The optimal strategy to control a non linear monotone system with unilateral interactions

Julien M. Hendrickx
Institute of Information and Communication Technologies, Electronics and Applied Mathematics
Université catholique de Louvain, Belgium
julien.hendrickx@uclouvain.be

Domenico Brunetto
Department of mathematics,
Politecnico di Milano, Italy
domenico.brunetto@polimi.it

1 Introduction

We consider opinion dynamics models that also allows for unilateral interactions instead of only bilateral ones. In particular we study and investigate the dynamics of the following non linear ODE system

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(t)\max(0, x_j(t) - x_i(t)), \tag{1}$$

where $x_i(t)$ represent the state of the agent i at time t and the adjacency matrix $\mathbf{A} = [a_{ij}]_{i,j=1}^{N}$ describes the connection between the agents. The convergence of systems of the form (1) has been the object of recent attention [1]. We focus here on choosing the (possibly time-varying) adjacency matrices that maximize the convergence speed under some structural constraints.

2 Application: student skills

One particular application we consider is an idealized model of the dynamics of skills of students who interact to solve a task in a classroom: We assume the evolution of the level of skill of $2n$ students is abstracted by a 1-dimensional real function $x_i(t) : \mathbf{R}^+ \to \mathbf{R}$, $i = 1,\ldots,2n$ and we suppose that students are repeatedly asked to work by pairs. During such a collaborations, we assume the skill level of the weaker student increases proportionally to the skill gap, with a proportionality constant assumed to be the same for all students: $\dot{x}_i = x_j - x_i$ if $x_i < x_j$. On the other hand, the skill level of the stronger student is assumed to remain constant $\dot{x}_j = 0$. By contrast, classical opinion dynamics model would have lead to a decrease of x_j. (More complex models could include some non-linearities of the skill gap is too large, or some constant gain for the stronger student.)

This model can be cast into system (1). Since the interactions are supposed to be pairwise, the adjacency matrix $\mathbf{A}$ is characterized by the following conditions

$$a_{ij}(t) \in \{0,1\} \tag{2}$$

$$a_{ij}(t) = a_{ji}(t) \tag{3}$$

$$a_{ij}(t) = 1 \Rightarrow a_{ik}(t) = 0 \quad \forall k \neq j \tag{4}$$

$\forall i, j = 1,\ldots,2n$ and $\forall t \in [0,T]$

3 Control Problem

We study the problem of choosing the interactions to maximize the average skill level

Problem 1 *Seek for $a_{ij}(t)$ in (1) which maximizes the object function*

$$J(\mathbf{x}) = \frac{1}{2n}\sum_{j=1}^{2n} x_j(T) \tag{5}$$

We present some intermediate steps which drive us towards the solution of the Problem 1 reporting some theoretical results and some numerical simulations.

Firstly, we solve a simpler problem where a_{ij} represents a local strategy in (1), hence the adjacency matrix is constant along the time $I = [0,T]$, that is the strategy used to make the pairs is fixed at the beginning of the dynamics. Then we introduce the piecewise function: $a_{ij}^{k} = a_{ij}(t)|_{I_k}$, where $\{I_k\}_{k=1}^{m}$ is a partition of I and $|I_k| = \Delta T$ such that $I = \cup_{k=1}^{m} I_k$. However, numerical examples show that the locally optimal strategy is usually not optimal on the long run: a strategy encoded by a matrix A^1 which is optimal for I could not be optimal if it is iterated for each I_k. We then study the behaviour of the dynamics if ΔT approaches zero, in particular we focus on a class of strategies in which we switch arbitrarily fast between two matrices A^1 and A^2, leading to an average behaviour described by $\frac{1}{2}(A^1 + A^2)$.

Moreover we present the analysis of the system (1) in two different cases: 1) in short time, namely when T is small, and 2) in long time. In the latter case we study the asymptotic behaviour reporting some counter-intuitive examples and proving that the optimal convergence rate is $-\frac{1}{2}$.

References

[1] Manfredi, Sabato, and David, Angeli. "Necessary and sufficient conditions for consensus in nonlinear monotone networks with unilateral interactions." Automatica 77 (2017): 51-60.