Stability and Stabilization of Discrete-Time Semi-Markov Jump Linear Systems via Semi-Markov Kernel Approach

Lixian Zhang, Yusong Leng, and Patrizio Colaneri, *Fellow, IEEE*

Abstract—This technical note is concerned with the problems of stability and stabilization for a class of discrete-time semi-Markov jump linear systems (S-MJLSs). The discrete-time semi-Markov kernel (SMK) is introduced, where the probability density function of *sojourn-time* is dependent on both current and next system mode. As a consequence, different types of distributions and/or different parameters in a same type of distribution of sojourn-time, depending on the target mode towards which the system jumps, can coexist in each mode of a SMK. The underlying S-MJLSs are therefore more general than those considered in existing studies. A new stability concept generalizing the traditional mean-square stability is proposed such that numerically testable criteria on the basis of SMK are obtained. Numerical examples are presented to illustrate the validity and advantage of the developed theoretical results.

I. Introduction

The past decades have seen a great advance in theories and applications of Markov jump linear systems (MJLSs). The systems can effectively model dynamical processes involved with stochastic switching (generally autonomous) subject to a Markov chain. Typical examples include fault-tolerant control systems where abrupt faults occur randomly and networked control systems where network-induced communication imperfections vary in a stochastic way, see for example [1] and [2]. The primary concern of MJLSs is the stability analysis that is relatively challenging due to the hybrid nature of the system. Up to date, quite a few important stability notions, stochastic stability, mean-square stability (MSS), almost sure stability and so on, are persistently utilized and many results are rather fundamental in the field of stochastic switching systems, see for example, [3] and [4]. Other issues including control, estimation, model reduction of MJLSs and the underlying systems with various complex dynamics, time-delays, uncertainties, positiveness, etc., have also been intensively studied, see for example [5]–[7] and the references therein.

However, as pointed out in [8]–[12], although Markov processes or Markov chain do have the ability in describing the mode switching in

many practical applications, they cannot cover all the scenarios yet. A key restriction in the underlying MJLSs is that the sojourn-time (the interval between two consecutive jumps) of each subsystem is subject to exponential distribution (geometric distribution in discrete-time domain, respectively). To relax the restriction, the concepts of nonhomogeneous Markov chain and semi-Markov chain, where the transition probabilities (TPs) are time-varying and memory, respectively, have been introduced in the control community, and studies on the corresponding systems have been gradually launched. As for the former case that is relatively tractable, the systems with dwell-time[1] switching TPs [10], piecewise homogeneous TPs [11], and polytopic time-varying TPs [12], etc., have been proposed within the past few years, and various methodologies have been explored for different situations. Such proposals have provided a basic foundation for further studies on the systems with nonhomogeneous TPs.

On the other hand, the developed theories on semi-Markov jump linear systems (S-MJLSs) are far away from maturity yet, although the systems have been investigated since 1960s, see for example, [8], [9], [13] and [14]. The inherent difficulties mainly lie in how the probability density function (PDF) information of the sojourn-time can be completely used in deriving criteria of stability analysis and control synthesis, and further, how the obtained criteria can be numerically tested. With the assumption that the PDF is dependent on the current system mode, the stabilization problem for a class of continuous-time S-MJLSs has been addressed in [8] by solving a set of coupled algebraic Riccati equations. Further improvements achieved in [9] offer a framework under which the control problems can be solved by the techniques of linear matrix inequalities (more conveniently to be checked), with *a priori* information of the upper and lower bounds of a PDF. However, it is noted that almost all the existing results presume that each mode possesses a single distribution of sojourn-time with certain parameters in a semi-Markov chain. It is very likely that the types of distributions and/or the parameters in a same type of distribution of sojourn-time can be different for each mode, depending on the target mode to which the system jumps from the current mode.

Motivated by the above observations, in this technical note, we aim at addressing the problems of stability and stabilization for a class of discrete-time S-MJLSs. There are two main contributions. *First*, the concept of discrete-time semi-Markov kernel (SMK) is introduced, and the PDF of sojourn-time in the SMK is allowed to depend on both the current and next system mode. As a result, different parameters in a same type of distribution or even different types of distributions of sojourn-time can be simultaneously considered in each mode of a SMK depending on the target mode, and the underlying S-MJLSs are therefore more general than those studied previously. *Second*, a new stability concept called σ-error mean square stability (σ-MSS) is proposed, where σ is used to characterize the degree of "approximation error" of σ-MSS to MSS (without the error the underlying system holds MSS). The stability concept generalizes the traditional MSS,

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[1]A term commonly used in the context of nondeterministic switched systems, where dwell-time is generally no greater than the running time (or sojourn-time called in this technical note) of an activated subsystem.
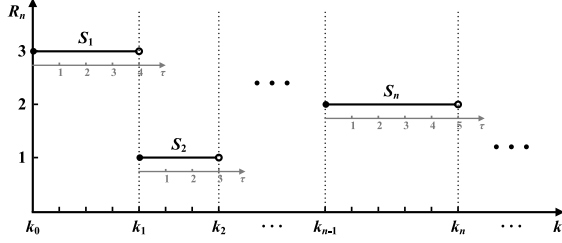
Fig. 1. Illustration of stochastic processes R_n, k_n and S_n ($M = 3$).

based on which the numerically testable stability and stabilization criteria that explicitly contains the PDF information of sojourn-time can be obtained.

Notations: In this technical note, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $\mathbb{R}_+$ and $\mathbb{Z}_+$ denote the set of non-negative real numbers and set of non-negative integers, respectively; $\mathbb{R}_{[s_1, s_2]}$, $\mathbb{Z}_{\geq s_1}$ and $\mathbb{Z}_{[s_1, s_2]}$ denote the sets $\{k \in \mathbb{R}_+ | s_1 \leq k \leq s_2\}$, $\{k \in \mathbb{Z}_+ | k \geq s_1\}$ and $\{k \in \mathbb{Z}_+ | s_1 \leq k \leq s_2\}$, respectively. For notation $(\Psi, \mathcal{F}, \mathrm{Pr})$, Ψ represents the sample space, $\mathcal{F}$ is the σ-algebra of subsets of the sample space and Pr is the probability measure on $\mathcal{F}$. $\mathcal{C}^1$ denotes the space of continuously differentiable functions, and a function $\kappa : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\kappa(0) = 0$. In addition, $diag\{\cdots\}$ stands for a block-diagonal matrix and $diag_{(n)}\{X\}$ a $n \times n$ block-diagonal matrix where all diagonal entries are X. Symbol $*$ is used as an ellipsis for the terms that are introduced by symmetry.

II. PRELIMINARIES AND PROBLEM FORMULATION

Fix the complete probability space $(\Psi, \mathcal{F}, \mathrm{Pr})$ and consider the following discrete-time stochastic switching systems:

$$x(k + 1) = A(r_k)x(k) + B(r_k)u(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$ are the system state and control input, respectively. $\{r_k\}_{k \in \mathbb{Z}_+}$ is a stochastic process, considered to be a semi-Markov chain, which takes values in a finite set $\mathcal{I} \triangleq \{1, 2, \ldots, M\}$, and governs the switching among M system modes. For $r_k = i \in \mathcal{I}$, the pair of matrices of the ith system mode is denoted by (A_i, B_i), which are real known matrices with appropriate dimensions.

To introduce the semi-Markov chain formally, we shall recall two concepts on Markov renewal chain and semi-Markov kernel, for which three following stochastic processes are first needed:

i) The stochastic process $\{R_n\}_{n \in \mathbb{Z}_+}$ taking values in $\mathcal{I}$, where R_n is the index of system mode at the nth jump;

ii) The stochastic process $\{k_n\}_{n \in \mathbb{Z}_+}$ taking values in $\mathbb{Z}_+$, where k_n denotes the time at the nth jump. It is noted that $k_0 = 0$ and k_n increases monotonically with n;

iii) The stochastic process $\{S_n\}_{n \in \mathbb{Z}_+}$ taking values in $\mathbb{Z}_+$, where $S_n = k_n - k_{n-1}$, $\forall n \in \mathbb{Z}_{\geq 1}$ denotes the sojourn-time of mode R_{n-1} between the $(n-1)$th jump and nth jump, and $S_0 = 0$.

These stochastic processes are illustrated in Fig. 1, and more details can be found in [15] and the references therein.

Definition 1 [15]: The stochastic process $\{(R_n, k_n)\}_{n \in \mathbb{Z}_+}$ is said to be a discrete-time homogeneous Markov renewal chain (MRC) if for any $j \in \mathcal{I}$, $\tau \in \mathbb{Z}_+$ and $n \in \mathbb{Z}_+$, $\mathrm{Pr}(R_{n+1} = j, S_{n+1} = \tau | R_0, \ldots, R_n = i; k_0, \ldots, k_n) = Pr(R_{n+1} = j, S_{n+1} = \tau | R_n = i) = \mathrm{Pr}(R_1 = j, S_1 = \tau | R_0 = i)$.

Then, let $\pi_{ij}(\tau) \triangleq \mathrm{Pr}(R_{n+1} = j, S_{n+1} = \tau | R_n = i), \forall i, j \in \mathcal{I}$, $\forall \tau \in \mathbb{Z}_+$, the matrix $\Pi(\tau) = [\pi_{ij}(\tau)]_{i,j \in \mathcal{I}}$ is called discrete-time *semi-*

Markov kernel (SMK), where $\pi_{ij}(\tau) \in \mathbb{R}_{[0,1]}$ and $\sum_{\tau=0}^{\infty} \sum_{j \in \mathcal{I}} \pi_{ij}(\tau) = 1$ with $\pi_{ij}(0) = 0$. In addition, from [15], $\{R_n\}_{n \in \mathbb{Z}_+}$ is called the embedded Markov chain (EMC) of MRC $\{(R_n, k_n)\}_{n \in \mathbb{Z}_+}$, and the TPs matrix $\Theta = [\theta_{ij}]_{i,j \in \mathcal{I}}$ of $\{R_n\}_{n \in \mathbb{Z}_+}$ is defined by $\theta_{ij} \triangleq \mathrm{Pr}(R_{n+1} = j | R_n = i)$, $\forall i, j \in \mathcal{I}$ with $\theta_{ii} = 0$.

With the above concepts, the definition of semi-Markov chain is given as below.

Definition 2 [15]: Consider a MRC $\{(R_n, k_n)\}_{n \in \mathbb{Z}_+}$. The chain $\{r_k\}_{k \in \mathbb{Z}_+}$ is said to be a semi-Markov chain (SMC) associated with MRC $\{(R_n, k_n)\}_{n \in \mathbb{Z}_+}$, if $r_k = R_{N(k)}$, $\forall k \in \mathbb{Z}_+$, where $N(k) \triangleq \max\{n \in \mathbb{Z}_+ | k_n \leq k\}$.

It is worth noting that the difference between EMC and SMC lies in that the stochastic variable varies with jump instant k_n in the former, whereas with the sampling instant k in the latter. From Definition 2, it is straightforward that the evolution of SMC is generated by the SMK $\Pi(\tau)$ that is dependent on sojourn-time τ. Thus, the knowledge on the probability density function (PDF) of sojourn-time is required to characterize a SMC. In this technical note, the PDF depending on both the current and next system mode is considered and defined as $\omega_{ij}(\tau) \triangleq \mathrm{Pr}(S_{n+1} = \tau | R_{n+1} = j, R_n = i)$, $\forall i, j \in \mathcal{I}, \forall \tau \in \mathbb{Z}_+$. As a consequence

$$\pi_{ij}(\tau) = \frac{\mathrm{Pr}(R_{n+1} = j, R_n = i)}{\mathrm{Pr}(R_n = i)} \frac{\mathrm{Pr}(R_{n+1} = j, S_{n+1} = \tau, R_n = i)}{\mathrm{Pr}(R_{n+1} = j, R_n = i)}$$
$$= \theta_{ij} \omega_{ij}(\tau). \tag{2}$$

Remark 1: Letting the PDF that only depends on the current system mode be denoted as $f_i(\tau) = \mathrm{Pr}(S_{n+1} = \tau | R_n = i)$, $\forall i \in \mathcal{I}$, $\forall \tau \in \mathbb{Z}_+$, it yields that $f_i(\tau) = \sum_{j \in \mathcal{I}} \pi_{ij}(\tau) = \sum_{j \in \mathcal{I}} \theta_{ij} \omega_{ij}(\tau)$. It can be observed that different $\omega_{ij}(\tau)$ may lead to a same $f_i(\tau)$. In almost all the previous studies on S-MJLSs, $f_i(\tau)$ is used to obtain the sojourn-time-dependent transition probability at mode i, say, denoted as $\lambda_i(\tau)$, and further the TPs $\lambda_{ij}(\tau)$ by multiplying θ_{ij}. Therefore the PDF $f_i(\tau)$ can be only one type, but $\omega_{ij}(\tau)$ that is used to form SMK in this technical note can be r_{k+1} dependent, having different types of distributions or different parameters in a same type of distribution for any $i \in \mathcal{I}$. Therefore, the PDF considered in the technical note is more specific and capable of describing the corresponding SMC accurately rather than $f_i(\tau)$.

Throughout the technical note, the cumulative density function (CDF) of sojourn-time for the ith system mode, $\forall i \in \mathcal{I}$, is denoted as $F_i(\tau) = \mathrm{Pr}(S_{n+1} \leq \tau | R_n = i) = \sum_{l=0}^{\tau} \sum_{j \in \mathcal{I}} \pi_{ij}(l)$, and it is assumed that $\omega_{ij}(0) = F_i(0) = 0$ without loss of generality. Now, to present the purposes of this technical note more precisely, the following stability definitions are required.

Definition 3: Consider a discrete-time stochastic switching nonlinear system $x_{k+1} = f(x_k, r_k)$, where r_k is a certain stochastic process governing the system switching and taking values in $\mathcal{I}$. The system is said to be mean-square stable if, for any initial conditions $x_0 \in \mathbb{R}^n, r_0 \in \mathcal{I}$, the following holds:

$$\lim_{k \to \infty} \mathbb{E}\left[\|x(k)\|^2\right]|_{x_0, r_0} = 0. \tag{3}$$

Remark 2: In Definition 3, when r_k is a Markov chain and the underlying system is linear, we can typically find the corresponding version of Definition 3 in the literature of MJLSs, see for example, [10] and [16]. Nonetheless, one drawback of Definition 3 is that it is established allowing for the random sojourn-time to be any length (even infinity), regardless of the fact that the practical sojourn-time is generally finite. Therefore, letting $T_{\max}^i$ denote the upper bound of sojourn-time for the ith mode of system (1), we generalize the MSS in this technical note to the following concept.

Definition 4: System (1) with $u(k) \equiv 0$, is said to be σ-error mean-square stable if, for any initial conditions $x_0 \in \mathbb{R}^n$, $r_0 \in \mathcal{I}$ and the upper bound of sojourn-time $T_{\max}^i \in \mathbb{Z}_{\geq 1}$, $\forall i \in \mathcal{I}$, the following holds:

$$\lim_{k \to \infty} \mathbb{E}\left[\|x(k)\|^2\right]\big|_{x_0, r_0, S_{n+1} \leq T_{\max}^i | R_n = i} = 0. \tag{4}$$

Further, σ is defined as

$$\sigma \triangleq \sum_{i \in \mathcal{I}} \left| \ln\left(F_i\left(T_{\max}^i\right)\right)\right|. \tag{5}$$

Remark 3: Note that in Definition 4, σ varies with $T_{\max}^i$ (σ decreases if all $T_{\max}^i$ are increased). In view of this, σ is actually capable to characterize the degree of "approximation error" of σ-MSS to MSS. Particularly, when $T_{\max}^i \to \infty$, $\forall i \in \mathcal{I}$, which implies that $F_i(T_{\max}^i) \to 1$, then, $\sigma \to 0$ and accordingly the σ-MSS approximates to MSS without any error.

Then, the objectives in this technical note are to derive the σ-MSS criterion for system (1), and to design a state-feedback stabilizing controller guaranteeing the σ-MSS of the resulting closed-loop system. The mode-dependent controller is considered here with the form

$$u(k) = K_i x(k), \quad \forall r_k = i \in \mathcal{I} \tag{6}$$

where K_i is the controller gain to be determined.

III. MAIN RESULTS

In this section, the numerically testable stability and stabilization criteria for S-MJLSs will be developed. A result on MSS of stochastic switching nonlinear systems is first given as below for later use.

Lemma 1: Consider a discrete-time stochastic switching nonlinear system $x_{k+1} = f(x_k, r_k)$, where x_k and r_k denote the system state and mode index, respectively. The switching instants are denoted by $k_0, k_1, \ldots, k_s, \ldots$ with $k_0 = 0$. The system is mean-square stable, if there exist a set of $\mathcal{C}^1$ functions $V(x_k, r_k) : \mathbb{R}^n \to \mathbb{R}$ and three class $\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$, such that for any initial conditions $x_0 \in \mathbb{R}^n$, $r_0 \in \mathcal{I}$ and a given finite $h_i > 0$, $\forall r_{k_s} = i \in \mathcal{I}$

$$\alpha_1\left(\|x_k\|\right) \leq V\left(x_k, r_{k_s}\right) \leq \alpha_2\left(\|x_k\|\right) \tag{7}$$

$$V\left(x_k, r_{k_s}\right) \leq h_i V\left(x_{k_s}, r_{k_s}\right), \quad k \in \mathbb{Z}_{(k_s, k_{s+1}]} \tag{8}$$

$$\mathbb{E}\left[V\left(x_{k_{s+1}}, r_{k_{s+1}}\right)\right]\big|_{x_0, r_0} - V(x_{k_s}, r_{k_s}) \leq -\alpha_3(\|x_{k_s}\|). \tag{9}$$

Proof: It follows from the proof of [16, Theorem 1] that (9) ensures $\sum_{s=0}^{\infty} \mathbb{E}[\alpha_3(\|x_{k_s}\|)]\big|_{x_0, r_0} \leq \mathbb{E}[V(x_0, r_0)] < \infty$ which implies $\lim_{s \to \infty} \mathbb{E}[\alpha_3(\|x_{k_s}\|)]\big|_{x_0, r_0} = 0$. Since $\alpha_3 \in \mathcal{K}_\infty$, $\lim_{s \to \infty} \mathbb{E}[\|x_{k_s}\|^2]\big|_{x_0, r_0} = 0$ holds. On the other hand, taking mathematical expectations at both sides of (7), (8), we have $\mathbb{E}[\alpha_1(\|x_k\|)]\big|_{x_0, r_0} \leq h_i \mathbb{E}[V(x_{k_s}, r_{k_s})]\big|_{x_0, r_0} \leq h_i \mathbb{E}[\alpha_2(\|x_{k_s}\|)]\big|_{x_0, r_0}$. As $s \to \infty$, $k \to \infty$, therefore $\lim_{k \to \infty} \mathbb{E}[\alpha_1(\|x_k\|)]\big|_{x_0, r_0} \leq h_i \lim_{s \to \infty} \mathbb{E}[\alpha_2(\|x_{k_s}\|)]\big|_{x_0, r_0}$, which implies (3).

Then, the following theorem gives a criterion of σ-MSS for unforced S-MJLSs.

Theorem 1: Consider S-MJLS (1) with $u(k) \equiv 0$ and a given finite constant $h_i > 0$. If, $\forall i \in \mathcal{I}$, there exist $T_{\max}^i \in \mathbb{Z}_{\geq 1}$ and matrices $P_i \succ 0$ such that $\forall t \in \mathbb{Z}_{[1, T_{\max}^i]}$

$$A_i'^t P_i A_i^t - h_i P_i \prec 0 \tag{10}$$

$$\sum_{\tau=1}^{T_{\max}^i} A_i'^\tau \mathcal{P}_i(\tau) A_i^\tau - P_i \prec 0 \tag{11}$$

where $\mathcal{P}_i(\tau) \triangleq \sum_{j \in \mathcal{I}} \pi_{ij}(\tau) P_j / \eta_i$ with $\eta_i \triangleq \sum_{\tau=1}^{T_{\max}^i} \sum_{j \in \mathcal{I}} \pi_{ij}(\tau)$, then the system is σ-error mean-square stable.

Proof: Construct the Lyapunov function as $V_i(x_k) \triangleq V(x_k, R_n)|_{R_n = i} = x_k' P_i x_k$, $\forall i \in \mathcal{I}$, where P_i satisfies (10) and (11). First, it is straightforward that

$$\inf_{i \in \mathcal{I}} \{\lambda_{\min}(P_i)\} \|x_k\|^2 \leq V_i(x_k) \leq \sup_{i \in \mathcal{I}} \{\lambda_{\max}(P_i)\} \|x_k\|^2 \tag{12}$$
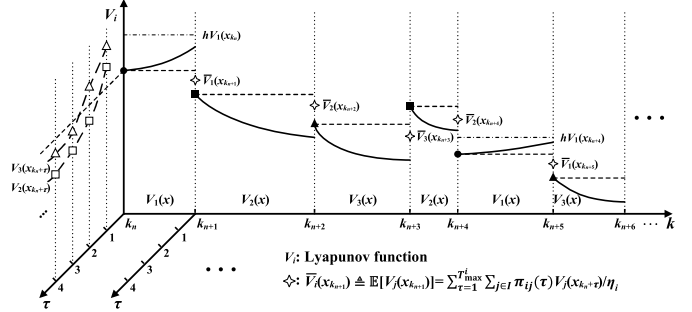

Fig. 2. Illustration of Theorem 1 for $M = 3$. In (k, V_i)-coordinate, solid line shows the real evolution of Lyapunov function, and in (τ, V_i)-coordinate, dashed line illustrates a possible evolution of the Lyapunov function starting from a fixed mode at a certain jumping instant; *circles* initiate V_1, *squares* initiate V_2 and *triangles* initiate V_3.

where $\lambda_{\min}(P_i)$ (respectively, $\lambda_{\max}(P_i)$) denotes the minimal (respectively, maximal) eigenvalue of P_i. In addition, for the case $R_n = i$, the following is ensured by (10) $\forall t \in \mathbb{Z}_{[1, T_{\max}^i]}$

$$V_i(x_{k_n+t}) - h_i V_i(x_{k_n}) = x_{k_n}' \left(A_i'^t P_i A_i^t - h_i P_i\right) x_{k_n} < 0. \tag{13}$$

On the other hand, for $R_n = i$, $R_{n+1} = j$, letting the sojourn-time $k_{n+1} - k_n$ be denoted by τ, it follows from (11) that

$$\mathbb{E}\left[V_j\left(x_{k_{n+1}}\right)\right]\big|_{x_0, r_0, S_{n+1} \leq T_{\max}^i | R_n = i} - V_i\left(x_{k_n}\right)$$

$$= x_{k_n}' \left[\sum_{\tau=1}^{T_{\max}^i} \sum_{j \in \mathcal{I}} \pi_{ij}(\tau) A_i'^\tau P_j A_i^\tau / \eta_i - P_i\right] x_{k_n}$$

$$\leq -\lambda_{\min}\left(-\sum_{\tau=1}^{T_{\max}^i} A_i'^\tau \mathcal{P}_i(\tau) A_i^\tau + P_i\right) \|x_{k_n}\|^2$$

$$\leq -\beta \|x_{k_n}\|^2 \tag{14}$$

where η_i is defined in (11) and $\beta \triangleq \inf_{i \in \mathcal{I}} \{\lambda_{\min}(-\sum_{\tau=1}^{T_{\max}^i} A_i'^\tau \mathcal{P}_i(\tau) A_i^\tau + P_i)\}$. Then, by (12), (13), (14) and Lemma 1, it follows that S-MJLS (1) is mean-square stable provided that the upper bound of sojourn-time is $T_{\max}^i$, i.e., (4) holds, thus the σ-MSS of the system is guaranteed.

Note that the Lyapunov function of each mode is not necessarily monotonically decreasing (h_i can be greater than 1), as shown in (10). An illustration of Theorem 1 for $M = 3$ is given in Fig. 2, where the system at k_n is supposed to run in mode 1. In (τ, V_i)-coordinate, $V_2(x_{k_n+\tau})$ and/or $V_3(x_{k_n+\tau})$ (the corresponding hollow square or triangle) can be greater than $V_1(x_{k_n})$ (the solid circle in (k, V_i)-coordinate). However, as long as the expectation $\bar{V}_i(x_{k_{n+1}})$ is lower than $V_i(x_{k_n})$, the value of Lyapunov function at jumping instants will tend to zero in stochastic sense, forcing $V_i(x_k) \to 0$, despite the allowable increase of Lyapunov function associated with mode 1 to some extent.

Remark 4: Combining with Remark 3, one can conclude from Theorem 1 that the unforced S-MJLSs will be mean-square stable when $T_{\max}^i \to \infty$, $\forall i \in \mathcal{I}$, but obviously the resulting conditions (10), (11) can not be numerically tested. Therefore, a finite $T_{\max}^i$ is necessitated in Theorem 1, which corresponds to the proposed σ-MSS with $\sigma > 0$.

Remark 5: In Theorem 1, the SMK $\Pi(\tau)$ is directly utilized to establish the stability criterion. In fact, $\Pi(\tau)$ can be first used to obtain the memory TPs (if denoted by $\lambda_{ij}(\tau)$), and the following stability criterion can be arrived at:

$$\sum_{j \in \mathcal{I}} \lambda_{ij}(\tau) A_i' P_j(\tau) A_i - P_i(\tau - 1) \prec 0 \tag{15}$$

where $P_i(\tau)$ is a time-varying Lyapunov matrix and $\tau \in \mathbb{Z}_{\geq 1}$. Note that when $\lambda_{ij}(\tau) \equiv \lambda_{ij}$ (the corresponding PDF of sojourn-time is

subject to geometric distribution with parameter μ_i, i.e., $\omega_{i1}(\tau) = \cdots = \omega_{iM}(\tau) = \mu_i(1 - \mu_i)^{\tau-1} \triangleq \omega_i(\tau))$ and $P_i(\tau) \equiv P_i$, (15) reduces to the usual stability criterion for MJLSs. It can be readily proved (cf. [4]) that (15) is a necessary and sufficient condition for the MSS of the unforced system (1). However, (15) is not numerically testable even conservatively setting $P_i(\tau) \equiv P_i$. Additional approximation techniques such as taking bounds for $\lambda_{ij}(\tau)$ are needed to make (15) be time-invariant, cf. [9]. Besides, (15) aims at MSS that is not "scaled" compared with σ-MSS.

Remark 6: It is also worth mentioning that if the sojourn-time is subject to "exponentially modulated periodic (EMP)" distribution (cf. [17]), say $\omega_i(\tau) = \bar{\omega}_i(\tau)(1 - \mu_i)^{\tau-1}$, where $\bar{\omega}_i(\tau)$ satisfies $\bar{\omega}_i(\tau) = \bar{\omega}_i(\tau + T)$ with T being the period, then the TPs obtained from SMK will be periodically time-varying. The corresponding criterion in (15) will be periodic and testable accordingly, i.e., only a finite set of $P_i(v)$, $v \in \mathbb{Z}_{[1,T]}$ is required. The benefit in this scenario therefore suggests that the EMP distribution can be considered to approximate to those general "exponential-like" distributions of sojourn-time, for the sake of a finite number of conditions.

Though the stability criterion explored in Theorem 1 can be used for analysis of the σ-MSS of S-MJLSs, the requirements (10), (11) impose a significant difficulty in deriving tractable conditions of controller design due to the existence of the power of A_i. To circumvent the difficulty, certain techniques will be further explored below to obtain sufficient conditions for Theorem 1.

Theorem 2: Consider the S-MJLS (1) with $u(k) \equiv 0$ and a given finite constant $h_i > 0$. If, $\forall i \in \mathcal{I}$, there exist $T_{\max}^i \in \mathbb{Z}_{\geq 1}$ and a set of matrices $O_i(t,m)$, $\forall t \in \mathbb{Z}_{[1,T_{\max}^i]}$, $\forall m \in \mathbb{Z}_{[0,t]}$ with $O_i \triangleq O_i(t,t) \succ 0$ and $\mathcal{O}_i(\tau,n)$, $\forall \tau \in \mathbb{Z}_{[1,T_{\max}^i]}$, $\forall n \in \mathbb{Z}_{[0,\tau-1]}$ such that $\forall t \in \mathbb{Z}_{[1,T_{\max}^i]}$, $\forall m \in \mathbb{Z}_{[0,t-1]}$ and $\forall n \in \mathbb{Z}_{[0,T_{\max}^i-1]}$

$$A_i'O_i(t,m+1)A_i - O_i(t,m) \prec 0 \quad (16)$$
$$O_i(t,0) - h_iO_i \prec 0 \quad (17)$$
$$\sum_{\tau=n+1}^{T_{\max}^i} \left[A_i'\mathcal{O}_i(\tau,n+1)A_i - \mathcal{O}_i(\tau,n) \right] \prec 0 \quad (18)$$
$$\sum_{\tau=1}^{T_{\max}^i} \mathcal{O}_i(\tau,0) - O_i \prec 0 \quad (19)$$

where $\mathcal{O}_i(l,l) \triangleq \sum_{j \in \mathcal{I}} \pi_{ij}(l)O_j/\eta_i$ with η_i defined in (11), then the unforced S-MJLS is σ-error mean-square stable.

Proof: It follows from (16) that $\sum_{m=0}^{t-1} A_i'^m(A_i'O_i(t,m+1)A_i - O_i(t,m))A_i^m \prec 0$ and accordingly

$$A_i'^tO_i(t,t)A_i^t - O_i(t,0) \prec 0. \quad (20)$$

By (17), (20), and bearing in mind $O_i = O_i(t,t)$, it can be obtained that $A_i'^tO_iA_i^t \prec h_iO_i$. Further, setting the positive definite matrix $O_i = P_i$, we have $A_i'^tP_iA_i^t - h_iP_i \prec 0$.

Likewise, (18) ensures that $\sum_{n=0}^{T_{\max}^i-1} A_i'^n[\sum_{\tau=n+1}^{T_{\max}^i}[A_i'\mathcal{O}_i(\tau,n+1)A_i - \mathcal{O}_i(\tau,n)]]A_i^n \prec 0$, which is equivalent to $\sum_{\tau=1}^{T_{\max}^i}\sum_{n=0}^{\tau-1} A_i'^n[A_i'\mathcal{O}_i(\tau,n+1)A_i - \mathcal{O}_i(\tau,n)]A_i^n \prec 0$ and implies

$$\sum_{\tau=1}^{T_{\max}^i} \left[A_i'^\tau\mathcal{O}_i(\tau,\tau)A_i^\tau - \mathcal{O}_i(\tau,0) \right] \prec 0. \quad (21)$$

Combining (19) and (21), we have $\sum_{\tau=1}^{T_{\max}^i} A_i'^\tau\mathcal{O}_i(\tau,\tau)A_i^\tau - P_i \prec 0$. Then setting the positive definite matrix $\mathcal{O}_i(\tau,\tau) = \mathcal{P}_i(\tau)$ gives rise to $\sum_{\tau=1}^{T_{\max}^i} A_i'^\tau\mathcal{P}_i(\tau)A_i^\tau - P_i \prec 0$. Thus, by Theorem 1, it can be concluded that the system is σ-error mean-square stable and this completes the proof.

Then, based on Theorem 2, the existence conditions of mode-dependent stabilizing controller (6) for S-MJLS (1) are presented in the following theorem.

Theorem 3: Consider the S-MJLS (1) with a given finite constant $h_i > 0$. If, $\forall i \in \mathcal{I}$, there exist $T_{\max}^i \in \mathbb{Z}_{\geq 1}$ and a set of matrices $H_i(t,m)$, $\forall t \in \mathbb{Z}_{[1,T_{\max}^i]}$, $\forall m \in \mathbb{Z}_{[0,t]}$ with $H_i \triangleq H_i(t,t) \succ 0$ and $\mathcal{H}_i(n)$, $\forall n \in \mathbb{Z}_{[0,T_{\max}^i]}, Z_i, U_i$ such that $\forall t \in \mathbb{Z}_{[1,T_{\max}^i]}$, $\forall m \in \mathbb{Z}_{[0,t-1]}$ and $\forall n \in \mathbb{Z}_{[0,T_{\max}^i-1]}$

$$\begin{bmatrix} H_i(t,m+1) - Z_i - Z_i' & A_iZ_i + B_iU_i \\ * & -H_i(t,m) \end{bmatrix} \prec 0 \quad (22)$$

$$H_i(t,0) - h_iH_i \prec 0 \quad (23)$$

$$\begin{bmatrix} \mathcal{H} - \mathcal{Z} - \mathcal{Z}' & 0 & (\mathcal{A}_i\mathcal{Z}_i + \mathcal{B}_i\mathcal{U}_i)\mathcal{L}_i(n+1) \\ * & \mathcal{H}_i(n+1) - Z_i - Z_i' & (A_iZ_i + B_iU_i)L_i(n+1) \\ * & * & -\mathcal{H}_i(n) \end{bmatrix} \prec 0 \quad (24)$$

$$\mathcal{H}_i(0) - H_i \prec 0 \quad (25)$$

where $\mathcal{A}_i \triangleq diag_{(M)}\{A_i\}$, $\mathcal{B}_i \triangleq diag_{(M)}\{B_i\}$, $\mathcal{Z}_i \triangleq diag_{(M)}\{Z_i\}$, $\mathcal{U}_i \triangleq diag_{(M)}\{U_i\}$, $\mathcal{H} \triangleq diag\{H_1, H_2, \ldots, H_M\}$, $\mathcal{Z} \triangleq diag\{Z_1, Z_2, \ldots, Z_M\}$, $L_i(n) = I$, $\forall n \in \mathbb{Z}_{[1,T_{\max}^i-1]}$ with $L_i(T_{\max}^i) = 0$, and $\mathcal{L}_i(n) \triangleq [\ell_{i1}(n)I, \ell_{i2}(n)I, \ldots, \ell_{iM}(n)I]'$ with $\ell_{ij}(n) \triangleq \sqrt{\pi_{ij}(n)/\eta_i}$ and η_i defined in (11), then a mode-dependent controller of form (6) can be obtained to guarantee the σ-MSS of the resulting closed-loop system. Moreover, the admissible controller gain is given by $K_i = U_iZ_i^{-1}$.

Proof: Letting $\bar{\mathcal{O}}_i(l) \triangleq \sum_{\tau=l+1}^{T_{\max}^i} \mathcal{O}_i(\tau,l)$, $\forall l \in \mathbb{Z}_{[0,T_{\max}^i-1]}$ and $\bar{\mathcal{O}}_i(T_{\max}^i) \triangleq 0$, we can rewrite (18) and (19) as

$$\bar{\mathcal{O}}_i(0) - O_i \prec 0 \quad (26)$$
$$A_i'\bar{\mathcal{O}}_i(n+1,n+1)A_i + A_i'\bar{\mathcal{O}}_i(n+1)A_i - \bar{\mathcal{O}}_i(n) \prec 0. \quad (27)$$

Consider (27), by Schur complement, it yields that

$$\begin{bmatrix} -\mathcal{O} & 0 & \mathcal{O}\mathcal{A}_i\mathcal{L}_i(n+1) \\ * & -\bar{\mathcal{O}}_i(n+1) & \bar{\mathcal{O}}_i(n+1)A_i \\ * & * & -\bar{\mathcal{O}}_i(n) \end{bmatrix} \prec 0 \quad (28)$$

where $\mathcal{A}_i, \mathcal{L}_i(n)$ are defined in (24) and $\mathcal{O} \triangleq diag\{O_1, O_2, \ldots, O_M\}$. Performing a congruence transformation to (28) by $diag\{\mathcal{O}^{-1}\mathcal{V},$ $\bar{\mathcal{O}}_i(n+1)^{-1}V_i, I\}$, where $\mathcal{V} \triangleq diag\{V_1, V_2, \ldots, V_M\}$, and since $(\mathcal{O} - \mathcal{V})'\mathcal{O}^{-1}(\mathcal{O} - \mathcal{V}) \succ 0$ ensures $\mathcal{O} - \mathcal{V} - \mathcal{V}' \succ -\mathcal{V}'\mathcal{O}^{-1}\mathcal{V}$, we can obtain

$$\begin{bmatrix} \mathcal{O} - \mathcal{V} - \mathcal{V}' & 0 & \mathcal{V}'\mathcal{A}_i\mathcal{L}_i(n+1) \\ * & \bar{\mathcal{O}}_i(n+1) - V_i - V_i' & V_i'A_i \\ * & * & -\bar{\mathcal{O}}_i(n) \end{bmatrix} \prec 0. \quad (29)$$

Then, apply the congruence transformation to (24) by $diag\{\mathcal{V}, V_i, V_i\}$, it gives (29) while replacing A_i by $A_i + B_iK_i$ and setting $Z_i \triangleq V_i^{-1}$, $U_i \triangleq K_iV_i^{-1}$, $\mathcal{H} \triangleq \mathcal{V}'^{-1}\mathcal{O}\mathcal{V}^{-1}$, $\mathcal{H}_i(n) \triangleq V_i'^{-1}\bar{\mathcal{O}}_i(n)V_i^{-1}$.

On the other hand, performing a congruence transformation to (25) by V_i, we have (26) accordingly. Then, it implies that (24) and (25) guarantee (18) and (19) in Theorem 2. Same techniques can be applied to the case that (22) and (23) ensures (16) and (17). Thus, it can be concluded from Theorem 2 that (22)–(25) can guarantee the σ-MSS of the resulting closed-loop system with the admissible controller gain given by $K_i = U_iZ_i^{-1}$.

Remark 7: In Theorems 1 and 3, for a given set of h_i, the $T_{\max}^i$ can be maximized while achieving a minimal σ-error and ensuring a feasible solution of (10), (11), and (22)–(25), for stability and stabilization problems, respectively. Besides, note also that if $h_i > 1$, the corresponding Lyapunov function in the ith mode can raise to some extent, which implies that mode i can be unstable/unstabilizable with $\tau \leq T_{\max}^i$. Nevertheless, it can be conjectured that if such an unstable/unstabilizable mode is expected to run longer (i.e., a larger $T_{\max}^i$), then probably a much larger h_i ($h_i \gg 1$) is required, which will result in a huge energy growth.

TABLE I
THE OPTIMAL $T^i_{\max}$ AND THE MINIMAL σ-ERROR FOR GIVEN DIFFERENT h_i, $i = 1, 2$

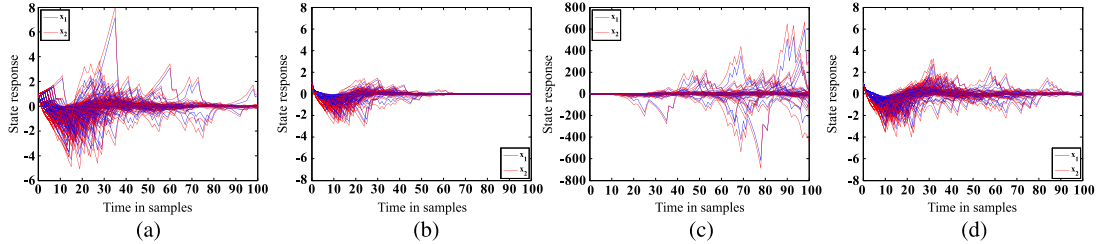| h_i	$\alpha = 1.07$		$\alpha = 1.15$		$\alpha = 1.20$		$\alpha = 1.27$		$\alpha = 1.28$
	$T^1_{\max}, T^2_{\max}$	σ	$T^1_{\max}, T^2_{\max}$	σ	$T^1_{\max}, T^2_{\max}$	σ	$T^1_{\max}, T^2_{\max}$	σ	
3	3, 6	0.26	3, 4	0.31	2, 2	0.73	1, 1	1.61	
5	5, 7	0.09	4, 3	0.27	2, 3	0.58	1, 1	1.61	
10	6, 10	0.05	4, 5	0.17	2, 5	0.48	1, 1	1.61	infeasible
10^2	12, 20	2.2×10^{-3}	4, 8	0.14	2, 5	0.48	1, 1	1.61	
10^3	18, 30	0.1×10^{-3}	4, 8	0.14	2, 5	0.48	1, 1	1.61	

Fig. 3. 100 realizations of state response of the system when generating different random jumping sequences. In (b) and (d), the sequences are further subject to $T^1_{\max} = 5$, $T^2_{\max} = 7$ and $T^1_{\max} = 4$, $T^2_{\max} = 3$, respectively. (a) $\alpha = 1.07$ (not MSS). (b) $\alpha = 1.07$ (σ-MSS with σ-error = 0.09). (c) $\alpha = 1.15$ (not MSS). (d) $\alpha = 1.15$ (σ-MSS with σ-error = 0.27).

IV. NUMERICAL EXAMPLES

In this section, both a MJLS and a S-MJLS will be provided to show the validity and advantage of the obtained theoretical results.

Example 1: (MJLS) Consider an unforced MJLS with two modes

$$A_1 = \alpha \begin{bmatrix} -0.36 & 0.69 \\ -1.81 & 1.97 \end{bmatrix} \quad A_2 = \alpha \begin{bmatrix} 0.34 & 0.62 \\ -0.37 & 1.36 \end{bmatrix}$$

where $\alpha > 0$ characterizes the distance of system eigenvalues and unit circle. The EMC is an anti-identity matrix, and the sojourn-time is subject to geometric distribution with $\omega_1(\tau) = 0.4(1 - 0.4)^{\tau-1}$ and $\omega_2(\tau) = 0.5(1 - 0.5)^{\tau-1}$.

Based on the usual criterion of MSS (the required TPs matrix can be straightforwardly obtained from $\omega_1(\tau)$ and $\omega_2(\tau)$), it can be readily checked that the system is not mean-square stable when $\alpha \geq 1.07$. Turning to Theorem 1, given different h_i, we can maximize the upper bound of sojourn-time for each mode (denoted as $[T^i_{\max}]_{opt}$, $i = 1, 2$) in achieving a minimal σ-error of approximation to MSS, as computed in Table I. It can be seen that when $\alpha > 1.27$, no matter how large the set of h_i is given, a finite σ-error for the MSS cannot be guaranteed (the corresponding system does not hold σ-MSS). As for $1.07 \leq \alpha \leq 1.27$, however, different minimal σ-error can be found for different h_i. Particularly, the smaller the α is assigned, the larger the $[T^i_{\max}]_{opt}$ can be found, and thus the smaller the minimal σ-error can be achieved.

Consider $\alpha = 1.07$, $x_0 = [1\ 1]'$, Fig. 3(a) illustrates the state response when randomly generating 100 realizations of jumping sequences satisfying the geometric distribution, and Fig. 3(b) presents the case where the jumping sequences are further subject to $T^1_{\max} = 5, T^2_{\max} = 7$. The scenario of $\alpha = 1.15$ is also given in Fig. 3(c) and (d) ($T^1_{\max} = 4, T^2_{\max} = 3$). It can be observed that although the two systems are not mean-square stable, they are σ-error mean-square stable with different approximation σ-error to be 0.09 and 0.27, respectively. Both Table I and Fig. 3 therefore demonstrate that the proposed σ-MSS is of the advantage of being well-scaled rather than MSS.

Example 2: (S-MJLS) Consider a dynamic system with possible failures in both structure and actuator as

$$\begin{cases} A_1 = \begin{bmatrix} -0.36 & 0.69 \\ -1.81 & 1.97 \end{bmatrix} B_1 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} & \text{(Normal Case)} \\ A_2 = \begin{bmatrix} 0.34 & 0.62 \\ -0.37 & 1.36 \end{bmatrix} B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} & \text{(Slight Fault)} \\ A_3 = \begin{bmatrix} 0.34 & 0.62 \\ -0.37 & 1.36 \end{bmatrix} B_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \text{(Serious Fault (unworkable actuator))} \end{cases}$$

The switching among the three modes is governed by a SMC, where the SMK is computed by (2) with $\Theta = [0\ 0.7\ 0.3; 0.4\ 0\ 0.6; 0.5\ 0.5\ 0]$ and, $\forall i, j \in \{1, 2, 3\}$

$$[\omega_{ij}(\tau)] = \begin{bmatrix} 0 & \frac{0.6^\tau \cdot 0.4^{10-\tau} \cdot 10!}{(10-\tau)!\tau!} & \frac{0.4^\tau \cdot 0.6^{10-\tau} \cdot 10!}{(10-\tau)!\tau!} \\ 0.9^{(\tau-1)^2} - 0.9^{\tau^2} & 0 & \frac{0.5^{10} \cdot 10!}{(10-\tau)!\tau!} \\ 0.4^{(\tau-1)^{1.3}} - 0.4^{\tau^{1.3}} & 0.3^{(\tau-1)^{0.8}} - 0.3^{\tau^{0.8}} & 0 \end{bmatrix}.$$

Note from $\omega_{ij}(\tau)$ that the "hybrid" PDF of sojourn-time in mode 1 and mode 3 are considered as Bernoulli distribution and Weibull distribution, respectively, with different parameters when the target modes are different, and the two types of distributions are supposed to coexist in mode 2.

First, it is checked that the open-loop system is not σ-error mean-square stable for any $h_i > 0$ by Theorem 1. Applying Theorem 3, the desired mode-dependent controllers can be designed such that the resulting closed-loop systems are σ-error mean-square stable for certain sets of h_i. Also, $[T^i_{\max}]_{opt}$ can be further obtained to achieve a minimal σ-error for the MSS of the system and ensuring a feasible solution of the controller. It can be verified that no matter how large h_1 and h_2 are assigned, no feasible controllers can be obtained if $h_3 < 1$. Thus, consider $h_3 \geq 1$ and change h_1, h_2, the different $[T^i_{\max}]_{opt}$ can be obtained. Fig. 4(a)–(c) present the two sets of $[T^i_{\max}]_{opt}$ that vary with $h_2 \in \mathbb{R}_{[0.001, 0.005]}$ and $h_3 \in \mathbb{R}_{[1.1, 1.5]}$, for given h_1 to be 0.01 and 0.001, respectively [the corresponding σ-error, which can be computed by (5), is shown in Fig. 4(d)].

Then, setting $h_1 = 0.01, h_2 = 0.005$ and $h_3 = 1.5$, the upper bound of sojourn-time can be optimized as $[T^1_{\max}]_{opt} = [T^2_{\max}]_{opt} = 9$ and $[T^3_{\max}]_{opt} = 3$, which guarantee the minimal σ-error to be 0.0467. Given the initial condition $x_0 = [-0.5\ 1]'$, Fig. 5 shows the 100 realizations of the state response of the closed-loop system for randomly generating jumping sequences satisfying (the required $T^i_{\max}$) $T^1_{\max} = T^2_{\max} = 9$ and $T^3_{\max} = 3$. It can be observed that the designed controller is valid in the presence of an unstabilizable mode and despite the coexistence of "hybrid" distributions of sojourn-time in a same SMK. In addition, as shown in Fig. 4(d), when fixing $T^1_{\max} = T^2_{\max} = 9$, the σ-error can be smaller if further increasing $T^3_{\max}$, which seemingly can be attained by raising h_i. However, it can be checked that even $h_i = 10^7$ can not give rise to an increase from $T^3_{\max} = 3$ to $T^3_{\max} = 4$, which verifies the conjecture in Remark 7 that only a proper sojourn-time of the unstabilizable mode can be allowed.
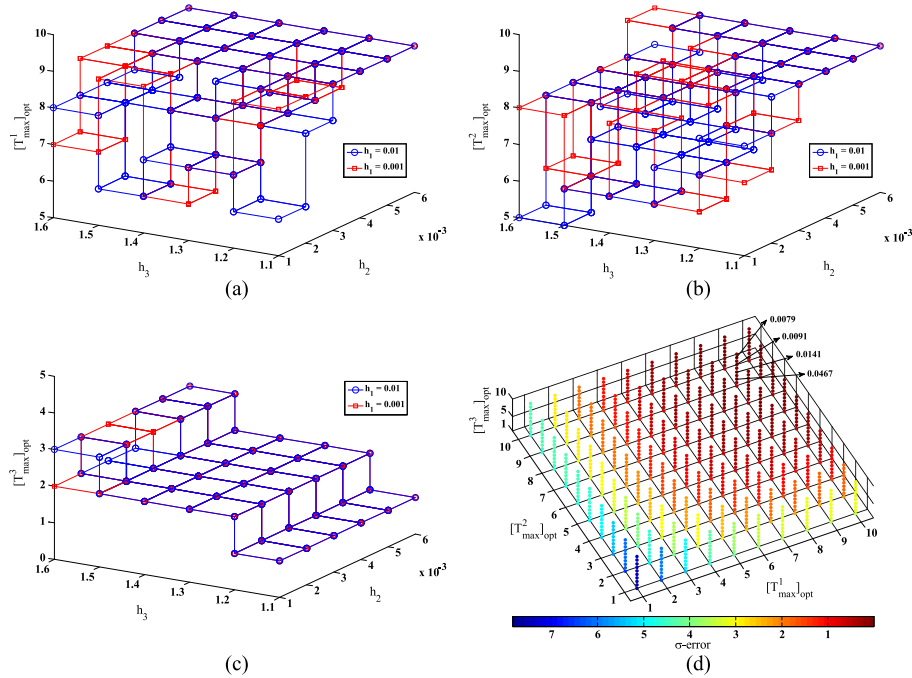
Fig. 4. Optimal $T_{\max}^i$ varying with h_i and the σ-error varying with $T_{\max}^i$, $i = 1,2,3$ (note that in (a)-(c), four *squares* or *circles* in the horizontal specify a face where the $T_{\max}^i$ is constant, e.g., in (a), for $0.001 \leq h_2 \leq 0.002$, and $1.5 \leq h_3 \leq 1.6$, $T_{\max}^1$ is 8 and 7 for $h_1 = 0.01$ and $h_1 = 0.001$, respectively). (a) Variation of optimal $T_{\max}^1$. (b) Variation of optimal $T_{\max}^2$. (c) Variation of optimal $T_{\max}^3$. (d) σ-error varying with $T_{\max}^i$.
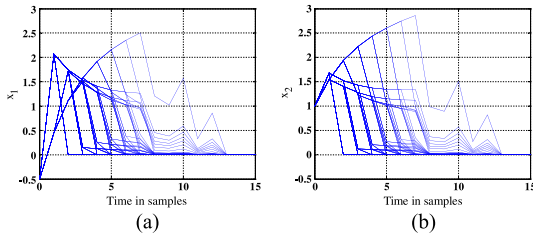

Fig. 5. 100 realizations of state response for different random jumping sequences satisfying $T_{\max}^1 = T_{\max}^2 = 9$ and $T_{\max}^3 = 3$. (a) State response x_1. (b) State response x_2.

V. CONCLUSION

In this technical note, the stability and stabilization problems of discrete-time S-MJLSs were investigated. The notion of discrete-time SMK is introduced, and the PDF of sojourn-time inside the SMK can be considered to be dependent on both current and next system mode. The underlying S-MJLSs are therefore more general than existing ones, since different parameters in a same type of distribution and/or different types of distributions can coexist in each mode of a SMK. The stability concept of σ-MSS was proposed, where σ is capable of characterizing the degree of approximation error of σ-MSS to MSS. Numerically testable stability and stabilization criteria that explicitly contain the PDF information of sojourn-time were obtained. A direct future work will be the developments of the S-MJLSs in continuous-time domain based on the SMK approach.

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