The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property

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Abstract

We consider the nonlocal Cahn–Hilliard equation with singular potential and constant mobility. Well-posedness and regularity of weak solutions are studied. Then we establish the validity of the strict separation property in dimension two. Further regularity results as well as the existence of regular finite dimensional attractors and the convergence of a weak solution to a single equilibrium are also provided. Finally, regularity results and the strict separation property are also proven for the two-dimensional nonlocal Cahn–Hilliard–Navier–Stokes system with singular potential.

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1 Introduction

The Cahn–Hilliard system was proposed in [15] (see also [14]) as a model which provides a macroscopic description of the formation and evolution of microstructures during the phase separation in a binary alloys system. Such a phenomenon is characterized by an early stage where the so-called spatial spinodal decomposition takes place, followed by the coarsening process. They occur when a homogeneous mixture undergoes a rapid cooling below a certain critical temperature. As a result, this process leads to the segregation of the system into spatial subdomains where one of the constituents prevails. Afterwards, the Cahn–Hilliard equation and its variants has been used to model different phenomena which are characterized by segregation-like processes (see, for instance, [10, 26, 46, 48, 49, 54, 58, 59]).

The general form of the Cahn–Hilliard system reads as follows

$$\begin{cases} \varphi_t = \mathrm{div}\left(m(\varphi)\nabla\mu\right), \\ \mu = -\kappa\Delta\varphi + F_0'(\varphi), \end{cases} \qquad \text{in } \Omega\times(0,T). \tag{1.1}$$

Here, Ω is a bounded smooth domain in $\mathbb{R}^d$, $d\leq 3$, φ represents the relative difference of the two phases (concentration), m is the concentration dependent mobility, μ is the chemical potential, $\kappa>0$ is a parameter related to the thickness of the interface and F_0' is the first derivative of a double well potential F_0. The physically relevant example is the logarithmic potential defined by

$$F_0(s) = \frac{\theta}{2}\Big[(1+s)\log(1+s) + (1-s)\log(1-s)\Big] - \frac{\theta_c}{2}s^2, \tag{1.2}$$

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for all $s \in (-1,1)$, with $0 < \theta < \theta_c$, where θ is the temperature of the system and θ_c the critical temperature, both assumed to be constants. Instead, significant examples of mobility function m are either a positive constant or the so-called degenerate case, namely, for all $s \in [-1,1]$,

$$m(s) = M, \quad \text{or} \quad m(s) = M(1-s^2). \tag{1.3}$$

System (1.1) is usually endowed with homogeneous Neumann boundary conditions

$$m(\varphi)\partial_n \mu = \partial_n \varphi = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.4}$$

where $n \in \mathbb{R}^d$ denotes the unit outward normal vector to the boundary $\partial\Omega$. The former condition means that no mass flux occurs at the boundary while the latter requires the interface to be orthogonal at the boundary.

The Cahn–Hilliard equation can be formally derived as the conserved dynamics generated by the variational derivative of the Ginzburg–Landau free energy with respect to φ (see, e.g., [24, 53] and references therein, cf. also [41] for a derivation based on the second law of thermodynamics). More precisely, (1.1) can be written as

$$\varphi_t = \operatorname{div}\left(m(\varphi)\nabla\mu\right) \quad \text{and} \quad \mu = \frac{\delta\mathcal{L}}{\delta\varphi},$$

being the free energy

$$\mathcal{L}(\varphi) = \mathcal{U}(\varphi) - \theta\mathcal{S}(\varphi),$$

where the (bulk) internal energy $\mathcal{U}$ is defined as

$$\mathcal{U}(\varphi) = \int_\Omega \left(\frac{\kappa}{2}|\nabla\varphi|^2 - \frac{\theta_c}{2}\varphi^2\right) dx$$

and the total entropy $\mathcal{S}$ is given by

$$\mathcal{S}(\varphi) = -\int_\Omega \frac{1}{2}\left[(1+\varphi)\log(1+\varphi) + (1-\varphi)\log(1-\varphi)\right] dx.$$

The internal energy $\mathcal{U}$ takes into account just short range interactions between particles and, for this reason, (1.1) is called the local Cahn–Hilliard system. Indeed, the gradient square term accounts for the fact that the local interaction energy is spatially dependent and varies across the interfacial surface due to spatial inhomogeneities in the concentration. However, there is no microscopic derivation of the system (1.1). This issue was observed by Giacomin and Lebowitz in [36]. There the authors rigorously derived an equation similar in structure to (1.1), which displays an analogous behavior during the late stage of the coarsening process. More precisely, starting from a discrete (microscopic) formulation on a lattice, they deduce through a stochastic argument (i.e., hydrodynamic limit) a macroscopic nonlocal evolution equation which takes into account long-range repulsive interactions between different species and short hard collisions between all particles as well (see [37, 38]).

In this work, we study the aforementioned nonlocal Cahn–Hilliard system proposed by Giacomin and Lebowitz describing the phase separation in a binary mixture. In order to emphasize the connection with the local Cahn–Hilliard (1.1)-(1.4), we rewrite the nonlocal Cahn–Hilliard equation in terms of the relative difference of two phases (cf. [36]). We refer to the concentrations of the two components as φ_1 and φ_2, subject to the natural constraint $\varphi_1 + \varphi_2 = 1$, and we also define their relative difference by setting

$$\varphi = \varphi_1 - \varphi_2 = 1 - 2\varphi_2 = 2\varphi_1 - 1.$$

The nonlocal Helmholtz free energy reads as (see, e.g., [8] and references therein)

$$\mathcal{E}(\varphi_1,\varphi_2) = \int_\Omega \theta\Big[\varphi_1\log(\varphi_1)+\varphi_2\log(\varphi_2)\Big]dx + \int_\Omega\int_\Omega J(x-y)\varphi_1(x)\varphi_2(y)dxdy,$$

where J is the interaction kernel such that $J(x)=J(-x)$. By the definition of φ and the symmetry of J, the energy functional can be rewritten (up to a constant) as

$$\mathcal{E}(\varphi) = \int_\Omega F(\varphi)dx - \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\varphi(x)\varphi(y)dxdy, \tag{1.5}$$

where

$$F(s) = \frac{\theta}{2}\Big[(1+s)\log(1+s)+(1-s)\log(1-s)\Big]. \tag{1.6}$$

Here, abusing the notation, $\frac{1}{2}J$ has been renamed as J. Then, the gradient flow associated to the Helmholtz free energy

$$\varphi_t = \operatorname{div}\left(m(\varphi)\nabla\mu\right) \quad \text{and} \quad \mu = \frac{\delta\mathcal{E}}{\delta\varphi}$$

leads to the nonlocal Cahn–Hilliard system

$$\begin{cases} \varphi_t = \operatorname{div}(m(\varphi)\nabla\mu), \\ \mu = F'(\varphi) - J*\varphi, \end{cases} \qquad \text{in } \Omega\times(0,T), \tag{1.7}$$

subject to the no mass flux boundary condition and the initial condition

$$\partial_n\mu = 0, \quad \text{on } \partial\Omega\times(0,T), \quad \varphi(\cdot,0)=\varphi_0, \quad \text{in } \ \Omega. \tag{1.8}$$

The formulation of the nonlocal system (1.7)-(1.8) is deeply connected with the local version (1.1)-(1.4). Firstly, as shown in [37], the Cahn–Hilliard system (1.1) can be seen as an approximation of the nonlocal one (1.7). Indeed, the Helmholtz free energy (1.5) is equivalent to

$$\mathcal{E}(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J(x-y)\Big(\varphi(x)-\varphi(y)\Big)^2 dxdy + \int_\Omega \tilde{F}(x,\varphi)dx, \tag{1.9}$$

where

$$\tilde{F}(x,s) = F(s) - \frac{1}{2}(J*1)(x)s^2. \tag{1.10}$$

Thus, it is easy to realize that (formally) the first approximation of the nonlocal interaction is $\frac{k}{2}|\nabla\varphi|^2$, for some $k>0$, provided that J is sufficiently peaked around 0. In the particular case $\Omega=\mathbb{T}^d$ (cf. [38]), note that $a=J*1$ is a constant, since the Lebesgue measure is translation invariant, and it is proportional to the critical temperature θ_c. Therefore, the form of $\mathcal{E}(\varphi)$ appears very similar to that of $\mathcal{L}(\varphi)$. On the other hand, a further close link between the local and the nonlocal models is related to their limit problem. More precisely, system (1.7) can be written as a second-order nonlocal equation in the unknown φ as

$$\varphi_t - \operatorname{div}\Big(m(\varphi)F''(\varphi)\nabla\varphi - m(\varphi)\nabla J*\varphi\Big) = 0. \tag{1.11}$$

Here, the second term in (1.11) is a nonlinear diffusion term provided that $m(\varphi)F''$ is strictly positive on $(-1,1)$ whereas the spatial convolution with a sufficiently smooth and fast decaying kernel J (e.g., Newtonian or Bessel potentials) models a nonlocal aggregation.

Then, it is worth noting that formally taking either $\kappa = 0$ in (1.1), or $J = \theta_c\delta_0$ in (1.11), δ_0 being the Dirac mass, one deduces the equation

$$\varphi_t - \operatorname{div}\big(m(\varphi)(F''(\varphi) - \theta_c)\nabla\varphi\big) = 0. \tag{1.12}$$

This is exactly the diffusion equation which was phenomenologically derived in [24]. However, under the assumption $\theta < \theta_c$ (i.e., when the phase separation takes place), equation (1.12) is ill posed due to its backward features. Thus, both models (1.1) and (1.11) can be interpreted as a well-posed approximation of (1.12).

After this overview on the modeling aspects, let us focus on the mathematical results. In this paper we will consider the case of constant mobility $m(s) = 1$ (the case of degenerate mobility will be treated in the forthcoming contribution [31]) leading to the study of the following nonlocal Cahn–Hilliard system (cf. (1.7))

$$\begin{cases} \varphi_t = \Delta\mu, \\ \mu = F'(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{1.13}$$

subject to the no mass flux boundary condition and the initial condition

$$\partial_n\mu = 0, \quad \text{on } \partial\Omega \times (0,T), \quad \varphi(\cdot,0) = \varphi_0, \quad \text{in } \ \Omega. \tag{1.14}$$

Note that, on account of (1.9) and (1.10), system (1.7) is equivalent to

$$\begin{cases} \varphi_t = \Delta\mu, \\ \ \mu = a(x)\varphi - J * \varphi + \tilde{F}'(x,\varphi), \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{1.15}$$

where $a(x) = (J * 1)(x)$. This is the form of the nonlocal equation which has been studied the most in the literature (see, for instance, [2, 6, 14, 19, 35, 39, 40, 45]) and, for binary fluids, [17, 20, 28, 29, 30, 32, 33, 34]). It is worth pointing out that $\tilde{F}$ has always been assumed to be independent of x by analogy with (1.2). Nevertheless, all the theoretical results established in the quoted papers can be straightforwardly rephrased for system (1.13)-(1.14) under less restrictive assumptions on J and F.

A good deal of references and results concerning the local system (1.1) with constant mobility can be found in [16]. More precisely, the well-posedness was first proven in [25] (see also [21, 47]). Then, regularity and global longtime behavior was analyzed in [21, 25, 51] while the longtime behavior of single solutions was studied in [3]. Regarding the nonlocal system (1.13), the existence of weak solutions and their uniqueness, and the existence of the connected global attractor were proven in [28] and [30] (see also [29]).

In this work, we first show a well-posedness result of weak solution by exploiting a different approximation argument to handle the singular potential (cf. [30]). Next, we prove our main results, namely, the regularity and the validity of the strict separation property in dimension two. In other words, we prove that if the initial state is not a pure phase (i.e. $\varphi_0 \equiv 1$ or $\varphi_0 \equiv -1$), then the corresponding solution stays away from the pure states in finite time, uniformly with respect to the initial datum. Consequently, we can improve the regularity properties of the solutions. In turn, this leads to the existence of a smooth global and exponential attractors, as well as the convergence to a single stationary state. The strict separation property for (1.1) in two dimensions with constant mobility was proven in [51]. However, even though in both cases one needs to find a global L^∞-bound of $F'(\varphi)$, the two techniques rely on different arguments. More precisely, in [51], the authors exploit the presence of the Laplace operator in the definition of the chemical potential. On the contrary, in the nonlocal case, we cannot exploit elliptic type arguments.

In the last part of this work we consider the nonlocal Cahn–Hilliard–Navier–Stokes system with a logarithmic potential F in dimension two. This is a nonlocal variant of the well-known model H which has been proposed to describe the evolution of a binary mixture of two incompressible and immiscible fluids (see [5, 42, 44, 43] and cf. [1, 11, 50] for theoretical issues). Denoting by $\mathbf{u}$ the volume-averaged fluid velocity and assuming, in addition, that the density and the viscosity are constants and equal to unity, the system reads as

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mu\nabla\varphi, \\ \operatorname{div}\,\mathbf{u} = 0, \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi = \Delta\mu, \\ \mu = F'(\varphi) - J*\varphi, \end{cases} \qquad \text{in } \Omega\times(0,T), \tag{1.16}$$

equipped with the boundary and initial conditions

$$\mathbf{u} = \partial_n\mu = 0, \quad \text{on } \partial\Omega\times(0,T), \quad \mathbf{u}(0,\cdot) = \mathbf{u}_0, \ \varphi(0,\cdot) = \varphi_0, \quad \text{in } \Omega. \tag{1.17}$$

This problem can be regarded as an important application of the results and techniques developed in this paper. Recalling that the existence of weak solutions and the uniqueness have recently been obtained in [28] and [30], we establish the strict separation property and we discuss its consequences on the regularity of weak solutions as well as their long-time behavior. It is worth noting that the validity of the strict separation property for the corresponding two-dimensional local Cahn–Hilliard–Navier–Stokes is an open issue. Nonetheless, a weaker version has been proven in [1]. That is, any solution stays eventually away from the pure states after a certain time depending on the single initial datum.

The paper is organized as follows. In Section 2 we introduce the functional setting and we recall some useful tools. Section 3 is devoted to the well-posedness of system (1.7) in a weak setting. Section 4 contains some regularity properties of the weak solution and the existence of the global attractor. The strict separation property is proven in Section 5. In the same section some of its consequences are analyzed. The final Section 6 deals with the nonlocal Cahn–Hilliard–Navier–Stokes system in dimension two.

2 Notation and Functional Spaces

We denote by $W^{\ell,p}(\Omega)$, $\ell\in\mathbb{N}$, the Sobolev space of functions in $L^p(\Omega)$ with distributional derivative of order less or equal to ℓ in $L^p(\Omega)$ and by $\|\cdot\|_{W^{\ell,p}(\Omega)}$ its norm. For an arbitrary $\ell\in\mathbb{N}$, $H^\ell(\Omega) = W^{\ell,2}(\Omega)$ is a Hilbert space with respect to the scalar product $(u,v)_\ell = \sum_{|\mathbf{k}|\le\ell}\int_\Omega D^{\mathbf{k}}u(x)D^{\mathbf{k}}v(x)dx$ ($\mathbf{k}$ being a multi-index) and the induced norm $\|u\|_\ell = \sqrt{(u,u)_\ell}$. We let $H = L^2(\Omega)$ and we denote the inner product as well as the norm in H by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. We also set $V = H^1(\Omega)$ equipped with the norm

$$\|u\|_V^2 = \|\nabla u\|^2 + \|u\|^2. \tag{2.1}$$

Denoting the total mass of a function by

$$\overline{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx,$$

we recall the well-known Poincaré–Wirtinger inequality

$$\|u-\overline{u}\| \le C\|\nabla u\|, \quad \forall\, u \in V. \tag{2.2}$$

We indicate by V' the dual space of V and by $\|\cdot\|_{V'}$ its norm. Then we introduce the spaces

$$V_0 = \{v \in V : \overline{v} = 0\}, \qquad V_0' = \left\{ f \in V' : \overline{f} = |\Omega|^{-1} \langle f, 1\rangle_{V',V} = 0 \right\},$$

and we consider the operator $A \in \mathcal{L}(V, V')$ defined by

$$\langle Au, v\rangle_{V',V} = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall\, u, v \in V.$$

The restriction of A on V_0 is an isomorphism from V_0 onto V_0' and we define the inverse map $\mathcal{N} : V_0' \to V_0$ such that

$$A\mathcal{N} f = f, \quad \forall\, f \in V_0', \quad \text{and} \quad \mathcal{N} A u = u, \quad \forall\, u \in V_0.$$

It is well known that for all $f \in V_0'$, $\mathcal{N} f$ is the unique $u \in V_0$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \langle f, v\rangle_{V',V}, \quad \forall\, v \in V.$$

On account of the above definitions, the following properties hold:

$$\langle Au, \mathcal{N} f\rangle_{V',V} = \langle f, u\rangle_{V',V}, \quad \forall\, u \in V,\ \forall\, f \in V_0', \tag{2.3}$$

$$\langle f, \mathcal{N} g\rangle_{V',V} = \langle g, \mathcal{N} f\rangle_{V',V} = \int_\Omega \nabla(\mathcal{N} f) \cdot \nabla(\mathcal{N} g)\, dx, \quad \forall\, f, g \in V_0', \tag{2.4}$$

Moreover, owing to (2.4), it is straightforward to prove that

$$\|f\|_* := \|\nabla \mathcal{N} f\|_H = \langle f, \mathcal{N} f\rangle_{V',V}^{\frac{1}{2}} \tag{2.5}$$

is an equivalent norm in V_0', and

$$\langle u_t(t), \mathcal{N} u(t)\rangle_{V',V} = \frac{1}{2}\frac{d}{dt}\|u(t)\|_*^2, \quad \text{for a.e. } t \in (0,T),\ \forall\, u \in H^1(0,T;V_0') \tag{2.6}$$

as well as

$$\|f-\overline{f}\|_*^2 + |\overline{f}|^2 \quad \text{is an equivalent norm in } V'. \tag{2.7}$$

We also introduce the solenoidal Hilbert spaces

$$G_{div} = \left\{ \mathbf{u} \in L^2(\Omega; \mathbb{R}^d) : \text{div}\, \mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}\,|_{\partial\Omega} = 0 \right\}$$

and

$$V_{div} = \left\{ \mathbf{u} \in H^1(\Omega; \mathbb{R}^d) : \text{div}\, \mathbf{u} = 0, \mathbf{u}\,|_{\partial\Omega} = 0 \right\}.$$

As customary, we denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm, respectively, in G_{div}, while V_{div} is equipped with inner product and induce norm

$$(\mathbf{u}, \mathbf{v})_{V_{div}} = (\nabla \mathbf{u}, \nabla \mathbf{v}), \quad \|\mathbf{u}\|_{V_{div}} = \|\nabla \mathbf{u}\|.$$

We define the trilinear V_{div}-continuous form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx, \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in V_{div}.$$

It is well-known that the following identity holds

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}), \quad \forall\ \mathbf{u},\mathbf{v},\mathbf{w}\in V_{div},$$

as well as the two-dimensional estimate

$$|b(\mathbf{u},\mathbf{v},\mathbf{w})| \le C\|\mathbf{u}\|^{\frac12}\|\nabla\mathbf{u}\|^{\frac12}\|\nabla\mathbf{v}\|\|\mathbf{w}\|^{\frac12}\|\nabla\mathbf{w}\|^{\frac12}. \tag{2.8}$$

We recall some classical Sobolev inequality which will be useful in the sequel (see, for instance, [12] and [52] and references therein):

- Gagliardo–Nirenberg inequality

$$\|u\|_{L^p(\Omega)} \le C\|u\|^{\frac2p}\|u\|_V^{1-\frac2p}, \quad \forall u\in V, \tag{2.9}$$

$$\|u\|_{L^p(\Omega)} \le C\|u\|^{\frac{6-p}{2p}}\|u\|_V^{\frac{3p-6}{2p}}, \quad \forall u\in V, \tag{2.10}$$

  if $p\ge 2$ when $d=2$ and $2\le p\le 6$ when $d=3$.

- Trudinger–Moser inequality

$$\int_\Omega e^{|u|}\,dx \le Ce^{C\|u\|_V^2}, \quad \forall\ u\in V. \tag{2.11}$$

We conclude the section by reporting an useful application of the Gagliardo–Nirenberg inequality in dimension two which will be needed in the following sections.

Lemma 2.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^2$ and $u\in V$. Then, for any $\varepsilon>0$ and any integer $1\le s<\infty$, there exists $C=C(s)>0$ such that*

$$\|u\|^2_{L^s(\Omega)} \le \varepsilon\|\nabla u\|^2 + \left(\varepsilon + \frac{C}{\varepsilon^{s-1}}\right)\|u\|^2_{L^1(\Omega)}. \tag{2.12}$$

Proof. We start from the following particular case of Gagliardo–Nirenberg inequality in dimension two (see [12][Comments on Chapter 9])

$$\|u\|_{L^s(\Omega)} \le C\|u\|^{\frac1s}_{L^1(\Omega)}\|u\|_V^{1-\frac1s},$$

valid for any $1\le s<\infty$. Exploiting the Poincaré–Wirtinger inequality, we have, for any $\varepsilon>0$,

$$\|u\|_{L^s(\Omega)} \le C\varepsilon^{1-\frac1s}\left(\|\nabla u\| + \|u\|_{L^1(\Omega)}\right)^{1-\frac1s}\left(\|u\|^{\frac1s}_{L^1(\Omega)}\varepsilon^{\frac1s-1}\right).$$

Now, applying the Young inequality with exponents $(s/(s-1),s)$ to the previous inequality, we obtain

$$\begin{aligned}\|u\|_{L^s(\Omega)} &\le \varepsilon\left(\|\nabla u\| + \|u\|_{L^1(\Omega)}\right) + \frac{C}{\varepsilon^{s-1}}\|u\|_{L^1(\Omega)}\\ &= \varepsilon\|\nabla u\| + \left(\varepsilon + \frac{C}{\varepsilon^{s-1}}\right)\|u\|_{L^1(\Omega)}.\end{aligned}$$

Then, by rescaling ε with $\sqrt{\varepsilon}$, we easily infer (2.12).

Throughout this work, C and K denote positive constants which may be estimated according to the parameters of the system and whose values may change even within the same line of a given equation. If necessary, their possible dependence on some quantity will be indicated explicitly. Moreover, given a metric space X, $B_H(0,R)$ denotes the closed ball of X centered in 0 with radius R.

3 Well-Posedness

The main assumptions on the interaction kernel J and the singular potential F are

(H.1) $J \in W^{1,1}(\mathbb{R}^d)$ with $J(x) = J(-x)$;

(H.2) $F \in \mathcal{C}([-1,1]) \cap \mathcal{C}^2(-1,1)$ such that

$$\lim_{s\to -1} F'(s) = -\infty, \quad \lim_{s\to 1} F'(s) = +\infty, \quad F''(s) \geq \alpha > 0.$$

We also extend $F(s) = +\infty$ for any $s \notin [-1,1]$. Notice that assumption (H.2) implies that there exists $\xi \in (-1,1)$ such that $F'(\xi) = 0$. There is no loss of generality in assuming $F(\xi) = 0$ since the potential is defined up to a constant. This also entails that $F(s) \geq 0$ for all $s \in [-1,1]$. Also, note that no assumption on the sign of J is made.

Remark 3.1. Hypothesis (H.2) is satisfied, in particular, by the logarithmic potential

$$F(s) = \frac{\theta}{2}\Big[(1+s)\log(1+s) + (1-s)\log(1-s)\Big].$$

We are now ready to give the definition of a weak solution to problem (1.13)-(1.14).

Definition 3.2. *Let φ_0 be a measurable function with $F(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A function φ is a weak solution to (1.13)-(1.14) on $[0,T]$ corresponding to φ_0 if*

$$\begin{aligned}
&\varphi \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\
&\varphi \in L^\infty(\Omega \times (0,T)) \textit{ with } |\varphi(x,t)| < 1 \textit{ a.e. } (x,t) \in \Omega \times (0,T),\\
&\mu = F'(\varphi) - J * \varphi \in L^2(0,T;V),
\end{aligned}$$

and satisfies the identity

$$\langle \varphi_t, v\rangle_{V',V} + (\nabla\mu, \nabla v) = 0 \quad \forall\, v \in V, \textit{ a.e. } t \in (0,T), \tag{3.1}$$

with $\varphi(0,\cdot) = \varphi_0$.

Remark 3.3. Let us observe that:

1. From $F(\varphi_0) \in L^1(\Omega)$ we deduce that $|\varphi_0(x)| \leq 1$, for almost any $x \in \Omega$.

2. The conservation of mass is a straightforward consequence of (3.1). Indeed, taking $v = 1$, we get $\langle \varphi_t, 1\rangle_{V',V} = 0$, so $\overline{\varphi}(t) = \overline{\varphi}_0$ for all $t \geq 0$.

3. Let $T > 0$ be arbitrary. Note that $\varphi \in L^\infty(\Omega \times (0,T))$ with $|\varphi(x,t)| < 1$ for almost any $(x,t) \in \Omega \times (0,T)$ implies $\varphi \in L^\infty(0,T;L^p(\Omega))$, for all $p \geq 1$, and $\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \leq |\Omega|^{\frac{1}{p}}$. Moreover, we observe that the function $t \mapsto \|\varphi(t)\|_{L^\infty(\Omega)}$ is measurable, essentially bounded and, for all $f \in L^1(0,T;L^1(\Omega))$, there holds

$$|(\varphi(t), f(t))| \leq \|f(t)\|_{L^1(\Omega)}, \qquad \text{a.e. } \ t \in (0,T).$$

   We refer the reader to [27].

4. As a direct consequence of Definition 3.2, we have $\varphi \in \mathcal{C}([0,T],H)$ and $F'(\varphi) \in L^2(0,T;V)$. The former property entails that the initial condition is well defined.

The well-posedness of system (1.13)-(1.14) is given by (cf. (1.5))

Theorem 3.4. *Let φ_0 be a measurable function with $F(\varphi_0) \in L^1(\Omega)$, $|\overline{\varphi}_0| < 1$ and $T > 0$ be given. Assume that hypotheses $(H.1) - (H.2)$ are satisfied. Then, there exists a unique weak solution φ to* (1.13)-(1.14) *which satisfies the dissipative inequality, for all $t \geq 0$,*

$$\mathcal{E}(\varphi(t)) + \omega \int_t^{t+1} \|\nabla \varphi(\tau)\|^2 + \|\nabla \mu(\tau)\|^2 \, d\tau \leq \mathcal{E}(\varphi_0)e^{-\omega t} + C(1 + F(\overline{\varphi}_0)), \tag{3.2}$$

where ω and C are positive constants independent of the initial condition. Moreover, for every two weak solutions φ_1 and φ_2 to (1.13)-(1.14) *on $[0,T]$ with initial data φ_{01} and φ_{02}, respectively, the following continuous dependence estimate holds for all $t \in [0,T]$*

$$\|\varphi_1(t) - \varphi_2(t)\|_{V'}^2 \leq \|\varphi_{01} - \varphi_{02}\|_{V'}^2 e^{CT} + K|\overline{\varphi}_{01} - \overline{\varphi}_{02}|e^{CT}, \tag{3.3}$$

where

$$K = C\left(\|F'(\varphi_1)\|_{L^1(0,T;L^1(\Omega))} + \|F'(\varphi_2)\|_{L^1(0,T;L^1(\Omega))}\right).$$

Remark 3.5. By virtue of the dissipative inequality (3.2) and $\varphi \in \mathcal{C}([0,T],H)$, the function $t \to \int_\Omega F(\varphi(t))\,dx$ is bounded for all $t \geq 0$. This immediately entails that

$$\sup_{t\geq 0} \|\varphi(t)\|_{L^\infty(\Omega)} \leq 1.$$

As a consequence, we deduce by interpolation that $\varphi \in \mathcal{C}([0,T],L^p(\Omega))$, for any $p \geq 2$.

The proof of Theorem 3.4 is based on a general method which consists in several steps. First, we provide a family of regular function defined on the whole $\mathbb{R}$ which approximates the singular potential. The existence of a weak solution to (1.13)-(1.14) with a regular potential is established via the Galerkin method (see [17]). Then, we show (uniform) estimates on the solutions of this approximate problem in order to pass to the limit via compactness. To the best of our knowledge, Theorem 3.4 ensures the existence and uniqueness of a weak solution in the most general framework. Indeed, it requires the convexity of the potential whereas other existence results (cf., for example, [30, Corollary 1]) require further monotonicity and sign conditions on higher derivatives (i.e. from the second one up) of F. For related results obtained within a more abstract framework see also [18].

Remark 3.6. We highlight that our analysis relies on the assumption $\overline{\varphi}_0 \in (-1,1)$ (see also, for instance, [47] for the standard Cahn–Hilliard equation). This is physically reasonable since $\overline{\varphi}_0 = 1$ (or $\overline{\varphi}_0 = -1$) means that the initial condition is a pure phase, so that no phase separation takes place in Ω.

3.1 Proof of Theorem 3.4

Approximation of F.
Let us consider the singular potential F. According to (H.2), it is immediate to prove that F is proper, convex and lower semicontinuous with domain $D(F) = [-1,1]$. Appealing to theory of maximal monotone operators (see, for instance, [7, 13, 56] and references therein), we define the subgradient of F as

$$\mathbb{A} = \partial F : D(\mathbb{A}) \subset \mathbb{R} \to \mathbb{R}.$$

For the reader's convenience, we report here below a result which establishes the action of the subgradient operator at regularity points (see [7, Chapter 1, Example 3]).

Lemma 3.7. *Let $\varphi : \mathbb{R} \to (-\infty, +\infty]$ be convex and differentiable at $s \in \mathbb{R}$. Then $\partial\varphi(s) = \varphi'(s)$.*

Since F is continuously differentiable in $(-1, 1)$, Lemma 3.7 yields

$$\mathbb{A}(s) = F'(s), \quad \forall\, s \in (-1, 1), \tag{3.4}$$

where F' stands for the standard derivative of F. Moreover, we also have the following characterization.

Lemma 3.8. *Let the potential F satisfy (H.2). Then $D(\mathbb{A}) \equiv (-1, 1)$.*

Proof. Observe that $\mathbb{A}$ is defined in $(-1, 1)$ by (3.4). Thanks to [56, Corollary 1.4, Chapter 4], we also know that

$$(-1, 1) \subset D(\mathbb{A}) \subset D(F) = [-1, 1].$$

We suppose by contradiction that $1 \in D(\mathbb{A})$ and we consider $z \in \mathbb{A}(1) \subset \mathbb{R}$. It is immediate to see that $1 + z \in 1 + \mathbb{A}(1) = (I + \mathbb{A})(1)$. Besides, the map

$$g : (-1, 1) \to \mathbb{R}, \quad g(s) = (I + \mathbb{A})(s) = s + F'(s)$$

is surjective since it is continuous, $\lim_{s\to 1^-} g(s) = +\infty$ and $\lim_{s\to -1^+} g(s) = -\infty$. Thus, there exists $\overline{s} \in (-1, 1)$ such that $g(\overline{s}) = 1 + z$. Since $\mathbb{A}$ is a maximal monotone operator, the inclusion $1 + z \in (I + \mathbb{A})s$ has at most one solution, so $1 \notin D(\mathbb{A})$. Repeating the same argument for -1, we conclude that $D(\mathbb{A}) = (-1, 1)$. $\square$

Thanks to the elementary properties of maximal monotone operators (see for instance [13] and [56]), we approximate F by means of the sequence of everywhere defined non-negative functions

$$F_\lambda(s) = \frac{\lambda}{2}|\mathbb{A}_\lambda s|^2 + F(J_\lambda(s)), \qquad s \in \mathbb{R},\ \lambda > 0, \tag{3.5}$$

where $J_\lambda = (I + \lambda\mathbb{A})^{-1}$ is the resolvent operator and $\mathbb{A}_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ is the Yosida approximation of $\mathbb{A}$. According to the general theory, the following main properties holds:

(i) F_λ is convex and $F_\lambda(s) \nearrow F(s)$, for all $s \in \mathbb{R}$, as λ goes to 0;

(ii) $F'_\lambda(s) = \mathbb{A}_\lambda(s)$ and F'_λ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$;

(iii) $|F'_\lambda(s)| \nearrow |F'(s)|$ for all $s \in (-1, 1)$ and $|F'_\lambda(s)| \nearrow \infty$, for all $|s| \geq 1$, as λ goes to 0;

(iv) $F_\lambda(\xi) = F'_\lambda(\xi) = 0$, for all $\lambda > 0$, where ξ is defined in (H.2).

Remark 3.9. We recall that, due to the convexity of F_λ (see (i)), we have

$$F_\lambda(s) \leq F_\lambda(w) + (s - w)F'_\lambda(s), \quad \text{for all } s, w \in \mathbb{R}. \tag{3.6}$$

Now we formulate and prove some uniform properties of F_λ.

Lemma 3.10. *For any $\lambda \in (0, 1]$, $F''_\lambda(s)$ exists for all $s \in \mathbb{R}$ and*

$$F''_\lambda(s) \geq \frac{\alpha}{1 + \alpha}, \quad \forall\, s \in \mathbb{R}. \tag{3.7}$$

Proof. We preliminarily note that J_λ is the inverse function of $g_\lambda(s) = (I + \lambda\mathbb{A})(s) : (-1,1) \to \mathbb{R}$ which is differentiable with $g'_\lambda(s) \geq 1 + \lambda\alpha > 0$. This entails that $\mathbb{A}_\lambda$ is differentiable in $\mathbb{R}$. Then, from the differentiation formula of the inverse function and the assumption (H.2), we deduce that

$$F''_\lambda(s) = \frac{1}{\lambda}\left[1 - \frac{1}{1 + \lambda F''(J_\lambda(s))}\right] \geq \frac{\alpha}{1 + \lambda\alpha}, \tag{3.8}$$

and, in particular, we get (3.7).

Lemma 3.11. *For any $0 < \lambda^* \leq 1$, we have*

$$F_\lambda(s) \geq \frac{1}{4\lambda^*}s^2 - K, \qquad \forall\ s \in \mathbb{R},\ \forall\ 0 < \lambda \leq \lambda^*, \tag{3.9}$$

where K depends only on λ^* but is independent of λ.

Proof. We infer from de L'Hôpital's rule that

$$\lim_{s\to\pm\infty} \frac{F_\lambda(s)}{s^2} = \lim_{s\to\pm\infty} \frac{F'_\lambda(s)}{2s} = \lim_{s\to\pm\infty} \frac{s - J_\lambda(s)}{2\lambda s} = \frac{1}{2\lambda} - \lim_{s\to\pm\infty} \frac{J_\lambda(s)}{2\lambda s} = \frac{1}{2\lambda},$$

where we have used that $Range(J_\lambda) = (-1,1)$. Setting $0 < \lambda^* \leq 1$, the above limit entails that there exists M_{λ^*} such that

$$F_{\overline{\lambda}}(s) \geq \frac{1}{4\overline{\lambda}}s^2, \qquad \forall\ |s| \geq M_{\lambda^*}.$$

On account of the monotonicity of F_λ with respect to λ, we have

$$F_\lambda(s) \geq \frac{1}{4\lambda^*}s^2, \qquad \forall\ |s| \geq M_{\lambda^*}, \quad \forall\ 0 < \lambda \leq \lambda^*.$$

On the other hand, since F_λ is non-negative, according to the last inequality, we conclude that

$$F_\lambda(s) \geq \frac{1}{4\lambda^*}s^2 - K, \qquad \forall\ s \in \mathbb{R}, \quad \forall\ 0 < \lambda \leq \lambda^*,$$

where $K = M^2_{\lambda^*}/(4\lambda^*)$ is independent of λ.

Lastly, we state an immediate result of convergence of F_λ to F.

Lemma 3.12. *For any set $[a,b] \subset (-1,1)$, F'_λ converges uniformly to F' on $[a,b]$.*

The approximating problem and the dissipative inequality.

For any fixed $\lambda > 0$, we consider the problem (1.13)-(1.14) replacing F with F_λ. The corresponding problem reads as follows

$$\begin{cases} \varphi_t = \Delta\mu, \\ \mu = F'_\lambda(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{3.10}$$

subject to

$$\partial_n \mu = 0, \quad \text{on } \partial\Omega \times (0,T), \quad \varphi(\cdot,0) = \varphi_0, \quad \text{in } \ \Omega. \tag{3.11}$$

Here, we simply use φ instead of φ_λ for the sake of simplicity. We denote the energy functional $\mathcal{E}_\lambda : H \to \mathbb{R}$ (cf. (1.5)) by

$$\mathcal{E}_\lambda(v) = \int_\Omega F_\lambda(v)\,dx - \frac{1}{2}\,(J * v, v)$$

and we show the dissipative nature of the system.

Lemma 3.13. *There exists $\overline{\lambda} > 0$ such that, for any $0 < \lambda \leq \overline{\lambda}$, any solution to* (3.10)-(3.11) *satisfies, for all $t \geq 0$,*

$$\mathcal{E}_\lambda(\varphi(t)) + \omega \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, d\tau \leq \mathcal{E}_\lambda(\varphi_0) e^{-\omega t} + C(1 + F_\lambda(\overline{\varphi}_0)). \tag{3.12}$$

Here, ω and C are positive constant that depend on J and α but are independent of the initial condition and λ.

We provide below a formal proof of Lemma 3.13. A rigorous argument can be done by performing the same computations within a Galerkin approximation scheme (see the proof of Theorem 3.15 reported below).

Proof. Let us consider $\mathcal{E}_\lambda$. By virtue of Lemma 3.11 and the Young inequality for convolution, for any $\lambda < \overline{\lambda}$, we obtain

$$\begin{aligned}\mathcal{E}_\lambda(v) &> \frac{1}{4\overline{\lambda}}\|v\|^2 - K|\Omega| - \frac{1}{2}\|J * v\|\|v\| \\ &\geq \left(\frac{1}{4\overline{\lambda}} - \frac{\|J\|_{L^1(\Omega)}}{2}\right)\|v\|^2 - K|\Omega|.\end{aligned}$$

Therefore, for any $\gamma > 0$ there exists K such that

$$\mathcal{E}_\lambda(v) \geq \gamma\|v\|^2 - K|\Omega| \tag{3.13}$$

provided that $\overline{\lambda}$ is small enough. It is also apparent from (ii), (iv) and (3.6) that

$$F_\lambda(s) \leq (s - \xi)F'_\lambda(s) \leq \frac{1}{\lambda}|s - \xi|^2.$$

Thus, we deduce that

$$\mathcal{E}_\lambda(v) \leq \left(\frac{2}{\lambda} + \frac{\|J\|_{L^1(\Omega)}}{2}\right)\|v\|^2 + \frac{2}{\lambda}|\Omega|. \tag{3.14}$$

Now, testing $(3.10)_1$ and $(3.10)_2$ by μ and φ_t, respectively, and adding the two equations, we obtain

$$\frac{d}{dt}\mathcal{E}_\lambda(\varphi) + \|\nabla\mu\|^2 = 0. \tag{3.15}$$

In order to reconstruct the energy functional on the left-hand side, we take the gradient of $(3.10)_2$ and we test by $\nabla\varphi$ yielding

$$\big(F''_\lambda(\varphi)\nabla\varphi, \nabla\varphi\big) = (\nabla\mu, \nabla\varphi) + (\nabla J * \varphi, \nabla\varphi).$$

According to Lemma 3.10 and the Young inequality for convolution, we get

$$\frac{\beta}{2}\|\nabla\varphi\|^2 \leq \frac{1}{2\beta}\|\nabla\mu\|^2 + \frac{1}{2\beta}\|\nabla J\|^2_{L^1(\Omega)}\|\varphi\|^2, \tag{3.16}$$

where $\beta = \alpha/(1+\alpha)$. On the other hand, testing again $(3.10)_2$ by $\varphi - \overline{\varphi}$ and using the Poincaré inequality, we obtain

$$\begin{aligned}\big(F'_\lambda(\varphi), \varphi - \overline{\varphi}\big) &= (J * \varphi, \varphi - \overline{\varphi}) + (\mu, \varphi - \overline{\varphi}) \\ &\leq C\|J * \varphi\|\|\nabla\varphi\| + C\|\nabla\mu\|\|\nabla\varphi\|.\end{aligned} \tag{3.17}$$

Exploiting (3.6) with $s = \varphi$, $w = \overline{\varphi}$, we find

$$\mathcal{E}_\lambda(\varphi) \le F_\lambda(\overline{\varphi})|\Omega| + \big(F'_\lambda(\varphi), \varphi - \overline{\varphi}\big) - \frac{1}{2}\,(J * \varphi, \varphi)\,. \tag{3.18}$$

Combining (3.17) with (3.18), and using the Young inequality, we infer that

$$\begin{aligned}\mathcal{E}_\lambda(\varphi) &\le F_\lambda(\overline{\varphi})|\Omega| + C\|J*\varphi\|\|\nabla\varphi\| + C\|\nabla\mu\|\|\nabla\varphi\| + \frac{1}{2}|(J*\varphi,\varphi)| \\ &\le F_\lambda(\overline{\varphi})|\Omega| + \frac{\beta}{4}\|\nabla\varphi\|^2 + \frac{C}{2\beta}\|\nabla\mu\|^2 + \Big(\frac{C}{2\beta}\|J\|^2_{L^1(\Omega)} + \frac{1}{2}\|J\|_{L^1(\Omega)}\Big)\|\varphi\|^2.\end{aligned}$$

Adding (3.16) to the above inequality, we reach

$$\begin{aligned}\mathcal{E}_\lambda(\varphi) + \frac{\beta}{4}\|\nabla\varphi\|^2 \le &\left(\frac{C+1}{2\beta}\right)\|\nabla\mu\|^2 + F_\lambda(\overline{\varphi})|\Omega| \\ &+ \Big(\frac{1}{2\beta}\|\nabla J\|^2_{L^1(\Omega)} + \frac{C}{2\beta}\|J\|^2_{L^1(\Omega)} + \|J\|_{L^1(\Omega)}\Big)\|\varphi\|^2.\end{aligned}$$

In light of the control from below (3.13), there exists $\overline{\lambda} > 0$ such that for any $0 < \lambda < \overline{\lambda}$ we have

$$\frac{1}{2}\mathcal{E}_\lambda(\varphi) + \frac{\beta}{4}\|\nabla\varphi\|^2 \le \frac{C+1}{2\beta}\|\nabla\mu\|^2 + F_\lambda(\overline{\varphi})|\Omega| + \frac{K}{2}|\Omega|. \tag{3.19}$$

Summing up, by (3.15) and (3.19) we find the differential inequality

$$\frac{d}{dt}\mathcal{E}_\lambda(\varphi) + \omega\Big(\mathcal{E}_\lambda(\varphi) + \|\nabla\varphi\|^2 + \|\nabla\mu\|^2\Big) \le C\,(1 + F_\lambda(\overline{\varphi}))\,,$$

for some $\omega > 0$ independent of λ. Finally, an application of the Gronwall lemma completes the argument. $\square$

Existence of an approximate solution.

By analogy with Definition 3.2, we recall the definition of weak solution.

Definition 3.14. *Let φ_0 be a measurable function with $F_\lambda(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A function φ is a weak solution to problem* (3.10)-(3.11) *on* $[0,T]$ *corresponding to* φ_0 *if*

$$\begin{aligned}&\varphi \in H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V),\\ &\mu = F'_\lambda(\varphi) - J*\varphi \in L^2(0,T;V),\end{aligned}$$

and φ satisfies the identity

$$\langle\varphi_t, v\rangle_{V',V} + (\nabla\mu, \nabla v) = 0 \quad \forall\, v \in V,\ \textit{a.e. } t \in (0,T),$$

with $\varphi(0,\cdot) = \varphi_0$.

It is immediate to see that points 2 and 4 of Remark 3.3 are valid in the regular potential case as well. We can thus prove the existence of a global weak approximating solution.

Theorem 3.15. *Let φ_0 be a measurable function with $F_\lambda(\varphi_0) \in L^1(\Omega)$ and $0 < \lambda \le \overline{\lambda}$. Then there exists a weak solution φ to problem* (3.10)-(3.11) *which fulfills the dissipative inequality* (3.12) *for all* $t \ge 0$.

Proof. The existence of a weak solution is established through a Galerkin scheme. Let us $n \in \mathbb{N}$ be fixed. We seek a function

$$\varphi_n(t) = \sum_{k=1}^{n} a_k(t)\psi_k$$

which solves for all $t \in (0,T)$

$$\langle \varphi_{n,t}, w\rangle_{V',V} + (\nabla\mu_n, \nabla w) = 0, \quad \forall\, w \in V_n, \tag{3.20}$$

where

$$\mu_n = \Pi_n \left[F'_\lambda(\varphi_n) - J * \varphi_n\right].$$

Here, $\{\psi_k\}_k$ are the eigenfunctions associated to the operator A, $V_n = \operatorname{span}\{\psi_1, ..., \psi_n\}$, Π_n is the projector operators from V onto V_n and $\varphi_{0n} = \Pi_n(\varphi_0)$. We observe that $\varphi_0 \in H$ due to $F_\lambda(\varphi_0) \in L^1(\Omega)$. Equation (3.20) is equivalent to a system of ordinary differential equations $\dot{\mathbf{a}}_n(t) = \mathcal{G}(\mathbf{a}_n(t))$, where $\mathbf{a}_n(t) = [a_1(t), ..., a_n(t)]$ is the unknown and $\mathcal{G}$ is a locally Lipschitz continuous function of $\mathbf{a}_n$. Then, the Cauchy–Lipschitz theorem entails the existence of a unique local solution $\mathbf{a}_n \in C^1([0,T^*), \mathbb{R}^n)$.
Since $w = 1$ is the first eigenfunction of A, we note that the conservation of mass holds for the approximated problem, namely, $\overline{\varphi}_n(t) = \overline{\varphi}_{0n}$. Thanks to Lemma 3.13, we derive some uniform estimates in order to guarantee that $T^* = \infty$ and recover compactness properties of the sequence φ_n. Indeed, the Galerkin approximation φ_n fulfills the following inequality for all $t \geq 0$,

$$\mathcal{E}_\lambda(\varphi_n(t)) + \omega \int_t^{t+1} \|\nabla\varphi_n(\tau)\|^2 + \|\nabla\mu_n(\tau)\|^2 \, d\tau \leq \mathcal{E}_\lambda(\varphi_{0n})e^{-\omega t} + C(1 + F_\lambda(\overline{\varphi}_{0n})).$$

By $\varphi_{0n} \to \varphi_0$ in H and (3.13) and (3.14), the right-hand side of can be controlled by a constant independent of n and we deduce that

$$\varphi_n \quad \text{is uniformly bounded in } L^\infty(0,T;H) \cap L^2(0,T;V), \tag{3.21}$$

$$\nabla\mu_n \quad \text{is uniformly bounded in } L^2(0,T;H). \tag{3.22}$$

On account of (iii) and the above boundedness properties, we have

$$|\overline{\mu}_n| \leq C\left(1 + \|\varphi_n\|_{L^1(\Omega)}\right) \leq C,$$

where C is independent by n. In turn, this combined with (2.2) entails that

$$\mu_n \quad \text{is uniformly bounded in } L^2(0,T;V). \tag{3.23}$$

By comparison we find

$$F'_\lambda(\varphi_n) \quad \text{is uniformly bounded in } L^2(\Omega \times (0,T)), \tag{3.24}$$

$$\varphi_{n,t} \quad \text{is uniformly bounded in } L^2(0,T;V'). \tag{3.25}$$

Thanks to (3.21)-(3.25) and standard compactness arguments, we infer that, up to subsequences,

$$\varphi_n \to \varphi \text{ weakly in } L^2(0,T;V),$$

$$
\begin{aligned}
&\varphi_n \to \varphi \text{ weakly star in } L^\infty(0,T;H),\\
&\varphi_{n,t} \to \varphi \text{ weakly in } L^2(0,T;V'),\\
&\mu_n \to \mu \text{ weakly in } L^2(0,T;V),\\
&F'_\lambda(\varphi_n) \to F'_\lambda(\varphi) \text{ weakly in } L^2(\Omega\times(0,T)).
\end{aligned}
$$

Hence, we can pass to the limit in the approximation problem achieving the existence of a weak solution to (3.10)-(3.11) in the sense of Definition 3.14. From $\varphi \in L^2(0,T;V)$ and $\varphi_t \in L^2(0,T;V')$, we also deduce that $\varphi \in \mathcal{C}([0,T],H)$. Furthermore, according to the above convergences properties, and passing to the limit in the dissipative inequality, the weak solution satisfies, for almost every $t \geq 0$,

$$
\mathcal{E}_\lambda(\varphi(t)) + \omega \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, d\tau \leq \mathcal{E}_\lambda(\varphi_0)e^{-\omega t} + C(1 + F_\lambda(\overline{\varphi}_0)).
$$

In particular, we have used the fact that $\varphi_{0n} \to \varphi_0$ in H entails that $\mathcal{E}(\varphi_{0n}) \to \mathcal{E}(\varphi_0)$, which easily follows from

$$
|F_\lambda(s) - F_\lambda(w)| \leq \frac{1}{\lambda}|s-w|\max\{|s-\xi|,|w-\xi|\}, \quad \forall\ s,w \in \mathbb{R}. \tag{3.26}
$$

We conclude by observing that the above dissipation inequality holds for every $t \geq 0$ by virtue of $\varphi \in \mathcal{C}([0,T],H)$. $\square$

We can now prove Theorem 3.4.

Passage to the limit.

First, we observe that $F(\varphi_0) \in L^1(\Omega)$ implies that $F_\lambda(\varphi_0) \in L^1(\Omega)$ for any $\lambda > 0$. Then, as a consequence of Theorem 3.15, for any $\lambda \in (0,\overline{\lambda}]$, there exists a weak solution φ_λ to problem (3.10)-(3.11) which satisfies for all $t \geq 0$,

$$
\mathcal{E}_\lambda(\varphi_\lambda(t)) + \omega \int_t^{t+1} \|\nabla\varphi_\lambda(\tau)\|^2 + \|\nabla\mu_\lambda(\tau)\|^2 \, d\tau \leq \mathcal{E}(\varphi_0)e^{-\omega t} + C(1 + F(\overline{\varphi}_0)).
$$

Here, we have used (i) to control the right-hand side. Hence, in light of (3.13), this entails that

$$
\begin{aligned}
\varphi_\lambda &\quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^\infty(0,T;H), && (3.27)\\
\varphi_\lambda &\quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;V), && (3.28)\\
\nabla\mu_\lambda &\quad \text{is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;H). && (3.29)
\end{aligned}
$$

By comparison we also obtain

$$
\varphi_{\lambda,t} \quad \text{is uniformly bounded w.r.t } \lambda \text{ in } L^2(0,T;V'). \tag{3.30}
$$

In order to pass to the limit we need to recover a uniform estimate for μ_λ in V. To this aim, we first control the $L^1(\Omega)-$norm of $F'_\lambda(\varphi_\lambda)$. We apply the argument devised in [47] (see also [30] for the details). Let us choose $m_1, m_2 \in (-1,1)$ in such a way that $m_1 \leq \xi \leq m_2$ and $m_1 < \overline{\varphi}_0 < m_2$. We also set $\delta := \min\{\overline{\varphi}_0 - m_1, m_2 - \overline{\varphi}_0\}$ and $\delta_1 := \max\{\overline{\varphi}_0 - m_1, m_2 - \overline{\varphi}_0\}$. Then, for almost every $t \in (0,T)$, we consider the sets

$$
\Omega_0 := \{m_1 \leq \varphi_\lambda(x,t) \leq m_2\}, \quad \Omega_1 := \{\varphi_\lambda(x,t) < m_1\}, \quad \Omega_2 := \{\varphi_\lambda(x,t) > m_2\}.
$$

Since F'_λ is monotone and $F'_\lambda(\xi) = 0$ for any λ, using the assumption $\overline{\varphi}_0 \in (-1, 1)$ and property (iii), we get

$$
\begin{aligned}
\delta \|F'_\lambda(\varphi_\lambda)\|_{L^1(\Omega)} &= \delta \int_{\Omega_0} |F'_\lambda(\varphi_\lambda)| \, dx + \delta \int_{\Omega_1} |F'_\lambda(\varphi_\lambda)| \, dx + \delta \int_{\Omega_2} |F'_\lambda(\varphi_\lambda)| \, dx \\
&\leq \delta \int_{\Omega_0} |F'_\lambda(\varphi_\lambda)| \, dx + \int_{\Omega_1} (\varphi - \overline{\varphi}_0) \, F'_\lambda(\varphi_\lambda) \, dx + \int_{\Omega_2} (\varphi - \overline{\varphi}_0) \, F'_\lambda(\varphi_\lambda) \, dx \\
&\leq (\delta + \delta_1) \int_{\Omega_0} |F'(\varphi_\lambda)| \, dx + \int_\Omega (\varphi_\lambda - \overline{\varphi}_0) F'_\lambda(\varphi_\lambda) \, dx \\
&\leq K(\overline{\varphi}_0) + \int_\Omega (\varphi_\lambda - \overline{\varphi}_0) F'_\lambda(\varphi_\lambda) \, dx
\end{aligned}
$$

where $K(\overline{\varphi}_0)$ is independent of λ. Now, arguing as in the proof of Lemma (3.13) (cf. (3.17)), we find

$$
\int_\Omega (\varphi_\lambda - \overline{\varphi_0}) F'_\lambda(\varphi_\lambda) \, dx \leq C\Big(1 + \|\nabla \mu_\lambda\|\Big),
$$

where C is independent of λ. Therefore, combining the above inequalities, we deduce from (3.30) that

$$
\int_0^T \|F'_\lambda(\varphi_\lambda)(\tau)\|^2_{L^1(\Omega)} \, d\tau \leq C \left(1 + \int_0^T \|\nabla \mu_\lambda(\tau)\|^2 \, d\tau\right) \leq C \tag{3.31}
$$

where C is independent of λ. In turn, by

$$
\int_\Omega \mu_\lambda \, dx = \int_\Omega F'_\lambda(\varphi_\lambda) \, dx + \int_\Omega J * \varphi_\lambda \, dx,
$$

we get

$$
\|\overline{\mu}_\lambda\|_{L^2(0,T)} \leq C.
$$

Thus, due to the Poincaré–Wirtinger inequality, we arrive at

$$
\mu_\lambda \text{ is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0, T; V). \tag{3.32}
$$

Accordingly, up to subsequences, we have the following convergences

$$
\varphi_\lambda \to \varphi \text{ weakly in } L^2(0, T; V), \tag{3.33}
$$

$$
\varphi_\lambda \to \varphi \text{ weakly star in } L^\infty(0, T; H), \tag{3.34}
$$

$$
\varphi_{\lambda,t} \to \varphi_t \text{ weakly in } L^2(0, T; V'), \tag{3.35}
$$

$$
\mu_\lambda \to \mu \text{ weakly in } L^2(0, T; V). \tag{3.36}
$$

Furthermore, compactness yields

$$
\varphi_\lambda \to \varphi \text{ strongly in } L^2(0, T; H). \tag{3.37}
$$

Also, (H.1) and (3.37) imply that

$$
J * \varphi_\lambda \to J * \varphi \text{ strongly in } L^2(0, T; V). \tag{3.38}
$$

Concerning the nonlinear term, we prove that the limit function φ fulfils

$$
|\varphi(x,t)| < 1 \quad \text{a.e. } (x,t) \text{ in } \Omega \times (0, T).
$$

Let η be such that $\xi \in (-1+\eta, 1-\eta)$. We introduce the sets

$$E_\eta^\lambda = \{(x,t) \in \Omega \times (0,T) : \ |\varphi_\lambda(x,t)| > 1-\eta\},$$
$$E_\eta = \{(x,t) \in \Omega \times (0,T) : \ |\varphi(x,t)| > 1-\eta\}.$$

Since $\varphi_\lambda \to \varphi$ a.e. $(x,t) \in \Omega \times (0,T)$, the Fatou Lemma entails

$$|E_\eta| \leq \liminf_{\lambda \to 0^+} |E_\eta^\lambda|.$$

Recalling that $F_\lambda'(x) \geq 0$ for $x \in [\xi, 1)$, $F_\lambda'(x) \leq 0$ for $x \in (-1, \xi]$ and F_λ' is monotone, we deduce

$$\min\{F'(1-\eta), -F'(-1+\eta)\}|E_\varrho^\varepsilon| \leq \|F_\lambda'(\varphi_\lambda)\|_{L^1(\Omega\times(0,T))} \leq C,$$

where C does not depends on λ and η. Therefore, we have

$$|E_\eta| \leq \frac{C}{\min\{F'(1-\eta), -F'(-1+\eta)\}}.$$

Passing to the limit as $\eta \to 0^+$, we deduce that

$$|\,\{(x,t) \in \Omega \times (0,T) : |\varphi(x,t)| \geq 1\}\,| = 0$$

which yields the desired conclusion. As a byproduct,

$$F_\lambda'(\varphi_\lambda) \to F'(\varphi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

where we have used the pointwise convergence of φ_λ and the uniform convergence of F_λ' to F' (see Lemma 3.12). Moreover, by the expression of μ_λ, we get

$$F_\lambda'(\varphi_\lambda) \text{ is uniformly bounded w.r.t. } \lambda \text{ in } L^2(0,T;H). \tag{3.39}$$

A standard argument implies that $F_\lambda'(\varphi_\lambda) \to F'(\varphi)$ weakly in $L^2(\Omega \times (0,T))$. On account of the above convergences, we easily find that

$$\langle \varphi_t, v\rangle_{V',V} + (\nabla\mu, \nabla v) = 0 \quad \forall\, v \in V, \text{ a.e. } t \in (0,T),$$

with

$$\mu = F'(\varphi) - J * \varphi \in L^2(0,T;V).$$

Now, by virtue of the regularity of φ and φ_t, we have $\varphi \in \mathcal{C}([0,T], H)$. By the above convergences, we pass to limit in the above dissipative inequality satisfied by φ_λ and we learn that, for almost every $t \geq 0$,

$$\mathcal{E}(\varphi(t)) + \omega \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 \, d\tau \leq \mathcal{E}(\varphi_0)e^{-\omega t} + C(1 + F(\overline{\varphi}_0)).$$

On the other hand, the above inequality holds for any $t \geq 0$ since $\varphi \in \mathcal{C}([0,T], H)$. Indeed, $J * \varphi \in \mathcal{C}([0,T], H)$, the integral terms on the left-hand side are continuous as well as the right-hand side. Let $t > 0$, there exists a sequence $\{t_j\}$ which tends to t and for which the above inequality holds. We show that

$$\lim_{t_j \to t} \int_\Omega F(\varphi(t_j))\, dx = \int_\Omega F(\varphi(t))\, dx.$$

On account of the continuity of φ, $\varphi(t_j) \to \varphi(t)$ strongly in H, so there exists a subsequence which converges for almost every $x \in \Omega$ and the limit necessarily satisfies $|\varphi(x,t)| \leq 1$ for almost every $x \in \Omega$. Since F is continuous on the compact set $[-1,1]$, using the Lebesgue theorem, we infer that (3.2) holds for all $t \geq 0$.

Continuous dependence on the initial data and uniqueness.
Let us consider two weak solutions φ_1 and φ_2 related to the initial conditions φ_{01} and φ_{02}, respectively. The function $\varphi(t) = \varphi_1(t) - \varphi_2(t)$ with $\varphi(0) = \varphi_{01} - \varphi_{02}$ solves

$$\langle \varphi_t, v \rangle_{V',V} + (\nabla \mu, \nabla v) = 0, \quad \forall\, v \in V,\ a.e.\ t \in (0,T),$$

where

$$\mu = F'(\varphi_1) - F'(\varphi_2) - J * \varphi.$$

Taking $v = \mathcal{N}(\varphi - \overline{\varphi})$ and exploiting (2.3), for almost every $t \in [0,T]$, we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi - \overline{\varphi}\|_*^2 + (\mu, \varphi - \overline{\varphi}) = 0. \tag{3.40}$$

According to the assumption (H.2) and the definition of the operator $\mathcal{N}$, we deduce that

$$(\mu, \varphi - \overline{\varphi}) \geq\ \alpha\|\varphi\|^2 - (F'(\varphi_1) - F'(\varphi_2), \overline{\varphi}) - (\nabla J * \varphi, \nabla \mathcal{N}(\varphi - \overline{\varphi})).$$

Moreover, we have

$$\begin{aligned} |(\nabla J * \varphi, \nabla \mathcal{N}(\varphi - \overline{\varphi}))| &\leq \|\nabla J * \varphi\| \|\varphi - \overline{\varphi}\|_* \\ &\leq \|\nabla J\|_{L^1(\Omega)} \|\varphi\| \|\varphi - \overline{\varphi}\|_* \\ &\leq \frac{\alpha}{2}\|\varphi\|^2 + C\|\varphi - \overline{\varphi}\|_*^2. \end{aligned}$$

Hence, we find the differential inequality for almost every $t \in [0,T]$,

$$\frac{d}{dt}\|\varphi - \overline{\varphi}\|_*^2 + \alpha\|\varphi\|^2 \leq C\|\varphi - \overline{\varphi}\|_*^2 + \Lambda|\overline{\varphi}|,$$

where

$$\Lambda = 2\|F'(\varphi_1)\|_{L^1(\Omega)} + 2\|F'(\varphi_2)\|_{L^1(\Omega)}.$$

Therefore, an application of the Gronwall Lemma yields, for all $t \in [0,T]$,

$$\|\varphi(t) - \overline{\varphi}\,(t)\,\|_*^2 \leq \|\varphi(0) - \overline{\varphi}\,(0)\,\|_*^2 e^{Ct} + |\overline{\varphi}\,(0)\,| e^{Ct} \int_0^t \Lambda(\tau) d\tau. \tag{3.41}$$

Finally, on account of (2.7), (3.3) follows. As a byproduct, we learn the uniqueness of weak solutions.

4 Regularity and the Global Attractor

In this section we study the regularity properties of the weak solutions which allow us, in particular, to establish the existence of the (smooth) global attractor for the dissipative dynamical system associated with (1.13)-(1.14) (cf. [30]).

We will derive some uniform higher order estimates which will be independent of the form of the initial datum, but only depend on its total mass and the value of the energy. Henceforth, the generic constant C may also depend on $m \in (0,1)$ and R such that

$$-1 + m \le \overline{\varphi}_0 \le 1 - m, \quad \text{and} \quad \mathcal{E}(\varphi_0) \le R.$$

As a consequence of the dissipative inequality (3.2), we have

$$\mathcal{E}(\varphi(t)) + \int_t^{t+1} \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2 + \|\varphi_t(\tau)\|_{V'}^2 \, d\tau \le C, \quad \forall\, t \ge 0. \tag{4.1}$$

Our first regularity result is

Theorem 4.1. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\varphi_t\|_{L^\infty(\sigma,t;V')} + \|\nabla\mu\|_{L^\infty(\sigma,t;H)} + \|\varphi_t\|_{L^2(t,t+1;H)} \le C, \quad \forall\, t \ge \sigma, \tag{4.2}$$

and

$$\sup_{t\ge\sigma} \|\varphi(t)\|_V \le C. \tag{4.3}$$

Proof. We provide below a formal estimate which can be easily justified by exploiting the Galerkin approximation scheme (see proof of Theorem 3.4). We differentiate system (1.13) with respect to time and we obtain

$$\varphi_{tt} = \Delta(F''(\varphi)\varphi_t - J * \varphi_t).$$

Testing by $\mathcal{N}\varphi_t$ and recalling that $\overline{\varphi_t} = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|_*^2 + (F''(\varphi)\varphi_t, \varphi_t) = (J * \varphi_t, \varphi_t).$$

By (H.2)

$$(F''(\varphi)\varphi_t, \varphi_t) \ge \alpha\|\varphi_t\|^2.$$

Reasoning as in the proof of the continuous dependence estimate, the right-hand side is controlled as follows

$$\begin{aligned}(J * \varphi_t, \varphi_t) &= (\nabla J * \varphi_t, \nabla\mathcal{N}\varphi_t)\\ &\le \frac{\alpha}{2}\|\varphi_t\|^2 + C\|\varphi_t\|_*^2.\end{aligned}$$

Here we have used the Young inequality for convolution and (2.5). Summing up, we find differential inequality

$$\frac{d}{dt}\|\varphi_t\|_*^2 + \alpha\|\varphi_t\|^2 \le C\|\varphi_t\|_*^2.$$

Therefore, exploiting (2.7), an application of the uniform Gronwall Lemma gives

$$\|\varphi_t(t)\|_*^2 + \int_t^{t+1} \|\varphi_t(\tau)\|^2 \, d\tau \le C, \quad \forall\, t \ge \sigma. \tag{4.4}$$

By comparison, we easily deduce that

$$\|\nabla\mu(t)\| \le C, \quad \forall\, t \ge \sigma. \tag{4.5}$$

Let us recover a uniform estimate of the weak solution in V. Applying the gradient operator to the chemical potential, and testing by $\nabla\varphi$, we get

$$(\nabla\mu, \nabla\varphi) = (F''(\varphi)\nabla\varphi, \nabla\varphi) - (\nabla J * \varphi, \nabla\varphi).$$

Recalling (H.2) and using Young and Cauchy–Schwarz inequalities, we arrive at

$$\alpha\|\nabla\varphi\|^2 \le \|\nabla\mu\|\|\nabla\varphi\| + \|\nabla J\|_{L^1(\Omega)}\|\varphi\|\|\nabla\varphi\|.$$

Then, on account of point 3 in Remark 3.3, the Young inequality gives

$$\|\nabla\varphi(t)\| \le C, \quad \forall\, t \ge \sigma. \tag{4.6}$$

Since (4.4) and (4.5) hold for the Galerkin aprroximation, from the lower semicontinuity of the norm we deduce (4.2). Finally, we infer (4.3) from (4.6), the continuity $\varphi \in \mathcal{C}([0,T], H)$ and the mass conservation. $\square$

In the following proposition we establish further regularity results and, in particular, a uniform V-bound of μ. These properties will be helpful in the next section.

Proposition 4.2. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|F'(\varphi)\|_{L^\infty(\sigma,t;V)} + \|\mu\|_{L^\infty(\sigma,t;V)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \le C, \quad \forall\, t \ge \sigma, \tag{4.7}$$

and

$$\|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;L^p(\Omega))} \le C, \qquad \text{if } \frac{p-2}{p} = \frac{2}{q},\ d = 2, \tag{4.8}$$

$$\|\nabla\mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla\varphi\|_{L^q(t,t+1;L^p(\Omega))} \le C, \qquad \text{if } \frac{3p-6}{2p} = \frac{2}{q},\ d = 3, \tag{4.9}$$

where $2 \le p < \infty$ if $d = 2$ and $2 \le p \le 6$ if $d = 3$.

Proof. Let us consider the identity

$$\mu - \overline{\mu} = -J * \varphi + \overline{J * \varphi} + F'(\varphi) - \overline{F'(\varphi)}.$$

By the Poincaré–Wirtinger inequality, we deduce that

$$\|F'(\varphi) - \overline{F'(\varphi)}\|_V \le C\|\nabla\mu\| + C\|\nabla J * \varphi\|.$$

Hence, according to Theorem 4.1, we have

$$\|F'(\varphi) - \overline{F'(\varphi)}\|_{L^\infty(\sigma,t;V)} \le C, \quad \forall\, t \ge \sigma.$$

In order to control the missing term $\overline{F'(\varphi)}$, arguing as in the proof of Theorem 3.4, we find

$$\|F'(\varphi)\|_{L^1(\Omega)} \le C\int_\Omega (\varphi - \overline{\varphi}_0)F'(\varphi)\,dx + C.$$

Then, testing μ by $\varphi - \overline{\varphi}_0$ and using (2.2) and (4.3), we obtain

$$\int_\Omega (\varphi - \overline{\varphi}_0)F'(\varphi)\,dx \le C(1 + \|\nabla\mu\|).$$

Therefore, the above inequalities yield

$$\|F'(\varphi)\|_{L^\infty(\sigma,t;L^1(\Omega))} \leq C, \quad \forall\, t \geq \sigma,$$

which, in turn, gives

$$\|\overline{\mu}\|_{L^\infty(\sigma,t)} \leq C, \quad \forall\, t \geq \sigma.$$

Thus, we end up with

$$\|F'(\varphi)\|_{L^\infty(\sigma,t;V)} + \|\mu(t)\|_{L^\infty(\sigma,t;V)} \leq C, \quad \forall\, t \geq \sigma. \tag{4.10}$$

Furthermore, notice that the regularity of φ_t in (4.2), (4.10) and classical elliptic regularity entail that the first equation of problem (1.13) is satisfied for almost every $(x,t) \in \Omega \times (\sigma,\infty)$ and

$$\|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \quad \forall\, t \geq \sigma. \tag{4.11}$$

Arguing now as in [32], we find a control of $\nabla\varphi$ in $L^p(\Omega)$ by means of the L^2-norm of φ_t. To this aim, we take the gradient of μ, multiply it by $|\nabla\varphi|^{p-2}\nabla\varphi$ and integrate over Ω. We observe that this estimate cannot be made rigorous within a Galerkin scheme. Nevertheless, the regularity of the weak solution is enough to compute it. Indeed, on account of (H.1), (H.2), by (4.11) we deduce that

$$\|F''(\varphi)\nabla\varphi\|_{L^2(t,t+1;L^p(\Omega))} \leq C, \quad \forall\, t \geq \sigma, \tag{4.12}$$

where $2 \leq p < \infty$ if $d = 2$ and $2 \leq p < 6$ if $d = 3$. This allows us to multiply by $|\nabla\varphi|^{p-2}\nabla\varphi$ yielding

$$\int_\Omega F''(\varphi)|\nabla\varphi|^p\, dx \leq \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\cdot\nabla\mu\, dx + \int_\Omega |\nabla\varphi|^{p-2}\nabla\varphi\cdot\nabla J * \varphi\, dx.$$

By (H.2) and the Young inequality, we have

$$\alpha\|\nabla\varphi\|^p_{L^p(\Omega)} \leq \|\nabla\mu\|_{L^p(\Omega)}\|\nabla\varphi\|^{p-1}_{L^p(\Omega)} + \|\nabla J\|_{L^1(\Omega)}\|\varphi\|_{L^p(\Omega)}\|\nabla\varphi\|^{p-1}_{L^p(\Omega)}.$$

Then, by (H.1) and (4.3) we get

$$\|\nabla\varphi\|_{L^p(\Omega)} \leq C(\|\nabla\mu\|_{L^p(\Omega)} + \|\varphi\|_V). \tag{4.13}$$

In order to estimate $\nabla\mu$ in $L^p(\Omega)$, the Gagliardo–Nirenberg inequality (2.9), together with (4.7), entails

$$\begin{aligned}\|\nabla\mu\|_{L^p(\Omega)} &\leq C\|\nabla\mu\|^{\frac{2}{p}}\|\nabla\mu\|_V^{1-\frac{2}{p}}\\ &\leq C\Big(\|\Delta\mu\|^{1-\frac{2}{p}} + \|\mu\|^{1-\frac{2}{p}}\Big)\\ &\leq C\Big(\|\varphi_t\|^{1-\frac{2}{p}} + 1\Big).\end{aligned}$$

Hence, setting q such that $\frac{p-2}{p} = \frac{2}{q}$, using (4.2) and (4.13), the estimate (4.8) easily follows. On the other hand, applying the Gagliardo–Nirenberg inequality (2.10) and arguing as before, we get

$$\begin{aligned}\|\nabla\mu\|_{L^p(\Omega)} &\leq C\|\nabla\mu\|^{\frac{6-p}{2p}}\|\nabla\mu\|_V^{\frac{3p-6}{2p}}\\ &\leq C\|\mu\|_{H^2(\Omega)}^{\frac{3p-6}{2p}}\\ &\leq C\Big(\|\varphi_t\|^{\frac{3p-6}{2p}} + 1\Big).\end{aligned}$$

Hence, (4.9) is obtained as a byproduct of (4.2) and (4.13). The proof is complete.

Remark 4.3. We infer from (4.8), (4.9) that $\varphi \in L^\infty(\sigma, t; L^\infty(\Omega))$ with $\|\varphi\|_{L^\infty(\sigma,t;L^\infty(\Omega))} \leq 1$ for all $t \geq \sigma$ and $d = 2, 3$. This is an immediate consequence of $\varphi \in L^6(\sigma, t; W^{1,3}(\Omega))$ for $d = 2$ and $\varphi \in L^{\frac{8}{3}}(\sigma, t; W^{1,4}(\Omega))$ for $d = 3$.

Let us now analyze the dynamical system associated with our problem (1.13)-(1.14). For any given $m \in (0, 1)$, we introduce the phase space

$$\mathcal{H}_m = \left\{\varphi \in L^\infty(\Omega) : \|\varphi(x)\|_{L^\infty(\Omega)} \leq 1 \text{ and } -1+m \leq \overline{\varphi} \leq 1-m\right\}, \tag{4.14}$$

endowed with the metric

$$\mathbf{d}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|. \tag{4.15}$$

It is easily seen that $\mathcal{H}_\kappa$ is a complete metric space. Thanks to Theorem 3.4, we can set

$$S(t) : \mathcal{H}_\kappa \to \mathcal{H}_\kappa, \quad S(t)\varphi_0 = \varphi(t), \quad \forall\ t \geq 0,$$

where φ is the weak solution in the sense of Definition 3.2 corresponding to the initial condition φ_0. The dynamical system $(\mathcal{H}_\kappa, S(t))$ is dissipative owing to (3.2). Moreover, $\{S(t)\}$ is a closed semigroup on the phase space $\mathcal{H}_\kappa$ because of (3.3) (see [55]).

The final result of this section is the existence of the global attractor. This is given by

Theorem 4.4. *The dynamical system $(\mathcal{H}_m, S(t))$ has a connected global attractor $\mathcal{A}_m$ which is bounded in $\mathcal{H}_m \cap V$.*

Proof. Let us set

$$\mathcal{B} = B_V(0, R) \cap \mathcal{H}_\kappa,$$

where $R > 0$ sufficiently large. We infer from Theorem 4.1 that $\mathcal{B}$ is a connected compact absorbing set for the dynamical system $(\mathcal{H}_m, S(t))$. Hence, the existence of the global attractor is an immediate consequence of [55, Corollary 6]. □

Remark 4.5. In the next section we will deduce more information on the asymptotic behavior of the weak solutions in dimension two. On the contrary, providing further results in the three dimensional case seems to be a difficult task. The global-in-time higher-order estimate in V established here is not enough to apply a compactness argument as in [3] to recover the result about the convergence to a single stationary state. Moreover, the finite-dimensionality of the global attractor is an open issue as well. In particular, we are not able to argue as in [51] whose technique is based on a comparison principle for parabolic equations (see, e.g., [51, Corollary 3.1]).

5 The Strict Separation Property and its Consequences

Hereafter we restrict our analysis to the two dimensional case, $d = 2$, and we prove the main result of this paper, namely the validity of the strict separation property for a class of singular potentials which includes the physically relevant logarithmic one. Some consequences of this property will also be analyzed.

In the sequel, the generic constant C is allowed to depend on m and R as in the previous section. Moreover, we will assume the following additional hypotheses on F:

(H.3) $F \in \mathcal{C}^3(-1, 1)$ satisfies

$$F''(s) \leq e^{C|F'(s)|+C}, \quad F'(s)F'''(s) \geq 0, \quad |F'''(s)| \leq C|F''(s)|^2, \quad \forall\ s \in (-1, 1).$$

Remark 5.1. It is easily seen that the logarithmic potential (1.6) fulfils (H.3).

Theorem 5.2. *Assume $d = 2$ and the singular potential F fulfills (H.2) and (H.3). Then, for any $\sigma > 0$, there exists $\delta = \delta(m, R, \sigma) > 0$ such that*

$$\|\varphi(t)\|_{L^\infty(\Omega)} \le 1 - \delta, \quad \forall\ t \ge 2\sigma. \tag{5.1}$$

Proof. We begin by proving some integrability properties of $F''(\varphi)$ and $F'''(\varphi)$. Let $p \ge 1$ be given. Thanks to the first assumption of (H.3), we have

$$\int_\Omega F''(\varphi)^p\, dx \le \int_\Omega e^{p[C|F'(\varphi)|+C]}\, dx = e^{Cp} \int_\Omega e^{Cp|F'(\varphi)|}\, dx.$$

Recalling that $F'(\varphi) \in V$ for almost every $t \in [\sigma, \infty)$, an application of the Trudinger–Moser inequality (2.11) to $CpF'(\varphi)$ gives

$$\|F''(\varphi)\|^p_{L^p(\Omega)} \le e^{Cp} e^{Cp^2\|F'(\varphi)\|_V^2}.$$

Then, on account of (4.7), we infer

$$\|F''(\varphi)\|_{L^\infty(\sigma,t;L^p(\Omega))} \le Ce^{Cp}, \quad \forall\ t \ge \sigma. \tag{5.2}$$

In turn, by (4.2), (4.7) and (5.2), we get

$$(F'(\varphi))_t = F''(\varphi)\varphi_t \in L^2(t, t+1; V'), \quad \forall\ t \ge \sigma.$$

Thus, we find $F'(\varphi) \in \mathcal{C}([\sigma, t], H)$ for all $t \ge \sigma$ and

$$\|F'(\varphi(t))\|_V \le C, \quad \|F''(\varphi(t))\|_{L^p(\Omega)} \le C, \quad \forall\ t \ge \sigma. \tag{5.3}$$

Consequently, according to (H.3) and (5.3), we easily deduce that

$$\|F'''(\varphi(t))\|_{L^p(\Omega)} \le Ce^{Cp}, \quad \forall\ t \ge \sigma. \tag{5.4}$$

Now, our aim is to show a uniform-in-time control of the L^∞-norm of $F'(\varphi)$. To this end, we perform a Alikakos–Moser iteration argument. Taking $v = |F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi)$ in (3.1), we have for almost every $t \ge \sigma$

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega |F'(\varphi)|^{p+1}\, dx + \int_\Omega F''(\varphi)\nabla\varphi \cdot \nabla(|F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi))\, dx \\ = \int_\Omega (\nabla J * \varphi) \cdot \nabla(|F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi))\, dx.\end{aligned} \tag{5.5}$$

Observe that

$$\nabla(|F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi)) = p|F'(\varphi)|^{p-1}F''(\varphi)^2\nabla\varphi + |F'(\varphi)|^{p-1}F'(\varphi)F'''(\varphi)\nabla\varphi.$$

Then, we can write

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega |F'(\varphi)|^{p+1}\, dx + \mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}_3 + \mathcal{I}_4, \tag{5.6}$$

where

$$\mathcal{I}_1 := p\int_\Omega F''(\varphi)\nabla\varphi \cdot |F'(\varphi)|^{p-1}F''(\varphi)^2\nabla\varphi\, dx,$$

$$
\begin{aligned}
\mathcal{I}_2 &:= \int_\Omega F''(\varphi)\nabla\varphi \cdot |F'(\varphi)|^{p-1}F'(\varphi)F'''(\varphi)\nabla\varphi\, dx,\\
\mathcal{I}_3 &:= p\int_\Omega (\nabla J * \varphi)\cdot |F'(\varphi)|^{p-1}F''(\varphi)^2\nabla\varphi\, dx,\\
\mathcal{I}_4 &:= \int_\Omega (\nabla J * \varphi)\cdot |F'(\varphi)|^{p-1}F'(\varphi)F'''(\varphi)\nabla\varphi\, dx.
\end{aligned}
$$

We point out that taking v in (3.1) is not formal. Indeed, it is easy to check that the regularities property $\nabla\varphi \in L^6(t,t+1;L^3(\Omega))$ in (4.8) and the uniform bounds (4.7), (5.2) and (5.4) entail $v \in L^2(t,t+1;V)$, for all $t \geq \sigma$. Then, since φ_t belong to $L^2(t,t+1;H)$, for any $t \geq \sigma$, and $s \mapsto |F'(s)|^{p+1}$ is convex, an application of [56, Chap.IV, Lemma 4.3] gives, for almost every $t \geq \sigma$,

$$
\frac{1}{p+1}\frac{d}{dt}\int_\Omega |F'(\varphi)|^{p+1}\, dx = \int_\Omega |F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi)\varphi_t\, dx.
$$

We now have to estimate all the terms $\mathcal{I}_i$, $i = 1,2,3,4$. By the identity

$$
|F'(\varphi)|^{p-1}F''(\varphi)^2|\nabla\varphi|^2 = \frac{4}{(p+1)^2}|\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2, \tag{5.7}
$$

and recalling (H.2), we have

$$
\mathcal{I}_1 \geq \alpha p\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2|\nabla\varphi|^2\, dx \geq \frac{4\alpha p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2\, dx. \tag{5.8}
$$

On the other hand, from (H.2) and (H.3), we obtain

$$
\mathcal{I}_2(t) = \int_\Omega |F'(\varphi)|^{p-1}F'(\varphi)F'''(\varphi)F''(\varphi)|\nabla\varphi|^2\, dx \geq 0.
$$

Hypotheses (H.1), (H.2) and (H.3) together with Young's inequality, Remark 3.5 and (5.7) allow us to control $\mathcal{I}_3$ and $\mathcal{I}_4$ as follows

$$
\begin{aligned}
\mathcal{I}_3 &\leq p\int_\Omega \left(|F'(\varphi)|^{\frac{p-1}{2}}F''(\varphi)|\nabla\varphi|\right)\left(|F'(\varphi)|^{\frac{p-1}{2}}F''(\varphi)|\nabla J * \varphi|\right)\, dx\\
&\leq \varepsilon p\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2|\nabla\varphi|^2 dx + \frac{p}{4\varepsilon}\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2|\nabla J * \varphi|^2\, dx\\
&\leq \frac{4\varepsilon p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 dx + \frac{p}{4\varepsilon}\|\nabla J * \varphi\|^2_{L^\infty(\Omega)}\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2\, dx\\
&\leq \frac{4\varepsilon p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 dx + \frac{Cp}{4\varepsilon}\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2\, dx\\
&\leq \frac{4\varepsilon p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 dx + \frac{Cp}{4\varepsilon}\|F''(\varphi)\|^2 + \frac{Cp}{4\varepsilon}\int_\Omega |F'(\varphi)|^{p+1}F''(\varphi)^2\, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_4| &\leq \int_\Omega \left(|F'(\varphi)|^{\frac{p-1}{2}}F''(\varphi)|\nabla\varphi|\right)\left(|F'(\varphi)|^{\frac{p+1}{2}}\frac{|F'''(\varphi)|}{F''(\varphi)}|\nabla J * \varphi|\right)\, dx\\
&\leq \varepsilon p\int_\Omega |F'(\varphi)|^{p-1}F''(\varphi)^2|\nabla\varphi|^2 dx + \frac{1}{4\varepsilon p}\int_\Omega |F'(\varphi)|^{p+1}\frac{|F'''(\varphi)|^2}{F''(\varphi)^2}|\nabla J * \varphi|^2\, dx
\end{aligned}
$$

$$\leq \frac{4\varepsilon p}{(p+1)^2}\int_\Omega |\nabla |F'(\varphi)|^{\frac{p+1}{2}}|^2 dx + \frac{C}{4\varepsilon p}\|\nabla J * \varphi\|^2_{L^\infty(\Omega)}\int_\Omega |F'(\varphi)|^{p+1}F''(\varphi)^2\,dx$$
$$\leq \frac{4\varepsilon p}{(p+1)^2}\int_\Omega |\nabla |F'(\varphi)|^{\frac{p+1}{2}}|^2 dx + \frac{C}{4\varepsilon p}\int_\Omega |F'(\varphi)|^{p+1}F''(\varphi)^2\,dx,$$

where $\varepsilon > 0$ is some arbitrary parameter. Choosing $\varepsilon = \frac{\alpha}{4}$ in the above estimates, from (5.3) and (5.6) we get, for almost every $t \geq \sigma$,

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega |F'(\varphi)|^{p+1}\,dx + \frac{2\alpha p}{(p+1)^2}\int_\Omega |\nabla |F'(\varphi)|^{\frac{p+1}{2}}|^2\,dx \tag{5.9}$$
$$\leq Cp + Cp\int_\Omega |F'(\varphi)|^{p+1}F''(\varphi)^2\,dx.$$

Taking now

$$\mathcal{J} = \int_\Omega |F'(\varphi)|^{p+1}|F''(\varphi)|^2\,dx$$

and applying the Hölder inequality, we find

$$\mathcal{J} \leq \|F''(\varphi)\|^2_{L^4(\Omega)}\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \leq C\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)},$$

where we have used (5.3) to control $F''(\varphi)$. Hence, (5.9) turns into

$$\frac{1}{p+1}\frac{d}{dt}\int_\Omega |F'(\varphi)|^{p+1}\,dx + \frac{2\alpha p}{(p+1)^2}\int_\Omega |\nabla |F'(\varphi)|^{\frac{p+1}{2}}|^2\,dx \leq Cp\Big(1 + \|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)}\Big).$$

Setting $w(t) = |F'(\varphi(t))|^{\frac{p+1}{2}}$, we rewrite the above differential inequality in terms of w as follows

$$\frac{d}{dt}\|w\|^2 + \frac{2\alpha p}{p+1}\|\nabla w\|^2 \leq Cp(p+1)\Big(1 + \|w\|^2_{L^4(\Omega)}\Big). \tag{5.10}$$

Exploiting Lemma 2.1 with $\varepsilon = \frac{\alpha}{C(p+1)^2}$

$$Cp(p+1)\|w\|^2_{L^4(\Omega)} \leq \frac{\alpha p}{p+1}\|\nabla w\|^2 + C\Big(1 + (p+1)^6\Big)\|w\|^2_{L^1(\Omega)}$$

and inserting the above estimate into (5.10), we obtain

$$\frac{d}{dt}\|w\|^2 + \frac{\alpha p}{p+1}\|\nabla w\|^2 \leq Cp^6\Big(1 + \|w\|^2_{L^1(\Omega)}\Big).$$

Then, noting that $\frac{p}{p+1} \geq \frac{1}{2}$, and using again Lemma 2.1 with $s = 2$ and $\varepsilon = \frac{\alpha}{2}$, we reach

$$\frac{d}{dt}\|w\|^2 + \|w\|^2 \leq Cp^6\Big(1 + \|w\|^2_{L^1(\Omega)}\Big). \tag{5.11}$$

for almost every $t \geq \sigma$ and any $p \geq 1$. We are now in a position to carry out an iterative argument (see [35] and references therein). To this aim, we preliminarily observe that $F'(\varphi) \in L^1(\sigma, 2\sigma; W^{1,3}(\Omega))$ (see, for instance, (4.12) and (4.10)) with

$$\|F'(\varphi)\|_{L^1(\sigma,2\sigma;W^{1,3}(\Omega))} \leq C,$$

where C only depends on σ. By the Sobolev embedding $W^{1,3}(\Omega) \hookrightarrow L^\infty(\Omega)$, we infer that there exists $\overline{\xi} \in (\sigma, 2\sigma)$ such that

$$\|F'(\varphi(\overline{\xi}))\|_{L^\infty(\Omega)} \leq C.$$

Hence, denoting

$$\eta = \max\Big\{ \|F'(\varphi(\bar{\xi}))\|_{L^\infty(\Omega)}, \max_{t\geq\bar{\xi}} \int_\Omega |F'(\varphi)|\, dx \Big\},$$

and according to (4.7), we find the estimate

$$1 \leq \eta \leq C. \tag{5.12}$$

Next, recalling the very definition of w, an application of the Gronwall Lemma to (5.11) gives

$$\max_{t\geq\bar{\xi}} \int_\Omega |F'(\varphi(t))|^{p+1}\, dx \leq \max\Big\{ \eta^{p+1}, Cp^6 \max_{t\geq\bar{\xi}} \Big(1 + \int_\Omega |F'(\varphi(t))|^{\frac{p+1}{2}}\, dx\Big)^2 \Big\}$$

for all $t \geq \bar{\xi}$ and $p \geq 1$. As customary, taking $p+1 = 2^k$, $k \in \mathbb{N}$, we rewrite the above inequality as

$$\begin{aligned} \max_{t\geq\bar{\xi}} \int_\Omega |F'(\varphi(t))|^{2^k}\, dx &\leq \max\Big\{ \eta^{2^k}, C2^{6k} \max_{t\geq\bar{\xi}} \Big(1 + \int_\Omega |F'(\varphi(t))|^{2^{k-1}} dx\Big)^2 \Big\} \\ &\leq \max\Big\{ \eta^{2^k}, C2^{6k+2} \max_{t\geq\bar{\xi}} \Big(\int_\Omega |F'(\varphi(t))|^{2^{k-1}} dx\Big)^2 \Big\}. \end{aligned}$$

Here, we have used the lower bound of η in (5.12). Setting $A_k = C2^{6k+2}$ and arguing by iteration, we arrive at

$$\begin{aligned} \max_{t\geq\bar{\xi}} \int_\Omega |F'(\varphi(t))|^{2^k} dx &\leq \eta^{2^k} A_k\ A_{k-1}^2\ A_{k-2}^{2^2} \ldots A_{k-(k-1)}^{2^{k-1}} \\ &\leq \eta^{2^k} C^{A2^k} 2^{B2^k}, \end{aligned} \tag{5.13}$$

where

$$A = \sum_{i=1}^\infty \frac{1}{2^i} < \infty, \quad B = \sum_{i=1}^\infty \frac{6i+2}{2^i} < \infty.$$

Finally, taking the 2^{-k}-power on both sides of (5.13), passing to the limit as $k \to +\infty$, and using (5.12), we end up with

$$\max_{t\geq\bar{\xi}} \|F'(\varphi(t))\|_{L^\infty} \leq C.$$

Therefore, (5.1) immediately follows from the above estimate. The proof is complete.

Remark 5.3. Suppose the third condition in (H.3) is replaced by the more general one

$$|F'''(s)| \leq CF''(s)^q, \quad \forall\ s \in (-1,1), \tag{5.14}$$

for some $q \geq 1$.Then, following line by line the above proof, and setting now

$$\mathcal{J} = \int_\Omega |F'(\varphi)|^{p+1} F''(\varphi)^{2(q-1)}\, dx,$$

we just need to control $\mathcal{J}$ in a slightly different way. Indeed, applying the Hölder inequality, we get

$$|\mathcal{J}| \leq \|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \|F''(\varphi)\|^{2(q-1)}_{L^{4(q-1)}(\Omega)} \leq C\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)}.$$

Thus the conclusion still follows arguing as above.

Remark 5.4. The validity of the strict separation property in dimension three was proven in [51] provided that F is algebraically unbounded at the endpoints. However, by using our technique, it is not clear how to extend this result to the nonlocal equation.

A first immediate consequence of Theorem 5.2 is

Corollary 5.5. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\mu(t)\|_{L^\infty(\Omega)} \leq C, \quad \forall\, t \geq 2\sigma.$$

Moreover, as a byproduct, we can also obtain the Hölder regularity of the weak solutions by means of [22, Corollary 4.2] (see also [35]). Indeed we have

Corollary 5.6. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ and $\alpha = \alpha(\sigma, \delta) \in (0,1)$ such that*

$$|\varphi(x_1,t_1) - \varphi(x_2,t_2)| \leq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}\right)$$
$$|\mu(x_1,t_1) - \mu(x_2,t_2)| \leq C\left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}\right),$$

for all $(x_1,t_1),(x_2,t_2) \in \Omega_t$, where $\Omega_t = [t,t+1] \times \overline{\Omega}$ and $t \geq 3\sigma$.

Leaning on the strict separation property (5.1), we are able to interpret the weak solutions to problem (1.13)-(1.14) as the weak solutions to a similar problem where F is replaced by a suitable regular potential. More precisely, we define the regular potential $\overline{F} \in C^3(\mathbb{R})$, which extends F outside of $[-1+\delta, 1-\delta]$, as follows

$$\begin{cases} \overline{F}(s) = \sum_{k=0}^3 \frac{F^{(k)}(1-\delta)}{k!}(s-1+\delta)^k, & \forall\, s \geq 1-\delta, \\ \overline{F}(s) = F(s), & \forall\, s \in (-1+\delta, 1-\delta) \\ \overline{F}(s) = \sum_{k=0}^3 \frac{F^{(k)}(-1+\delta)}{k!}(s+1-\delta)^k, & \forall\, s \leq -1+\delta. \end{cases} \tag{5.15}$$

According to the assumptions (H.2) and (H.3) and taking into account the sign of F and its derivatives at $s = 1-\delta$ and $s = -1+\delta$, we deduce the following properties:

(A.1) for any $\Lambda > 0$, there exists $K > 0$ such that

$$\overline{F}(s) \geq \Lambda s^2 - K, \quad \forall\, s \in \mathbb{R};$$

(A.2) there exists $N > 0$ such that

$$|\overline{F}'(s)| \leq N(1+s^2), \quad \forall\, s \in \mathbb{R};$$

(A.3) there exists $N > 0$ such that

$$\alpha \leq \overline{F}''(s) \leq N(1+|s|), \quad |\overline{F}'''(s)| \leq N, \quad \forall\, s \in \mathbb{R}.$$

Here, α is the same value defined in assumption (H.2). Instead, K and N can be easily estimated in terms of δ.

Let us now set $\varphi_1 = \varphi(3\sigma)$, which is a function in V such that $\|\varphi_1\|_{L^\infty(\Omega)} \leq 1-\delta$, $\overline{\varphi}_1 \in [-1+m, 1-m]$. We consider the problem

$$\begin{cases} \tilde{\varphi}_t = \Delta\tilde{\mu}, \\ \tilde{\mu} = \overline{F}'(\tilde{\varphi}) - J * \tilde{\varphi}, \end{cases} \quad \text{in } \Omega \times (0,T), \tag{5.16}$$

subject to the boundary and initial conditions

$$\partial_n \tilde{\mu} = 0, \quad \text{on } \partial\Omega \times (0,T), \quad \tilde{\varphi}(0,\cdot) = \varphi_1, \quad \text{in } \Omega. \tag{5.17}$$

Combining Lemma 3.13 and Theorem 3.15, it follows immediately that problem (5.16)-(5.17) has a unique weak solution in the sense of Definition (3.14) obtained as a limit of a Galerkin sequence. On the other hand, from the separation property, the definition of $\overline{F}$ and the uniqueness of (5.16)-(5.17), we easily infer that φ is also a weak solution to (5.16) so $\tilde{\varphi}(t) \equiv \varphi(t+3\sigma)$ for all $t \geq 0$. According to this equivalence, the idea is to compute some higher-order estimates on the Galerkin sequence due to its regularity. Note that the Galerkin sequence does not satisfy the separation property. Nevertheless, we can take advantage of the specific form of $\overline{F}$.

Lemma 5.7. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\varphi_t\|_{L^\infty(5\sigma,t;H)} + \|\varphi_t\|_{L^2(t,t+1;V)} + \|\nabla \mu_t\|_{L^2(t,t+1;H)} \leq C, \quad \forall\, t \geq 5\sigma. \tag{5.18}$$

Proof. Let us consider the Galerkin sequence $\tilde{\varphi}_n$ which converges to $\tilde{\varphi}$. Due to the regularity of $\tilde{\varphi}_n$ and the properties of $\overline{F}$, we can repeat line by line the proof of Theorem 4.1. In particular, we have

$$\|\tilde{\varphi}_n(t)\|_V^2 + \|\tilde{\varphi}_{n,t}(t)\|_{V'}^2 + \int_t^{t+1} \|\tilde{\varphi}_{n,t}(\tau)\|^2\, d\tau \leq C, \quad \forall\, t \geq \sigma, \tag{5.19}$$

where C is independent of n. Now, arguing as in [32], we differentiate the system with respect to time and we test by $\tilde{\mu}_{n,t}$ getting

$$(\tilde{\varphi}_{n,tt}, \tilde{\mu}_{n,t}) + \|\nabla \tilde{\mu}_{n,t}\|^2 = 0.$$

Hence, exploiting the form of $\tilde{\mu}_n$, we obtain

$$(\tilde{\varphi}_{n,tt}, \overline{F}''(\tilde{\varphi}_n)\tilde{\varphi}_{n,t}) - (\tilde{\varphi}_{n,tt}, J * \tilde{\varphi}_{n,t}) + \|\nabla \tilde{\mu}_{n,t}\|^2 = 0.$$

Using the fist equation of (5.16), we can rewrite the above equality as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\, dx + \|\nabla \tilde{\mu}_{n,t}\|^2 = (\Delta \tilde{\mu}_{n,t}, J * \tilde{\varphi}_{n,t}) + \frac{1}{2}\int_\Omega \overline{F}'''(\tilde{\varphi}_n)\tilde{\varphi}_{n,t}^3\, dx.$$

After an integration by parts in the right-hand side, we get

$$\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\, dx + \|\nabla \tilde{\mu}_{n,t}\|^2 = -(\nabla \tilde{\mu}_{n,t}, \nabla J * \tilde{\varphi}_{n,t}) + \frac{1}{2}\int_\Omega \overline{F}'''(\tilde{\varphi}_n)\tilde{\varphi}_{n,t}^3\, dx.$$

By the Young inequality, assumption (H.1) and the properties of $\overline{F}$, we deduce

$$\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2 dx + \frac{1}{2}\|\nabla \tilde{\mu}_{n,t}\|^2 \leq C\|\tilde{\varphi}_{n,t}\|^2 + C\int_\Omega |\tilde{\varphi}_{n,t}|^3 dx. \tag{5.20}$$

On account of the Gagliardo–Nirenberg inequality (2.9), we control the last term on the right-hand side as

$$\|\tilde{\varphi}_{n,t}\|_{L^3(\Omega)}^3 \leq \gamma\|\nabla \tilde{\varphi}_{n,t}\|^2 + C\|\tilde{\varphi}_{n,t}\|^4,$$

for any $\gamma > 0$ and $C > 0$ depending on γ but independent of n. In order to reconstruct the L^2-norm of the gradient of $\tilde{\varphi}_{n,t}$ on the left-hand side, we multiply the gradient of $\tilde{\mu}_{n,t}$ by $\nabla\tilde{\varphi}_{n,t}$

$$\int_\Omega \nabla\tilde{\mu}_{n,t}\cdot\nabla\tilde{\varphi}_{n,t}\,dx = \int_\Omega \overline{F}''(\tilde{\varphi}_n)|\nabla\tilde{\varphi}_{n,t}|^2\,dx + \int_\Omega \overline{F}'''(\tilde{\varphi}_n)\nabla\tilde{\varphi}_n\cdot\nabla\tilde{\varphi}_{n,t}\,dx - \int_\Omega \nabla J * \tilde{\varphi}_{n,t}\cdot\nabla\tilde{\varphi}_{n,t}\,dx.$$

Using again the Young inequality, assumption (H.1) and the properties of $\overline{F}$, we obtain

$$\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\nabla\tilde{\varphi}_{n,t}|^2\,dx \le \frac{\alpha}{2}\|\nabla\tilde{\varphi}_{n,t}\|^2 + C\|\nabla\tilde{\mu}_{n,t}\|^2 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\tilde{\varphi}_{n,t}\|^2.$$

According to the bound from below of $\overline{F}''$, the above inequality yields

$$\frac{1}{2}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\nabla\tilde{\varphi}_{n,t}|^2\,dx \le C\|\nabla\tilde{\mu}_{n,t}\|^2 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\tilde{\varphi}_{n,t}\|^2. \tag{5.21}$$

Gathering together (5.20) and (5.21), there exists $\omega > 0$ such that

$$\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx + \omega\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\nabla\tilde{\varphi}_{n,t}|^2\,dx + \frac{1}{4}\|\nabla\tilde{\mu}_{n,t}\|^2 \le \gamma\|\nabla\tilde{\varphi}_{n,t}\|^2 + C\|\tilde{\varphi}_{n,t}\|^4 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\tilde{\varphi}_{n,t}\|^2.$$

Setting $\gamma = \frac{\omega\alpha}{2}$, we have

$$\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx + \gamma\int_\Omega |\nabla\tilde{\varphi}_{n,t}|^2\,dx + \frac{1}{4}\|\nabla\tilde{\mu}_{n,t}\|^2 \le C\|\tilde{\varphi}_{n,t}\|^4 + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\tilde{\varphi}_{n,t}\|^2.$$

Noting that the first term on the right-hand side can be controlled as follows

$$\|\tilde{\varphi}_{n,t}\|^4 \le C\|\tilde{\varphi}_{n,t}\|^2\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx,$$

we get

$$\frac{d}{dt}\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx + \gamma\int_\Omega |\nabla\tilde{\varphi}_{n,t}|^2 dx + \frac{1}{4}\|\nabla\tilde{\mu}_{n,t}\|^2 \le C\|\tilde{\varphi}_{n,t}\|^2\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx + C\|\nabla\tilde{\varphi}_n\|^2 + C\|\tilde{\varphi}_{n,t}\|^2. \tag{5.22}$$

In order to apply the uniform Gronwall lemma, we need to find a bound of

$$\int_t^{t+1}\int_\Omega \overline{F}''(\tilde{\varphi}_n(\tau))|\tilde{\varphi}_{n,t}(\tau)|^2\,dxd\tau, \quad \forall\ t \ge \sigma.$$

To this aim, we observe that

$$\int_\Omega \overline{F}''(\tilde{\varphi}_n)|\tilde{\varphi}_{n,t}|^2\,dx = (J * \tilde{\varphi}_{n,t}, \tilde{\varphi}_{n,t}) + (\tilde{\mu}_{n,t}, \tilde{\varphi}_{n,t}) = (J * \tilde{\varphi}_{n,t}, \tilde{\varphi}_{n,t}) - \frac{1}{2}\frac{d}{dt}\|\tilde{\varphi}_{n,t}\|_*^2.$$

Integrating in time from t to $t+1$ and exploiting (5.19), we get

$$\int_t^{t+1}\int_\Omega \overline{F}''(\tilde{\varphi}_n(\tau))|\tilde{\varphi}_{n,t}(\tau)|^2\,dxd\tau \le C\int_t^{t+1}\|\tilde{\varphi}_{n,t}(\tau)\|^2\,d\tau + \frac{1}{2}\|\tilde{\varphi}_{n,t}(t)\|_*^2 \le C.$$

Therefore, due to the above estimate and (5.19), we apply the uniform Gronwall lemma to (5.22) deducing

$$\|\tilde{\varphi}_{n,t}(t)\|^2 + \int_t^{t+1}\|\tilde{\varphi}_{n,t}(\tau)\|_V^2 + \|\nabla\tilde{\mu}_{n,t}(\tau)\|^2\,d\tau \le C, \quad \forall\, t\ge 2\sigma.$$

Passing to the limit as n goes to ∞, using the lower-semicontinuity of the norm and the equivalence between $\tilde{\varphi}$ and φ, we obtain (5.18). □

Remark 5.8. Notice that by comparison, we also infer that for all $t\ge 5\sigma$,

$$\|\nabla\mu\|_{L^\infty(5\sigma,t;L^p(\Omega))} + \|\mu\|_{L^\infty(t,t+1;H^2(\Omega))} + \|\nabla\varphi\|_{L^\infty(5\sigma,t;L^p(\Omega))} + \|\varphi_{tt}\|_{L^2(t,t+1;V')} \le C.$$

If we strengthen a bit the assumptions on the interaction kernel J, we can say more about the regularity of the solution. More precisely, let us introduce the following assumption.

(H.J) Either $J\in W^{2,1}(\mathcal{B}_\rho)$, where $\mathcal{B}_\rho=\{x\in\mathbb{R}^d : |x|<\rho\}$ with $\rho\sim diam(\Omega)$ such that $\overline{\Omega}\subset\mathcal{B}_\rho$, or J is admissible in the sense of [9, Definition 1].

Remark 5.9. We recall that Newtonian and Bessel potentials satisfy assumption (H.J).

Then we can prove the following (see [28, Theorem 5] and [34, Lemma 3.6]).

Lemma 5.10. *Assume that J satisfies (H.J). For any $\sigma>0$, there exists $C=C(\sigma,\rho)>0$ such that*

$$\sup_{t\ge 5\sigma}\|\varphi(t)\|_{H^2(\Omega)} \le C.$$

Finally, we also have

Lemma 5.11. *Assume J satisfies (H.J). For any $\sigma>0$, there exists $C=C(\sigma,p)>0$ such that*

$$\|\varphi_t(t)\|_{L^p(\Omega)} + \|\mu(t)\|_{W^{2,p}(\Omega)} + \|\varphi(t)\|_{W^{2,p}(\Omega)} \le C, \quad \forall\, t\ge 6\sigma, \tag{5.23}$$

for any $p\in[2,\infty)$.

The foregoing estimate is deduced by making use of the maximal regularity of the Neumann Laplacian (see [34] for further details). The regularity properties implied by the strict separation property can be exploited in the analysis of the longtime behavior of solutions. Indeed, we have

Theorem 5.12. *Let the assumptions of Theorem 5.2 hold. Then, for every $m>0$, there exists an exponential attractor $\mathcal{M}_m$ bounded in $V\cap\mathcal{C}^\alpha(\overline{\Omega})$ for the dynamical system $(\mathcal{H}_m,S(t))$ defined in Section 4, namely,*

(i) $S(t)\mathcal{M}_m\subset\mathcal{M}_m$, $\forall\, t\ge 0$;

(ii) $\mathcal{M}_m$ *exponentially attracts the bounded subset of* $\mathcal{H}_m$*, i.e. there exist* C *and* ω *such that for every* $\mathcal{B}$ *bounded set of* $\mathcal{H}_m$

$$dist_{\mathcal{C}^\alpha(\overline{\Omega})\cap H^{1-\nu}(\Omega)}(S(t)\mathcal{B}, \mathcal{M}_m) \leq Ce^{-\omega t}, \quad \forall\, t \geq 0,$$

for any $\nu \in (0,1)$ *and some* $\alpha \in (0,1)$*;*

(iii) *the fractal dimension of* $\mathcal{M}_m$ *is finite, that is,*

$$dim_F(\mathcal{M}_m, \mathcal{C}^\alpha(\overline{\Omega})) \leq C,$$

where C *depends on* α *and* m.

As consequences of Theorem 5.12 we have

Corollary 5.13. *Let the assumptions of Theorem 5.2 hold, then the global attractor is a bounded subset of* $V \cap \mathcal{C}^\alpha(\overline{\Omega})$ *and has finite fractal dimension, that is,*

$$dim_F(\mathcal{A}_m, \mathcal{C}^\alpha(\overline{\Omega})) \leq C.$$

Corollary 5.14. *Let the assumptions of Theorem 5.2 hold. If* J *satisfies (H.J) then, the global attractor* $\mathcal{A}_m$ *and the exponential attractor* $\mathcal{M}_m$ *are bounded in* $\mathcal{H}_m \cap W^{2,p}(\Omega)$*, for any* $p \in [2,\infty)$.

Theorem 5.12 and Corollary 5.13 and 5.14 are byproducts of the separation property. Indeed, we recall that the weak solutions to (1.13)-(1.14) coincide in finite time with the weak solutions to (1.13)-(1.14) with a smooth F (see above). Hence we can use [35, Theorem 2.8] to guarantee the existence of an exponential attractor and its consequences.

We conclude this section by stating a result on the convergence of single trajectories. More precisely, we have that any weak solution does converge to a single stationary state. This result also follows from the argument mentioned above which is based on the strict separation property. More precisely, it can be proven arguing as in [35, Theorem 2.21] where the regular potential case is considered. Thus, we also have

Corollary 5.15. *Let the assumptions of Theorem 5.2 hold. If* F *is real analytic on* $[-1+\delta(m), 1-\delta(m)]$*. Then any weak solution* φ *to problem* (1.13)-(1.14) *is such that*

$$\lim_{t\to\infty} \|\varphi(t) - \varphi_*\|_{L^\infty(\Omega)} = 0, \tag{5.24}$$

where $\varphi_* \in V \cap \mathcal{C}^\alpha(\overline{\Omega})$ *solves* $-J * \varphi_* + F'(\varphi_*) = C_*$ *for some* $C_* > 0$*, and* $\overline{\varphi}_0 = \overline{\varphi}_*$.

Remark 5.16. Our proof of the strict separation property can be adapted to other related models characterized by a logarithmic potential. For instance, the nonlocal version of the so-called Cahn–Hilliard–Oono equation (see [19] and references therein, cf. also [50]) with conservation of total mass, namely, the system

$$\begin{cases} \varphi_t + \varepsilon(\varphi - \overline{\varphi}_0) = \Delta\mu, \\ \mu = F'(\varphi) - J * \varphi, \end{cases} \qquad \text{in } \Omega \times (0,T), \tag{5.25}$$

subject to (1.14), where $\varepsilon > 0$. Indeed, observe that there exists a unique v which solves $-\Delta v = \varepsilon(\varphi - \overline{\varphi}_0)$ with homogenous Neumann boundary condition. This v has the form

$G*\varepsilon(\varphi-\overline{\varphi}_0)$ where G is the Green function associated with the problem. Therefore system (5.25) can be rewritten as follows

$$\begin{cases}\varphi_t = \Delta\mu_\varepsilon, \\ \mu_\varepsilon = F'(\varphi) - J*\varphi - G*\varepsilon(\varphi-\overline{\varphi}_0),\end{cases} \qquad \text{in } \Omega\times(0,T),$$

subject to

$$\partial_n\mu_\varepsilon = 0, \quad \text{on } \partial\Omega\times(0,T), \quad \varphi(\cdot,0)=\varphi_0, \quad \text{in } \Omega.$$

Thus the original problem can be rewritten in a form which is quite similar to the one we have analyzed.

6 The Nonlocal Cahn-Hilliard-Navier-Stokes System

This section is devoted to extend the regularity results and the validity of the strict separation property to problem (1.16)-(1.17) in dimension two. Let us introduce first the definition of weak solution (see [30]).

Definition 6.1. *Let $\mathbf{u}_0 \in G_{div}$, φ_0 be a measurable function with $F(\varphi_0)\in L^1(\Omega)$ and $T>0$ be given. A couple $[\mathbf{u},\varphi]$ is a weak solution to problem* (1.16)-(1.17) *on* $[0,T]$ *corresponding to* $[\mathbf{u}_0,\varphi_0]$ *if*

$$\begin{aligned}&\mathbf{u}\in H^1(0,T;V'_{\mathrm{div}})\cap L^\infty(0,T;G_{div})\cap L^2(0,T;V_{div}),\\ &\varphi\in H^1(0,T;V')\cap L^\infty(0,T;H)\cap L^2(0,T;V),\\ &\varphi\in L^\infty(\Omega\times(0,T)) \text{ with } |\varphi(x,t)|<1 \text{ a.e. } (x,t)\in\Omega\times(0,T),\\ &\mu = F'(\varphi)-J*\varphi\in L^2(0,T;V)\end{aligned}$$

such that

$$\begin{aligned}\langle\mathbf{u}_t,\mathbf{v}\rangle_{V'_{div},V_{div}} + b(\mathbf{u},\mathbf{u},\mathbf{v}) + (\nabla\mathbf{u},\nabla\mathbf{v}) &= (\mu\nabla\varphi,\mathbf{v}) \qquad \forall\ \mathbf{v}\in V_{div}, \text{ a.e. } t\in(0,T),\\ \langle\varphi_t,v\rangle_{V',V} + (\mathbf{u}\cdot\nabla\varphi,v) + (\nabla\mu,\nabla v) &= 0 \qquad \forall\ v\in V, \text{ a.e. } t\in(0,T),\end{aligned}$$

and satisfies the initial conditions

$$\mathbf{u}(0,\cdot)=\mathbf{u}_0, \quad \varphi(0,\cdot)=\varphi_0 \qquad \text{in } \Omega.$$

Recalling the energy associated to system (1.16)

$$E(\mathbf{u},\varphi) = \frac{1}{2}\|\mathbf{u}\|^2 + \int_\Omega F(\varphi)dx - \frac{1}{2}(J*\varphi,\varphi),$$

we state the well-posedness result related to problem (1.16)-(1.17).

Theorem 6.2. *Let $\mathbf{u}_0\in G_{div}$, φ_0 be a measurable function with $F(\varphi_0)\in L^1(\Omega)$ and $\overline{\varphi}_0\in(-1,1)$ and $T>0$ be given. Assume that (H.1)−(H.2) hold. Then, there exists a unique weak solution $[\mathbf{u},\varphi]$ in the sense of Definition 6.1 which satisfies the dissipative estimate for all $t\geq 0$*

$$E(\mathbf{u}(t),\varphi(t)) + \omega\int_t^{t+1} \|\nabla\mathbf{u}(\tau)\|^2 + \|\nabla\varphi(\tau)\|^2 + \|\nabla\mu(\tau)\|^2\,d\tau \leq E(\mathbf{u}_0,\varphi_0)e^{-\omega t} + C,$$

where ω and C are positive constants independent of the initial condition.

The proof of existence is based on the approximation technique established in the proof of Theorem 3.4 and the standard Galerkin scheme for the Navier–Stokes system (see [57]). We also refer to [30, Section 2, Theorem 1] for a different approximation technique. Instead, uniqueness has been proven arguing as in [28, 3.1].
Let us fix $m \in (0,1)$ and $R \geq 0$. We consider trajectories such that

$$-1 + m \leq \overline{\varphi}_0 \leq 1 - m \quad \text{and} \quad E(\mathbf{u}_0, \varphi_0) \leq R.$$

Accordingly, the generic constant C may depend on R and m but is independent of the specific form of the initial datum. Moreover, thanks to the above result, we have that any weak solution fulfills for all $t \geq 0$

$$E(\mathbf{u}(t), \varphi(t)) + \int_t^{t+1} \|\nabla \mathbf{u}(\tau)\|^2 + \|\nabla \varphi(\tau)\|^2 + \|\nabla \mu(\tau)\|^2 + \|\varphi_t(\tau)\|_{V'}^2 \, d\tau \leq C. \tag{6.1}$$

We begin to report a regularity result for the Navier–Stokes system in dimension two.

Lemma 6.3. *For any $\sigma > 0$, there exists $C = C(\sigma)$ such that*

$$\|\mathbf{u}(t)\|_{L^\infty(\sigma,t;V_{div})} + \|\mathbf{u}\|_{L^2(t,t+1;H^2(\Omega))} + \|\mathbf{u}_t\|_{L^2(t,t+1;L^2(\Omega))} \leq C, \quad \forall\ t \ \geq \sigma. \tag{6.2}$$

Proof. We observe that the Korteweg force can be rewritten as

$$\mu \nabla \varphi = \nabla p^* - (J * \varphi) \nabla \varphi.$$

On the other hand, we have for all $t \geq 0$

$$\int_t^{t+1} \|(J * \varphi(\tau)) \nabla \varphi(\tau)\|^2 \, d\tau \leq C.$$

Thus (6.2) follows from [57, Theorem 3.10].

Thanks to Lemma 6.3, we can prove the following lemma.

Lemma 6.4. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\varphi_t\|_{L^\infty(2\sigma,t;V')} + \|\varphi_t\|_{L^2(t,t+1;H)} \leq C, \quad \forall\ t \geq 2\sigma. \tag{6.3}$$

Proof. We provide below a formal computation. Nonetheless, a rigorous proof can be done by arguing as in Lemma 6.8 .We differentiate the nonlocal Cahn–Hilliard equation with respect to time and we test the equation by $\mathcal{N}\varphi_t$. Then, arguing as in the proof of Lemma 4.1, we obtain

$$\frac{d}{dt}\|\varphi_t\|_*^2 + \alpha \|\varphi_t\|^2 \leq C\|\varphi_t\|_*^2 + 2|(\mathbf{u}_t \varphi, \nabla \mathcal{N}\varphi_t)| + 2|(\mathbf{u}\varphi_t, \nabla \mathcal{N}\varphi_t)|.$$

By the Hölder inequality and the properties of $\mathcal{N}$, we deduce that

$$|(\mathbf{u}_t \varphi, \nabla \mathcal{N}\varphi_t)| \leq \|\mathbf{u}_t\| \|\varphi_t\|_*$$

and

$$|(\mathbf{u}\varphi_t, \nabla \mathcal{N}\varphi_t)| \leq \|\mathbf{u}\|_{L^\infty(\Omega)} \|\varphi_t\| \|\varphi_t\|_*.$$

Collecting together the above estimates and using the Young inequality, we get

$$\frac{d}{dt}\|\varphi_t\|_*^2 + \frac{\alpha}{2}\|\varphi_t\|^2 \leq C\left(1 + \|\mathbf{u}\|_{L^\infty(\Omega)}^2\right)\|\varphi_t\|_*^2 + \|\mathbf{u}_t\|^2.$$

Now, exploiting the Gronwall Lemma together with (6.1) and (6.2), we easily infer (6.3).

As an immediate consequence, we deduce two additional regularity results whose proofs can be performed following line by line the proofs of their previous counterparts (namely, Theorem 4.1 and Proposition 4.2).

Lemma 6.5. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that*

$$\|\nabla \mu\|_{L^\infty(\sigma,t;H)} \leq C, \quad \forall\ t \geq 2\sigma, \tag{6.4}$$

and

$$\sup_{t\geq 2\sigma} \|\varphi(t)\|_V \leq C. \tag{6.5}$$

Lemma 6.6. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ such that for all $t \geq 2\sigma$*

$$\|F'(\varphi)\|_{L^\infty(2\sigma,t;V)} + \|\mu(t)\|_{L^\infty(2\sigma,t;V)} + \|\mu\|_{L^2(t,t+1;H^2(\Omega))} \leq C, \tag{6.6}$$

$$\|\nabla \mu\|_{L^q(t,t+1;L^p(\Omega))} + \|\nabla \varphi\|_{L^q(t,t+1;L^p(\Omega))} \leq C, \quad \text{if } \frac{p-2}{p} = \frac{2}{q} \text{ and } d = 2. \tag{6.7}$$

We now have all the ingredients to establish the strict separation property.

Theorem 6.7. *Given $\sigma > 0$. Suppose that F also fulfills (H.3). Then, there exists $\delta = \delta(R, m, \sigma) > 0$ such that*

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall\ t \geq 3\sigma. \tag{6.8}$$

Proof. We apply the same argument of Theorem 5.1. We need to handle the following term

$$\mathcal{Z} = \int_\Omega \mathbf{u}\varphi\nabla\left(|F'(\varphi)|^{p-1}F'(\varphi)F''(\varphi)\right) dx.$$

Using the boundedness of φ, we have

$$\begin{aligned}
|\mathcal{Z}| &\leq \int_\Omega |\mathbf{u}||F'(\varphi)^{p-1}F'(\varphi)F'''(\varphi)\nabla\varphi| \, dx + p\int_\Omega |\mathbf{u}|F''(\varphi)^2|F'(\varphi)|^{p-1}|\nabla\varphi| \, dx \\
&\leq \mathcal{Z}_1 + \mathcal{Z}_2.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
\mathcal{Z}_1 &\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + \frac{1}{4Cp}\int_\Omega |F'(\varphi)|^{p+1}|F'''(\varphi)|^2\mathbf{u}^2 \, dx \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + \|F'''(\varphi)^2\mathbf{u}^2\|_{L^2(\Omega)}\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + \|F'''(\varphi)\|^2_{L^8(\Omega)}\|\mathbf{u}\|^2_{L^8(\Omega)}\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)} \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + C\|F'(\varphi)\|^{p+1}_{L^{2(p+1)}(\Omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Z}_2 &\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + \frac{1}{4C}\|\mathbf{u}^2F''(\varphi)^2\|_{L^2(\Omega)}\|F'(\varphi)\|^{p-1}_{L^{2(p+1)}(\Omega)} \\
&\leq \frac{4p}{(p+1)^2}\int_\Omega |\nabla|F'(\varphi)|^{\frac{p+1}{2}}|^2 \, dx + C\|F'(\varphi)\|^{p-1}_{L^{2(p-1)}(\Omega)}.
\end{aligned}$$

Arguing as in Theorem 5.2 and using a Moser type iteration procedure, we obtain (6.8).

Thanks to the strict separation property we can also prove some Hölder continuity. Indeed we have

Lemma 6.8. *For any $\sigma > 0$, there exists $C = C(\sigma) > 0$ and $\alpha \in (0,1)$, depending on δ such that*

$$\sup_{t\in[4\sigma,\infty)} \|\mathbf{u}(t)\|_{C^{\frac{1}{2}}(\overline{\Omega})} \leq C \tag{6.9}$$

and

$$|\varphi(x_1,t_1) - \varphi(x_2,t_2)| \leq C \left(|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}} \right), \tag{6.10}$$

for all $(x_1,t_1),(x_2,t_2) \in [t,t+1] \times \overline{\Omega}$ and any $t > 4\sigma$.

Proof. We observe that the Korteweg force can be also rewritten in the following form

$$\mu\nabla\varphi = \nabla\tilde{p} - (\nabla J * \varphi)\varphi.$$

Thanks to Lemma 6.4 and the boundedness of φ, we deduce that

$$\begin{aligned}&\|\partial_t((\nabla J * \varphi)\varphi)\|_{L^2(t,t+1;H)} \\ &\leq \|(\nabla J * \varphi_t)\varphi\|_{L^2(t,t+1;H)} + \|(\nabla J * \varphi)\varphi_t\|_{L^2(t,t+1;H)} \leq C, \quad \forall\, t \geq 3\sigma.\end{aligned}$$

Therefore, we can consider the Navier–Stokes equation

$$\langle \mathbf{u}_t, \mathbf{v}\rangle_{V_{div}', V_{div}} + b(\mathbf{u},\mathbf{u},\mathbf{v}) + (\nabla\mathbf{u},\nabla\mathbf{v}) = (f,\mathbf{v}) \qquad \forall\ \mathbf{v} \in V_{div},\ \text{a.e. } t \in (0,T),$$

where f is a vector-field bounded in $L^2(t,t+1;H)$ with f_t bounded in $L^2(t,t+1;H)$. Setting $\partial_t^h u = \frac{1}{h}(u(t+h) - u(t))$, we take the difference of the above equation for $t+h$ and t and we test by $\partial_t^h\mathbf{u}$. This gives

$$\frac{1}{2}\frac{d}{dt}\|\partial_t^h\mathbf{u}\|^2 + \|\nabla\partial_t^h\mathbf{u}\|^2 + b(\partial_t^h\mathbf{u}, \mathbf{u}(t+h), \partial_t^h\mathbf{u}) + b(\mathbf{u}(t), \partial_t^h\mathbf{u}, \partial_t^h\mathbf{u}) = (\partial_t^h f, \partial_t^h\mathbf{u}).$$

Notice that the last term on the left-hand side is equal to zero. We exploit the standard estimate for the trilinear term (2.8) and the Young inequality. Thus we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t^h\mathbf{u}\|^2 + \|\nabla\partial_t^h\mathbf{u}\|^2 \leq \|\mathbf{u}(t+h)\|_{V_{div}}\|\partial_t^h\mathbf{u}\|\|\partial_t^h\mathbf{u}\|_{V_{div}} + \|\partial_t^h f\|\|\partial_t^h\mathbf{u}\|.$$

Due to the Poincaré inequality, we deduce

$$\begin{aligned}\frac{d}{dt}\|\partial_t^h\mathbf{u}\|^2 + \|\nabla\partial_t^h\mathbf{u}\|^2 &\leq C\|\mathbf{u}(t+h)\|_{V_{div}}^2\|\partial_t^h\mathbf{u}\|^2 + C\|\partial_t^h f\|^2 \\ &\leq C\|\partial_t^h\mathbf{u}\|^2 + C\|\partial_t^h f\|^2,\end{aligned}$$

where we have used Lemma 6.3. On account of the inequality

$$\|\partial_t^h v\|_{L^2(t,t+1;H)} \leq \|v_t\|_{L^2(t,t+1;H)},$$

applying the uniform Gronwall lemma, we infer that

$$\|\partial_t^h\mathbf{u}\|_{L^\infty(4\sigma,t;L^2(\Omega))} + \|\nabla\partial_t^h\mathbf{u}\|_{L^2(t,t+1;H)} \leq C, \quad \forall\ t \geq 4\sigma.$$

Hence, we conclude that

$$\|\mathbf{u}_t\|_{L^\infty(4\sigma,t,L^2(\Omega))} + \|\nabla\mathbf{u}_t\|_{L^2(t,t+1;H)} \leq C, \quad \forall\ t \geq 4\sigma. \tag{6.11}$$

Let us now prove that $\mathbf{u}$ is eventually bounded. To this end, according to (6.3) and (6.11), an application of [4, Theorem 1.1] yields

$$\|\mathbf{u}(t)\|_{W^{1,4}(\Omega)} \leq C, \quad \forall\, t \geq 4\sigma,$$

and we conclude that (6.9) holds. Thus we can apply [22, Corollary 4.2] to the nonlocal Cahn–Hilliard equation with convective term and infer (6.10). □

Remark 6.9. We recall that Remark 5.3 holds. Moreover, as we observed in Section 5 we can still identify the weak solutions to problem (1.16)-(1.17) with the weak solutions to a similar problem with a regular potential. Then, we can generalize the results on the longtime behavior contained in Section 5. More precisely, we know from [28] that (1.16)-(1.17) generates a dissipative dynamical systems which possesses a global attractor. Then the regularity of the global attractor as well as the convergence of any weak solution to a single equilibrium proved in [32] for a regular potential can be extended to the present case. The same can be told for results on the existence of an exponential attractor proven in [28, Section 5]. The details are left to the interested reader to check and extend to that situation.

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