mathbb{Z}}$ be a bounded Hermitian complex-valued sequence on $\mathbb{Z}$. The sequence a has κ negative squares if and only if there exist measures μ_+ and μ_- on $[0,2\pi]$ and mutually distinct points $t_1,\dots,t_\kappa \in [0,2\pi]$ satisfying $\mu_+(t_j) = 0$ for $j = 1,\dots,\kappa$ and $\operatorname{supp}(\mu_-) = \{t_1,\dots,t_k\}$ and such that*

$$a(n) = \int_0^{2\pi} e^{int} d\mu_+(t) - \int_0^{2\pi} e^{int} d\mu_-(t), \qquad n \in \mathbb{Z}. \tag{7.2}$$

Proof. A proof for this result when $\mathbb{Z}$ is replaced by an arbitrary locally compact Abelian group can be found in [19]. $\square$

It will be our goal in this section to obtain a direct analogue of Theorem 7.1 when $(a(n))_{n\in\mathbb{Z}}$ is a bounded Hermitian $\mathbb{H}^{s\times s}$-valued sequence. To achieve this goal, we will first generalize Theorem 7.1 to the case when $(a(n))_{n\in\mathbb{Z}}$ is $\mathbb{C}^{s\times s}$-valued sequence and then the desired result will follow.

Definition 7.2. *Let $M = \sum_{q=1}^k P_q \delta_{t_q}$ be a $\mathbb{C}^{s\times s}$-valued measure on $[0,2\pi]$, where δ_t denotes the usual Dirac point measure at t. We let*

$$\operatorname{card}\operatorname{supp} M = \sum_{q=1}^k \operatorname{rank} P_q.$$

Theorem 7.3. *Let $A = (A(n))_{n\in\mathbb{Z}}$ be a bounded Hermitian $\mathbb{C}^{s\times s}$-valued sequence on $\mathbb{Z}$. A has κ negative squares if and only if there exist positive $\mathbb{C}^{s\times s}$-valued measures M_+ and M_- on $[0,2\pi]$ and mutually distinct points $t_1,\dots,t_k \in [0,2\pi]$ satisfying $M_+(t_j) = 0$ for $j = 1,\dots,k$, $\operatorname{supp}(M_-) = \{t_1,\dots,t_k\}$ and $\operatorname{card}\operatorname{supp} M_- = \kappa$ and such that*

$$A(n) = \int_0^{2\pi} e^{int} dM_+(t) - \int_0^{2\pi} e^{int} dM_-(t), \qquad n \in \mathbb{Z}. \tag{7.3}$$

Proof. If A has κ negative squares, then $a_v = (v^* A(n) v)_{n\in\mathbb{Z}}$ will be a complex-valued sequence with at most κ negative squares for any $v \in \mathbb{C}^s$. It follows then from Theorem 7.1 that there exist measures $\mu_+^{(v)}$ and $\mu_-^{(v)}$ on $[0,2\pi]$ and mutually distinct points $t_1^{(v)},\dots,t_{k_v}^{(v)} \in [0,2\pi]$ satisfying $\mu_+^{(v)}(t_j^{(v)}) = 0$ for $j = 1,\dots,k_v$ and $\operatorname{supp}(\mu_-^{(v)}) = \{t_1^{(v)},\dots,t_{k_v}^{(v)}\}$, where $k_v \le \kappa$, and such that

$$a_v(n) = \int_0^{2\pi} e^{int} d\mu_+^{(v)}(t) - \int_0^{2\pi} e^{int} d\mu_-^{(v)}(t), \qquad n \in \mathbb{Z}. \tag{7.4}$$

Let

$$4\mu_\pm^{(v,w)} = \mu_\pm^{(v+w)} - \mu_\pm^{(v-w)} + i\mu_\pm^{(v+iw)} - i\mu_\pm^{(v-iw)}, \qquad v,w \in \mathbb{C}^s.$$

Then there exist positive $\mathbb{C}^{s\times s}$-valued measures $M_\pm$ such that $\langle M_\pm v, w\rangle = \mu_\pm^{(v,w)}$ and

$$A(n) = \int_0^{2\pi} e^{int} dM_+(t) - \int_0^{2\pi} e^{int} dM_-(t), \qquad n \in \mathbb{Z}. \tag{7.5}$$

It follows from (7.5) together with the fact that A has κ negative squares that

$$\operatorname{card}\operatorname{supp} M_- = \kappa.$$

By construction, $M_+(t_j) = 0$ for all $t_j \in \operatorname{supp} M_+$.
Conversely, suppose (7.3) is in force. It is easy to check that A is a bounded Hermitian sequence with at most κ negative squares. The fact that A has exactly κ negative follows from the uniqueness of the measure $M = M_+ - M_-$ in (7.7) (see Theorem 5.1). $\square$

Definition 7.4. *Let $\nu = \nu_1 + \nu_2 j$ be a q-positive measure on $[0, 2\pi]$ with finite support. We let*

$$\operatorname{card}\operatorname{supp}\nu = (1/2)\operatorname{card}\operatorname{supp}\mu,$$

where μ is an in (5.2).

Theorem 7.5. *Let $a = (a(n))_{n\in\mathbb{Z}}$ be a bounded Hermitian $\mathbb{H}^{s\times s}$-valued sequence on $\mathbb{Z}$. The sequence a has κ negative squares if and only if there exist q-positive measures ν_+ and ν_- on $[0, 2\pi]$ and mutually distinct points $t_1, \ldots, t_k \in [0, 2\pi]$ satisfying satisfying $\nu_+(t_j) = 0$ for $j = 1, \ldots, k$, $\operatorname{supp}(d\nu_-) = \{t_1, \ldots, t_k\}$ and $\operatorname{card}\operatorname{supp}\nu_- = \kappa$ and such that*

$$a(n) = \int_0^{2\pi} e^{int} d\nu_+(t) - \int_0^{2\pi} e^{int} d\nu_-(t), \qquad n \in \mathbb{Z}. \tag{7.6}$$

Proof. If a is a bounded Hermitian $\mathbb{H}^{s\times s}$-valued sequence with κ negative squares, then $A = (A(n))_{n\in\mathbb{Z}}$, where $A(n) = \chi\, a(n)$, has 2κ negative squares (see Proposition 11.4 in [5]). Thus, Theorem 7.3 guarantees the existence of positive $\mathbb{C}^{2s\times 2s}$-valued measures M_+ and M_- on $[0, 2\pi]$ and mutually distinct points $t_1, \ldots, t_k \in [0, 2\pi]$ satisfying satisfying $M_+(t_j) = 0$ for $j = 1, \ldots, k$, $\operatorname{supp}(M_-) = \{t_1, \ldots, t_k\}$ and $\operatorname{card}\operatorname{supp} M_- = 2\kappa$ and such that

$$A(n) = \int_0^{2\pi} e^{int} dM_+(t) - \int_0^{2\pi} e^{int} dM_-(t), \qquad n \in \mathbb{Z}. \tag{7.7}$$

If we write

$$dM_\pm = \begin{pmatrix} dM_\pm^{(11)} & dM_\pm^{(12)} \\ (dM_\pm^{(12)})^* & dM_\pm^{(22)} \end{pmatrix} \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix} \to \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix}$$

and proceed as in the proof of Theorem 5.5 we get that

$$dM_+^{(11)}(t) - dM_-^{(11)}(t) = dM_+^{(22)}(2\pi - t) - dM_-^{(22)}(2\pi - t), \qquad t \in [0, 2\pi)$$

and

$$dM_+^{(12)}(t) - dM_-^{(12)}(t) = -(dM_+^{(12)}(2\pi - t)^T - dM_-^{(12)}(2\pi - t)^T), \qquad t \in [0, 2\pi).$$

Consequently, it follows from $dM_+(t_j) = 0$ for $j = 1, \ldots, k$ and $\operatorname{supp}(dM_-) = \{t_1, \ldots, t_k\}$ that

$$dM_-^{(11)}(t) = dM_-^{(22)}(2\pi - t), \qquad t \in [0, 2\pi)$$

and

$$dM_-^{(12)}(t) = -dM_-^{(12)}(2\pi - t)^T, \qquad t \in [0, 2\pi).$$

Thus,

$$dM_+^{(11)}(t) = dM_+^{(22)}(2\pi - t), \qquad t \in [0, 2\pi)$$

and

$$dM_+^{(12)}(t) = -dM_+^{(12)}(2\pi - t)^T, \qquad t \in [0, 2\pi).$$

Taking advantage of the above equalities we can obtain

$$a(n) = \int_0^{2\pi} e^{int} d\nu_+(t) - \int_0^{2\pi} e^{int} d\nu_-(t), \qquad n \in \mathbb{Z},$$

where $\nu_\pm(t) = M_\pm^{(11)}(t) + M_\pm^{(12)}(t)j$. It is readily checked that $\nu_\pm$ are q-positive measures. Moreover, a has κ negative squares since A has 2κ negative squares and $\nu_+(t_j) = 0$ for all $t_j \in \operatorname{supp}\nu_-$ and $\operatorname{card}\operatorname{supp}\nu_- = \kappa$.

Conversely, suppose that a is a $\mathbb{H}^{s\times s}$-valued sequence which obeys (7.6). Consequently, a is bounded. The fact that a is Hermitian follows from Remark 5.4. To see that a has κ negative squares, one can consider the $\mathbb{C}^{2s\times 2s}$-valued sequence $(A(n))_{n\in\mathbb{Z}}$, where $A(n) = \chi\, a(n)$, $n \in \mathbb{Z}$ and use the converse statement in Theorem 7.3 to see that A has 2κ negative squares. The fact that a has κ negative squares then follows by definition. □

8. Herglotz's integral representation theorem in the scalar case

In this section we present an analogue of Herglotz's theorem in the quaternionic scalar case. Even though this is a byproduct of the preceding discussion, it may be useful to have the result stated for scalar valued slice hyperholomorphic functions. We begin by proving an integral representation formula which holds on $\mathbb{B}_r = \{p \in \mathbb{H} \ : \ |p| < r\}$, namely on the quaternionic open ball centered at 0 and with radius $r > 0$. A similar formula which is based on a different representation of a slice hyperholomorphic function, less useful to determine the real part of a function, is discussed in [11]. Note also that, unlike what happens in the complex case, the real part of a slice hyperholomorphic function is not harmonic.

Lemma 8.1. *Let $f : \mathbb{B}_{1+\varepsilon} \to \mathbb{H}$ be a slice hyperholomorphic function, for some $\varepsilon > 0$. Let $I, J \in \mathbb{S}$ with J orthogonal to I and let $F, G : \mathbb{B}_{1+\varepsilon} \cap \mathbb{C}_I \to \mathbb{C}_I$ be holomorphic functions such that for any $z = x + Iy$ the restriction f_I can be written as $f_I(z) = F(z) + G(z)J$. Then, on $\mathbb{B}_1 \cap \mathbb{C}_I$ the following formula holds:*

$$f_I(z) = I[\mathrm{Im}F(0) + \mathrm{Im}G(0)J] + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{It}+z}{e^{It}-z}\,[\mathrm{Re}(F(e^{It})) + \mathrm{Re}(G(e^{It}))J]dt.$$

Moreover

$$\mathrm{Re}\Big(\frac{e^{It}+z}{e^{It}-z}\,[\mathrm{Re}(F(e^{It})) + \mathrm{Re}(G(e^{It}))J]\Big) = \frac{1-|z|^2}{|e^{It}-z|^2}\mathrm{Re}(F(e^{It})). \tag{8.1}$$

Proof. The proof is an easy consequence of the Splitting Lemma 2.4: for every fixed I, $J \in \mathbb{S}$ such that J is orthogonal to I, there are two holomorphic functions $F, G : \mathbb{B}_{1+\varepsilon} \cap \mathbb{C}_I \to \mathbb{C}_I$ such that for any $z = x + Iy$, it is $f_I(z) = F(z) + G(z)J$. It is immediate that these two holomorphic functions F, G satisfy (see p. 206 in [1])

$$F(z) = I\mathrm{Im}\,F(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{It}+z}{e^{It}-z}\,\mathrm{Re}(F(e^{It}))dt \quad z \in \mathbb{B}_1 \cap \mathbb{C}_I,$$

$$G(z) = I\mathrm{Im}\,G(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{It}+z}{e^{It}-z}\,\mathrm{Re}(G(e^{It}))dt \quad z \in \mathbb{B}_1 \cap \mathbb{C}_I,$$

and the first part of the statement follows. The second part is a consequence of the equality

$$\frac{e^{It}+z}{e^{It}-z} = \frac{1-|z|^2}{|e^{It}-z|^2} + 2I\frac{y\cos t - x\sin t}{|e^{It}-z|^2}$$

leading to

$$\mathrm{Re}\Big(\frac{e^{It}+z}{e^{It}-z}\,[\mathrm{Re}(F(e^{It})) + \mathrm{Re}(G(e^{It}))J]\Big) = \frac{1-|z|^2}{|e^{It}-z|^2}\mathrm{Re}(F(e^{It})).$$

Remark 8.2. Let $f : \Omega \to \mathbb{H}$ be a slice hyperholomorphic function and write

$$f(p) = f_0(x_0, \ldots, x_3) + f_1(x_0, \ldots, x_3)i + f_2(x_0, \ldots, x_3)j + f_3(x_0, \ldots, x_3)k,$$

with $f_\ell : \Omega \to \mathbb{R}$, $\ell = 0, \ldots, 3$, $p = x_0 + x_1 i + x_2 j + x_3 k$. It is easily seen that the restriction $f_i = f_{|\mathbb{C}_i}$ can be written as

$$\begin{aligned} f_i(x+iy) &= (f_0(x+iy) + f_1(x+iy)i) + (f_2(x+iy) + f_3(x+iy)i)j \\ &= F(x+iy) + G(x+iy)j \end{aligned}$$

and so

$$\mathrm{Re}(f_{|\mathbb{C}_i})(x+iy) = f_{0|\mathbb{C}_i}(x+iy)) = \mathrm{Re}(F)(x+iy).$$

More in general, consider $I, J \in \mathbb{S}$ with I orthogonal to J, and rewrite i, j, k in terms of the imaginary units $I, J, IJ = K$. Then

$$f(p) = f_0(x_0, \dots, x_3') + \tilde{f}_1(x_0, \dots, x_3')I + \tilde{f}_2(x_0, \dots, x_3')J + \tilde{f}_3(x_0, \dots, x_3')K,$$

where $p = x_0 + x_1'I + x_2'J + x_3'K$ and the x_ℓ' are linear combinations of the x_ℓ, $\ell = 1, 2, 3$. The restriction of f to the complex plane $\mathbb{C}_I$ is then $f_I(x + Iy) = \tilde{F}(x + Iy) + \tilde{G}(x + Iy)J$ and reasoning as above we have

$$\mathrm{Re}(f_I(x + Iy)) = f_{0|\mathbb{C}_I}(x_0, \dots, x_3') = \mathrm{Re}(\tilde{F}(x + Iy)).$$

We conclude that the real part of the restriction f_I of f to a complex plane $\mathbb{C}_I$ is the restriction of f_0 to the given complex plane. Thus if $\mathrm{Re}(f)$ is positive also the real part of the restriction f_I to any complex plane is positive.

Theorem 8.3 (Herglotz's theorem on a slice)**.** *Let $f : \mathbb{B}_1 \to \mathbb{H}$ be a slice hyperholomorphic function with $\mathrm{Re}(f(p)) \geq 0$ in $\mathbb{B}_1$. Fix $I, J \in \mathbb{S}$ with J be orthogonal to I. Let f_I be the restriction of f to the complex plane $\mathbb{C}_I$ and let $F, G : \mathbb{B}_1 \cap \mathbb{C}_I \to \mathbb{C}_I$ be holomorphic functions such that for any $z = x + Iy$, it is $f_I(z) = F(z) + G(z)J$. Then f_I can be written in $\mathbb{B}_1 \cap \mathbb{C}_I$ as*

$$f_I(z) = I[\mathrm{Im}F(0) + \mathrm{Im}G(0)J] + \int_0^{2\pi} \frac{e^{It} + z}{e^{It} - z}\, d\mu_J(t), \tag{8.2}$$

where $\mu_J(t) = \mu_1 + \mu_2 J$ is a finite variation complex measure on $\mathbb{C}_J$ with μ_1 positive and μ_2 real and of finite variation on $[0, 2\pi]$.

Proof. The proof follows [1, p. 207]. First, we note that by Remark 8.2, $\mathrm{Re}(f(p)) \geq 0$ in $\mathbb{B}_1$ implies that $\mathrm{Re}(f_I(z)) \geq 0$ for $z \in \mathbb{B}_1 \cap \mathbb{C}_I$. Let ρ be a real number such that $0 < \rho < 1$. Then $f_I(\rho z)$ is slice hyperholomorphic in the disc $|z| < 1/\rho$ and so by Lemma 8.1 the restriction $f_I(z)$ may be written in $|z| < 1$ as

$$f_I(\rho z) = I[\mathrm{Im}F(0) + \mathrm{Im}G(0)J] + \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{It} + z}{e^{It} - z} [\mathrm{Re}(F(\rho e^{It})) + \mathrm{Re}(G(\rho e^{It}))J] dt.$$

where

$$d\mu_J(t, \rho) = \frac{1}{2\pi} [\mathrm{Re}(F(\rho e^{It})) + \mathrm{Re}(G(\rho e^{It}))J] dt$$

has real positive part, since it is immediate that $\mathrm{Re}(f_I(\rho e^{It})) = \mathrm{Re}(F(\rho e^{It}))$ and

$$\int_0^{2\pi} d\mu_J(t, \rho) = [\mathrm{Re}F(0) + \mathrm{Re}G(0)J].$$

Let us set

$$\Lambda_I(z, t) := \frac{e^{It} + z}{e^{It} - z}$$

and consider

$$\int_0^{2\pi} \Lambda_I(z, t) d\mu_J(t; \rho).$$

Let $\{\rho_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers with $0 < \rho_n < 1$ such that $\rho_n \to 1$ when n goes to infinity. To conclude the proof we need Helly's theorem in the complex case. This result assures that the family of finite variation real-valued $d\nu(t; \rho_n)$ contains a convergent subsequence which tends to $d\nu(t)$ which is of finite variation, in the sense that

$$\lim_{n \to \infty} \int_0^{2\pi} \lambda(w, t) d\nu(t, \rho_n) = \int_0^{2\pi} \lambda(w, t) d\nu(t)$$

for every continuous complex-valued function $\lambda(w, t)$. In the slice hyperholomorphic setting the integrand is the product of the continuous $\mathbb{C}_I$-valued function $\Lambda_I(z, t) = \Lambda_1(z, t) + I\Lambda_2(z, t)$ where Λ_1 and Λ_2 are real valued, and of the $\mathbb{C}_J$-valued $d\mu_J(t, \rho_n) = d\mu_1(t, \rho_n) +$

$d\mu_2(t,\rho_n)J$ (since both $d\mu_1(t,\rho_n)$ and $d\mu_2(t,\rho_n)$ are real-valued).
Then $\Lambda_I(z,t)d\mu_J(t)$ can be split in components to which we apply Helly's theorem. The positivity of $d\mu_1$ follows from the positivity of $d\mu_1(t,\rho_n)$, and this completes the proof.

Corollary 8.4. *Let f be slice hyperholomorphic function on $\mathbb{B}_1$ such that $f(0)=1$. Suppose that f has real positive part on $\mathbb{B}_1$. Then its restriction f_I can be represented as*

$$f_I(z) = \int_0^{2\pi} \frac{e^{It}+z}{e^{It}-z}\, d\mu_J(t),$$

where $\mu_J(t)=\mu_1(t)+\mu_2(t)J$ is a finite variation complex measure on $\mathbb{C}_J$ with $\mu_1(t)$ positive for $t\in[0,2\pi]$. Moreover, the power series expansion of f

$$f(p) = 1+\sum_{n=1}^{\infty} p^n a_n$$

is such that $|a_n|\le k$, for some $k\in\mathbb{R}$, for every $n\in\mathbb{N}$.

Proof. The first part of the corollary immediately follows from Theorem 8.3. Then observe that

$$\frac{e^{It}+z}{e^{It}-z} = 1+2\sum_{n=1}^{\infty} z^n e^{-Int}$$

and so the coefficients a_n in the power series expansion are given by

$$a_n = 2\int_0^{2\pi} e^{-Int} d\mu_J(t). \tag{8.3}$$

Moreover

$$|a_n| \le 2\int_0^{2\pi} |d\mu_J(t)| \le k,$$

for some $k\in\mathbb{R}$ since $d\mu_J(t)$ is of finite variation and so it is bounded.

Remark 8.5. Formula (8.3) expresses a_n in integral form. However, there is an infinite number of ways of writing a_n with a similar expression, depending on the choices of I and J made to write (8.2). An important difference with the result in Section 5 is that the measure $d\mu$ in formula (5.3) is quaternionic valued, while in this case it is complex valued (with values in $\mathbb{C}_J$). One may wonder is there are choices of I,J for which formula (8.3) would allow to define a_{-n} via (8.3) and then obtain $a_{-n}=\bar{a}_n$. Since

$$a_{-n} = 2\int_0^{2\pi} e^{Int} d\mu_J(t), \qquad \bar{a}_n = 2\int_0^{2\pi} \overline{d\mu_J(t)} e^{Int},$$

and

$$\begin{aligned}
\bar{a}_n &= 2\int_0^{2\pi} (d\mu_1(t) - d\mu_2(t)J)e^{Int} \\
&= 2\int_0^{2\pi} e^{Int} d\mu_1(t) - e^{-Int} d\mu_2(t)J \\
&= 2\int_0^{2\pi} e^{Int}(d\mu_1(t) - d\mu_2(2\pi-t)J),
\end{aligned}$$

the condition $a_{-n}=\bar{a}_n$ translates into $\mathrm{Re}(G)(e^{It}) = -\mathrm{Re}(G)(e^{I(2\pi-t)})$. If one writes the power series expansion of f_I in the form

$$f_I(x+Iy) = \sum_{n=0}^{\infty} (x+Iy)^n a_n = \sum_{n=0}^{\infty} (x+Iy)^n (a_{0n}+Ia_{1n}+(a_{2n}+Ia_{3n})J)$$

then it follows that

$$F(x+Iy)=\sum_{n=0}^{\infty}(x+Iy)^n(a_{0n}+Ia_{1n})\quad\text{and}\quad G(x+Iy)=\sum_{n=0}^{\infty}(x+Iy)^n(a_{2n}+Ia_{3n}).$$

Then

$$\mathrm{Re}(G)(x+Iy)=\sum_{n=0}^{\infty}u_n(x,y)a_{2n}-v_n(x,y)a_{3n}$$

where

$$(x+Iy)^n=u_n(x,y)+Iv_n(x,y)$$

and

$$u_n(x,y)=\sum_{k=0,\,k\,even}^{n}\binom{n}{k}(-1)^{k/2}x^{n-k}y^k,$$

$$v_n(x,y)=\sum_{k=1,\,k\,odd}^{n}\binom{n}{k}(-1)^{(k-1)/2}x^{n-k}y^k.$$

It is immediate that u_n and v_n are even and odd in the variable y, respectively, thus $\mathrm{Re}(G)$ is odd in the variable y if and only if $a_{2n}=0$ for all $n\in\mathbb{N}$. In general, given a slice hyperholomorphic function f on $\mathbb{B}_1$ there is no change of basis for which one can have all the coefficients $a_{2n}=0$ for all $n\in\mathbb{N}$. Thus, formula (8.3) does not allow to define a_{-n} in order to obtain the desired equality $a_{-n}=\bar{a}_n$. The formula is however one of the several possibilities to assign the coefficients of f in integral form.

We conclude this section with a global integral representation.

Theorem 8.6. *Let $f:\mathbb{B}_1\to\mathbb{H}$ be a slice hyperholomorphic function. Let f_I be the restriction of f to the complex plane $\mathbb{C}_I$ and let $F,G:\mathbb{B}_1\cap\mathbb{C}_I\to\mathbb{C}_I$ be holomorphic functions $f_I(z)=F(z)+G(z)J$, $z=x+Iy$. Then*

$$f(q)=I[\mathrm{Im}F(0)+\mathrm{Im}G(0)J]+\frac{1}{2\pi}\int_0^{2\pi}K(q,e^{It})\,d\mu_J(t),$$

where $\mu_J(t)$ is a finite variation complex measure on $\mathbb{C}_J$ for $t\in[0,2\pi]$ and

$$\begin{aligned}K(q,e^{It})&=\frac{1}{2}\left(\frac{e^{It}+z}{e^{It}-z}+\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}\right)+\frac{1}{2}I_qI\left(\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}-\frac{e^{It}+z}{e^{It}-z}\right)\\&=(1+q^2-2q\mathrm{Re}(e^{It}))^{-1}(1+2q\mathrm{Im}(e^{It})-q^2).\end{aligned}$$

Proof. From Theorem 8.3 the restriction of f to the complex plane $\mathbb{C}_I$ is

$$f_I(z)=I[\mathrm{Im}F(0)+\mathrm{Im}G(0)J]+\int_0^{2\pi}\frac{e^{It}+z}{e^{It}-z}\,d\mu_J(t),$$

where $\mu_J(t)$ is a finite variation complex measure on $\mathbb{C}_J$. Consider

$$f_I(z)+f_I(\overline{z})=2[\mathrm{Im}F(0)+\mathrm{Im}G(0)J]\left(\frac{e^{It}+z}{e^{It}-z}+\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}\right)\,d\mu_J(t)$$

and

$$f_I(\overline{z})-f_I(z)=\int_0^{2\pi}\left(\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}-\frac{e^{It}+z}{e^{It}-z}\right)\,d\mu_J(t);$$

by applying the Representation Formula we obtain the kernel

$$K(q,e^{It})=\frac{1}{2}\left(\frac{e^{It}+z}{e^{It}-z}+\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}\right)+\frac{1}{2}I_qI\left(\frac{e^{It}+\overline{z}}{e^{It}-\overline{z}}-\frac{e^{It}+z}{e^{It}-z}\right).$$

written in the first form. Now we write it in an equivalent way observing that the slice hyperholomorphic extension of the function

$$K(z, e^{It}) = \frac{e^{It} + z}{e^{It} - z}, \quad z = x + Iy$$

is (for the $\star$-inverse see Ch. 4 in [10]) and the

$$K(q, e^{It}) = (e^{It} - q)^{-*} * (e^{It} + q)$$

so that

$$K(q, e^{It}) = (1 + q^2 - 2q\mathrm{Re}(e^{It}))^{-1}(1 + 2q\mathrm{Im}(e^{It}) - q^2),$$

and the statement follows.

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