On 3 new principles in non-equilibrium statistical mechanics and their equivalence

L. Accardi, F. Fagnola, R. Quezada

Abstract

The 3 new principles mentioned in the title are:
– the **nonlinear** Boltzmann-Gibbs prescription;
– the **local** KMS condition;
– the **dynamical** detailed balance condition.
We prove their equivalence under general conditions and we generalize some of the properties that enter into their definitions.
We also introduce the notion of irreversible (H, β)–KMS condition and prove its equivalence, under additional conditions, with the local (H, β)–KMS condition.

Contents

1 Introduction

The notion of equilibrium states of physical systems is sufficiently well understood and there exist several characterizations of this class of states which, although based on different ideas, when applicable to the same class of systems, define the same objects. For discrete systems, i.e. with a pure point spectrum Hamiltonian H, the most explicit description of an equilibrium state at inverse temperature β is the Boltzmann–Gibbs prescription. The KMS condition is more general because not restricted to discrete systems. In addition to this there are various types of principles (variational, based on stability conditions, . . .) applicable to different classes of systems. An instructive and concise discussion of several characterizations of equilibrium states is contained in the paper [20]. A discussion of the various characterizations of Gibbs states by means of stability conditions is contained in [21]. A further characterization, called Quantum Detailed Balance (QDB) and characterizing equilibrium distributions as stationary states of special classes of Markov semi–groups, was proposed by Kossakowski, Frigerio, Gorini and Verri [22], see also [12], [23] and motivated by the singular coupling limit of open systems (see section (5)). The classical analogue of this notion was well known in the theory of classical Markov processes under the name of *time reversibility* and expresses the stochastic reversibility of the trajectories of such processes.

For non equilibrium phenomena the situation was, until recently, quite different. Like nonlinearity, non equilibrium is a negative connotation that covers an infinity of totally inequivalent situations.
Therefore any attempt to characterize such a variety of behaviors in terms of a few qualitative properties would be naive and probably doomed to failure. A more realistic program is to look for some interesting candidates that, within the class of stationary states for a given Hamiltonian, singles out some special sub-class of states with properties that are rich enough to go beyond the equilibrium situation, but specific enough to avoid vagueness.
The possibility to realize such a program begun to emerge in the late

1990's with the discovery that the stochastic limit of quantum systems can be performed also starting from non–equilibrium states of the environment [10]: according to the *similarity principle* of the stochastic limit approach, the invariant states of the reduced dynamics for the system should *reflect* the properties of the initial state of the environment. This intuition was substantiated by the results of the analysis of a number of specific models arising in concrete physical situations [8], [7], [2], [9], [6].

The analysis of these concrete examples, combined with the possibility to perform the stochastic limit forward and backward in time (hence to compare the two limit Heisenberg evolutions and their irreversible reductions on the system), led to the abstraction of three general principles which generalize in a natural way three of the above mentioned characterizations of equilibrium states (see [3]) namely:

(i) the **nonlinear** Boltzmann-Gibbs prescription (see section 2 for this and for (ii) below).

(ii) the **local** KMS condition

(iii) the **dynamical** detailed balance condition.

In the same paper it was proved that the relationships among these 3 conditions are natural extensions of those among the 3 corresponding equilibrium notions namely: when restricted to some $\mathcal{B}(\mathcal{H})$ (algebra of all operators on some Hilbert space) the first two conditions characterize the same class of states and these states are invariant for the Markov generators which satisfy the third condition (and under some conditions they are the only states whith this property).
Moreover, in the non equilibrium case the Kossakowski, Frigerio, Gorini, Verri quantum detailed balance condition is modified by the emergence, in the expression of the adjoint of the forward Markov generator with respect to an invariant measure, of the so–called *current operator* (see sections 6 and 7) whose name is justified by the possibility, which distinguishes the stochastic limit from the old Markovian (weak coupling and low density) limit, to explicitly calculate the micro–currents of energy from the environment to the system (see [3]).

These results were obtained under some special conditions on the system Hamiltonian (genericity: see [7] for a discussion).
In the present paper we prove that this restriction can be dropped

(see section 7). Moreover, based on previous results of Fagnola and Umanità [17, 18, 19] who introduced the notion of *privileged representation of a Markov generator with respect to a state*, which will play a crucial role in the present paper, we generalize the notion of dynamical detailed balance into that of **weighted** detailed balance (see section 6) which gives the possibility to include Markov generators not necessarily of stochastic limit type (see section 7), for example *Markov generator associated to an Hamiltonian* (see section 4) or even more general classes whose mutual relationships will be discussed elsewhere. We also introduce the notions of *infinitesimal and irreversible KMS condition* and prove that in some cases they are equivalent to the *local KMS condition* (see section 3).

A preliminary version of the present paper has appeared in the special issue of *Busseikenkyu* in commemoration of Shuichi Tasaki [2]. The paper [2], in addition to the material included in the present one, includes a discussion of the connection between weighted detailed balance and the cycle description of Markov generators used in the Qian–Kalpazidou approach (see [24], [28]) as well as the construction of an example of a non-equilibrium steady state for a quantum spin chain coupled to two reservoirs at different temperatures, including a discussion of its cycle dynamics and entropy production.

Finally an Appendix (see section 8) recalls some standard notions and results of the stochastic limit frequently used in the present paper.

2 The local KMS condition

We denote $\mathcal{B}(\mathcal{H})$ the von Neumann algebra of all bounded operator on a separable Hilbert space $\mathcal{H}$ and $\mathrm{Tr}(\mathcal{H})$ the corresponding space of trace class operators. In the following we will be mostly concerned with bounded generators, but we try to state the main definitions and problems so that the extension to unbounded ones becomes as transparent as possible.

Definition 1. Let be given a von Neumann algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, a self–adjoint operator H affiliated with $\mathcal{A}$ and a Borel function

$$\beta : \mathbb{R}_+ \to \mathbb{R}_+$$

Denote

$$u_t : \mathcal{A} \ni x \to u_t(x) := e^{itH} x e^{-itH} =: x(t) \in \mathcal{A} \tag{1}$$

the 1–parameter automorphism group of $\mathcal{A}$ generated by H (Heisenberg evolution). A normal state φ on $\mathcal{A}$ is said to satisfy the *local KMS condition with respect to the function β and the Heisenberg dynamics* (1) (simply the (H, β)–KMS condition, or the local KMS condition, if no confusion is possible), if for each $x, y \in \mathcal{A}$:
(i) The map

$$\mathbb{R} + i\beta(\mathrm{spec}(H)) \ni t + i\beta(\lambda) \mapsto \varphi(xy(t + i\beta(\lambda))) \tag{2}$$

is well defined by analytic continuation of the map $t \in \mathbb{R} \mapsto \varphi(xy(t))$.
(ii) Denoting $E_H(\,\cdot\,)$ the spectral measure of H and introducing the complex valued measure

$$\mathbb{R}_+ \times \mathbb{R}_+ \supseteq I \times J \mapsto \varphi_{x,y,H}(I, J) := \varphi(xE_H(I)yE_H(J)) \tag{3}$$

for each $t \in \mathbb{R}$ the integral

$$\int_{\mathbb{R}_+ \times \mathbb{R}_+} e^{it(\lambda-\mu)} e^{\beta(\mu)\mu - \beta(\lambda)\lambda} \varphi_{x,y,H}(d\lambda, d\mu) =: \varphi\left(xy(t + i\beta(H))\right) \tag{4}$$

exists.
(iii) In the notations (1), (4) for all $t \in \mathbb{R}$ the following identity holds:

$$\varphi\left(xy(t + i\beta(H))\right) = \varphi\left(y(t)x\right) \tag{5}$$

Remark. If H is bounded and β is a locally bounded function (bounded on bounded sets), then the operator $\exp(\beta(H)H)$ is bounded and for all $x \in \mathcal{A}$ and $t \in \mathbb{R}$ one has:

$$\begin{aligned} y(t + i\beta(H)) &= e^{i(t+i\beta(H))H} x e^{-i(t+i\beta(H))H} = e^{-\beta(H)H} e^{itH} x e^{-itH} e^{\beta(H)H} \\ &= e^{-\beta(H)H} x(t) e^{\beta(H)H} \quad ; \qquad x \in \mathcal{B}(\mathcal{H}) \end{aligned} \tag{6}$$

In the general case the operator $\exp(\beta(H)H)$ is well defined by the spectral theorem and affiliated to $\mathcal{A}$. Moreover the two maps

$$\mathcal{B}(\mathcal{H}) \ni x \mapsto e^{-\beta(H))H} x e^{\beta(H))H} \;,\; e^{\beta(H))H} x e^{-\beta(H))H} \tag{7}$$

are well defined on the finite rank operators, which are a weakly dense sub–$*$–algebra of $\mathcal{B}(\mathcal{H})$, and on this domain they can be shown to be linear, multiplicative, trace preserving and mutually inverse.

Therefore the maps (7) are densely defined on $\mathcal{B}(\mathcal{H})$ and, in the following, whenever these maps will be used, it will always be understood that their arguments are in their domains.
With these notations the local KMS condition (5) can be re–written in the more intuitive form:

$$\varphi\left(xe^{-\beta(H)H}y(t)e^{\beta(H)H}\right) = \varphi\left(y(t)x\right) \qquad ; \qquad \forall t \in \mathbb{R} \tag{8}$$

for x, y in a dense subspace of $\mathcal{B}(\mathcal{H})$. From now on we fix the choice:

$$\mathcal{A} = \mathcal{B}(\mathcal{H})$$

Notice that the identity (8) makes sense for any 1–parameter family $T_t y := y(t)$ and, if in addition

$$T_0 y = y \qquad ; \qquad \forall y \in \mathcal{B}(\mathcal{H}) \tag{9}$$

then (8) becomes equivalent to:

$$\varphi\left(xe^{-\beta(H)H}ye^{\beta(H)H}\right) = \varphi\left(yx\right) \tag{10}$$

for x, y in a dense subspace of $\mathcal{B}(\mathcal{H})$.
For reasons that will be clear in section 3 we formulate the following Theorem in greater generality than needed for the goals of the present section.

Theorem 1. Let H be a positive self–adjoint operator on a Hilbert space $\mathcal{H}$, $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel function and ρ a normal state on $\mathcal{B}(\mathcal{H})$. For any map $(y, t) \in \mathcal{B}(\mathcal{H}) \times \mathbb{R} \;\to\; T_t y := y(t) \in \mathcal{B}(\mathcal{H})$ satisfying (9) and for any state $\varphi := \mathrm{Tr}(\rho\,\cdot\,)$ the following statements are equivalent:

(i) ρ satisfies the local (H, β)–KMS condition (8);

(ii) ρ satisfies the local (H, β)–KMS condition (10) at $t = 0$;

(iii) $e^{-\beta(H)H}$ is trace class and

$$\rho = {Z_\beta}^{-1} e^{-\beta(H)H}, \qquad Z_\beta := \mathrm{Tr}\left(e^{-\beta(H)H}\right) \tag{11}$$

Proof. *(i) ⇒ (ii).* Obvious.
(ii) ⇒ (iii). Using the cyclicity of the trace (10) becomes

$$\mathrm{Tr}\left(xe^{-\beta(H)H}ye^{\beta(H)H}\rho\right) = \mathrm{Tr}\left(x\rho y\right) \qquad ; \qquad \forall x, y \in \mathcal{B}(\mathcal{H})$$

Since $x \in \mathcal{B}(\mathcal{H})$ is arbitrary in a dense subspace, this is equivalent to

$$e^{-\beta(H)H} y e^{\beta(H)H} \rho = \rho y \tag{12}$$

which holds if and only if

$$y e^{\beta(H)H} \rho = e^{\beta(H)H} \rho y \qquad ; \ \forall y \in \mathcal{B}(\mathcal{H}) \tag{13}$$

Since also $y \in \mathcal{B}(\mathcal{H})$ is arbitrary in a dense subspace, this implies that, for some scalar λ ($\neq 0$ because $\mathrm{Tr}(\rho) = 1$), one has:

$$e^{\beta(H)H} \rho = \lambda 1 \tag{14}$$

In particular ρ is invertible and the condition $\mathrm{Tr}(\rho) = 1$ implies (11).
(iii) $\Rightarrow$ *(i)*. This follows because the identity (8), given (11), can be rewritten in the form:

$$\mathrm{Tr}\left(\rho x e^{-\beta(H)H} y(t) e^{\beta(H)H}\right) = Z_\beta{}^{-1} \mathrm{Tr}\left(x e^{-\beta(H)H} y(t)\right) = \ \mathrm{Tr}\left(\rho y(t) x\right)$$

Corollary 1. In the notations and assumptions of Theorem 1 the positive self–adjoint operator (Hamiltonian) H has necessarily the form

$$H = \sum_{\epsilon \in Spec(H)} \epsilon P_\epsilon =: \sum_{m \in \mathbb{N}} \epsilon_m P_m \tag{15}$$

with

$$\mathrm{Tr}(P_\epsilon) < +\infty \qquad ; \qquad \forall \epsilon \in Spec(H) \tag{16}$$

Proof. Writing for simplicity $F(\lambda) := \beta(\lambda)\lambda$ ($\lambda \in \mathbb{R}_+$) we know from Theorem (1) that the operator $\exp\big(-F(H)\big)$ is trace class, hence it has the form

$$e^{-F(H)} = \int_{\mathbb{R}} e^{-F(\lambda)} E_H(d\lambda) = \sum_{F_n \in Range(F)} e^{-F_n} E_n$$

where E_H is the spectral measure of H, the range of F is countable and

$$E_n := E_H(F^{-1}(F_n)) \qquad ; \qquad \forall F_n \in \mathrm{Range}(F) \tag{17}$$

$$Tr(E_n) < +\infty \tag{18}$$

Since (E_n) is an orthogonal resolution of the identity, (17) and (18) imply that H has pure point spectrum, so it must be of the form (15). Finally condition (16) follows from the fact that, if $\epsilon \in Spec(H)$, and $F(\epsilon) = F_n$, then

$$Tr(P_\epsilon) \leq Tr(E_n) < +\infty$$

Remark.

(i) Notice that when β is an affine function:

$$\beta(\lambda) = \beta \cdot \lambda + \mu \qquad ; \qquad \lambda \in \mathbb{R}_+$$

with $\beta \geq 0$ and μ constants, the state (11) is the usual Gibbs state at inverse temperature β and chemical potential μ.

(ii) Our condition that $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ excludes the case that some value of β could be $+\infty$. This means that every state of the form (11) is faithful. Some equivalent formulations of the local KMS condition are meaningful also for non faithful states. However in the present paper we will restrict or attention to faithful states.

(iii) Any invertible density operator ρ which is a function of the Hamiltonian H can be written in the form (11) for some function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and, if one allows the value $+\infty$ in the range of β the invertibility condition can be dropped. Thus the local KMS condition distinguishes, among the invariant states of a dynamics ($\rho \in \{H\}'$), those which are functions of the dynamics, i.e. in $\rho \in \{H\}''$.

3 The infinitesimal and the irreversible (H, β)–KMS condition

In the following, by a *Markov semigroup* we mean a weak*–continuous semigroup of completely positive, normal, identity preserving maps on $\mathcal{B}(\mathcal{H})$. We will use the term *Markov generator* to denote any conditionally completely positive linear operator densely defined on $\mathcal{B}(\mathcal{H})$ and equal to zero on the identity.

We emphasize that, unless explicitly said otherwise, we do not assume that such an operator effectively generates a Markov semi–group.

In the present section we introduce an infinitesimal and an irreversible variant of the local KMS condition, relating a Markov semigroup with a discrete Hamiltonian H and a state (see the identity (41) in [3]). We prove that, under additional conditions on the Markov generator or on the Hamiltonian, the two conditions characterize the same family of states.

In the notations of Theorem (1), if T_t is a strongly continuous 1–parameter semi–group with generator $\mathcal{L}$, then differentiating (8) at $t = 0$ one finds the condition:

$$\varphi\left(xe^{-\beta(H)H}\mathcal{L}(y)e^{\beta(H)H}\right) = \varphi\left(\mathcal{L}(y)x\right) \tag{19}$$

valid on the same dense subspace as in (8) with the additional condition that y is in the domain of $\mathcal{L}$.

Definition 2. Let $\mathcal{L}$ be a Markov generator, φ a state on $\mathcal{B}(\mathcal{H})$, and (H, β) as in section (2). The pair $(\varphi, \mathcal{L})$ is said to satisfy the infinitesimal form of the local (H, β)–KMS condition if condition (19) holds.

Remark. Heisenberg evolutions are included in the above formulation and in this case the Markov generator $\mathcal{L}$ is of Hamiltonian type, i.e. of the form

$$\mathcal{L}(y) := i[H, y] \qquad ; \qquad y \in \mathcal{B}(\mathcal{H}) \cap \text{Domain}(i[H, \, \cdot \,]) \tag{20}$$

If the dissipative part of the generator is non zero we speak of the *irreversible* (H, β)–KMS condition.
In some cases condition (19), in general weaker than (8) is in fact equivalent to it. The simplest example is provided by the generators of the form (20).

Proposition 1. Given a Borel function β, suppose that the Hamiltonian H given by (15) has non degenerate spectrum (i.e. mutually distinct eigenvectors with 1–dimensional eigenspaces). Then, for the Markov generator (20) and a state φ on $\mathcal{B}(\mathcal{H})$ the following statements are equivalent:
(i) The pair $(\varphi, \mathcal{L})$ satisfies the infinitesimal (instead irreversible) (H, β)–KMS condition (19) (instead (8));
(ii) The state φ satisfies the local (H, β)–KMS condition (8).

Proof. We have only to prove that (19) implies (8). To this goal denote $\varphi := \text{Tr}(\rho \, \cdot \,)$ and $|\epsilon_m\rangle$, $|\epsilon_n\rangle$ the eigenvalues corresponding to the eigenvectors ϵ_m and ϵ_m. The rank-one operator $|\epsilon_m\rangle\langle\epsilon_n|$ satisfies $\delta(|\epsilon_m\rangle\langle\epsilon_n|) = i(\epsilon_m - \epsilon_n)|\epsilon_m\rangle\langle\epsilon_n|$. It follows that the range of the map δ contains all rank-one operators $|\epsilon_m\rangle\langle\epsilon_n|$ with $n \neq m$.
By the arbitrariness of $x \in \mathcal{B}(\mathcal{H})$ in this dense set, the *infinitesimal* local (H, β)–KMS condition (19) yields

$$|\epsilon_m\rangle\langle\epsilon_n| \, e^{\beta(H)H} \rho = e^{\beta(H)H} \rho \, |\epsilon_m\rangle\langle\epsilon_n|$$

and this implies that $e^{\beta(H)H}\rho$ is a multiple of the identity operator, i.e. the thesis. $\square$

In the following section we will prove that the thesis of Proposition (1) is also true for an important class of non Hamiltonian Markov generators.

4 Markov generators associated with a given Hamiltonian

The infinitesimal form (19) of the (H, β)–KMS condition is clearly a strong relationship between H, β and $\mathcal{L}$.
The stochastic limit of quantum theory gives rise to a class of Markov semigroups which are "strongly related" with a discrete Hamiltonian operator H of the form (15) (which in the stochastic limit is interpreted as the Hamiltonian of the *small* system coupled to the environment).
In the present section we begin to discuss the connections of some of the properties which determine the above mentioned "strong relationship" with the local KMS condition.
The first of these properties consists in leaving invariant the commutant of the algebra of all Borel functions of H.
Recall that the commutant algebra $\{H\}'$ of a self–adjoint operator H is, by definition, the commutant of the (abelian) von Neumann algebra generated by the spectral projections of H.
In the following, operators commuting with a self–adjoint operator H will be called *H–diagonal* (simply diagonal if no confusion is possible).

Definition 3. Let H be a self–adjoint operator. A Markov semigroup $(\mathcal{T}_t)$ (resp. generator $\mathcal{L}$) is called *associated to H* if:

$$\mathcal{T}_t\left(\{H\}'\right) \subseteq \{H\}' \qquad \forall t \geq 0 \tag{21}$$

respectively

$$\mathcal{L}\left(\text{Domain}(\mathcal{L}) \cap \{H\}'\right) \subseteq \{H\}' \tag{22}$$

From now on, in this section, we fix the Hamiltonian (15). Since it has discrete spectrum, the map

$$\mathcal{B}(\mathcal{H}) \ni x \mapsto E_0(x) := \sum_{n\in\mathbb{N}} P_n x P_n$$

is a normal Umegaki conditional expectation (completely positive norm one projection) onto the commutant of H, also called *the diagonal algebra.* Therefore:

$$\{H\}' = \{x \in \mathcal{B}(\mathcal{H}) \ : \ [x, H] = 0\} = \{x \in \mathcal{B}(\mathcal{H}) \ : \ E_0(x) = x\} \tag{23}$$

The operator space

$$\mathcal{B}(\mathcal{H})_{off} := \{x - E_0(x) \ : \ x \in \mathcal{B}(\mathcal{H})\} = \tag{24}$$

$$= \{x \in \mathcal{B}(\mathcal{H}) \ : \ E_0(x) = 0\} = \{x \in \mathcal{B}(\mathcal{H}) \ : \ x = \sum_{m \neq n} P_m x P_n\}$$

will be called the *off–diagonal space*.
One easily verifies that a Markov generator $\mathcal{L}$ is associated with H if and only if:

$$\mathcal{L} \circ E_0 = E_0 \circ \mathcal{L} = E_0 \circ \mathcal{L} \circ E_0 \tag{25}$$

and that this is equivalent to say that

$$x \in \text{Domain}(\mathcal{L}) \ \Leftrightarrow \ E_0(x) \ , \ x - E_0(x) \in \text{Domain}(\mathcal{L}) \tag{26}$$

and

$$\mathcal{L}(\text{Domain}(\mathcal{L}) \cap \mathcal{B}(\mathcal{H})_{off}) \subseteq \mathcal{B}(\mathcal{H})_{off} \tag{27}$$

Lemma 1. Let H and β be as in Theorem 1 and suppose that ρ is a function of H. Then, if x is diagonal and y off–diagonal one has

$$\text{Tr}(\rho x y) = 0 \tag{28}$$

Proof. It is sufficient to prove the statement in the case in which x has the form $x = P_N z P_N$ for some $N \in \mathbb{N}$ and $z \in \mathcal{B}(\mathcal{H})$ because the generic diagonal x (instead $x \in \mathcal{B}(\mathcal{H})_{off}$) is a sum of terms of this form. With this choice of x the left hand side of (28) becomes

$$\text{Tr}(\rho x y) = \text{Tr}(\rho P_N z P_N y) = \rho_N \text{Tr}(P_N z P_N y) = \rho_N \text{Tr}(z P_N y P_N) = 0.$$

Lemma 2. Let H and β be as in Theorem 1 and suppose that:
(i) $\mathcal{L}$ is a Markov generator satisfying (25).
(ii) ρ is a function of H.
Then if either x or y are diagonal, the identity (19) (infinitesimal form of the irreversible (H, β)–KMS condition) holds for any choice of the function β.

Proof.
If $y \in \{H\}'$, (25) implies that also $\mathcal{L}(y) \in \{H\}'$. Therefore, since ρ is a function of H we conclude that:

$$\text{Tr}(\rho x e^{-\beta(H)H} \mathcal{L}(y) e^{\beta(H)H}) = \ \text{Tr}(\rho x \mathcal{L}(y))$$

which is (19). If $x \in \{H\}'$, since ρ is a function of H, we have:

$$\text{Tr}(\rho e^{\beta(H)H} \mathcal{L}(y) e^{-\beta(H)H} x) = \ \text{Tr}(\rho \mathcal{L}(y) x) = \ \text{Tr}(\rho x \mathcal{L}(y))$$

which is again (19).
Remark. Notice that the above proof of Lemma 2 cannot be applied in general if, instead of assuming that ρ is a function of H, one only assumes that $\rho \in \{H\}'$.

Definition 4. A Markov generator $\mathcal{L}$ is said to have simple range with respect to an Hamiltonian H of the form (15), (16) if:

$$\forall (M,N) \in \mathbb{N}^2 \ , \ M \neq N \ , \ \exists y \in \text{Domain}(\mathcal{L}) \ : \ P_N y P_M \neq 0. \tag{29}$$

Theorem 2. Let H and β be as in Theorem (1) and let be given:
(i) a Markov generator $\mathcal{L}$ associated with H, i.e. satisfying (25),
(ii) a density operator ρ which is a function of H,
Then:
(II) If ρ satisfies the (H,β)–KMS condition, i.e. it has the form (11), then the pair $(\mathcal{L}, \rho)$ satisfies (19) (infinitesimal form of the irreversible (H,β)–KMS condition).
(I) Conversely, if the pair $(\mathcal{L}, \rho)$ satisfies (19) and if $\mathcal{L}$ has simple range with respect to H, then ρ has the form (11).

Proof. *(II)* $\Rightarrow$ *(I)*. Suppose that, in the notation (11), the ρ_M $(M \in \mathbb{N})$ have the form

$$\rho_M = \frac{e^{-\beta(\varepsilon_M)\varepsilon_M}}{Z_\beta} \tag{30}$$

Then

$$\text{Tr}(\rho x e^{-\beta(H)H}\mathcal{L}(y)e^{\beta(H)H}) = \sum_n \frac{1}{Z_\beta} \ \text{Tr}(P_n x e^{-\beta(H)H}\mathcal{L}(y))$$

Because of (26) we can consider separately the case in which x is diagonal and the case in which x is off–diagonal. If x is diagonal, then

$$\sum_n \frac{1}{Z_\beta} \ \text{Tr}(P_n x e^{-\beta(H)H}\mathcal{L}(y)) = \sum_n \frac{1}{Z_\beta} \ \text{Tr}(x P_n e^{-\beta(H)H}\mathcal{L}(y))$$

$$= \sum_n \frac{e^{-\beta(\varepsilon_n)\varepsilon_n}}{Z_\beta} \ \text{Tr}(x P_n \mathcal{L}(y)) = \sum_n \rho_n \ \text{Tr}(P_n \mathcal{L}(y) x) = \ \text{Tr}(\rho \mathcal{L}(y) x)$$

hence (19) holds. If x is off–diagonal, then

$$\sum_n \frac{1}{Z_\beta} \ \text{Tr}(P_n x e^{-\beta(H)H}\mathcal{L}(y)) = \sum_n \frac{1}{Z_\beta} \sum_{m\neq n} \ \text{Tr}(P_n x P_m e^{-\beta(H)H}\mathcal{L}(y))$$

$$= \sum_n \sum_{m \neq n} \frac{e^{-\beta(\varepsilon_m)\varepsilon_m}}{Z_\beta} \ \mathrm{Tr}(P_n x P_m \mathcal{L}(y)) = \sum_m \rho_m \sum_{n \neq m} \ \mathrm{Tr}(P_n x P_m \mathcal{L}(y))$$

$$= \sum_m \rho_m \ \mathrm{Tr}(x P_m \mathcal{L}(y)) = \sum_m \rho_m \ \mathrm{Tr}(P_m \mathcal{L}(y) x) = \ \mathrm{Tr}(\rho \mathcal{L}(y) x)$$

Therefore also in this case (19) holds.

$(II) \Rightarrow (I)$. Since any element of $\mathcal{B}(\mathcal{H})$ can be written in a unique way as a sum of a diagonal and an off diagonal part, Lemma (2) allows to reduce the proof of (19) to the case in which either one is in the diagonal space or both x and y are in the off–diagonal space. In the former case the validity of (19) is guaranteed by Lemma 2 independently of the function β. In the latter case we argue as follows. Let $x, y \in \mathcal{B}(\mathcal{H})_{off}$. It is sufficient to prove the statement in the case in which x has the form $x = P_M z P_N$ for some $M \neq N \in \mathbb{N}$ and $z \in \mathcal{B}(\mathcal{H})$ because the generic $x \in \mathcal{B}(\mathcal{H})_{off}$ is a sum of terms of this form. With this choice of x the left hand side of (19) becomes

$$\mathrm{Tr}(\rho x e^{-\beta(H)H} \mathcal{L}(y) e^{\beta(H)H}) = \ \mathrm{Tr}(\rho P_M z P_N e^{-\beta(H)H} \mathcal{L}(y) e^{\beta(H)H}) =$$

$$= \rho_M e^{\beta(\varepsilon_M)\varepsilon_M - \beta(\varepsilon_N)\varepsilon_N} \ \mathrm{Tr}(\mathcal{L}(y) P_M z P_N) \tag{31}$$

and the right hand side of (19) becomes

$$\mathrm{Tr}(\rho \mathcal{L}(y) x) = \ \mathrm{Tr}(\rho \mathcal{L}(y) P_M z P_N) = \rho_N \ \mathrm{Tr}(\mathcal{L}(y) P_M z P_N) \tag{32}$$

Choosing $z = \mathcal{L}(y)^*$ and using the identity

$$\mathrm{Tr}((P_N \mathcal{L}(y) P_M)(P_M \mathcal{L}(y)^* P_N)) = \ \mathrm{Tr}(|P_N \mathcal{L}(y) P_M|^2) \tag{33}$$

The identity between (31) and (32) becomes equivalent to:

$$\rho_M e^{\beta(\varepsilon_M)\varepsilon_M - \beta(\varepsilon_N)\varepsilon_N} \ \mathrm{Tr}(|P_N \mathcal{L}(y) P_M|^2) = \rho_N \ \mathrm{Tr}(|P_N \mathcal{L}(y) P_M|^2) \tag{34}$$

Define:

$$\mathbb{N}^2_{\neq 0} := \{(M, N) \in \mathbb{N}^2 \ : \ M \neq N \ , \ \exists y \in \mathrm{Domain}(\mathcal{L}) \ , \ P_N y P_M \neq 0\}$$

Then (34) implies that

$$\rho_M e^{\beta(\varepsilon_M)\varepsilon_M - \beta(\varepsilon_N)\varepsilon_N} = \rho_N \qquad ; \qquad \forall (M, N) \in \mathbb{N}^2_{\neq 0} \tag{35}$$

Since the identity (35) is trivially verified for $M = N$, the assumption that $\mathcal{L}$ has simple range with respect to H implies that

$$\mathbb{N}^2_{\neq 0} = \mathbb{N}^2$$

Therefore one can sum over $N \in \mathbb{N}$ obtaining

$$\rho_M e^{\beta(\varepsilon_M)\varepsilon_M} \sum_{N\in\mathbb{N}} e^{-\beta(\varepsilon_N)\varepsilon_N} = \sum_{N\in\mathbb{N}} \rho_N = 1$$

which is equivalent to (30). Thus (I) holds and this end the proof.

Remark.

(i) It should be emphasized that the above theorem does not require the invertibility of ρ. Because of (35) this is a consequence of the irreversible (H,β)–KMS condition if $\mathcal{L}$ has simple range. In general the landscape can be more complex and this kind of complexity depends only on the interaction between system and environment, not on the temperature function β.

The following theorem provides a different approach to the problem of determining the structure of the pairs $(\mathcal{L},\rho)$ satisfying the infinitesimal form of the irreversible (H,β)–KMS condition

Theorem 3. Let H,β be as in Theorem 2, let ρ be a state on $\mathcal{B}(\mathcal{H})$ and $\mathcal{L}$ a Markov generator (not necessarily associated with H).
Then the pair $(\rho,\mathcal{L})$ satisfies the infinitesimal form (19), of the irreversible (H,β)–KMS condition, if and only if:

$$e^{\beta(H)H}\rho \in \{\mathrm{Range}(\mathcal{L})\}' \tag{36}$$

Proof. Since the pair $(\rho,\mathcal{L})$ satisfies the infinitesimal form (19), one has for all $x,y \in \mathcal{B}(\mathcal{H})$:

$$\mathrm{Tr}(e^{\beta(H)H}\rho x e^{-\beta(H)H}\mathcal{L}(y)) = \mathrm{Tr}(\rho\mathcal{L}(y)x)$$

if and only if

$$\mathrm{Tr}(e^{-\beta(H)H}\mathcal{L}(y)e^{\beta(H)H}\rho x) = \mathrm{Tr}(\rho\mathcal{L}(y)x).$$

Since $x \in \mathcal{B}(\mathcal{H})$ is arbitrary, this is equivalent to:

$$e^{-\beta(H)H}\mathcal{L}(y)e^{\beta(H)H}\rho = \rho\mathcal{L}(y) \quad \text{if and only if} \quad \mathcal{L}(y)e^{\beta(H)H}\rho = e^{\beta(H)H}\rho\mathcal{L}(y)$$

Since $y \in \mathcal{B}(\mathcal{H})$ is arbitrary, this is equivalent to (36).

Corollary 2. If the Markov generator $\mathcal{L}$ satisfies (19) and the commutant of the range of $\mathcal{L}$ is trivial, i.e.

$$\{\mathrm{Range}(\mathcal{L})\}' = \mathbb{C}\cdot 1 \tag{37}$$

then ρ has the form (11).

Proof. The thesis follows because (36) implies that $e^{\beta(H)H}\rho$ is a multiple of the identity and we have seen that this implies that ρ has the form (11).

Remark. Recently Bolaños and Fagnola [13] have shown that the commutant of the range of the infinitesimal generator of a quantum Markov semigroup on the algebra of $d \times d$ matrices is always an Abelian subagebra. Exploiting this fact, they have proved that the local (H, β)–KMS condition (8) for the pair $(\mathcal{L}, \rho)$ is equivalent to the infinitesimal form of the local (H, β)–KMS (19). instead: the irreversible $(H; \beta)$–KMS condition is equivalent to the local irreversible $(H; \beta)$–KMS condition in infinitesimal form.

5 Time reversed and adjoints of a Markov generator

The theory of stochastic limit allows us to associate in a canonical way to a system with free Hamiltonian H, interacting with an *environment*, two Markov processes: the *forward* and the *backward* process, obtained by taking the stochastic limit respectively in the forward and backward time direction.

Like all Markov processes also these ones are canonically associated to Markov semigroups, the forward and the backward (or *time reversed*) semigroup, whose structure depends not only on H but also on the free Hamiltonian of the environment, on the interaction and on the initial state of the environment.

The generators of the forward and the backward semigroup are related by a kind of duality relation introduced in [3] and called *dynamical detailed balance condition*.

If the initial state of the environment is an equilibrium one, this reduces to Kossakowski, Frigerio, Gorini, Verri detailed balance.

In the following sections we will analyze the connections between the above mentioned duality and some known operator–theoretical duality notions between Markov semigroups or their generators. For this reason, in the present section, we recall some of these duality notions and their properties.

If $\mathcal{L} : \mathcal{D} \subseteq \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is any linear operator, with a dense domain $\mathcal{D}$, the *trace dual* of $\mathcal{L}$ is by definition the linear operator

$\mathcal{L}_* : \mathcal{D}_* \subseteq \mathrm{Tr}(\mathcal{H}) \to \mathrm{Tr}(\mathcal{H})$, with domain $\mathcal{D}_*$, defined by the relation

$$\mathrm{Tr}\Big(\rho\mathcal{L}(x)\Big) = \mathrm{Tr}\Big(\mathcal{L}_*(\rho)x\Big) \qquad ; \qquad \rho \in \mathcal{D}_* \ , \ x \in \mathcal{D} \tag{38}$$

A density operator $\rho \in \mathcal{D}_*$ is called $\mathcal{L}$–stationary if

$$\mathcal{L}_*(\rho) = 0 \tag{39}$$

For Markov generators the following notions of duality with respect to a fixed state ρ is often used.

Definition 5. Given a linear operator Φ defined on a dense domain $\mathrm{Dom}(\Phi) \subseteq \mathcal{B}(\mathcal{H})$ and a normal state ρ on $\mathcal{B}(\mathcal{H})$, the linear operator $\big(\Phi^*_\rho, \mathrm{Dom}(\Phi^*_\rho)\big)$ is the adjoint of Φ with respect to the scalar product induced by ρ on $\mathcal{B}(\mathcal{H})$, i.e.

$$\langle x, y\rangle_\rho =: \mathrm{Tr}\,(\rho\, x^* y) \qquad ; \qquad x, y \in \mathcal{B}(\mathcal{H}) \tag{40}$$

More explicitly, the pair $\big(\Phi^*_\rho, \mathrm{Dom}(\Phi^*_\rho)\big)$, where $\mathrm{Dom}(\Phi^*_\rho)$ is

$$\{x \in \mathcal{B}(\mathcal{H}) \ : \exists z \in \mathcal{B}(\mathcal{H}), \forall y \in \mathrm{Dom}(\Phi) \ , \ \ \mathrm{Tr}\big(\rho z y\big) = \mathrm{Tr}\big(\rho x \Phi(y)\big\}$$

and

$$\mathrm{Tr}\left(\rho\Phi^*_\rho(x)y\right) = \mathrm{Tr}\left(\rho x \Phi(y)\right), \ \ \forall \ \ y \in \mathrm{Dom}(\Phi), \tag{41}$$

is called the ρ-adjoint of Φ and we denote it simply by Φ^*_ρ.

Lemma 3. Suppose that $\mathrm{Dom}(\mathcal{L})$ is dense and consider the following statements:
(i) ρ is $\mathcal{L}$–stationary
(ii) $\mathcal{L}^*_\rho$ satisfies

$$\mathcal{L}^*_\rho(1) = 0 \tag{42}$$

Then (ii) implies (i) and, if ρ is invertible, (i.e., it has a dense range, therefore its inverse is densely defined, but not necessarily bounded), then (i) implies (ii).

Proof. (41) implies the following identities:

$$\mathrm{Tr}\Big(\mathcal{L}^*_\rho(1)y\rho\Big) = \mathrm{Tr}\Big(\rho\mathcal{L}^*_\rho(1)y\Big) = \mathrm{Tr}\Big(\rho\mathcal{L}(y)\Big) = \mathrm{Tr}\Big(\mathcal{L}_*(\rho)y\Big) \ ; \ \forall y \in \mathrm{Dom}(\mathcal{L})$$

Thus if (ii) holds then, for all $y \in \mathrm{Dom}(\mathcal{L})$, $tr\Big(\mathcal{L}_*(\rho)y\Big) = 0$ and (i) follows from the density of $\mathrm{Dom}(\mathcal{L})$. Conversely if (i) holds then, with $x = 1$ (41) implies that

$$tr\Big(\rho\mathcal{L}^*_\rho(1)y\Big) = tr\left(\rho\mathcal{L}(y)\right) = tr\left(\mathcal{L}_*(\rho)y\right) = 0$$

for all $y \in \mathrm{Dom}(\mathcal{L})$.
Since $\mathrm{Dom}(\mathcal{L})$ is dense and the map $y \mapsto y\rho$ is invertible and bounded because such is ρ, this implies that also $\rho\mathrm{Dom}(\mathcal{L})$ is dense and therefore (42) holds.
The pairs $(\rho, \mathcal{L})$ such that $\mathcal{L}^*_\rho$ is a Markov generator can be characterized, if $\mathcal{L}$ is uniformly bounded, as follows (see e.g. [17] Theorem 3.1 p. 341).

Theorem 4. If $\mathcal{L}$ is uniformly bounded and ρ is faithful, then the following statements are equivalent:

(i) $\mathcal{L}^*_\rho$ is a Markov generator (in this case it is uniformly bounded),

(ii) denoting

$$\sigma_t(a) = \rho^{it} a \rho^{-it}$$

the modular group of ρ, $\mathcal{L}$ commutes with σ_t, i.e. $\mathcal{L}\,\sigma_t = \sigma_t\,\mathcal{L}$, $\forall t \geq 0$,

(iii) $\mathcal{L}$ commutes with σ_{-i}, i.e. $\mathcal{L}\,\sigma_{-i} = \sigma_{-i}\,\mathcal{L}$.

Definition 6. Any representation of a Markov generator $\mathcal{L}$ of the form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\sum_{k\in I} \left(L_k^* L_k x - 2L_k^* x L_k + x L_k^* L_k\right) \tag{43}$$

where the triple $(I, H, (L_k)_{k\in I})$ satisfies:
(i) I is an at most countable set,
(ii) $H = H^* \in \mathcal{B}$,
(iii) $L_k \in \mathcal{B}(\mathcal{H})$ for all $k \in I$ and the series $\sum_{k\in I} L_k^* L_k$ is strongly convergent on a dense sub–set of $\mathcal{H}$, is called a Gorini–Kossakowski–Sudarshan–Lindblad (GKSL) representation of $\mathcal{L}$.

Remark. Given a GKSL representation (6) of a Markov generator $\mathcal{L}$, if ρ is an invertible density matrix, then $\forall x \in \mathcal{B}(\mathcal{H})$ the formal expression for $\mathcal{L}^*_\rho(x)$ is given by

$$\mathcal{L}^*_\rho(x) = \sum_k \left(\rho^{-1} L_k \rho x L_k^* - \frac{1}{2}\left(x L_k^* L_k + \rho^{-1} L_k^* L_k \rho x\right)\right) \tag{44}$$

From (44) it is clear that the ρ–adjoint of a Markov generator in general does not need to be densely defined or to map the bounded operators into themselves. Furthermore, even if either of these properties holds, in general $\mathcal{L}^*_\rho(x)$ will not be a Markov generator.

6 Weighted detailed balance for Markov generators

In the paper [3] it was shown that the dynamical detailed balance condition implies a very special relation, which is a natural generalization of the quantum detailed balance condition of Frigerio, Kossakowski, Gorini, Verri ([22]), between a Markov generator with an invariant measure ρ and its ρ–adjoint.
In this section we introduce the notion of *weighted detailed balance*, which generalizes the dynamical detailed balance condition.

Definition 7. A quantum Markov generator $\mathcal{L}$ is said to satisfy a **weighted detailed balance** condition with respect to a faithful normal state ρ, if:
(i) the ρ–adjoint of $\mathcal{L}$ is a bounded Markov generator;
(ii) $\mathcal{L}$ admits a GKSL representation (6) with the following property: there exists a sequence of positive numbers $q := (q_k)_{k\in I}$ such that

$$\mathcal{L}^*_\rho - \mathcal{L} = -2i[K, \cdot] + \Pi \tag{45}$$

where $K \in \mathcal{B}(\mathcal{H})$ is a self–adjoint operator and

$$\Pi(x) := \sum_{k\in I}(q_k - 1)L^*_k x L_k \qquad ; \qquad x \in \mathcal{B}(\mathcal{H}) \tag{46}$$

Notice that (45) and (46) imply that Π is completely bounded (What is the meaning of completely bounded?) and

$$\Pi(1) = 0 \tag{47}$$

7 Markov generators of stochastic limit type with respect to an Hamiltonian H

The notion of weighted detailed balance generalizes the notion of dynamical detailed balance introduced in [3] for a special form of the coefficients (q_k) and a special class of Markov generators. These generators are given through a GKSL representation of very special type whose origins, from the stochastic limit approach, which suggests the intuitive interpretation of the operator (46) as *current operator*, is described in section 7 (see also Appendix II of the paper [2]).

In the following we will freely use the notations introduced in Appendix I (see section (8)).

Let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator (Hamiltonian) with discrete spectral decomposition

$$H = \sum_{\epsilon_m \in \mathrm{Spec}(H)} \epsilon_m P_m \tag{48}$$

and denote B_+ the set of its strictly positive Bohr frequencies (i.e. the set of strictly positive eigenvalues of $e^{itH}(\ \cdot\ )e^{-itH}$):

$$B_+ := B_+(H) := \{\omega = \varepsilon_r - \varepsilon_{r'} > 0 \ : \ \varepsilon_r, \varepsilon_{r'} \in \mathrm{Spec}\,(H)\} = \tag{49}$$

$$= \mathrm{Spec}_+\,(\mathrm{Ad}(e^{itH}))$$

Definition 8. A Markov generator $\mathcal{L}$ on $\mathcal{B}(\mathcal{H})$ is said to be of of stochastic limit type with respect to the Hamiltonian (48) if it has the form:

$$\mathcal{L}(x) = i[\Delta, x] - \tag{50}$$

$$\sum_{\omega \in B_+} \left(\Gamma_{-,\omega} \left(\frac{1}{2}\{D_\omega^\dagger D_\omega, x\} - D_\omega^\dagger x D_\omega \right) + \Gamma_{+,\omega} \left(\frac{1}{2}\{D_\omega D_\omega^\dagger, x\} - D_\omega x D_\omega^\dagger \right) \right)$$

where, in the notations of Appendix I below (section (8)), for each $\omega \in B$:

$$\Gamma_{\pm,\omega} \in \mathbb{R}_+ \tag{51}$$

$$\Delta = \Delta^* \in \{H\}' \tag{52}$$

$$D_\omega = E_\omega(D) \qquad ; \qquad D \in \mathcal{B}(\mathcal{H}) \tag{53}$$

The numerical coefficients (51) have a special structure deduced from the stochastic limit and described in Appendix II of the paper [2] (see also [1], [7]).

7.1 Canonical form of Markov generators of stochastic limit type

Introducing the set

$$\hat{B}_+ := \{\omega \in B_+ \ : \ D_\omega \neq 0 \text{ and either } \Gamma_{-,\omega} \neq 0 \text{ or } \Gamma_{+,\omega} \neq 0\} \tag{54}$$

it is convenient to write the generator (50) in the form

$$\mathcal{L}(x) = i[\Delta, x] - \sum_{\omega \in \hat{B}_+} \mathcal{L}_\omega(x) \tag{55}$$

with

$$\mathcal{L}_\omega(x) = \left(\Gamma_{-,\omega}\left(\frac{1}{2}\{D^\dagger_\omega D_\omega, x\} - D^\dagger_\omega x D_\omega\right) + \Gamma_{+,\omega}\left(\frac{1}{2}\{D_\omega D^\dagger_\omega, x\} - D_\omega x D^\dagger_\omega\right)\right) \tag{56}$$

Remark. This shows that notation (54) has been introduced to eliminate from the sum (56) all the $\mathcal{L}_\omega$ which are identically zero.
Notice that, while B_+ depends only on H, $\hat{B}_+$ depends also on $\mathcal{L}$.
For $\omega \in \hat{B}_+$, the operators D_ω in (55) have the form (see also Appendix (8))

$$D_\omega := E_\omega(D) = \sum_{\{(\varepsilon_m, \varepsilon_n) \in B_+(\omega)\}} P_{\varepsilon_m} D P_{\varepsilon_n} \tag{57}$$

$$= \sum_{\{(\varepsilon_m, \varepsilon_n) \in B_+(\omega, D)\}} P_{\varepsilon_m} D P_{\varepsilon_n} = \sum_{\{(\varepsilon_m, \varepsilon_n) \in B_+(\omega, D)\}} D_{(\varepsilon_m, \varepsilon_n)}$$

where by definition

$$B_+(\omega) := \{(\varepsilon_m, \varepsilon_n) \in (\mathrm{Spec}\,(H))^2 \ : \ \varepsilon_n - \varepsilon_m = \omega\} \tag{58}$$

and for some $D \in \mathcal{B}(\mathcal{H})$, denoting $\forall (\varepsilon_m, \varepsilon_n) \in B_+(\omega)$:

$$D_{(\varepsilon_m, \varepsilon_n)} := P_{\varepsilon_m} D P_{\varepsilon_n} \tag{59}$$

$$B_+(\omega, D) := \{(\varepsilon_m, \varepsilon_n) \in B_+(\omega) \ : \ P_{\varepsilon_m} D P_{\varepsilon_n} \neq 0\}$$

where, for any operator $D \in \mathcal{B}(\mathcal{H})$, $E_\omega(D)$ is given by (57).
Now for each $\omega \in \hat{B}_+$ consider the generator (see (50))

$$\mathcal{L}_\omega(x) =: \Gamma_{-,\omega}\left(\frac{1}{2}\{D^\dagger_\omega D_\omega, x\} - D^\dagger_\omega x D_\omega\right) + \Gamma_{+,\omega}\left(\frac{1}{2}\{D_\omega D^\dagger_\omega, x\} - D_\omega x D^\dagger_\omega\right) =$$

$$= \frac{1}{2}\Gamma_{-,\omega}\{D^\dagger_\omega D_\omega, x\} + \frac{1}{2}\Gamma_{+,\omega}\{D_\omega D^\dagger_\omega, x\} - \left(\Gamma_{-,\omega} D^\dagger_\omega x D_\omega + \Gamma_{+,\omega} D_\omega x D^\dagger_\omega\right) \tag{60}$$

Using (53), for each $x \in \mathcal{B}(\mathcal{H})$ one finds

$$\Gamma_{-,\omega} D^\dagger_\omega x D_\omega + \Gamma_{+,\omega} D_\omega x D^\dagger_\omega = \tag{61}$$

$$= \sum_{\{(\varepsilon_m,\varepsilon_n)\in B_+(\omega,D)\}} \sum_{\{(\varepsilon_M,\varepsilon_N)\in B_+(\omega,D)\}} \left(\Gamma_{-,\omega} D^\dagger_{(\varepsilon_m,\varepsilon_n)} x D_{(\varepsilon_M,\varepsilon_N)} + \Gamma_{+,\omega} D_{(\varepsilon_m,\varepsilon_n)} x D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)$$

$$= \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}} \left(\Gamma_{-,\omega} D^\dagger_{(\varepsilon_m,\varepsilon_n)} x D_{(\varepsilon_M,\varepsilon_N)} + \Gamma_{+,\omega} D_{(\varepsilon_m,\varepsilon_n)} x D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)$$

With these notations

$$\begin{aligned} \mathcal{L}_\omega(x) = & \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}} \frac{1}{2}\Gamma_{-,\omega}\{D^\dagger_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_M,\varepsilon_N)}, x\} \\ & + \frac{1}{2}\Gamma_{+,\omega}\{D_{(\varepsilon_m,\varepsilon_n)} D^\dagger_{(\varepsilon_M,\varepsilon_N)}, x\} \\ & - \left(\Gamma_{-,\omega} D^\dagger_{(\varepsilon_m,\varepsilon_n)} x D_{(\varepsilon_M,\varepsilon_N)} + \Gamma_{+,\omega} D_{(\varepsilon_m,\varepsilon_n)} x D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right) \end{aligned} \tag{62}$$

7.2 Dynamical detailed balance

In this section we prove that, if ρ is any faithful density operator which is a function of the Hamiltonian H, then every Markov generator of stochastic limit type with respect to H admits a ρ–adjoint and satisfies the dynamical detailed balance condition in the sense of [3] which is a particular case of Definition 7. To this goal the following Lemma plays an important role.

Lemma 4.

$$(\varepsilon,\varepsilon'),(\varepsilon,\varepsilon'')\in B_+(\omega) \Rightarrow \varepsilon'=\varepsilon'' \tag{63}$$

$$(\varepsilon',\varepsilon),(\varepsilon'',\varepsilon)\in B_+(\omega) \Rightarrow \varepsilon'=\varepsilon'' \tag{64}$$

$$(\varepsilon,\varepsilon')\neq(\varepsilon'',\varepsilon''')\in B_+(\omega) \Rightarrow \varepsilon\neq\varepsilon'' \quad \text{and} \quad \varepsilon'\neq\varepsilon''' \tag{65}$$

Proof. (63) follows from:

$$\varepsilon'-\varepsilon=\omega=\varepsilon''-\varepsilon \Rightarrow 0=(\varepsilon'-\varepsilon)-(\varepsilon''-\varepsilon)=\varepsilon'-\varepsilon''$$

(64) follows from:

$$\varepsilon-\varepsilon'=\omega=\varepsilon-\varepsilon'' \Rightarrow 0=(\varepsilon-\varepsilon')-(\varepsilon-\varepsilon'')=\varepsilon''-\varepsilon'$$

Finally (64) implies that, if $\varepsilon=\varepsilon''$, then one must have also $\varepsilon'=\varepsilon'''$ against the assumption. Similarly (63) implies that, if $\varepsilon'=\varepsilon'''$, then one must have also $\varepsilon=\varepsilon''$ against the assumption. Thus (65) follows.

In view of the following result, Lemma 4 is of crucial importance for the thesis of the present paper.

Lemma 5. For any $\omega \in B_+$ and for any $(\varepsilon_m, \varepsilon_n)$ and $(\varepsilon_M, \varepsilon_N)$ in $B_+(\omega)$ one has:

$$D^+_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_M,\varepsilon_N)} \in \{H\}' \tag{66}$$

Proof. We know, from (59), that

$$D_{(\varepsilon_m,\varepsilon_n)} := P_{\varepsilon_m} D P_{\varepsilon_n} \qquad ; \qquad D^+_{(\varepsilon_m,\varepsilon_n)} = P_{\varepsilon_n} D^+ P_{\varepsilon_m}$$

Therefore, if $(\varepsilon_m, \varepsilon_n) = (\varepsilon_M, \varepsilon_N)$, then

$$D^+_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_m,\varepsilon_n)} = P_{\varepsilon_n} D^+ P_{\varepsilon_m} P_{\varepsilon_m} D P_{\varepsilon_n} = P_{\varepsilon_n} D^+ P_{\varepsilon_m} D P_{\varepsilon_n} \in \{H\}'.$$

Now, if $(\varepsilon_m, \varepsilon_n) \neq (\varepsilon_M, \varepsilon_N)$, then

$$D^+_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_M,\varepsilon_N)} = P_{\varepsilon_n} D^+ P_{\varepsilon_m} P_{\varepsilon_M} D P_{\varepsilon_N}$$

From (65) we know that

$$(\varepsilon_m, \varepsilon_n) \neq (\varepsilon_M, \varepsilon_N) \in B_+(\omega) \Rightarrow \varepsilon_m \neq \varepsilon_M \qquad \text{and} \qquad \varepsilon_n \neq \varepsilon_N$$

Therefore, if $(\varepsilon_m, \varepsilon_n) \neq (\varepsilon_M, \varepsilon_N)$, then

$$D^+_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_M,\varepsilon_N)} = 0$$

in both cases (66) holds.

Lemma 6. Suppose that $h' \in \{H\}'$ and ρ is a function of H. Then the linear map $x \mapsto \{x, h'\}$ is self–addjoint with respect to the scalar product (40) induced by ρ. (instead ρ–scalar product.)

Proof.

$$\begin{aligned} \mathrm{Tr}(\rho\{h', x\}y) = {} & \mathrm{Tr}(\rho h' xy) + \mathrm{Tr}(\rho x h' y) = \mathrm{Tr}(\rho xy h') + \mathrm{Tr}(\rho x h' y) \\ = {} & \mathrm{Tr}(\rho x \{y, h'\}). \end{aligned}$$

Corollary 3. For any $\omega \in \hat{B}_+$ and for any $(\varepsilon_m, \varepsilon_n)$ and $(\varepsilon_M, \varepsilon_N)$ in $\hat{B}_+(\omega)$, if ρ is a function of H, then the linear operators

$$x \mapsto \{x, D^+_{(\varepsilon_m,\varepsilon_n)} D_{(\varepsilon_M,\varepsilon_N)}\} \qquad ; \qquad x \mapsto \{x, D_{(\varepsilon_m,\varepsilon_n)} D^+_{(\varepsilon_M,\varepsilon_N)}\}$$

are self–adjoint with respect to the scalar product (40) induced by ρ. (instead: ρ–self–adjoint.)

Proof. Since

$$D^{+}_{(\varepsilon_m,\varepsilon_n)}D_{(\varepsilon_M,\varepsilon_N)}\in\{H\}' \tag{67}$$

the thesis is an immediate consequence of Lemma 6.

Corollary 3 implies that the anticommutator part of the generator (62) is ρ–self–adjoint for any state ρ which is a function of H. Let us consider the completely positive part of (62), i.e.

$$\Psi(x):=\left(\Gamma_{-,\omega}D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}xD_{(\varepsilon_M,\varepsilon_N)}+\Gamma_{+,\omega}D_{(\varepsilon_m,\varepsilon_n)}xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}\right)$$

From the identity

$$\mathrm{Tr}(\rho D_{(\varepsilon_m,\varepsilon_n)}xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}y)=\rho_m\ \mathrm{Tr}(D_{(\varepsilon_m,\varepsilon_n)}xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}y)$$

$$=\rho_m\ \mathrm{Tr}(xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}yD_{(\varepsilon_m,\varepsilon_n)})=\rho_m\ \mathrm{Tr}(xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}yD_{(\varepsilon_m,\varepsilon_n)}\rho^{-1}\rho)$$

$$=\rho_m\rho_n^{-1}\ \mathrm{Tr}(xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}yD_{(\varepsilon_m,\varepsilon_n)}\rho)=\rho_m\rho_n^{-1}\ \mathrm{Tr}(\rho xD^{\dagger}_{(\varepsilon_M,\varepsilon_N)}yD_{(\varepsilon_m,\varepsilon_n)})$$

one deduces that

$$\left(D_{(\varepsilon_m,\varepsilon_n)}\ \cdot\ D^{\dagger}_{(\varepsilon_M,\varepsilon_N)}\right)^*_{\rho}=\rho_m\rho_n^{-1}\left(D^{\dagger}_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)$$

where $(X)^*_\rho$ denotes the ρ-adjoint of X. Similarly

$$\mathrm{Tr}(\rho D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}xD_{(\varepsilon_M,\varepsilon_N)}y)=\rho_n\ \mathrm{Tr}(D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}xD_{(\varepsilon_M,\varepsilon_N)}y)$$

$$=\rho_n\ \mathrm{Tr}(xD_{(\varepsilon_M,\varepsilon_N)}yD^{\dagger}_{(\varepsilon_m,\varepsilon_n)})=\rho_n\ \mathrm{Tr}(xD_{(\varepsilon_M,\varepsilon_N)}yD^{\dagger}_{(\varepsilon_m,\varepsilon_n)}\rho^{-1}\rho)$$

$$=\rho_n\rho_m^{-1}\ \mathrm{Tr}(xD_{(\varepsilon_M,\varepsilon_N)}yD^{\dagger}_{(\varepsilon_m,\varepsilon_n)}\rho)=\rho_n\rho_m^{-1}\ \mathrm{Tr}(\rho xD_{(\varepsilon_M,\varepsilon_N)}yD^{\dagger}_{(\varepsilon_m,\varepsilon_n)})$$

Therefore

$$\left(D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}\ \cdot\ D_{(\varepsilon_M,\varepsilon_N)}\right)^*_{\rho}=\rho_m^{-1}\rho_n\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}\right)$$

In conclusion, the ρ-adjoint of $\Psi(\cdot)$ is

$$(\Psi)^*_\rho(\cdot)=\rho_m^{-1}\rho_n\Gamma_{-,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^{\dagger}_{(\varepsilon_m,\varepsilon_n)}\right)+\rho_m\rho_n^{-1}\Gamma_{+,\omega}\left(D^{\dagger}_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)$$

and we conclude that the adjoint of $\mathcal{L}_\omega$ is:

$$(\mathcal{L}_\omega)^*_\rho=$$

$$= \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}} \left\{\frac{1}{2}\Gamma_{-,\omega}\{D^\dagger_{(\varepsilon_m,\varepsilon_n)}D_{(\varepsilon_M,\varepsilon_N)},x\} + \frac{1}{2}\Gamma_{+,\omega}\{D_{(\varepsilon_m,\varepsilon_n)}D^\dagger_{(\varepsilon_M,\varepsilon_N)},x\}\right.$$

$$\left. - \rho_m^{-1}\rho_n\Gamma_{-,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right) - \rho_m\rho_n^{-1}\Gamma_{+,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D_{(\varepsilon_m,\varepsilon_n)}\right)\right\}$$

$$= \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}} \left\{\frac{1}{2}\Gamma_{-,\omega}\{D^\dagger_{(\varepsilon_m,\varepsilon_n)}D_{(\varepsilon_M,\varepsilon_N)},x\} + \frac{1}{2}\Gamma_{+,\omega}\{D_{(\varepsilon_m,\varepsilon_n)}D^\dagger_{(\varepsilon_M,\varepsilon_N)}, \ \cdot \ \}\right.$$

$$\left. -\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D_{(\varepsilon_M,\varepsilon_N)}\right) - \Gamma_{+,\omega}\left(D_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)\right\}$$

$$+\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D_{(\varepsilon_M,\varepsilon_N)}\right) + \Gamma_{+,\omega}\left(D_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)$$

$$- \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\left(\rho_m^{-1}\rho_n\Gamma_{-,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right) + \rho_m\rho_n^{-1}\Gamma_{+,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D_{(\varepsilon_m,\varepsilon_n)}\right)\right)\right\}$$

$$= \mathcal{L}_\omega(\ \cdot \ ) +$$

$$+ \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D_{(\varepsilon_M,\varepsilon_N)}\right) + \Gamma_{+,\omega}\left(D_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)\right\}$$

$$- \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\left(\rho_m^{-1}\rho_n\Gamma_{-,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right) + \rho_m\rho_n^{-1}\Gamma_{+,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D_{(\varepsilon_m,\varepsilon_n)}\right)\right)\right\}$$

Using the identity:

$$\sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D_{(\varepsilon_m,\varepsilon_n)}\right) + \Gamma_{+,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)} \ \cdot \ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right)\right\} =$$

$$= \sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D_{(\varepsilon_M,\varepsilon_N)}\right) + \Gamma_{+,\omega}\left(D_{(\varepsilon_m,\varepsilon_n)} \ \cdot \ D^\dagger_{(\varepsilon_M,\varepsilon_N)}\right)\right\} =$$

One finds

$$(\mathcal{L}_\omega)^*_\rho = \mathcal{L}_\omega(\ \cdot\ )\ +\sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$+\left\{\Gamma_{-,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)+\Gamma_{+,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right)\right\}$$

$$-\sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\left(\rho_m^{-1}\rho_n\Gamma_{-,\omega}\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right)+\rho_m\rho_n^{-1}\Gamma_{+,\omega}\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)\right)\right\}$$

$$=\mathcal{L}_\omega(\ \cdot\ )\ -\sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}}$$

$$\left\{\left((\rho_m^{-1}\rho_n\Gamma_{-,\omega}-\Gamma_{+,\omega})\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right)+(\rho_m\rho_n^{-1}\Gamma_{+,\omega}-\Gamma_{-,\omega})\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)\right)\right\}$$

Thus, introducing the ω–**current operator**

$$\Pi_{\omega,\rho} := -\sum_{\{((\varepsilon_m,\varepsilon_n),(\varepsilon_M,\varepsilon_N))\in B_+(\omega,D)^2\}} \tag{68}$$

$$\left\{\left((\rho_m^{-1}\rho_n\Gamma_{-,\omega}-\Gamma_{+,\omega})\left(D_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D^\dagger_{(\varepsilon_m,\varepsilon_n)}\right)+(\rho_m\rho_n^{-1}\Gamma_{+,\omega}-\Gamma_{-,\omega})\left(D^\dagger_{(\varepsilon_M,\varepsilon_N)}\ \cdot\ D_{(\varepsilon_m,\varepsilon_n)}\right)\right)\right\}$$

and recalling that $\Delta = \sum_{\omega\in F}\Delta_\omega$ (see equation (??)), (Δ_ω has not been defined) one obtains

$$(\mathcal{L}_\omega)^*_\rho = \mathcal{L}_\omega + \Pi_{\omega,\rho}$$

In conclusion the ρ–adjoint of the generator (55) has the form

$$(\mathcal{L})^*_\rho = -i[\Delta,\ \cdot\ ] + \mathcal{L} + \sum_{\omega\in B_+}\Pi_{\omega,\rho} = -i[\Delta,\ \cdot\ ] + \mathcal{L} + \Pi_\rho$$

More explicitly the current operator Π_ρ takes the form

$$\Pi_\rho(x) = \sum_{\{(m,n):\epsilon_m-\epsilon_n>0\}}\left(\left(\frac{\rho_m q_{mn}}{\rho_n q_{nm}}-1\right)q_{nm}D^*_{mn}xD_{mn}+\left(\frac{\rho_n q_{nm}}{\rho_m q_{mn}}-1\right)q_{mn}D^*_{nm}xD_{nm}\right)$$

$$=\sum_{\{(m,n):\epsilon_m-\epsilon_n>0\}}\left(J_{mn}\rho_n^{-1}D^*_{mn}xD_{mn}+J_{nm}\rho_m^{-1}D^*_{nm}xD_{nm}\right) \tag{69}$$

where

$$J_{mn} = \rho_m q_{mn} - \rho_n q_{nm},\quad J_{nm} = -J_{mn} \tag{70}$$

This is the current operator defining the notion of *dynamical detailed balance* deduced by Accardi and Imafuku from the stochastic limit (see [3]). In the same paper it was proved that the quantities J_{mn} have a natural interpretation as **micro–currents** of quanta from the level ϵ_m to the level ϵ_n and that the case of all currents equal zero, corresponds to the notion of quantum detailed balance in the sense of (81).

In section (**??**) we will produce examples of simple physical situations which can give rise to Markov generators with non–zero currents.(Remove this paragraph.)

7.3 ρ–privileged GKSL representations of a Markov generator $\mathcal{L}$

In a GKSL representation (43), of a Markov generator $\mathcal{L}$, the triple $(I, H, (L_k)_{k\in I})$ is in general not unique. However, fixing arbitrarily a normal state ρ on $\mathcal{B}(\mathcal{H})$, one can introduce special, ρ–dependent, classes of GKSL representations defined by triples $(I, H, (L_k)_{k\in I})$ which are simply related among themselves in the sense described by the following theorem (see [27], Theorem 30.16 for the proof).

Theorem 5. Let $\mathcal{L}$ be a norm–continuous Markov generator on $\mathcal{B}(\mathcal{H})$ and let ρ be a normal state on $\mathcal{B}(\mathcal{H})$. Then there exists a GKSL representation (43) of $\mathcal{L}$, (insert: hereafter called special,) whose triple $(I, H, (L_k)_{k\in I})$, in addition to the above listed properties (i), (ii), (iii), satisfies:
(iv) for each $k \in I$,

$$\mathrm{tr}\,(\rho L_k) = 0 \tag{71}$$

(v) if $\sum_{k\in I} |c_k|^2 < \infty$ and $c_0 + \sum_{k\in I} c_k L_k = 0$ for complex scalars $(c_k)_{k\in I\cup\{0\}}$ ((by definition $0 \notin I$), then $c_k = 0$ for every $k \geq 0$.
If $(\tilde{H}, \tilde{I},\ (\tilde{L}_k)_{k\in\tilde{I}}$ is another GKSL triple with the above five properties, then:
– the cardinalities of I and $\tilde{I}$ are equal,

– there exists a scalar $c \in \mathbb{R}$ such that

$$\tilde{H} = H + c$$

– there exists a unitary matrix $(u_{lj})_{l,j\in I}$ such that

$$\tilde{L}_l = \sum_j u_{lj} L_j \qquad ; \qquad \forall l \in I$$

In the notations of Theorem 5 the *multiplicity space* of the generator $\mathcal{L}$ is defined, up to unitary isomorphisms, to be an Hilbert space $\mathcal{K}$ whose dimension is equal to the cardinality of I.
The following notion was introduced in [17].

Definition 9. A GKSL representation $(H, I, \ (L_k)_{k\in I})$ of a bounded Markov generator $\mathcal{L}$ is called *privileged with respect to a faithful state* ρ if, in addition to (i), ..., (v), the following conditions are satisfied:
(vi) ρ commutes with H:

$$H\rho = \rho H \tag{72}$$

(vii) for some sequence of positive real numbers (λ_k) one has

$$\rho L_k = \lambda_k L_k \rho \tag{73}$$

Remark Conditions (72) and (73) imply that ρ is $\mathcal{L}$–invariant. Multiplying on the right by L_k^* and taking trace one sees that the faithfulness of ρ implies that

$$\lambda_k > 0 \tag{74}$$

Conditions (73) and (74) imply that (71) becomes equivalent to

$$\mathrm{tr}\,(L_k) = 0 \tag{75}$$

In the privileged case the ρ–adjoint of $\mathcal{L}$ is

$$\mathcal{L}^*_\rho(x) = \tag{76}$$

$$= \sum_k \left(L_k^* x L_k - \frac{1}{2}\{x, L_k^* L_k\} \right) + \sum_k (\lambda_k^{-1} - 1) L_k^* x L_k - i[H, \ \cdot\ ]$$

Thus, by definition, a Markov generator which has a ρ–privileged representation automatically satisfies the weighted detailed balance condition of Definition 7.
The stochastic limit type Markov generators, described in Section 7 satisfy conditions (72), (73) and (71), but boundedness and condition (v) of Theorem 5 might be difficult to be verified in general.
Privileged representations characterize those bounded GKSL generators whose ρ–adjoint $\mathcal{L}^*_\rho$ is also the generator of a uniformly continuous QMS. Moreover, to every privileged, representation of a Markov generator $\mathcal{L}$ corresponds a privileged representation of its adjoint $\mathcal{L}^*_\rho$. More precisely:

Theorem 6. Let $\mathcal{L}$ be a bounded Markov generator with faithful invariant state ρ.
(i) The ρ–adjoint of $\mathcal{L}$ is the generator of a uniformly continuous QMS if and only if $\mathcal{L}$ admits a privileged representation with respect to ρ.
(ii) If a privileged representation of $\mathcal{L}$ is given by the triple $(H, I,\ (L_k)_{k\in I})$ then there exist $\alpha \in \mathbb{R}$ and $\lambda_k > 0$ $(k \in I)$ such that, defining

$$\tilde{H} := -H - \alpha \tag{77}$$

$$\tilde{L}_k = \lambda_k^{-\frac{1}{2}} L_k^* \tag{78}$$

the triple $(\tilde{H}, I,\ (\tilde{L}_k)_{k\in I})$ is a privileged representation of the ρ-adjoint $\mathcal{L}_\rho^*$ of $\mathcal{L}$.

Proof. For (i) see Theorem 4.3 in [17]. For (ii) see Theorem 4.4 in [17].

Theorem 7. Let $(\mathcal{T}_t)_{t\geq 0}$ be a norm continuous QMS with (bounded) generator $\mathcal{L}$ and faithful invariant state ρ.
Then the following are equivalent:

(i) There exist:
– a sequence of positive numbers $q := (q_k)_{k\in I}$,
– a ρ–special representation of $\mathcal{L}$ (in the sense of Theorem 5) defined by a triple $(H, I,\ (L_k)_{k\in I})$,
– a bounded operator $K = K^*$
such that, defining the operator $\Pi(x)$ by (46), the weighted detailed balance condition (45) is satisfied.

(ii) $\mathcal{L}_\rho^*$ is a bounded Markov generator and the triple $(H, I,\ (L_k)_{k\in I})$ yields a privileged representation of $\mathcal{L}$ with $\tilde{H}, \tilde{L}_k$ the operators in the corresponding privileged GKSL representation of $\mathcal{L}_\rho^*$, given by Theorem 6, and there exists a sequence of positive weights $q := (q_k)$ and operators H'', L_k'' of a (possibly another) special representation of $\mathcal{L}$ such that,

$$\tilde{L}_k = q_k^{\frac{1}{2}} L_k'', \ \ \forall k \geq 1 \tag{79}$$

Proof. $(i) \Rightarrow (ii)$. Since the q_k are real, any Π of the form (46) is a $*$–map, i.e. $\Pi(x)^* = \Pi(x^*)$. It follows that $\mathcal{L}_\rho^*$ is also a $*$–map, being a sum of maps with these properties. Since, by assumption, $\mathcal{L}$, Π and K are bounded and ρ is faithful, we can apply a result of Majewski and Streater (see Theorem 6, p. 7985 in [25]) and conclude that $\mathcal{L}_\rho^*$

is a Markov generator. Hence, by Theorem 6, $\mathcal{L}$ admits a privileged representation with respect to ρ.
Then we have that H and $\sum_k L_k^* L_k$ commutes with ρ and (45), (46) imply that

$$\sum_k \tilde{L}_k^* x \tilde{L}_k = \sum_k L_k^* y L_k + \sum_k (q_k - 1) L_k^{'*} x L_k^{'} \tag{80}$$

with $L_k^{'}$ operators of a special representation of $\mathcal{L}$. By Theorem 5, we can write $L_k^{'} = \sum_l u_{kl} L_l$ with $u = (u_{kl})$ unitary operator on $\mathcal{K}$. Now a direct computation shows that

$$\sum_k L_k^{'*} x L_k^{'} = \sum_{j,\ell} \left(\sum_k \overline{u}_{kj} u_{kl} \right) L_j^* x L_l = \sum_j L_j^* x L_j$$

Therefore we can simplify the right-hand side of (80) and find

$$\sum_k \tilde{L}_k^* x \tilde{L}_k = \sum_k q_k L_k^{'*} x L_k^{'} = \sum_k \left(q_k^{1/2} L_k^{'} \right)^* x \left(q_k^{1/2} L_k^{'} \right)$$

Then we can apply Theorem 30.16 in [27] on Kraus' representations of normal completely positive maps to conclude that there exists a unitary operator $v = (v_{kl})$ on the multiplicity space of the ρ–special representation of $\mathcal{L}$ such that

$$\tilde{L}_k = q_k^{\frac{1}{2}} \sum_j v_{kj} L_k^{'} = q_k^{\frac{1}{2}} L_k^{''}$$

with $L_k^{''} = \sum_j v_{kj} L_k^{'}$. This proves (ii).

(ii) $\Rightarrow$ *(i)* Conversely, assume (ii) holds and let us compute the ρ–adjoint of $\Phi(x) = \sum_k L_k^* x L_k$, the CP part of $\mathcal{L}$. Since the GKSL representation of $\mathcal{L}$ by means of the operators H, L_k is privileged, by Theorem (6) the ρ–adjoint of the CP part $\Phi(x) = \sum_k L_k^* x L_k$ of $\mathcal{L}$ is

$$\tilde{\Phi}(x) = \sum_k \tilde{L}_k^* x \tilde{L}_k$$

where $\tilde{L}_k = \lambda_k^{-\frac{1}{2}} L_k^*$. A direct computation using (79) with $L_k^{''} = \sum_l u_{kl} L_l$ yields

$$\begin{aligned}
\tilde{\Phi}(x) &= \sum_k \tilde{L}_k^* x \tilde{L}_k = \sum_k q_k L_k^{''*} x L_k^{''} \\
&= \sum_k L_k^{''*} x L_k^{''} + \sum_k (q_k - 1) L_k^{''*} x L_l^{''} \\
&= \Phi(x) + \sum_k (q_k - 1) L_k^{''*} x L_k^{''}
\end{aligned}$$

Since H and $\sum_k L_k^* L_k$ commute with ρ, we obtain (45) and (46) with $L_k' = L_k''$. This proves (i).

Corollary 4. Assume that $H,\ L_k$ are operators of a privileged representation of the bounded Markov generator $\mathcal{L}$ with respect to a faithful invariant state ρ.
Then the following are equivalent:

(i) the generator $\mathcal{L}$ satisfies the quantum detailed balance condition of Frigerio, Kossakowski, Gorini, Verri [22]

$$\mathcal{L} - \mathcal{L}_\rho^* = 2i[H,\ \cdot\ ] \tag{81}$$

(ii) $\mathcal{L}$ satisfies a weighted detailed balance condition with respect to the faithful invariant state ρ with weights

$$q = (1, 1, \cdots)$$

i.e., $q_k = 1,\ \forall k$.

Proof. The thesis is an immediate consequence of Theorem 7 combined with Theorem 5.1 in [17].

7.4 Generic Markov generators of stochastic limit type satisfy a weighted detailed balance condition

The simplest class of Markov generators on $\mathcal{B}(\mathcal{H})$, of stochastic limit type with respect to a discrete spectrum Hamiltonian H is obtained when the Hamiltonian H is *generic* in the sense of [7], i.e.

Definition 10. A Markov generator (8), of stochastic limit type with respect to a discrete spectrum Hamiltonian H is called generic if:
(i) H has a simple spectrum
(ii) for any $\omega \in B_+$ (see (49)), there exists a unique ordered pair (ϵ_m, ϵ_n) of eigenvalues of H such that

$$\epsilon_m - \epsilon_n = \omega > 0$$

(this is equivalent to say that the strictly positive eigenvalues of $e^{itH}(\ \cdot\ )e^{-itH}$ are simple).

In the present subsection we shall prove that, for this special class of generators condition (v) of Theorem (5) can be easily verified, hence for it the notions of weighted detailed balance and of dynamical detailed balance coincide.
It is convenient, for simplicity of notations, to rewrite the Markov generator (50) exploiting the genericity assumption and simplifying the set of indices, so to make the multiplicity space clear. To this goal we denote

$$\hat{B}_+ := \{\omega_j \in B_+ \ : \ \text{either } \Gamma_{-,\omega} \neq 0 \text{ or } \Gamma_{-,\omega} \neq 0\}$$

Since the set $\hat{B}_+$ is at most countable, we can write

$$\hat{B}_+ = \{\omega_j \ : \ 0 \leq j \leq |\hat{B}_+|\} \subseteq \mathbb{N} \tag{82}$$

hence denoting

$$\Gamma_{\pm,j} = \Gamma_{\pm,\omega_j} \qquad \text{and} \qquad D_j = D_{\omega_j}$$

the generator (50) can be written in the form

$$\mathcal{L}(x) = i[\Delta, x] - \tag{83}$$

$$\sum_{j\in\hat{B}_+} \left(\Gamma_{-,j}\left(\frac{1}{2}\{D_j^\dagger D_j, x\} - D_j^\dagger x D_j\right) + \Gamma_{+,j}\left(\frac{1}{2}\{D_j D_j^\dagger, x\} - D_j x D_j^\dagger\right)\right)$$

Defining, for each $j \in \hat{B}_+$:

$$\gamma_{2j} := \Gamma_{-,j} \ ; \ \gamma_{2j+1} := \Gamma_{+,j} \ ; \ L_{2j} := \gamma_{2j}^{\frac{1}{2}} D_j \ ; \ L_{2j+1} := \gamma_{2j+1}^{\frac{1}{2}} D_j^* \tag{84}$$

we have that

$$j \in \{0 \leq j \leq |\hat{B}_+| - 1\} =: I \tag{85}$$

finally write the generator (50) in the form

$$\mathcal{L}(x) = \Phi(x) + G^* x + xG \tag{86}$$

where

$$\Phi(x) = \sum_{j\in I} L_j^* x L_j \qquad ; \qquad G = -\frac{1}{2}\Phi(I) - i\Delta$$

Recalling the definition of the operators D_j (see Appendix I), if the index $j \in I$ corresponds to the ordered pair (ϵ_m, ϵ_n) of eigenvalues of H, then we can write:

$$\begin{aligned} L_{2j} &= \gamma_{2j}^{\frac{1}{2}} D_j = \gamma_{2j}^{\frac{1}{2}} E_j(D) = \gamma_{2j}^{\frac{1}{2}} \langle \epsilon_n | D | \epsilon_m \rangle |\epsilon_n\rangle\langle\epsilon_m| \\ L_{2j+1} &= \gamma_{2j+1}^{\frac{1}{2}} D_{j+1} = \gamma_{2j+1}^{\frac{1}{2}} E_j(D)^* = \gamma_{2j+1}^{\frac{1}{2}} \overline{\langle \epsilon_n | D | \epsilon_m \rangle} |\epsilon_m\rangle\langle\epsilon_n| \end{aligned} \tag{87}$$

since by genericity $\epsilon_n \neq \epsilon_m$, one has from (87):

$$tr\Big(\rho L_{2j}\Big) = \gamma_{2j}^{\frac{1}{2}} \langle \epsilon_n | D | \epsilon_m \rangle tr\Big(\rho |\epsilon_n\rangle\langle\epsilon_m|\Big) = \gamma_j^{\frac{1}{2}} \langle \epsilon_n | D | \epsilon_m \rangle \rho_n tr\Big(|\epsilon_n\rangle\langle\epsilon_m|\Big) = 0$$

Similarly $tr\Big(\rho L_{2j}\Big) = 0$. (87) also implies that, if $0 = c_0 + \sum_j c_j L_j$ then

$$0 = tr\Big((c_0 + \sum_j c_j L_j)^*(c_0 + \sum_{j'} c_{j'} L_{j'})\Big) = \sum_j |c_j|^2,$$

it follows that $c_j = 0$ for all $j \geq 0$. Therefore the set $\{1, (L_k)_{k\in I}\}$ is linearly independent, hence I is the multiplicity space of $\mathcal{L}$.

Theorem 8. The expression (86) of a Markov generator $\mathcal{L}$, of stochastic limit type associated with a generic Hamiltonian H and with a faithful invariant state of the form

$$\rho = \sum_{\epsilon_k \in \mathrm{Spec}(H)} \rho_k |\epsilon_k\rangle\langle\epsilon_k| \in \{H\}' \equiv \{H\}'' \tag{88}$$

is a privileged decomposition with eigenvalues (λ_j) defined as follows: if the index $j \in I$ corresponds to the ordered pair (ϵ_m, ϵ_n), then:

$$\lambda_j := \begin{cases} \rho_n {\rho_m}^{-1}, & \text{if} \quad j \text{ is even} \\ {\rho_n}^{-1} \rho_m, & \text{if} \quad j \text{ is odd} \end{cases} \tag{89}$$

Moreover the Markov generator $\mathcal{L}$, in the expression (86), satisfies a weighted detailed balance condition in which $\tilde{L}_k = q_k^{\frac{1}{2}} L_k''$ where $L_k'' = \sum_j u_{kj} L_j$ with $u \equiv (u_{k,j})_{k,j\in I}$ is the unitary (permutation) operator whose elements are defined by

$$u_{k,j} := \begin{cases} \delta_{k+1,j}, & \text{if} \quad k \text{ is even} \\ \delta_{k-1,j}, & \text{if} \quad k \text{ is odd} \end{cases} \tag{90}$$

and the sequence of weights $q \equiv (q_k)$ is given by (89) and, in the notation (84):

$$q_k := \begin{cases} \lambda_k^{-1} \gamma_k \gamma_{k+1}^{-1}, & \text{if} \quad k \text{ is even} \\ \lambda_k^{-1} \gamma_k^{-1} \gamma_{k+1}, & \text{if} \quad k \text{ is odd} \end{cases} \tag{91}$$

Proof. For j even one has, if the (λ_j) are defined as in (89):

$$\rho L_j = \gamma_j^{\frac{1}{2}} \langle \epsilon_n | D | \epsilon_m \rangle \rho | \epsilon_n \rangle \langle \epsilon_m | = \gamma_j^{\frac{1}{2}} \rho_n {\rho_m}^{-1} \langle \epsilon_n | D | \epsilon_m \rangle | \epsilon_n \rangle \langle \epsilon_m | \rho = \lambda_j L_j \rho$$

$$\rho L_{j+1} = \gamma_{j+1}^{\frac{1}{2}} \overline{\langle \epsilon_n | D | \epsilon_m \rangle} \rho | \epsilon_m \rangle \langle \epsilon_n | = \gamma_{j+1}^{\frac{1}{2}} \overline{\langle \epsilon_n | D | \epsilon_m \rangle} \rho_m \rho_n^{-1} | \epsilon_m \rangle \langle \epsilon_n | \rho = \lambda_{j+1} L_{j+1} \rho$$

This implies that the representation of $\mathcal{L}$ by means of operators $(L_j)_j$ and Δ is privileged with eigenvalues (λ_j) given by (89).
Finally, let us verify that condition (ii) in Theorem 7 holds.
From (87) we see that for every $j \in I$ we have

$$L_j^* = \begin{cases} \gamma_j^{\frac{1}{2}} \gamma_{j+1}^{-\frac{1}{2}} L_{j+1} & , \text{ if } \quad j \text{ is even} \\ \gamma_j^{\frac{1}{2}} \gamma_{j-1}^{-\frac{1}{2}} L_{j-1} & , \text{ if } \quad j \text{ is odd} \end{cases}$$

Hence denoting for $2j, 2j+1 \in I$

$$q_{2j} := \lambda_{2j}^{-1} \gamma_{2j} \gamma_{2j+1}^{-1} \qquad ; \qquad q_{2j+1} := \lambda_{2j+1}^{-1} \gamma_{2j}^{-1} \gamma_{2j+1}$$

and, using (90) to define the unitary operator $u \equiv (u_{k,j})_{k,j \in I}$, one obtains the relation:

$$\tilde{L}_k = \lambda_k^{-1} L_k^* = q_k^{\frac{1}{2}} L_k'' \tag{92}$$

This finishes the proof.
Remark. Notice that, if the index $k \in I$ corresponds to the ordered pair (ϵ_m, ϵ_n) in the sense of Definition 10, then the identity (91) implies that, for generic Markov generators of stochastic limit type, one has:

$$q_k = \lambda_k^{-1} \gamma_k \gamma_{k+1}^{-1} = \lambda_k^{-1} \Gamma_{-,\epsilon_m - \epsilon_n} (\Gamma_{+,\epsilon_m - \epsilon_n})^{-1} =: \lambda_k^{-1} q_{mn} q_{nm}^{-1} \tag{93}$$

8 Appendix (I): Eigenoperators of $Ad(e^{itH})$

In the present Appendix we recall some useful notions from [7].

Theorem 9. (see Theorem 34 in [7]) Let $H = H^* \in \mathcal{B}(\mathcal{H})$ be a pure point spectrum Hamiltonian

$$H = \sum_{\epsilon \in spec(H)} \epsilon P_\epsilon = \sum_m \epsilon_m P_m \tag{94}$$

Consider the associated 1–parameter automorphism group;

$$u_t(\cdot) := e^{itH} (\cdot) e^{-itH} \tag{95}$$

and the associated set of Bohr frequencies.

$$B = B_H := \{\omega = \varepsilon_r - \varepsilon_{r'} : \varepsilon_r, \varepsilon_{r'} \in \text{Spec}\,(H)\} = \text{Spec}\,(u_t) \tag{96}$$

Then one has:

$$u_t(x) = \sum_{\omega \in B} e^{-it\omega} E_\omega(x) \qquad ; \ \forall x \in \mathcal{B}(\mathcal{H}) \ , \ \forall t \in \mathbb{R}_+ \tag{97}$$

where, for each $\omega \in B$, the operator E_ω is defined by (58) and

$$D_\omega := E_\omega(D) = \sum_{\{(\varepsilon_m, \varepsilon_n) \in B_+(\omega)\}} P_{\varepsilon_m} D P_{\varepsilon_n} \tag{98}$$

The operators E_ω satisfy the identities

$$E_\omega(x)^* = E_{-\omega}(x^*) \qquad ; \qquad \forall x \in \mathcal{B}(\mathcal{H}) \tag{99}$$

$$E_\omega E_{\omega'} = \delta_{\omega,\omega'} E_\omega \qquad \text{(mutual orthogonality)} \tag{100}$$

$$\sum_{\omega \in B} E_\omega(\cdot) = id_{\mathcal{B}(\mathcal{H})} \qquad \text{(normalization)} \tag{101}$$

Finally the operator

$$E_0(\cdot) := \sum_{\{\varepsilon_n \in \text{ Spec } (H)\}} P_{\varepsilon_n}(\cdot) P_{\varepsilon_n} \tag{102}$$

is the Umegaki conditional expectation onto $\{H\}'$.

Remark. One easily verifies (see Proposition 33 of [7]) that

$$E_\omega(\mathcal{B}(\mathcal{H})) = \{x \in \mathcal{B}(\mathcal{H}) : e^{itH} x e^{-itH} = e^{-it\omega} x\} \tag{103}$$

Any element of this subspace will be called an ω–eigen–operator of $Ad(e^{itH})$. (101) is equivalent to

$$\mathcal{B}(\mathcal{H}) = \bigoplus_\omega E_\omega(\mathcal{B}(\mathcal{H})) \tag{104}$$

the sum being orthogonal in the sense that, if $\omega' \neq \omega$, then

$$E_{\omega'} E_\omega(x) = 0 \qquad ; \qquad \forall x \in \mathcal{B}(\mathcal{H})$$

The sum (104) is orthogonal also in another sense, specified by the following Lemma.

Lemma 7. *If $\rho \in \{H\}'$ (in particular if $\rho \in \{H\}''$), then for any $\omega, \omega' \in B$ one has:*

$$Tr\,(\rho E_\omega(x)^* E_{\omega'}(y)) = \delta_{\omega,\omega'} Tr\,(\rho E_\omega(x)^* E_\omega(y)) \qquad ; \qquad \forall\, x, y \in \mathcal{B}(\mathcal{H})$$

Lemma 8. *For any $\omega, \omega' \in B$ one has:*

$$E_\omega(\mathcal{B}(\mathcal{H})) \cdot E_{\omega'}(\mathcal{B}(\mathcal{H})) \subseteq E_{\omega+\omega'}(\mathcal{B}(\mathcal{H})) \qquad \textit{(grading)} \tag{105}$$

$$E_\omega(\mathcal{B}(\mathcal{H}))^* = E_{-\omega}(\mathcal{B}(\mathcal{H})) \tag{106}$$

Corollary 5. *For all $\omega \in B$ and any operator $A \in \{H\}' \equiv E_0(\mathcal{B}(\mathcal{H}))$, one has*

$$[A, E_\omega(\mathcal{B}(\mathcal{H}))] \subseteq E_\omega(\mathcal{B}(\mathcal{H})) \tag{107}$$

$$\{A, E_\omega(\mathcal{B}(\mathcal{H}))\} \subseteq E_\omega(\mathcal{B}(\mathcal{H})) \tag{108}$$

Moreover, $\forall \omega, \omega' \in B$ and $\forall D_\omega \in E_\omega(\mathcal{B}(\mathcal{H}))$, one has:

$$D_\omega^* D_\omega, D_\omega D_\omega^* \in \{H\}' \equiv E_0(\mathcal{B}(\mathcal{H})) \tag{109}$$

$$D_\omega^* E_{\omega'}(\mathcal{B}(\mathcal{H})) D_\omega \subseteq E_{\omega'}(\mathcal{B}(\mathcal{H})) \tag{110}$$

Lemma 9. *Let $F : spec(H) \to \mathbb{R}$ be a Borel function. Then $\forall y$ and $\forall m, n \in \mathbb{N}$*

$$F(H) P_m y = F(\varepsilon_m) P_m y$$

$$y P_n F(H) = F(\varepsilon_n) y P_n$$

in particular, if y has the form

$$y = P_m z P_n \quad ; \quad z \in \mathcal{B}(\mathcal{H}) \quad ; \quad m, n \in \mathbb{N} \tag{111}$$

Then

$$e^{\beta(H)H} y e^{-\beta(H)H} = e^{\beta(\varepsilon_m)\varepsilon_m - \beta(\varepsilon_n)\varepsilon_n} y \tag{112}$$

8.1 The generic case

The generic case is characterized by the condition

$$\text{cardinality of } \; B_\omega =: |B_\omega| = 1 \qquad ; \qquad \forall \omega \in B_+ \tag{113}$$

Let $\omega \in B_+$. Condition (113) is characterized by the existence of a unique pair $(\varepsilon_\omega^+, \varepsilon_\omega^-)$ such that $\varepsilon_\omega^+, \varepsilon_\omega^- \in spec(H)$ and

$$\varepsilon_\omega^- := \varepsilon_\omega^+ - \omega \in spec(H)$$

or equivalently

$$\varepsilon_\omega^+ - \varepsilon_\omega^- = \omega > 0$$

In this case the spectrum of H is non degenerate so that

$$P_\varepsilon = |\varepsilon\rangle\langle\varepsilon|$$

Therefore:

$$E_0(x) = \sum_n \langle\varepsilon_n, x\varepsilon_n\rangle|\varepsilon_n\rangle\langle\varepsilon_n|$$

and, for $\omega > 0$

$$E_\omega(D) = \langle\varepsilon_\omega^+, D\varepsilon_\omega^-\rangle|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-| =: \delta_\omega|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-|$$

Then recalling that $E_\omega^*(D) = E_{-\omega}(D^*)$, one has

$$E_\omega(D)E_\omega(D)^* = |\delta_\omega|^2|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^+| =: q_\omega P_{\varepsilon_\omega^+}$$

$$E_\omega(D)^*E_\omega(D) = |\delta_\omega|^2 P_{\varepsilon_\omega^-} = q_\omega P_{\varepsilon_\omega^-}$$

$$E_\omega(D)^* x E_\omega(D) = q_\omega\langle\varepsilon_\omega^+, x\varepsilon^+\rangle P_{\varepsilon_\omega^-}$$

$$E_\omega(D) x E_\omega(D)^* = q_\omega\langle\varepsilon_\omega^-, \lambda\varepsilon_\omega\rangle P_{\varepsilon_\omega^+}$$

$$\Delta|\varepsilon\rangle = i\hat{d}_\varepsilon|\varepsilon\rangle \qquad ; \qquad d_\omega := \hat{d}_{\varepsilon_\omega^+} - \hat{d}_{\varepsilon_\omega^-}$$

$$\begin{aligned}
&\mathcal{L}(|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-|) \\
&= id_\omega|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-| + \Gamma_{\omega,-}\left(-\frac{1}{2}\,q_\omega|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-|\right) + \Gamma_{\omega,+}\left(-\frac{1}{2}\,q_\omega|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-|\right) \\
&= \left[-\left(\frac{\Gamma_{\omega,-} + \Gamma_{\omega,+}}{2}\right) q_\omega + id_\omega\right]|\varepsilon_\omega^+\rangle\langle\varepsilon_\omega^-|
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}(|\varepsilon_\omega^-\rangle\langle\varepsilon_\omega^+|) &= -id_\omega|\varepsilon^-\rangle\langle\varepsilon^+| - \left(\frac{\Gamma_{\omega,-} + \Gamma_{\omega,+}}{2}\right) q_\omega|\varepsilon^-\rangle\langle\varepsilon^+| \\
&= \left[-\left(\frac{\Gamma_{\omega,-} + \Gamma_{\omega,+}}{2}\right) - id_\omega\right]|\varepsilon^-\rangle\langle\varepsilon^+|
\end{aligned}$$

Under our assumptions the right hand side of (41) is equal to

$$e^{\beta(\varepsilon_n)\varepsilon_m - \beta(\varepsilon_n)\varepsilon_n} Tr(\rho\mathcal{L}(y)x)$$

Let

$$\omega = \varepsilon_m - \varepsilon_n$$

Therefore the right hand side of (41) is equal to

$$e^{\beta(\varepsilon_m)\varepsilon_m-\beta(\varepsilon_n)\varepsilon_n}\left[-\frac{1}{2}\left(\Gamma_{\omega,-}+\Gamma_{\omega,+}\right)q_\omega+id_\omega\right]Tr(\rho|\varepsilon_n\rangle\langle\varepsilon_n|)$$

and, using that:

$$\rho=\sum F(\varepsilon_n)|\varepsilon_n\rangle\langle\varepsilon_n|$$

this is equal to

$$e^{\beta(\varepsilon_m)\varepsilon_m-\beta(\varepsilon_n)\varepsilon_n}F(\varepsilon_m)\left[-\frac{1}{2}\left(\Gamma_{\omega,-}+\Gamma_{\omega,+}\right)q_\omega+id_\omega\right]$$

Acknowledgments

The financial support from CONACYT-Mexico (Grant 49510-F) and the project Mexico-Italia “Dinámica Estocástica con Aplicaciones en Física y Finanzas”, are gratefully acknowledged.

References

[1] L. Accardi, Y.G. Lu, I. Volovich, *Quantum Theory and Its Stochastic Limit*, Springer-Verlag 2002.

[2] L. Accardi, F. Fagnola, R. Quezada, Weighted Detailed Balance and Local KMS Condition for Non-Equilibrium Stationary States, in: *Perspectives of Nonequilibrium Statistical Physics*, Busseikenkyu 2011, special issue in commemoration of Shuichi Tasaki, ed. Shigeru Ajisaka, **97** (3) (2011) 318–356

[3] L. Accardi and K. Imafuku, Quantum Dynamical Detailed Balance and Local KMS Condition for Non-Equilibrium States, *International Journal of Modern Physics B* **18** (4-5) (2004), 435-467. `quant-ph/0209088`

[4] L. Accardi, S. Tasaki, Nonequilibrium steady states for a harmonic oscillator interacting with two Bose fields - stochastic limit approach and C* algebraic approach.
in: *Quantum Information and Computing, Proceeding International conference QBIC08*, (2003) Eds: L. Accardi, M. Ohya, N.

Watanabe WS (2006)
http://www.worldscibooks.com/mathematics

[5] L. Accardi, K. Imafuku, Y.G. Lu, (Write the title) in *Fundamental Aspects of Quantum Physics, Proceedings of the "Japan-Italy Joint workshop on Quantum open systems and quantum measurement"*, Waseda University, 27–29 September (2001) eds. L. Accardi, S. Tasaki, World Scientific, (2003)

[6] Luigi Accardi, Shuichi Tasaki (eds.),
Fundamental Aspects of Quantum Physics,
http://www.wspc.com.sg/books/physics/5213.html
QP–PQ XVII World Scientific (2003) 306–321, Proceedings of the "Japan-Italy Joint workshop on Quantum open systems and quantum measurement", Waseda University, 27–29 September (2001)

[7] L. Accardi, S. Kozyrev, Lectures on Quantum Interacting Particle Systems, in *Quantum Interacting Particle Systems, QP-PQ Volume XIV*, L. Accardi and F. Fagnola (eds.), (World Scientific, 2002) 1-195.

[8] Accardi L., Imafuku K., Kozyrev S.V., Stimulated emission with a non-equilibrium state of radiation, in: *Irreversible processes and dynamical systems, Keihanna, Proc. XXI–st Solvay Conference on Physics*, I.Antoniou, V.A. Sadovnichy, H.Walther (eds.) World Scientific
The Physics of Communication, (2003) 157–175
quant-ph/0104119 Preprint Volterra, N. 492 (2001)

[9] Accardi L., Imafuku K., Kozyrev S.V., Interaction of 3-level atom with radiation, *Optika i Fotonika* **94** (6) 969–975, Optics and Spectroscopy (1) (2003)
quant-ph/01-04118 Preprint Volterra N. 473 (2001) (Doppio Preprint) Preprint Volterra N. 520 July (2002) http://xxx.lanl.gov/abs/quant-ph/0104119

[10] Accardi L., Aref'eva I.Ya., Volovich I.V.,
Non-Equilibrium Quantum Field Theory and Entangled Commutation Relations, *Proceedings of the Steklov Institute of Mathematics* **228** (2000) 106–125, Special Issue dedicated to the 90th birthday of N. N. Bogoliubov, Translated from Trudy Matematicheskogo Instituta imeni V.A. Steklova 228 (2000) 116–135, Report-no: SMI-15-99 hep-th/9905035, Preprint Volterra N. 368

(1999)
http://xxx.lanl.gov/hep-th/9905035

[11] L. Accardi and A. Mohari, Time reflected Markov processes, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **2** (3) (1999), 397–425.

[12] R. Alicki, On detailed balance condition for non-Hamiltonian systems, *Rep. Math. Phys.* **10** (1976), 249-258.

[13] J. Bolaños, F. Fagnola, On the KMS condition for irreversible dynamics, work in progress.

[14] R. Carbone, F. Fagnola and S. Hachicha, Generic Quantum Markov Semigroups: the Gaussian Gauge Invariant Case, *Open Sys. and Information Dynamics* **14** (2007), 425-444.

[15] F. Fagnola, Quantum Markov semigroups and quantum flows, *Proyecciones* **18** (1999), 1-144.

[16] F. Fagnola and R. Rebolledo, From classical to quantum entropy production. In H. Ouerdiane, A. Barhoumi (eds.) *Quantum Probability and Infinite Dimensional Analysis*, Proceedings of the 29-th Conference on Quantum Probability and Infinite Dimensional Analysis, Hammamet (Tunisia), October 13-18, 2008, QP-PQ: Quantum Probability and White Noise Analysis - Vol. 25 p. 245-261. World Scientific 2010.

[17] F. Fagnola and V. Umanità, Generators of detailed balance quantum Markov semigroups, *IDA–QP Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **10** no.3 (2007) 335 - 363

[18] F. Fagnola and V. Umanità, On two quantum versions of the detailed balance condition, in: *Non-commutative harmonic analysis with applications to probability II*, Banach Center Publications **89**, 105-119, Polish Acad. Sci. Inst. Math., Warszawa, 2010.

[19] F. Fagnola and V. Umanità, Generators of KMS Symmetric Markov Semigroups on $\mathcal{B}(h)$ Symmetry and Quantum Detailed Balance. *Commun. Math. Phys.* **298** (2), (2010) 523-547. DOI: 10.1007/s00220-010-1011-1

[20] Fannes M., Verbeure A., Equilibrium Conditions for Coupled Classical-Quantum Systems, *J. Phys. A: Math. Gen.* **20** (17) (1987) 6037 doi:10.1088/0305-4470/20/17/036

[21] Frigerio A., Gorini V., Verri M.: The zeroth law of thermodynamics, *Physica* **137** A (1986) 573-602

[22] A. Frigerio, A. Kossakowski, V. Gorini, M. Verri, Quantum detailed balance and KMS condition, *Commun. Math. Phys.* **57** (1977), 97-110. Erratum: *Commun. Math. Phys.* **60** (1978), 96–98.

[23] A. Frigerio, V. Gorini, Markov dilations and quantum detailed balance, *Commun. Math. Phys.* **93** (1984), 517-532.

[24] S.L. Kalpazidou, *Cycle Representations of Markov Processes*, Springer, 2006.

[25] W.A. Majewski and R. Streater, Detailed balance and quantum dynamical maps, *J. Phys. A: Math. Gen.* **31** (1998), 7981-7995.

[26] Pantaleón-Martínez L. and Quezada R., The asymmetric exclusion quantum Markov semigroup, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **12** (2009), 367–385.

[27] Parthasarathy K.R., *An Introduction to Quantum Stochastic Calculus*, Birkhäuser-Verlag 1992.

[28] Qian M-P., Qian M., Jiang D-J., *Mathematical Theory of Nonequilibrium Steady States*, Springer, 2003.

[29] E. Shamarova, A mathematical approach to Jarzynski's identity in non-equilibrium statistical mechanics, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **12**, 2009, 213-229.

[30] Tasaki S., Current fluctuations in nonequilibrium steady states for a one–dimensional lattice conductor, in *Quantum Information III*, T. HIda, K. Saito (eds.) World Scientific (2000) 157–176

Luigi Accardi, Centro V. Volterra, Università di Roma Tor Vergata, Via della Columbia, 00133 Roma (Italy). `accardi@volterra.uniroma2.it`

Franco Fagnola, Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, I-20133 Milano (Italy). `franco.fagnola@polimi.it`

Roberto Quezada, Dep. de Matemáticas, UAM-Iztapalapa, Av. San Rafael Atlixco 186, Col Vicentina, 09340 México D.F., (México). `roqb@xanum.uam.mx`