Switched Positive Linear Systems

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Abstract

In this monograph we consider the class of continuous-time positive switched systems. We discuss several problems, including stability, performance analysis, stabilization via switching control, and optimization. The monograph starts with a chapter where several application examples are provided, to motivate the interest in this class of systems. The rest of the monograph is dedicated to the theory of stability, stabilization and performance optimization of positive switched systems. The main existing results are recalled, but also new challenging problems are proposed and solved. Special attention has been devoted to point out those results that specifically pertain to positive (linear) switched systems and do not find a counterpart for the wider class of (nonpositive) linear switched systems.

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Introduction

Positive systems are an important class of systems that frequently arise in application areas, such as in the chemical process industry, electronic circuit design, communication networks and biology.

Stability problems arising in the study of positive systems differ from those pertaining to standard systems. The main difference stems from the fact that the state variables are confined to the positive orthant. Thus, the whole analysis of these systems focuses only on the trajectories generated under positivity constraints, and consequently stability can be deduced from the existence of copositive Lyapunov functions whose derivatives are required to be negative only along the system trajectories in the positive orthant.

Switched positive systems also arise in a variety of applications. Examples can be found in TCP congestion control, in processes described by non-homogeneous Markov chains, in image processing, in biochemical networks, etc... Differently from general switched systems, that have received a lot of attention in the past years, the theory for positive switched systems is still in a relative infancy.

In this monograph we study the stability, performance evaluation, stabilization via switching control and optimal control of (continuous-time and linear) positive switched systems. We discuss results that have already been established in the literature, but other results, especially those regarding norm computation and optimization, are new and integrated with the previous ones.

In Chapter 2 we present many examples and motivations for studying positive switched systems. These examples include thermal systems, fluid networks, traffic systems, biological and epidemiological models and transmission networks. We present some specific problems that should be inspirational (at least we hope they are) for the subsequent chapters.

In Chapter 3, we consider the stability problem, namely the problem of determining stability under arbitrary switching. We show that this problem can be generalized to the problem of establishing a convergence (or divergence) rate. We characterize the stability property in terms of the existence of convex homogeneous Lyapunov functions. In general these functions can be extremely complex, so we provide some special classes of Lyapunov functions, including copositive linear and copositive quadratic Lyapunov functions, which are conservative but simpler to be computed. We also discuss a famous conjecture, now disproved in the general case, regarding the equivalence between stability under arbitrary switching and Hurwitz robustness, namely the fact that all the matrices in the convex hull of the family of system matrices are Hurwitz. The statement is true for 2-dimensional systems and false in general, since Hurwitz robustness is only necessary when the system dimension n is greater than 2. We also investigate the case when dwell time is imposed on the switching signals, namely a minimum amount of time has to elapse between any pair of consecutive switching times.

In Chapter 4 we discuss the performance evaluation of positive switched systems in terms of several input-output induced norms. Notwithstanding the fact that, for positive systems, it is often easy to establish the worst-case signal, namely the one providing the largest output norm, computing these norms is in general hard. Then we provide computationally tractable ways to generate upper bounds, for both arbitrary switching signals and dwell-time constrained ones.

In Chapter 5 we consider the stabilization problem for systems for which the switching signal represents a control input. This problem has some interesting properties that are the counterpart of some properties established in the stability analysis case. Stabilizability is equivalent to the existence of a concave homogeneous copositive control Lyapunov function. Again, finding any such function is in general hard, so we investigate special classes including linear and quadratic copositive functions. We also provide some sufficient stabilizability conditions in terms of Lyapunov Metzler inequalities. The disproved conjecture about stability has a counterpart for the stabilizabity case: is stabilizability equivalent to the existence of at least one Hurwitz element in the convex hull of the matrices? Again the is true for 2-dimensional systems and false in general, as the existence of a Hurwitz convex combination is only a sufficient condition for stabilization. It is interesting to note that the existence of a Hurwitz convex combination is a necessary and sufficient condition for the existence of a smooth homogeneous control Lyapunov function.

Finally, in Chapter 6, we consider the optimal control problem for positive systems with a controlled switching signal. We show how some of the material presented in the previous chapters, such as the Lyapunov Metzler inequalities technique, can be successfully exploited to derive some conditions that allow to design a guaranteed cost control.

In addition to the simple numerical examples provided in Chapters 3-5 to illustrate the developed theory, in this chapter simulations are provided for a couple of "realistic" examples presented in Chapter 2, dedicated to the motivational part, and specifically: the optimal therapy scheduling for mitigation of the HIV viral load, and the disease free control applied to a SIS (Susceptible-Infective-Susceptible) epidemiological system.

This survey does not aim at providing an exhaustive account of all the research problems investigated in the literature and concerned with positive switched systems. Important issues have been omitted here, due to page constraints. Among them, it is worth quoting the following ones: controllability/reachability (see Fornasini and Valcher [2011], Santesso and Valcher [2008], Valcher and Santesso [2010], Valcher

[2009], Xie and Wang [2006]), observability of positive switched systems (Li et al. [2014]), positive switched systems with delays (Li et al. [2013a,b], Liu and Dang [2011], Liu and Lam [2012], Xiang and Xiang [2013]), and interesting characterizations like joint spectral properties and asymptotic properties of matrix semigroups (Guglielmi and Protasov [2013], Protasov et al. [2010], Jungers [2012], and extremal norms for linear inclusions, Mason and Wirth [2014]). For all these topics we refer the interested Reader to the previous references. On the other hand, for the topics specifically addressed in this monograph, no references are provided in this introduction, being them appropriately quoted when needed within the text.

1.1 Notation

The notation used throughout the monograph is standard for positive systems. The sets of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively, while $\mathbb{R}_+$ is the set of nonnegative real numbers. Capital letters denote matrices, small (bold face) letters denote vectors. For matrices or vectors, $(^\top)$ indicates transpose. The (ℓ, j)th entry of a matrix A is denoted by $[A]_{\ell,j}$, while the ith entry of a vector $\mathbf{x}$ is x_i or $[\mathbf{x}]_i$. When the vector $\mathbf{x}$ is obtained as the result of some mathematical operation, e.g. $\mathbf{x} = A\mathbf{y}$, we will generally adopt the latter notation $[A\mathbf{y}]_i$. The symbol $\mathbf{e}_i$ denotes the ith canonical vector in $\mathbb{R}^n$, where n is always clear from the context, while $\mathbf{1}_n$ denotes the n-dimensional vector with all entries equal to 1. The symbol I_n denotes the identity matrix of order n.

A (column or row) vector $\mathbf{x} \in \mathbb{R}^n$ is said to be *nonnegative*, $\mathbf{x} \geq 0$, if all its entries x_i, $i = 1, 2, \ldots, n$, are nonnegative. It is *positive* if nonnegative and at least one entry is positive. In this case, we will use $\mathbf{x} > 0$. It is said to be strictly positive if all its entries are greater than 0, and in this case, we will use the notation $\mathbf{x} \gg 0$. The set of all n-dimensional nonnegative vectors is denoted by $\mathbb{R}^n_+$ and referred to as the *positive orthant*. The expressions $\mathbf{x} \gg \mathbf{y}$, $\mathbf{x} > \mathbf{y}$ and $\mathbf{x} \geq \mathbf{y}$ mean that the difference $\mathbf{x} - \mathbf{y}$ is strictly positive, positive and nonnegative, respectively. Similar notation is used for the (real) matrices.

The set of n-dimensional nonnegative vectors whose entries sum up to 1 is the simplex

$$\mathcal{A}_n := \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ : \sum_{i=1}^{n} \alpha_i = 1\}.$$

A square matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Metzler*[1] if its off-diagonal entries $[A]_{ij}$, $i \neq j$, are nonnegative. Every Metzler matrix A has a real dominant eigenvalue $\lambda_F \in \sigma(A)$ satisfying $\mathrm{Re}(\lambda_F) > \mathrm{Re}(\lambda)$ for every $\lambda \in \sigma(A), \lambda \neq \lambda_F$. λ_F is called the Frobenius eigenvalue of A, see Farina and Rinaldi [2000]. Also, associated with λ_F there is always both a left and a right positive eigenvector, known as (left/right) Frobenius eigenvectors.

An $n \times n$ Metzler matrix A is *reducible* if there exists a permutation matrix Π such that

$$\Pi^\top A \Pi = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square (nonvacuous) matrices, otherwise it is *irreducible*. It follows that 1×1 matrices are always irreducible.

A linear state space model described by the linear differential equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where A is a Metzler matrix, is called a *positive system*, see Berman et al. [1989], Farina and Rinaldi [2000], Kaczorek [2002], Krasnoselskii [1964], Luenberger [1979], because it enjoys the property that any trajectory starting in the positive orthant remains confined in it.

A square matrix is *Hurwitz* if all its eigenvalues lie in the open left half plane. A Metzler matrix is Hurwitz if and only if there exists a vector $\mathbf{v} \gg 0$ such that $\mathbf{v}^\top A \ll 0$, or, equivalently if and only there exists a vector $\mathbf{w} \gg 0$ such that $A\mathbf{w} \ll 0$, see e.g. Farina and Rinaldi [2000].

Given two matrices $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{p \times q}$, the expression $C = A \otimes B \in \mathbb{R}^{np \times mq}$ stands for the usual *Kronecker product*. If $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{p \times p}$, their *Kronecker sum* is defined as $A \oplus B = A \otimes I_p +$

[1] A Metzler matrix is also known in the literature as "essentially nonnegative matrix" (see Berman et al. [1989], Horn and Johnson [1985]) or as the opposite of a "Z-matrix" (see Horn and Johnson [1991]).

$I_n \otimes B \in \mathbb{R}^{np\times np}$. Properties of Kronecker operators can be found in Graham [1981].

The symbols $\succ, \succeq, \prec$ and $\preceq$ are used to denote order relations induced by definiteness properties. For instance, the expression $P = P^\top \succ 0 \in \mathbb{R}^{n\times n}$ means that P is a (symmetric and) positive definite matrix, i.e. $\mathbf{x}^\top P\mathbf{x} > 0$ for every $\mathbf{x} \neq 0$. $P_1 \succeq P_2$ means that $P_1 - P_2$ is a (symmetric and) positive semi-definite matrix.

1.2 Continuous-time positive switched systems

A *continuous-time positive switched system* is described by the following equation

$$\dot{\mathbf{x}}(t) = A_{\sigma(t)}\mathbf{x}(t), \qquad t \in \mathbb{R}_+, \tag{1.1}$$

where $\mathbf{x}(t)$ denotes the value of the n-dimensional state variable at time t, and $\sigma(t)$ is a right-continuous and piece-wise constant mapping from $\mathbb{R}_+$ into the finite set $\{1, ..., M\}$. This latter property ensures that in any bounded time interval the map σ has always a finite number of discontinuities, known as *switching instants* and denoted in the following by $0 = t_0 < t_1 < t_2 < \dots$. This amounts to saying that $\sigma(t)$ takes some constant value $i_k \in \{1, 2, \dots, M\}$ at every $t \in [t_k, t_{k+1})$ and that $\sigma(t_k) \neq \sigma(t_{k+1})$. In the sequel, when we will refer to an "arbitrary switching signal" σ we will always mean an arbitrary switching signal endowed with the aforementioned properties and we will denote the set of such switching signals by the symbol $\mathcal{D}_0$. The reason for this notation will be clarified later on.

A function $\mathbf{x} : \mathbb{R}_+ \mapsto \mathbb{R}^n$ is a solution of (1.1) if, see Shorten et al. [2007], it is continuous and piecewise continuously differentiable and if there is a switching signal σ such that (1.1) holds at every $t \in \mathbb{R}_+$, except at the switching instants. For every value i taken by the switching signal σ (at t), $\dot{\mathbf{x}}(t) = A_i\mathbf{x}(t)$ is a (autonomous[2]) continuous-time positive system, which means that A_i is an $n \times n$ Metzler matrix. This ensures that if $\mathbf{x}(0)$ belongs to the positive orthant $\mathbb{R}^n_+$, then, for every choice of σ, the state evolution $\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}(0), \sigma)$ belongs to $\mathbb{R}^n_+$ for

[2]In this monograph by an autonomous system we will always mean a system with no inputs, see Khalil [2002], Sun and Ge [2005], Willems [1970].

every $t \in \mathbb{R}_+$. It is worth noticing that also for switched systems the Metzler property of the matrices $A_i, i \in \{1, 2, \ldots, M\}$, is both necessary and sufficient to ensure that all the state trajectories starting in the positive orthant remain in $\mathbb{R}^n_+$ at all subsequent times, for every choice of the switching signal. Given any initial state $\mathbf{x}_i \in \mathbb{R}^n_+$, any switching signal $\sigma : \mathbb{R}_+ \mapsto \{1, 2, \ldots, M\}$, and any pair of time instants $t \geq \tau \geq 0$, the state at time t can be expressed as

$$\mathbf{x}(t) = \Phi(t, \tau, \sigma)\mathbf{x}_i,$$

where $\Phi(t, \tau, \sigma)$ represents the *state transition matrix* of system (1.1) corresponding to the time interval $[\tau, t]$ and the switching signal σ. Clearly, if we denote by $\tau = t_1 < t_2 < \cdots < t_k < t_{k+1} = t$ the switching instants in the time interval $[\tau, t]$ and by i_h the value of the switching signal σ in the time interval $[t_h, t_{h+1}), h \in \{1, 2, \ldots, k\}$, then

$$\Phi(t, \tau, \sigma) = e^{A_{i_k}(t-t_k)} \ldots e^{A_{i_2}(t_3-t_2)} e^{A_{i_1}(t_2-\tau)}.$$

In the following, we will also consider non-autonomous positive switched systems, described, for instance (but not only), by the following equations:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_{\sigma(t)}\mathbf{x}(t) + B_{\sigma(t)}\mathbf{u}(t), \\ \mathbf{y}(t) &= C_{\sigma(t)}\mathbf{x}(t) + D_{\sigma(t)}\mathbf{u}(t), \qquad t \in \mathbb{R}_+, \end{aligned} \tag{1.2}$$

where $\mathbf{x}(t), \mathbf{u}(t)$ and $\mathbf{y}(t)$ are the n-dimensional state variable, the m-dimensional input variable and the p-dimensional output variable, respectively, at time t. For every value i taken by σ (at t), A_i is an $n \times n$ Metzler matrix, while B_i, C_i and D_i are nonnegative matrices. Under these conditions, the nonnegativity of the input at every time $t \geq 0$ and the nonnegativity of the initial condition $\mathbf{x}(0)$ ensure the nonnegativity of the state and output trajectories at every $t \geq 0$.

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