ERGODIC BSDES WITH MULTIPLICATIVE AND DEGENERATE NOISE

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Abstract. In this paper we study an Ergodic Markovian BSDE involving a forward process X that solves an infinite dimensional forward stochastic evolution equation with multiplicative and possibly degenerate diffusion coefficient. A concavity assumption on the driver allows us to avoid the typical quantitative conditions relating the dissipativity of the forward equation and the Lipschitz constant of the driver. Although the degeneracy of the noise has to be of a suitable type we can give a stochastic representation of a large class of Ergodic HJB equations; morever our general results can be applied to get the synthesis of the optimal feedback law in relevant examples of ergodic control problems for SPDEs.

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1. Introduction. In this paper we study the following BSDE of ergodic type

$$Y_t^x = Y_T^x + \int_t^T [\widehat{\psi}(X_s^x, Z_s^x, U_s^x) - \lambda]\, ds - \int_t^T Z_s^x\, dW_s^1 - \int_t^T U_s^x\, dW_s^2, \qquad 0 \le t \le T < \infty,$$

where the processes (Y^x, Z^x, U^x) and the constant λ are the unknowns of the above equation while the diffusion X is the (mild) solution of the infinite dimensional (forward) SDE:

$$\begin{cases} dX_s^x = AX_s^x ds + F(X_s^x)ds + QG(X_s^x)dW_s^1 + DdW_s^2, \\ X_t^x = x. \end{cases} \tag{1.1}$$

In the above equation X takes values in an Hilbert space H and W^1, W^2 are independent cylindrical Wiener processes (see **(A.1)-(A.6)** in Section 3 and **(B.1)** in Section 4 for precise description of the other terms). We just stress that we will assume that $G(x)$ is invertible for all $x \in H$ while Q and D will be general, possibly degenerate, linear operators.

Ergodic BSDEs have been introduced in [19] in relation to optimal stochastic ergodic control problems and as a tool to study the asymptotic behaviour of parabolic HJB equations and consequently to give a stochastic representation to the limit semilinear elliptic PDEs (see equation (5.1) below).

In [19] the same class of BSDEs have been introduced, already in an infinite dimensional framework, but only in the case in which the noise coefficient was constant ($Q = 0$ in our notation). Successive works, see [15] and [7] weakened the assumptions and refined the results in the same *additive noise* case. Then in [24], in a finite dimensional framework, the case of 'multiplicative noise ($Q \neq 0$ and G depending on x in our notation) is treated under quantitative conditions relating the dissipativity constant of the forward equation to the Lipscitz norm of $\widehat{\psi}$ with respect to Z. Afterwards, in [21], still in finite dimensions, such quantitative assumptions are dropped in the case of a non degenerate and bounded diffusion coefficient ($Q = I$ and G bounded and invertible in our notation) by a careful use of smoothing properties of the Kolmogorov semigroup associated to the non-degenerate underlying diffusion X. Finally in [14] the result is extended to the case of non degenerate but unbounded (linearly growing) diffusion coefficients ($Q = I$ and G invertible and linearly

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growing in our notation). To complete the picture we mention, [2], [3], [4] and [13] where Ergodic BSDEs are studied in various frameworks different from the present one: namely, respectively when they are driven by a Markov chain, in the context (see [17]) of randomized control problems and BSDEs with constraints on the martingale term both in finite and in infinite dimensions and finally in the context of G- expectations theory.

In this paper we propose an alternative approach that works well in the infinite dimensional case and allows to consider degenerate multiplicative noise (Q in general non invertible and G bounded invertible but depending on x). On the other side we have to assume that $\widehat{\psi}$ has the form:

$$\widehat{\psi}(x, z, u) := \psi(x, zG^{-1}(x), u),$$

where ψ is Lipschitz and *concave* function with respect to (z, u). Although not standard, our assumptions allow to give a stochastic representation of a relevant class of Ergodic HJB equations in Hilbert spaces (see Section 5) and of ergodic stochastic control problems for SPDEs (see Example 7.1 and Example 7.2). Notice that ψ defined above is exactly the function that naturally appears in the related HJB equation and in the applications to ergodic control.

As in all the literature devoted to the problem the main point is to prove a uniform gradient estimate (independent on α) for $v^\alpha(x) := Y^{\alpha,x}$ where $(Y^{\alpha,x}, Z^{\alpha,x}, U^{\alpha,x})$ is the solution of the discounted BSDE with infinite horizon:

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T [\widehat{\psi}(X_s^x, Z_s^{\alpha,x}, U_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}]\, ds - \int_t^T Z_s^{\alpha,x}\, dW_s^1 - \int_t^T U_s^{\alpha,x}\, dW_s^2, \quad 0 \le t \le T < \infty.$$

Such estimate can be obtained by a change of probability argument when the noise is additive (see [19]), by energy type estimates under quantitative assumptions on the exponential decay of the forward equation (see [24]) or by regularizing properties of the Kolmogorov semigroup when the noise in multiplicative but non degenerate (see [14] and [21]).

Here we exploit concavity of ψ to introduce an auxiliary control problem and eventually obtain the gradient estimate using a decay estimate on the difference between states starting from different initial conditions, see Assumption **(A.6)** and, in particular, requirement (3.5). We stress the fact that the estimate in (3.5) is only in mean and not uniform (with respect to the stochastic parameter) as in the additive noise case. Moreover, as we show in Proposition 3.2, Assumption **(A.6)** is verified if we impose a *joint dissipativity* condition on the coefficients, see Assumption **(A.7)**. As a matter of fact, in this case, the stronger formulation in which L^2 replaces L^1 norm holds. On the other side **(A.6)** allows to cover a wider class of interesting examples, see for instance Example 7.1 in which Assumption **(A.7)** does not seem to hold.

The structure of the paper in the following: in Section 2 we introduce the function spaces that will be used in the following, Section 3 is devoted to the infinite dimensional forward equation; in particular we state and discuss the key stability assumption **(A.6)**. In Section 4 we present the main contribution of this work introducing the auxiliary control problem, proving the gradient estimate and the consequent existence of the solution to the ergodic BSDEs. In Section 5 we relate our ergodic BSDE to a semilinear PDE in infinite dimensional spaces (the ergodic HJB equation). In Section 6 we discute the regularity of the solution of the ergodic BSDE, in particular we state that under quantitative conditions on the dissipativity of the forward equation similar to the ones assumed in [24], when all coefficients are differentiable then the solution of the ergodic BSDE is differentiable with respect to the initial data as well. The proof of such result adapts a similar argument in [16] and is rather technical, we have postponed it in the Appendix Section 7 we use our ergodic BSDE to obtain an optimal ergodic control problem (that is with cost depending only on the asymptotic behaviour of the state) for an infinite dimensional equation. We close, see Section 7.1, by two examples of controlled SPDEs to which our results can be applied. In both we consider a stochastic heat equation in one dimension with additive white noise. In the first, Example 7.1 the system is controlled through one Dirichlet boundary condition (on which multiplicative noise also acts) while, in the second one, Example 7.2, the control enters the system through a finite dimensional process that affects the coefficients of the SPDE. In this last case we also give conditions guaranteeing differentiability of the related solution to the Ergodic BSDE.

2. General notation. Let Ξ, H and U be real separable Hilbert spaces. In the sequel, we use the notations $|\cdot|_\Xi$, $|\cdot|_H$ and $|\cdot|_U$ to denote the norms on Ξ, H and U respectively; if no confusion arises, we simply write $|\cdot|$. We use similar notation for the scalar products. We denote the dual spaces of Ξ, H and U by Ξ^*, H^*, and U^* respectively. We also denote by $L(H,H)$ the space of bounded linear operators from H to H, endowed with the operator norm. Moreover, we denote by $L_2(\Xi,H)$ the space of Hilbert-Schmidt operators from Ξ to H. Finally, a map $f : H \to \Xi$ is said to belong to the class $\mathcal{G}^1(H,\Xi)$ if it is continuous and Gateaux differentiable with directional derivative $\nabla_x f(x)h$ in $(x,h) \in H \times H$ and we denote by $\mathcal{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ.

Given a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ together with a filtration $(\mathcal{F}_t)_{t\geq 0}$ (satisfying the usual conditions of $\mathbb{P}$-completeness and right-continuity) and an arbitrary real separable Hilbert space V we define the following classes of processes for fixed $0 \leq t \leq T$ and $p \geq 1$:

- $L^p_{\mathcal{P}}(\Omega\times[t,T];V)$ denotes the set of (equivalence classes) of $(\mathcal{F}_s)$-predictable processes $Y \in L^p(\Omega\times[t,T];V)$ such that the following norm is finite:

$$|Y|_p \; = \; \left(\mathbb{E}\int_t^T |Y_s|^p\, ds\right)^{1/p}.$$

- $L^{p,loc}_{\mathcal{P}}(\Omega\times[0,+\infty[;V)$ denotes the set of processes defined on $\mathbb{R}^+$, whose restriction to an arbitrary time interval $[0,T]$ belongs to $L^p_{\mathcal{P}}(\Omega\times[0,T];V)$.
- $L^p_{\mathcal{P}}(\Omega;C([t,T];V))$ denotes the set of $(\mathcal{F}_s)$-predictable processes Y on $[t,T]$ with continuous paths in V, such that the norm

$$\|Y\|_p \; = \; \big(\mathbb{E}\sup_{s\in[t,T]}|Y_s|^p\big)^{1/p}.$$

  is finite. The elements of $L^p_{\mathcal{P}}(\Omega;C([t,T];V))$ are identified up to indistinguishability.
- $L^{p,loc}_{\mathcal{P}}(\Omega;C([0,+\infty[;V))$ denotes the set of processes defined on $\mathbb{R}^+$, whose restriction to an arbitrary time interval $[0,T]$ belongs to $L^p_{\mathcal{P}}(\Omega;C([0,T];V))$.

We consider on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ two independent cylindrical Wiener processes $W^1 = (W^1_t)_{t\geq 0}$ with values in Ξ and $W^2 = (W^2_t)_{t\geq 0}$ with values in H. By $(\mathcal{F}_t)_{t\geq 0}$, we denote the natural filtration of (W^1,W^2), augmented with the family $\mathcal{N}$ of $\mathbb{P}$-null sets of $\mathcal{F}$. The filtration $(\mathcal{F}_t)$ satisfies the usual conditions of right-continuity and $\mathbb{P}$-completeness.

3. Forward equation. Given $x \in H$ and a uniformly bounded progressively measurable process $\mathfrak{g}$ with values in H, we consider the stochastic differential equation for $t \geq 0$

$$(3.1) \qquad dX^{x,\mathfrak{g}}_t \; = \; AX^{x,\mathfrak{g}}_t dt + F(X^{x,\mathfrak{g}}_t)dt + QG(X^{x,\mathfrak{g}}_t)dW^1_t + DdW^2_t + \mathfrak{g}(t)\,dt, \qquad X^{x,\mathfrak{g}}_0 \; = \; x.$$

On the coefficients A, F, G, Q, D we impose the following assumptions.

($\mathbf{A.1}$) $A\colon \mathcal{D}(A) \subset H \to H$ is a linear, possibly unbounded operator generating a C_0 semigroup $\{e^{tA}\}_{t\geq 0}$.

($\mathbf{A.2}$) $F\colon H \to H$ is continuous and there exists $L_F > 0$ such that

$$|F(x) - F(x')|_H \ \leq \ L_F|x - x'|_H,$$

for all $x, x' \in H$.

($\mathbf{A.3}$) $G\colon H \to L(\Xi)$ is a bounded Lipschitz map. Moreover, for every $x \in H$, $G(x)$ is invertible. Thus there exists three positive constants L_G, M_G and $M_{G^{-1}}$ such that for all $x, x' \in H$:

$$|G(x)|_{L(\Xi)} \ \leq M_G, \qquad |G(x) - G(x')|_{L(\Xi)} \ \leq L_G, |x - x'|_H, \qquad \left|G^{-1}(x)\right|_{L(\Xi)} \leq M_{G^{-1}}.$$

We notice that the above yields Lipschitzianity of G^{-1}, namely :

$$|G^{-1}(x) - G^{-1}(x')]|_{L(\Xi)} \ \leq M_{G^{-1}}^2 L_G\,|x - x'|_H,$$

($\mathbf{A.4}$) Q is an Hilbert-Schmidt operator from Ξ to H.

($\mathbf{A.5}$) D is a linear and bounded operator from H to H and there exist constants $L > 0$ and $\gamma \in [0, \frac{1}{2}[$:

$$|e^{sA}D|_{L_2(H)} \leq L\left(s^{-\gamma} \wedge 1\right), \quad \forall s \geq 0. \tag{3.2}$$

Proposition 3.1. *Under ($\mathbf{A.1} - -\mathbf{A.5}$), for any $x \in H$ and any $\mathfrak{g}$ bounded and progressively measurable process with values in H, there exists a unique (up to indistinguishability) process $X^{x,\mathfrak{g}} = (X_t^{x,\mathfrak{g}})_{t\geq 0}$ that belongs to $L_{\mathcal{P}}^{p,loc}(\Omega; C([0, +\infty[; H))$ for all $p \geq 1$ and is a mild solution of (3.1), that is it satisfies for every $t \geq 0$, $\mathbb{P}$-a.s.:*

$$\begin{aligned} X_t^{x,\mathfrak{g}} &= e^{tA}x + \int_0^t e^{(t-s)A}F(X_s^{x,\mathfrak{g}})\,ds + \int_0^t e^{(t-s)A}\mathfrak{g}(s)\,ds + \int_0^t e^{(t-s)A}QG(X_s^{x,\mathfrak{g}})\,dW_s^1 \\ &\quad + \int_0^t e^{(t-s)A}D\,dW_s^2. \end{aligned}$$

Moreover there exists a positive constant $\kappa_{\mathfrak{g},T}$ such that

$$\mathbb{E}|X_t^{x,\mathfrak{g}}|^2 \ \leq \ \kappa_{\mathfrak{g},T}(1 + |x|^2), \qquad \forall t \in [0,T] \text{ and } x \in H. \tag{3.3}$$

Our main result will be obtained under the following exponential stability in L^1 norm requirement. We stress the fact that such assumption is much weaker in comparison with the uniform decay holding when noise is addittive (see [19]).

($\mathbf{A.6}$) There exist positive constants $\kappa_{\mathfrak{g}}$, κ and μ, independent from $\mathfrak{g}$, such that

$$\sup_{t\geq 0} \mathbb{E}|X_t^{x,\mathfrak{g}}| \leq \kappa_{\mathfrak{g}}(1 + |x|); \tag{3.4}$$

$$\mathbb{E}|X_t^{x,\mathfrak{g}} - X_t^{x',\mathfrak{g}}| \ \leq \kappa e^{-\mu t}|x - x'|; \tag{3.5}$$

for any $x, x' \in H$ and for all $t \geq 0$.

Below we show that hypothesis ($\mathbf{A.6}$) (as a matter of fact the stronger condition obtained replacing L^1 norm by L^2 norm) is verified under the usual *joint dissipative condition* ($\mathbf{A.7}$) (see [5]). We have preferred to keep the weaker, but less intrinsic, form ($\mathbf{A.6}$) since it allows to cover a wider class of examples, see for instance Example 7.1.

(A.7) - *Joint dissipative conditions*

A is dissipative i.e. $< Ax, x > \leq \rho|x|^2$, for all $x \in \mathcal{D}(A)$, and for some $\rho \in \mathbb{R}$, moreover there exists $\mu > 0$ such that for all $x, x' \in D(A)$:

$$(3.6)\quad 2\langle A(x-x') + F(x) - F(x'), x - x'\rangle_H + ||Q[G(x) - G(x')]||^2_{L_2(\Xi,H)} \;\leq\; -\mu|x-x'|^2_H,$$

Notice that, by adding a suitable constant to F and subtracting it from A we can always assume that ρ above is strictly negative.

Indeed we have that following holds

PROPOSITION 3.2. *Assume* $(\mathbf{A.1}--\mathbf{A.5})$ *and* $(\mathbf{A.7})$ *then the following estimates hold for the solution* $X^{x,\mathfrak{g}}$ *of equation* (3.1)*:*

$$(3.7)\qquad \sup_{t\geq 0} \mathbb{E}|X_t^{x,\mathfrak{g}}|^2 \leq \kappa_{\mathfrak{g}}(1+|x|^2);$$

$$(3.8)\qquad \mathbb{E}|X_t^{x,\mathfrak{g}} - X_t^{x',\mathfrak{g}}|^2 \;\leq e^{-\mu t}|x-x'|^2;$$

for any $x, x' \in H$ *and for all* $t \geq 0$*. In particular, hypothesis* $(\mathbf{A.6})$ *is verified.*

Proof.

The proof of these estimates follows rather standard arguments, see for instance [5] where dissipative systems are widely treated. □

We end this section noticing that will be mainly interested in the special case where $\mathfrak{g} \equiv 0$:

$$(3.9)\qquad dX_t \;=\; AX_t dt + F(X_t)dt + QG(X_t)dW_t^1 + DdW_t^2, \qquad X_0^x \;=\; x,$$

and we will denote by X^x its solution through the whole paper.

4. Ergodic BSDEs .

In this section we study the following equation:

$$(4.1)\qquad Y_t^x = Y_T^x + \int_t^T [\psi(X_s^x, Z_s^x G^{-1}(X_s^x), U_s^x) - \lambda]\, ds - \int_t^T Z_s^x\, dW_s^1 - \int_t^T U_s^x\, dW_s^2, \qquad 0 \leq t \leq T < \infty,$$

where, we recall, λ is a real number and it is part of the unknowns, and the equation has to hold for every t and every T, see for instance [19, section 4]. On the function $\psi : H \times \Xi^* \times H^* \to \mathbb{R}$ we assume:

(B.1) $(z, u) \to \psi(x, z, u)$ is a concave function at every fixed $x \in H$.

Moreover there exist $L_x, L_z, L_u > 0$ such that

$$(4.2)\qquad |\psi(x,z,u) - \psi(x',z'.u')| \leq L_x|x-x'| + L_z|z-z'| + L_u|u-u'|,\;\; x, x' \in H,\; z, z' \in \Xi^*,\; u, u' \in H^*.$$

Moreover $\psi(\cdot, 0.0)$ is bounded. We denote $\sup_x |\psi(x, 0.0)|$ by M_ψ.

We associate to ψ its Legendre transformation (modified according to the fact that we are dealing with concave functions):

$$(4.3)\qquad \psi^*(x,p,q) = \inf_{z\in\Xi^*, u\in H^*}\{-zp - uq - \psi(x,z,u)\}, \qquad x \in H, p \in \Xi, q \in H.$$

Clearly ψ^* is concave w.r.t to (p, q).

We collect some other properties of ψ and ψ^* we will use in the future:

PROPOSITION 4.1. *Under hypothesis* ($\mathbf{B.1}$) *we have that*

$$\psi(x,z,u) = \inf_{(p,q)\in\mathcal{D}^*(x)} \{-zp - uq - \psi^*(x,p,q)\}.$$

where $\mathcal{D}^*(x) = \{(p,q) : \psi^*(x,p,q) \neq -\infty\} \subset \{(p,q) \in \Xi \times H : |p| \leq L_z,\ |q| \leq L_u\}$.

Moreover $\mathcal{D}^*(x) = \mathcal{D}^*$ *does not depend on* $x \in H$ *and the following holds*

$$|\psi^*(x,p,q) - \psi^*(x',p,q)| \leq L_x|x - x'|, \qquad x, x \in H,\ (p,q) \in \mathcal{D}^*. \tag{4.4}$$

Finally we remark that the above implies that for every $x \in H, z \in \Xi^*, u \in H^*$ *:*

$$\sup_{(p,q)\in\mathcal{D}} \{\psi(x,z,u) + zp + uq + \psi^*(x,p,q)\} = 0.$$

Proof. Since $\psi(x,\cdot,\cdot)$ is concave its double Legendre transform coincides with the function itself and the first relation follows immediately (see [1]).

Then, by the definition of ψ^*:

$$|\psi^*(x,p,q) - \psi^*(x',p,q)| \leq \sup_{z\in\Xi^*,\, u\in H^*} |-zp - uq - \psi(x,z,u) + zp + uq + \psi(x',z,u)| \leq L_x|x - x'|,$$

thus we deduce that $\mathcal{D}^*$ doesn't depend on $x \in H$ and (4.4) holds. □

As in [19] we introduce, for each $\alpha > 0$, the infinite horizon equation:

$$Y_t^{x,\alpha} = Y_T^{x,\alpha} + \int_t^T [\psi(X_s^x, Z_s^{x,\alpha} G^{-1}(X_s^x), U_s^{x,\alpha}) - \alpha Y_s^{x,\alpha}]\, ds - \int_t^T Z_s^{x,\alpha}\, dW_s^1 - \int_t^T U_s^{x,\alpha}\, dW_s^2, \tag{4.5}$$

where $0 \leq t \leq T < \infty$.

The next result was proved in [25, Theorem 2.1] in finite dimensions, the extension to the infinite dimensional case is straightforward, see also [19, Lemma 4.2]. Notice that the random function, $\widehat{\psi}(t,z,u) := \psi(X_t, G^{-1}(X_t)z, u)$, inherits the following properties:

$$|\widehat{\psi}(t,0,0)| = |\psi(X_t,0,0)| \leq M_\psi, \quad t \geq 0,\ \mathbb{P}\text{- a.s.}. \tag{4.6}$$

$$|\widehat{\psi}(t,z,u) - \widehat{\psi}(t,z',u')| \leq L_z M_{G^{-1}}|z - z'| + L_u|u - u'| \quad t \geq 0, \quad z, z' \in \Xi^*,\ u, u' \in H^*\ . \tag{4.7}$$

therefore it satisfies the assumptions in [19, Lemma 4.2].

THEOREM 4.1. *Let us assume* ($\mathbf{A.1} - -\mathbf{A.5}$) *and* ($\mathbf{B.1}$). *Then for every* $\alpha > 0$ *there exists a unique solution* $(Y^{x,\alpha}, Z^{x,\alpha}, U^{x,\alpha})$ *to the BSDE* (4.5) *such that* $Y^{x,\alpha}$ *is a bounded continuous process,* $Z^{\alpha,x} \in L^{2,loc}_{\mathcal{P}}(\Omega \times [0,+\infty[;\Xi^*)$ *and* $U^{\alpha,x} \in L^{2,loc}_{\mathcal{P}}(\Omega \times [0,+\infty[;H^*)$.

Moreover

$$|Y_t^{x,\alpha}| \leq \frac{M_\psi}{\alpha},\ \mathbb{P}\text{-a.s., for all } t \geq 0. \tag{4.8}$$

and

$$\mathbb{E}\int_0^\infty |e^{-\alpha s} Z_s^{x,\alpha}|^2\, ds + \mathbb{E}\int_0^\infty |e^{-\alpha s} U_s^{x,\alpha}|^2\, ds < \infty. \tag{4.9}$$

We define

$$v^\alpha(x) = Y_0^{\alpha,x}. \tag{4.10}$$

The following is the main estimate of the paper.

PROPOSITION 4.2. *Under* (**A.1**$--$**A.6**) *and* (**B.1**) *one has that for any* $\alpha > 0$*:*

$$|v^\alpha(x) - v^\alpha(x')| \leq \frac{C}{\mu}|x - x'|, \qquad x, x' \in H. \tag{4.11}$$

where C *depends on the constants in* (**A.1**$--$**A.5**) *and* (**B.1**) *but not on* α *(nor on* μ*).*

Proof. Since, instead of the pathwise decay estimate holding for $|X_t^x - X_t^{x'}|$ in the additive noise case (see [19, Theorem 3.2]), only the mean bound (3.5) is true here we cannot proceed as in [19, Theorem 4.4]. Moreover, being the diffusion X, in general, degenerate, it is not possible to rely on the smoothing properties of its Kolmogorov semigroup (see [21]). On the contrary, concavity assumption **(B.1)** allows us to use control theoretic arguments.

First we notice that

$$Y_0^{x,\alpha} = e^{-\alpha t}Y_t^{x,\alpha} + \int_0^t e^{-\alpha s}\psi(X_s^x, Z_s^{x,\alpha}G^{-1}(X_s^x), U_s^{x,\alpha})\, ds - \int_0^t e^{-\alpha s} Z_s^{x,\alpha}\, dW_s^1 - \int_0^t e^{-\alpha s} U_s^{x,\alpha}\, dW_s^2.$$

Thus we have, taking also into account (4.8) and (4.9), that

$$Y_0^{x,\alpha} = \int_0^{+\infty} e^{-\alpha s}\psi(X_s^x, Z_s^{x,\alpha}G^{-1}(X_s^x), U_s^{x,\alpha})\, ds - \int_0^{+\infty} e^{-\alpha s} Z_s^{x,\alpha}\, dW_s^1 - \int_0^{+\infty} e^{-\alpha s} U_s^{x,\alpha}\, dW_s^2. \tag{4.12}$$

Moreover being $Y_0^{x,\alpha}$ deterministic, the uniqueness in law for the system formed by equations (3.9) -(4.5) yields that it doesn't depend on the specific independent Wiener processes.

We fix any stochastic setting $(\hat\Omega, \hat{\mathcal{E}}, (\hat{\mathcal{F}}_t), \hat{\mathbb{P}}, (\hat W_t^{\,1}), (\hat W_t^{\,2}))$ where $((\hat W_t^{\,1}), (\hat W_t^{\,2}))$ are independent $(\hat{\mathcal{F}}_t)$ Wiener processes with values in Ξ and H respectively.

Given any $(\hat{\mathcal{F}}_t)$ progressively measurable process $\mathfrak{p} := (p_t, q_t)$ with values in $\mathcal{D}^*$ by $(\hat X_t^{x,\mathfrak{p}})$ we denote the unique mild solution of the forward equation:

$$d\hat X_t^{x,\mathfrak{p}} \;=\; A\hat X_t^{x,\mathfrak{p}}dt + F(\hat X_t^{x,\mathfrak{p}})dt + Dq_t dt + QG(\hat X_t^{x,\mathfrak{p}})p_t dt + QG(\hat X_t^{x,\mathfrak{p}})d\hat W_t^1 + Dd\hat W_t^2, \qquad \hat X_0^{x,\mathfrak{p}} = x. \tag{4.13}$$

Clearly $(\hat X_t^{x,\mathfrak{p}})$ is also the unique mild solution of the forward equation:

$$d\hat X_t^{x,\mathfrak{p}} \;=\; A\hat X_t^{x,\mathfrak{p}}dt + F(\hat X_t^{x,\mathfrak{p}})dt + QG(\hat X_t^{x,\mathfrak{p}})d\hat W_t^{1,\mathfrak{p}} + Dd\hat W_t^{2,\mathfrak{p}}, \qquad \hat X_0^{x,\mathfrak{p}} \;=\; x. \tag{4.14}$$

where

$$\hat W_t^{1,\mathfrak{p}} := \hat W_t^1 + \int_0^t G^{-1}(\hat X_s^{x,\mathfrak{p}})p_s\, ds, \quad \hat W_t^{2,\mathfrak{p}} := \hat W_t^2 + \int_0^t q_s\, ds, \tag{4.15}$$

and we know that under a suitable probability $\hat{\mathbb{P}}^{\mathfrak{p}}$ the processes $((\hat W_t^{\,1,\mathfrak{p}}), (\hat W_t^{\,2,\mathfrak{p}}))$ are independent Wiener processes with values in Ξ and H respectively.
Let now $(\hat Y^{x,\alpha,\mathfrak{p}}, \hat Z^{x,\alpha,\mathfrak{p}}, \hat U^{x,\alpha,\mathfrak{p}})$ be the solution to:

$$\hat Y_t^{x,\alpha,\mathfrak{p}} = \hat Y_T^{x,\alpha,\mathfrak{p}} + \int_t^T [\psi(\hat X_s^{x,\mathfrak{p}}, \hat Z_s^{x,\alpha,\mathfrak{p}}G^{-1}(\hat X_s^{x,p}), \hat U_s^{x,\alpha,\mathfrak{p}}) - \alpha Y_s^{x,\alpha,\mathfrak{p}}]\, ds$$

$$- \int_t^T \hat{Z}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^{1,\mathfrak{p}} - \int_t^T \hat{U}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^{2,\mathfrak{p}},$$

where $0 \le t \le T < \infty$.

By previous considerations one has, recalling that $\{\psi(x,z) + zp + uq + \psi^*(x,p)\} \le 0, \forall x \in H, z \in \Xi^*, u \in H^*, (p,q) \in \mathcal{D}^*$, that for every $x \in H$

$$\begin{aligned}
&Y_0^{x,\alpha} = \hat{Y}_0^{x,\alpha,\mathfrak{p}}\\
&= \int_0^\infty e^{-\alpha s} \Big[\psi(\hat{X}_s^{x,\mathfrak{p}}, \hat{Z}_s^{x,\alpha,\mathfrak{p}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}}), \hat{U}_s^{x,\alpha,\mathfrak{p}}) + \hat{Z}_s^{x,\alpha,\mathfrak{p}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}}) p_s + \hat{U}_s^{x,\alpha,\mathfrak{p}} q_s + \psi^*(\hat{X}_s^{x,\mathfrak{p}}, p_s, q_s)\Big] ds\\
&- \int_0^{+\infty} e^{-\alpha s} \hat{Z}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^1 - \int_0^{+\infty} e^{-\alpha s} \hat{U}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^2 - \int_0^\infty \psi^*(\hat{X}_s^{x,\mathfrak{p}}, p_s, q_s)\, ds\\
&\le - \int_0^{+\infty} e^{-\alpha s} \hat{Z}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^1 - \int_0^{+\infty} e^{-\alpha s} \hat{U}_s^{x,\alpha,\mathfrak{p}}\, d\hat{W}_s^2 - \int_0^\infty \psi^*(\hat{X}_s^{x,\mathfrak{p}}, p_s, q_s)\, ds.
\end{aligned}$$

So:

$$Y_0^{x,\alpha} \le -\hat{\mathbb{E}} \int_0^\infty e^{-\alpha s} \psi^*(\hat{X}_s^{x,\mathfrak{p}}, p_s, q_s)\, ds, \tag{4.16}$$

for arbitrary stochastic setting and arbitrary progressively measurable $\mathcal{D}^*$ valued control $\mathfrak{p} = (p,q)$.

Then we fix $x \in H$ and assume, for the moment, that $\forall \varepsilon > 0$ there exists a stochastic setting

$$(\hat{\Omega}^{\varepsilon,x}, \hat{\mathcal{E}}^{\varepsilon,x}, (\hat{\mathcal{F}}_t^{\varepsilon,x}), \hat{\mathbb{P}}^{\varepsilon,x}, (\hat{W}_t^{1,\varepsilon,x}), (\hat{W}_t^{2,\varepsilon,x})),$$

and a couple of predictable processes $\mathfrak{p}^{\varepsilon,x} = (p^{\varepsilon,x}, q^{\varepsilon,x})$ with values in $\mathcal{D}^*$ such that (with the notations introduced above) the following holds $\mathbb{P}$ - a.s. for a.e. $s \ge 0$:

$$\begin{aligned}
\psi(\hat{X}_s^{x,\mathfrak{p}^\varepsilon}, \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}), \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}}) + \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}) p_s^\varepsilon + \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} q_s^{\varepsilon,x}&\\
+ \psi^*(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x}) \ge -\varepsilon.&
\end{aligned} \tag{4.17}$$

Proceeding as before we get:

(4.18)

$$\begin{aligned}
Y_0^{x,\alpha} =& \hat{Y}_0^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} =\\
&= \int_0^\infty e^{-\alpha s} \Big[\psi(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}), \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}})\\
&\qquad + \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} G^{-1}(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}) p_s^{\varepsilon,x} + \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}} q_s^{\varepsilon,x} + \psi^*(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x})\Big]\, ds\\
&\quad - \int_0^{+\infty} e^{-\alpha s} \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}}\, d\hat{W}_s^{1,\varepsilon,x} - \int_0^{+\infty} e^{-\alpha s} \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}}\, d\hat{W}_s^{2,\varepsilon,x} - \int_0^\infty \psi^*(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x})\, ds\\
&\ge -\frac{\varepsilon}{\alpha} - \int_0^{+\infty} e^{-\alpha s} \hat{Z}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}}\, d\hat{W}_s^{1,\varepsilon,x} - \int_0^{+\infty} e^{-\alpha s} \hat{U}_s^{x,\alpha,\mathfrak{p}^{\varepsilon,x}}\, d\hat{W}_s^{2,\varepsilon,x} - \int_0^\infty \psi^*(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x})\, ds.
\end{aligned}$$

Thus by (4.16) taking into account (4.18) and (4.4) we have:

$$Y_0^{x',\alpha} - Y_0^{x,\alpha} \le \int_0^\infty e^{-\alpha s} \hat{\mathbb{E}}^{\mathfrak{p}^{\varepsilon,x}} |\psi^*(\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x}) - \psi^*(\hat{X}_s^{x',\mathfrak{p}^{\varepsilon,x}}, p_s^{\varepsilon,x}, q_s^{\varepsilon,x})|\, ds + \varepsilon$$

$$\leq \int_0^\infty e^{-\alpha s}\, \hat{\mathbb{E}}^{\mathfrak{p}^{\varepsilon,x}} |\hat{X}_s^{x,\mathfrak{p}^{\varepsilon,x}} - \hat{X}_s^{x',\mathfrak{p}^{\varepsilon,x}}|\, ds + \frac{\varepsilon}{\alpha},$$

we stress the fact that we keep the stochastic setting $(\hat{\Omega}^{\varepsilon,x}, \hat{\mathcal{E}}^{\varepsilon,x}, (\hat{\mathcal{F}}_t^{\varepsilon,x}), \hat{\mathbb{P}}^{\varepsilon,x}, (\hat{W}_t^{1,\varepsilon,x}), (\hat{W}_t^{2,\varepsilon,x}))$ and control $\mathfrak{p}^{\varepsilon,x}$ corresponding to the initial datum x and just replace the initial state x with a different one x'.

Noticing now that both $(\hat{X}^{x,\mathfrak{p}^{\varepsilon,x}})$ and $(\hat{X}^{x',\mathfrak{p}^{\varepsilon,x}})$ satisfy (only the initial conditions differ):

$$d\hat{X}_t \; = \; A\hat{X}_t dt + F(\hat{X}_t)dt + Dq_t^{\varepsilon,x}dt + QG(\hat{X}_t)p_t^{\varepsilon,x}dt + QG(\hat{X}_t^{x,\mathfrak{p}})d\hat{W}_t^{1,\varepsilon,x} + Dd\hat{W}_t^{2,\varepsilon,x}.$$

and taking into account (3.5) we can conclude that:

$$Y_0^{x',\alpha} - Y_0^{x,\alpha} \leq L_x \int_0^\infty e^{-(\alpha+\frac{\mu}{2})s} |x-x'|\, ds + \frac{\varepsilon}{\alpha} \leq \frac{C}{\mu}|x-x'| + \frac{\varepsilon}{\alpha}.$$

Interchanging the role of x with x' one gets:

$$\left| Y_0^{x,\alpha} - Y_0^{x',\alpha} \right| \leq \frac{C}{\mu}|x-x'| + \frac{\varepsilon}{\alpha}. \tag{4.19}$$

where the constant C is independent of α, μ and ε and is able to conclude (4.11) being $\varepsilon > 0$ arbitrary.

We are left with the construction, for any fixed $x \in H$ and $\varepsilon > 0$ of a stochastic setting $(\hat{\Omega}^{\varepsilon,x}, \hat{\mathcal{E}}^{\varepsilon,x}, (\hat{\mathcal{F}}_t^{\varepsilon,x}), \hat{\mathbb{P}}^{\varepsilon,x}, (\hat{W}_t^{1,\varepsilon,x}), (\hat{W}_t^{2,\varepsilon,x}))$ and control $\mathfrak{p}^{\varepsilon,x}$ for which (4.17) holds.

We start from an arbitrary stochastic setting: $(\Omega, \mathcal{E}, (\mathcal{F}_t), \mathbb{P}, (W_t^1), (W_t^2))$. Let (X^x) be the corresponding mild solution of equation (3.1) and $(Y^{x,\alpha}, Z^{x,\alpha}, U^{x,\alpha})$ the solution of (4.5). By a measurable selection argument see [22, Theorem 4] we can find a couple of progressive measurable process $\mathfrak{p}^{\varepsilon,x} = (p^{\varepsilon,x}, q^{\varepsilon,x})$, (possibly depending on α as well), such that:

$$\psi(X_s^x, Z_s^{x,\alpha}G^{-1}(X_s^x), U_s^{x,\alpha}) + Z_s^{x,\alpha}G^{-1}(X_s^x)p_s^{\varepsilon,x} + U_s^{x,\alpha}q_s^{\varepsilon,x} + \psi_*(X_s^x, p_s^{\varepsilon,x}, q_s^{\varepsilon,x}) \geq -\varepsilon.$$

Then it is enough to set:

$$\hat{W}_t^{1,\varepsilon,x} := W_t^1 - \int_0^t G^{-1}(X_s^x)p_s^{\varepsilon,x}\, ds, \quad \hat{W}_t^{2,\varepsilon,x} := W_t^2 - \int_0^t q_s^{\varepsilon,x}\, ds, \tag{4.20}$$

and choose $\hat{\Omega}^{\varepsilon,x} = \Omega$, $\hat{\mathcal{E}}^{\varepsilon,x} = \mathcal{E}$, $(\hat{\mathcal{F}}_t^{\varepsilon,x})) = (\mathcal{F}_t)$ and as $\hat{\mathbb{P}}^{\varepsilon,x}$ the (unique) probability measure under which $((\hat{W}_t^{1,\varepsilon,x}), (\hat{W}_t^{2,\varepsilon,x}))$ are independent Wiener processes. The claim then follows selecting the above control $\mathfrak{p}^{\varepsilon,x}$ and noticing that, by construction, $(\hat{X}^{x,\mathfrak{p}^{\varepsilon,x}}) = (X^x)$. □

Following [19] we can find a function $\bar{v}$ and a number λ such that:

$$[v^{\alpha_m}(x) - v^{\alpha_m}(0)] \to \bar{v}(x), \qquad \forall x \in H, \tag{4.21}$$

$$\alpha_m v^{\alpha_m}(0) \to \lambda. \tag{4.22}$$

where $\{\alpha_m\}_{m\in\mathbb{N}}$ is a suitable subsequence constructed using a diagonal method.

We can then proceed as in [19] to deduce from above the existence of a solution to (4.1) and the uniqueness of λ.

THEOREM 4.2. *Assume* $(\mathbf{A.1})-(\mathbf{A.6})$ *and* $(\mathbf{B.1})$*, let* λ *the number defined in* (4.22) *and set* $\bar Y^x_t := \bar v(X^x_t)$*, where* $\bar v$ *is defined in* (4.21)*. Then there exists* $\bar Z^x$ *in* $L^{2,loc}_{\mathcal P}(\Omega\times[0,+\infty[;\Xi^*)$ *and* $\bar U^x$ *in* $L^{2,loc}_{\mathcal P}(\Omega\times[0,+\infty[;H^*)$ *such that* $(\bar Y^x,\bar Z^x,\bar U^x,\lambda)$ *solves equation* (4.1)*,* $\mathbb P$ *-a.s. for all* $0\le t\le T$*.*

Moreover suppose that another quadruple (Y',Z',U',λ) *where* Y' *is a progressively measurable continuous process verifying* $|Y'_t|\le c(1+|X^x_t|)$*,* $Z'\in L^{2,loc}_{\mathcal P}(\Omega\times[0,+\infty[;\Xi^*)$ *,* $U'\in L^{2,loc}_{\mathcal P}(\Omega\times[0,+\infty[;H^*)$ *and* $\lambda'\in\mathbb R$*, satisfies* (4.1)*. Then* $\lambda'=\lambda$*.*

Finally there exists a measurable function $\bar\zeta: H\to\Xi^*\times H^*$ *such that* $(\bar Z^x_t,\bar U^x_t)=\bar\zeta(X^x_t)$*.*

Proof.

Once (4.11), (4.21) and (4.22) are obtained, the proof as far the first two statements is concerned follows exactly as in [19, Theorem 4.4].

To get the existence of a function $\bar\zeta$, we proceed in the following way. For arbitrary fixed $0\le t\le T$ let $(\bar Y^{x,t,T},\bar Z^{x,t,T},\bar U^{x,t,T})$ be the solution to:

$$\left\{\begin{array}{l} dX^{t,x}_s \;=\; AX^{t,x}_s ds+F(X^{t,x}_s)ds+QG(X^{t,x}_s)dW^1_s+DdW^2_s,\\ X^{t,x}_t \;=\; x,\\ -dY^{x,t,T}_s=\widehat\psi(X^{x,t}_s,Z^{x,t,T}_s,U^{x,t,T}_s)\,ds-Z^{x,t,T}_s\,dW^1_s-U^{x,t,T}_s\,dW^2_s-\lambda\,ds,\\ Y^{x,t,T}_T=\bar v(X^{x,t}_T).\end{array}\right. \tag{4.23}$$

Then we clearly have that $(\bar Y^x,\bar Z^x,\bar U^x)$, restricted on $[0,T]$, coincide with $(\bar Y^{x,0,T},\bar Z^{x,0,T},\bar U^{x,0,T})$, for all $T>0$. By [8, Prop. 3.2] we know that there exists a measurable function $\zeta^T:[0,T]\times H\to\Xi^*\times H^*$, such that $(\bar Z^{x,t,T}_s,\bar U^{x,t,T}_s)=\zeta^T(s,X^{x,t}_s)$, $s\in[t,T]$. Moreover, see also [8, Remark 3.3], the map $[0,T]\ni(\tau,x)\to\zeta^T(\tau,x)$ is characterized in terms of the laws of $(\int_\tau^{\tau+\frac1n}\bar Z^{\tau,x,T}_s\,ds,\int_\tau^{\tau+\frac1n}\bar U^{\tau,x,T}_s\,ds)$, $n\in\mathbb N$.

The uniqueness in law of the solutions to the system (4.23) together with the fact that its coefficients are time autonomous, we get:

$$\int_\tau^{\tau+\frac1n}\bar Z^{\tau,x,T}_s\,ds\sim\int_0^{\frac1n}\bar Z^{0,x,T-\tau}_s\,ds\sim\int_0^{\frac1n}\bar Z^x_s\,ds,$$

and

$$\int_\tau^{\tau+\frac1n}\bar U^{\tau,x,T}_s\,ds\sim\int_0^{\frac1n}\bar U^{0,x,T-\tau}_s\,ds\sim\int_0^{\frac1n}\bar U^x_s\,ds.$$

So far we've proved that $\zeta^T(\tau,\cdot)$ does not depend neither from T nor from τ, thus we can define $\zeta^T(\tau,\cdot)=:\bar\zeta(\cdot)$ and observe that $(\bar Z^x_t,\bar U^x_t)=(\bar Z^{x,0,T}_t,\bar U^{x,0,T}_t)=\zeta^T(t,X^{x,0}_t)=\bar\zeta(X^x_t)$. □

REMARK 4.1. *Concerning the uniqueness of the Markovian solution to the Ergodic BSDE* (4.1) *and consequently of the mild solution to the ergodic HJB equation* (5.1) *only partial results are available even in the additive case (beside the obvious consideration that adding a constants to* Y *and consequently to* v *transforms solutions into solutions). In particular an argument based on recurrence of the solution* X *to* (1.1) *is developed in [12] (see also [19]) to obtain a control theoretic representation of* v *and consequently its uniqueness up to an additive constant. Such arguments seem inapplicable in the present context due to possible degeneracy of the noise.*

5. Ergodic Hamilton-Jacobi-Bellman. We wish now to prove that function $\bar v$ satisfies, in a suitable way, the following Hamilton Jacobi Bellman elliptic partial differential equation:

$$\frac12\mathrm{tr}[QG(x)G^*(x)Q\nabla^2\bar v(x)]+\frac12\mathrm{tr}[DD^*(x)Q\nabla^2\bar v(x)]+\langle Ax+F(x),\nabla\bar v(x)\rangle= \tag{5.1}$$

$$- \psi(x, \nabla \bar{v}(x) Q, \nabla \bar{v}(x) D) + \lambda$$

Since the prof of differentiability of $\bar{v}$ requires quantitative conditions that we were able to avoid in Theorem 4.2 we firstly formulate the PDE in a weaker sense involving the *Generalized directional gradient* introduced in [11]. The following is the version of Theorem 3.1 in [11] adapted to the present autonomous and Lipschitz case. The proof is identical to the one in [11] and is omitted.

THEOREM 5.1. *Given any Lipschitz function v on H there exists a couple of bounded and Borel measurable functions $\zeta^1 : H \to \Xi^*$, $\zeta^2 : H \to H^*$ such that denoting, for all $\xi = (\xi^1, \xi^2) \in \Xi \times H$, by $W^\xi_s := \langle (W^1_s, W^2_s), \xi \rangle$ the real Brownian Motion obtained projecting (W^1_s, W^2_s) along direction ξ, then we have the following relation, for any $x \in H$ and any $\rho > 0$*

$$\langle v(X^x_\cdot), W^\xi_\cdot \rangle_{[0,\rho]} = \int_0^\rho \zeta^1(X^x_t)\xi^1 \, dt + \int_0^\rho \zeta^2(X^x_t)\xi^2 \, dt, \qquad \mathbb{P} - a.s.$$

DEFINITION 5.1. *The family of functions $\zeta = (\zeta^1, \zeta^2)$ satisfying the above will be called the generalized (QG, D) directional gradients of u (denoted by $\tilde{\nabla}^{QG,D}$).*

REMARK 5.1. *Concerning uniqueness we can only say that if ζ and $\hat{\zeta}$ both belong to $\tilde{\nabla}^{QG,D}$ then $\zeta^1(X^x_t) = \hat{\zeta}^1(X^x_t)$ and $\zeta^2(X^x_t) = \hat{\zeta}^2(X^x_t)$, $\mathbb{P}$-a.s. for almost every $t \geq 0$. See [11]. It is also clear that, by Ito rule, if u is regular enough, including twice continuously differentiable, then $(\nabla u(\cdot) QG(\cdot), \nabla u(\cdot) D)$ is in $\tilde{\nabla}^{QG,D}$.*

We are therefore led to the following definition of generalized solution to HJB equation. see [11, Section 5]:

DEFINITION 5.2. *A pair (v, λ) is a mild solution in the sense of generalized directional gradient of the HJB equation (5.1) if $v : H \to \mathbb{R}$ is Lipschitz and, for every $T > 0$ and for all $0 \leq t \leq T$ and $x \in H$ it holds*

$$v(x) = P_{T-t}[v](x) + \int_t^T (P_{s-t}[\psi(\cdot, \zeta^1(\cdot) G^{-1}, \zeta^2(\cdot))](x) - \lambda) \, ds. \tag{5.2}$$

where $\zeta = (\zeta^1, \zeta^2)$ is an arbitrary element of the generalized gradient $\tilde{\nabla}^{(QG,D)}$ and $(P_t)_{t \geq 0}$ is the transition semigroup corresponding to the diffusion X^x, see equation (3.9), that is:

$$P_t[\phi](x) := \mathbb{E}\, \phi(X^x_t), \qquad \phi : H \to \mathbb{R} \text{ measurable and bounded.} \tag{5.3}$$

We notice that function $\bar{v}$ defined in (4.21) is Lipschitz. Moreover recalling, seeTheorem 4.2, that $(\bar{Y}^x_t, \bar{Z}^x_t, \bar{U}^x_t) = (\bar{v}(X^x_t), \bar{\zeta}^1(X^x_t), \bar{\zeta}^2(X^x_t))$ we have that then equation (4.1) is satisfied, in particular, for $t = 0$ and all $T > 0$ we immediately deduce that $\bar{\zeta} = (\bar{\zeta}^1, \bar{\zeta}^2)$ is in $\tilde{\nabla}^{(QG,D)}$. Finally recalling once more equation (4.1) now interpreted as a finite horizon BSDE:

$$-d\bar{Y}^x_t = \psi(X^x_s, \bar{Z}^x_s G^{-1}(X^x_s), \bar{U}^x_s) \, ds - \lambda \, ds - \bar{Z}^x_s \, dW^1_s - \bar{U}^x_s \, dW^2_s, \qquad \bar{Y}^x_T = \bar{v}(X^x_T)$$

we can conclude the following, proceeding exactly as in [11] Theorem 5.1

THEOREM 5.2. *Assume $(\mathbf{A.1} - -\mathbf{A.6})$, $(\mathbf{B.1})$ then the couple $(\bar{v}, \lambda)$, characterized in Theorem 4.2, is a mild solution, in the sense of the generalized directional gradient of equation (5.1).*

Whenever $\bar{v}$ is differentiable then we can switch to the more classical notion of mild solution to equation (5.1) $(\bar{v}, \lambda)$, see [20, Section 6]:

Definition 5.3. *A pair (v, λ) is a mild solution to the HJB equation (5.1) if $v \in \mathcal{G}^1(H, \mathbb{R})$ with bounded derivative and, for all $0 \le t \le T$, $x \in H$ it holds:*

$$v(x) = P_{T-t}[v](x) + \int_t^T (P_{s-t}[\psi(\cdot, \nabla v(\cdot)Q, \nabla v(\cdot)D)](x) - \lambda)\, ds. \tag{5.4}$$

We have the following result.

Theorem 5.3. *Assume $(\mathbf{A.1}--\mathbf{A.6})$, $(\mathbf{B.1})$ and that $\bar{v}$ is of class $\mathcal{G}^1$. Then $(\bar{v}, \lambda)$, defined in (4.21) is a* mild *solution of the HJB equation (5.1). On the other hand if (v', λ') is a mild solution of (5.1) then setting $Y_t^x := v'(X_t^x)$, $Z_t^x = \nabla v'(X_t^x)QG(X_t^x)$ and $U_t^x = \nabla v'(X_t^x)D$, we obtain that (Y^x, Z^x, U^x, λ) is a solution to equation (4.1).*

Moreover if (v', λ') is another solution with v' Gateaux differentiable with linear growth then $\lambda = \lambda'$.

Proof. The existence part follows from [10, Theorem 6.2], while the uniqueness of λ in the class of solutions that are Gateaux differentiable with linear growth follows as [20, Theorem 4.6]. □

Remark 5.2. *The differentiability of function $\bar{v}$ is proved in Theorem 6.1 under quantitative assumptions on the coefficients. Although the argument essentially follows the classical paths of L^2 estimates on infinite horizon see, for instance, [6] it is not completely standard since exploits in several points an apriori L^∞ estimate on Z and U descending from Proposition 4.2. In particular the uniform bounds for Z is essential in getting (A.9).*

We conclude this section proving the following asymptotic expansion result for parabolic solutions to the HJB equation.

Proposition 5.1. *Let $v(\cdot, \cdot)$ be a mild solution of the parabolic HJB equation:*

$$\begin{cases} \partial_t v(t,x) = & \frac{1}{2}[tr[QG(x)G^*(x)Q\nabla^2 v(t,x)] + tr[DD^*(x)Q\nabla^2 v(t,x)) + \langle Ax + F(x), \nabla v(t,x)\rangle \\ & + \psi(x, \nabla v(t,x)Q, \nabla v(t,x)D), \\ v(0,x) = & \phi(x). \end{cases} \tag{5.5}$$

where $\phi : H \to \mathbb{R}$ is function of class $\mathcal{G}^1$ with bounded derivative and by mild solution of equation 5.5 we mean a function $v : \mathbb{R}^+ \times H \to \mathbb{R}$ of class $\mathcal{G}^{0,1}$ (see [10]) verifying for all $t > 0$, $x \in H$:

$$v(t,x) = P_t[\phi](x) + \int_0^t P_{t-s}[\psi(\cdot, \nabla v(\cdot)Q, \nabla v(\cdot)D)](x)\, ds. \tag{5.6}$$

Then

$$\lim_{T\to\infty} \frac{v(T,x)}{T} = \lambda. \tag{5.7}$$

Proof. We fix $T > 0$ and consider the following finite horizon BSDE

$$\begin{cases} -d\bar{Y}_s^{T,x} = \psi(X_s^x, G^{-1}(X_s^x)Z_s^{T,x}, U_s^{T,x})\, ds - Z_s^{T,x}\, dW_s^1 - U_s^{T,x}\, dW_s^2 - \lambda\, ds \\ \bar{Y}_T^{T,x} = \phi(X_T^x). \end{cases} \tag{5.8}$$

By standard results on finite horizon BSDEs and mild solution of parabolic Hamilton-Jacobi-Bellman equations (see [10]) we have that $\bar{Y}_s^{T,x} = v(T-s, X_s^x) - \lambda(T-s)$, $s \in [0,T]$.

Set $\tilde{Y}_t^{T,x} = \bar{Y}_t^x - \bar{Y}_t^{T,x}$, for all $t \in [0,T]$, then $\tilde{Y}^{T,x}$ verifies:

$$
\left\{
\begin{array}{l}
-d\tilde{Y}_s^{T,x} = [\psi(X_s^x, G^{-1}(X_s^x)\bar{Z}_s^x, \bar{U}_s^x) - \psi(X_s^{x,t}, G^{-1}(X_s^x)Z_s^{T,x}, U_s^{T,x})]\, ds - (\bar{Z}_s^x - Z_s^{T,x})\, dW_s^1 \\
\qquad\qquad -(\bar{U}_s^x - U_s^{T,x})\, dW_s^2 - \lambda\, ds \\
\quad \tilde{Y}_T^{T,x} = \bar{v}(X_T^x) - \phi(X_T^x).
\end{array}
\right. \tag{5.9}
$$

We rewrite (5.9) as:

$$
\left\{
\begin{array}{l}
-d\tilde{Y}_s^{T,x} = \gamma_t^1(\bar{Z}_s^x - Z_s^{T,x})\, ds + \gamma_t^2(\bar{U}_s^x - U_s^{T,x})\, ds - (\bar{Z}_s^x - Z_s^{T,x})\, dW_s^1 \\
\qquad\qquad -(\bar{U}_s^x - U_s^{T,x})\, dW_s^2 - \lambda\, ds \\
\quad \tilde{Y}_T^{T,x} = \bar{v}(X_T^x) - \phi(X_T^x).
\end{array}
\right. \tag{5.10}
$$

where γ^1 and γ^2 are the typical uniformly bounded processes that arise in the linearization trick.

Hence, by a Girsanov argument, we get that

$$\tilde{Y}_0^{T,x} = \mathbb{E}^{\gamma^1,\gamma^2}(\bar{v}(X_T^x)), \tag{5.11}$$

where the probability measure $\mathbb{P}^{\gamma^1,\gamma^2}$ is the one under which $W_t^{\gamma^1,\gamma^2} = (W_t^1 - \int_0^t \gamma_s^1\, ds, W_t^2 - \int_0^t \gamma_s^2\, ds)$ is a cylindrical Wiener process in $\Xi \times H$ in $[0,T]$. Therefore by (3.4) and having $\bar{v}$ Lipschitz, we get that

$$\tilde{Y}_0^{T,x} = \mathbb{E}^{\gamma^1,\gamma^2}(\bar{v}(X_T^x) - \phi(X_T^x)) \leq \kappa_{\gamma_1,\gamma_2}(1+|x|), \tag{5.12}$$

for some constant $\kappa_{\gamma_1,\gamma_2}$ independent of T. Thus, noticing that $\tilde{Y}_0^{T,x} = \bar{v}(x) - v(T,x) + \lambda T$ we get that:

$$\lim_{T\to\infty} \frac{v(T,x)}{T} = \lim_{T\to\infty} \frac{\bar{v}(x)}{T} + \lambda = \lambda. \tag{5.13}$$

□

Remark 5.3. *A more precise description of the asymptotic behaviour of $\frac{v(T,x)}{T}$ is obtained in [15] when the noise is non degenerate by techniques involving Girsanov change of probability and coupling estimates. Due to the possible non-invertibility of Q we do not know whether similar results can be true in the present framework. We do think that, in any case, the proof of such results would require different arguments.*

6. Differentiability with respect to initial data. In this section we wish to present sufficient conditions under which the function $\bar{v}$ defined in the section above is differentiable.

Throughout the section we assume the following:

(**C.1**) F is of class $\mathcal{G}^1(H,H)$ and G is of class $\mathcal{G}^1(H, L(\Xi,H))$

We start from a straightforward result in the non-degenerate case.

Proposition 6.1. *Beside* (**A.1** − −**A.6**)*,* (**B.1**) *and* (**C.1**) *assume that the operator $\mathcal{Q} := (Q,D) : \Xi \times H \to H$ admits a right inverse $\mathcal{Q}^{-1}$ then $\bar{v}$ belongs to class $\mathcal{G}^1(H)$.*

Proof. We fix $T > 0$ and notice that $(\bar{Y}, \bar{Z}, \bar{U}, \lambda)$ satisfies (see (4.1) and the definition of $\bar{Y}_t$ in Theorem 4.2):

$$Y_t^x = \bar{v}(X_T^x) + \int_t^T [\widehat{\psi}(X_s^x, \bar{Z}_s^x, \bar{U}_s^x) - \lambda]\, ds - \int_t^T \bar{Z}_s^x\, dW_s^1 - \int_t^T \bar{U}_s^x\, dW_s^2, \qquad 0 \leq t \leq T < \infty,$$

where, we recall $\widehat{\psi}(x,z,u) = \psi(x, zG^{-1}(x), u)$ is lipschitz with respect to z and u. Moreover the forward equation (3.9) solved by X^x can be rewritten as

$$dX_t^x \;=\; AX_t^x dt + F(X_t^x)dt + \tilde{\mathcal{Q}}(X_t^x)d\mathcal{W}_t, \qquad X_0^x \;=\; x.$$

where $\mathcal{W}_t := \begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}$ is a $\Xi \times H$ valued Wiener process and $\tilde{\mathcal{Q}}(x) = (QG(x), D)$.

Under the present assumptions $\tilde{\mathcal{Q}}(x)$ turns out to be invertible with bounded right inverse:

$$[\tilde{\mathcal{Q}}(x)]^{-1} = \begin{pmatrix} G^{-1}(x) & 0 \\ 0 & I \end{pmatrix} \mathcal{Q}^{-1}.$$

It is then straight forward to verify that all the assumptions in [9, Theorem 3.10] are satisfied and consequently $\bar{v}$ (that coincides with the map $x \to \bar{Y}_x$) is in class $\mathcal{G}^1$ □

When the noise in the diffusion can be degenerate the situation is less simple and we will need quantitative conditions on the coefficients (see, for instance, [24]).
We will now work under the *joint dissipative condition* (**A**.**7**) that, taking into account differentiability of F and G becomes:

$$2\langle Ay + \nabla_x F(x)y, y\rangle_H + ||Q\nabla_x G(x)y||^2_{L_2(\Xi,H)} \;\le\; -\mu|y|^2_H, \quad \forall y \in D(A),\, \forall x \in H. \tag{6.1}$$

Under the above assumptions the following well known differentiability result for the forward equation (3.1) holds:

Lemma 6.1. *Under* (**A**.**1** − −**A**.**5**), (**A**.**7**) *and* (**C**.**1**) *the map $x \to X^x$ is Gâteaux differentiable. Moreover, for every $h \in H$, the directional derivative process $\nabla_x X^x h$, solves, $\mathbb{P}-$ a.s., the equation*
(6.2)

$$\nabla_x X_t^x h \;=\; e^{tA}h + \int_0^t e^{(t-s)A}\nabla_x F(X_s^x)\nabla_x X_s^x h\, ds + \int_0^t e^{(t-s)A} Q\nabla_x G(X_s^x)\nabla_x X_s^x h\, dW_s, \qquad t \ge 0,$$

Moreover

$$\mathbb{E}|\nabla_x X_t^x h|^2 \le e^{-\mu t}|h|^2. \tag{6.3}$$

Proof. Our hypotheses imply the Hypotheses 3.1 of [10], therefore we can apply [10, Prop 3.3]. The estimate (6.3) follows applying the Itô formula to $|\nabla_x X_t^x h|^2$ and arguing as in Proposition 3.1. □

We will need the following additional assumption to state the last result
(**C**.**2**) G and G^{-1} are of class $\mathcal{G}^1(H, L(\Xi))$ and ψ is of class $\mathcal{G}^1(H \times \Xi^*, \mathbb{R})$
We eventually have:

Theorem 6.1. *Assume that* (**A**.**1** − −**A**.**5**), (**A**.**7**) *and* (**B**.**1**) *hold with $\mu > 2(L_z^2 M_{G^{-1}}^2 + L_u^2)$, moreover we assume* (**C**.**1**) *and* (**C**.**2**). *Then the function $\bar{v}$ defined in* (4.21) *is of class $\mathcal{G}^1(H, \mathbb{R})$.*

Proof. The proof is detailed in the Appendix. □

7. Application to optimal control.

Let Γ be a separable metric space, an admissible control γ is any $\mathcal{F}_t$ - progressively measurable Γ-valued process. The cost corresponding to a given control is defined as follows. Let $R_1 : \Gamma \to \Xi$, $R_2 : \Gamma \to H$ and $L : H \times \Gamma \to \mathbb{R}$ measurable functions such that, for some constant $c > 0$, for all $x, x' \in H$ and $\gamma \in \Gamma$:

(E.1) $\quad |R_1(\gamma)| \leq c, \quad |R_2(\gamma)| \leq c, \quad |L(x,\gamma)| \leq c, \quad |L(x,\gamma) - L(x',\gamma)| \leq c|x-x'|.$

Let for every $x \in H$ be X^x the solution to (3.9), then for every $T > 0$ and every control γ we consider the Girsanov density:

$$\rho_T^\gamma = \exp\left(\int_0^T G^{-1}(X_s^x)R_1(\gamma_s)dW_s^1 + \int_0^T R_2(\gamma_s)\,dW_s^2 - \frac{1}{2}\int_0^T [|G^{-1}(X_s^x)R_1(\gamma_s)|_\Xi^2 + |R_2(\gamma_s)|_H^2]\,ds\right)$$

and we introduce the following ergodic cost corresponding to x and γ:

$$J(x,\gamma) = \limsup_{t\to\infty} \frac{1}{T}\,\mathbb{E}^{\gamma,T}\int_0^T L(X_s^x,\gamma_s)\,ds,$$

where $\mathbb{E}^{\gamma,T}$ is the expectation with respect to $\mathbb{P}^\gamma := \rho_T^\gamma \mathbb{P}$. Notice that with respect to $\mathbb{P}^\gamma$ the processes

$$W_t^{1,\gamma} := -\int_0^t G^{-1}(X_s^x)R_1(\gamma_s)ds + dW_s^1, \quad W_t^{2,\gamma} := -\int_0^t R_2(\gamma_s)ds + dW_s^2,$$

are independent cylindrical Wiener processes and with respect to them X^x verifies:

$$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + QR_1(\gamma_s)ds + DR_2(\gamma_s)ds + QG(X_t^x)dW_t^{1,\gamma} + DdW_t^{2,\gamma}, \quad t \geq 0, \\ X_0^x = x, \end{cases}$$

and this justifies the above (weak) formulation of the control problem.

We introduce the *usual* Hamiltonian:

(7.1) $$\psi(x,z,u) = \inf_{\gamma\in\Gamma}\{L(x,\gamma) + zR_1(\gamma) + uR_2(\gamma)\}, \qquad x \in H, z \in \Xi^*, u \in H^*,$$

that by construction is a concave function and, under **(E.1)**, fullfils assumption **(B.1)**. The forward backward system associated to this problem, is the following:

(7.2) $$\begin{cases} dX_t^x = AX_t^x dt + F(X_t^x)dt + QG(X_t^x)dW_t^1 + DdW_t^2, \qquad t \geq 0, \\ X_0^x = x, \\ -dY_t^x = [\psi(X_t^x, Z_t^x G^{-1}(X_t^x), U_t^x) - \lambda]\,dt - Z_t^x\,dW_t^1 - U_t^x\,dW_t^2. \end{cases}$$

By Theorem 4.2 under $(\mathbf{A.1}--\mathbf{A.6})$ and $(\mathbf{E.1})$ for every $x \in H$ there exists a solution:

(7.3) $$(\bar{Y}^x, \bar{Z}^x, \bar{U}^x, \lambda) = (\bar{v}(X^x), \bar{\zeta}_1(X^x), \bar{\zeta}_2(X^x), \lambda),$$

where $\bar{Y}$ is a progressive measurable continuous process, $\bar{Z} \in L_{\mathcal{P}}^{2,loc}(\Omega\times[0,+\infty[;\Xi^*)$, $\bar{U} \in L_{\mathcal{P}}^{2,loc}(\Omega\times[0,+\infty[;H^*)$, $\lambda \in \mathbb{R}$, $\bar{v}$ is Lipschitz and $\bar{\zeta}_1$, $\bar{\zeta}_2$ are measurable.

Once we have solved the above ergodic BSDE the proof of the following result containing the synthesis of the optimal control for the ergodic cost is identical to the one of [19, Theorem 7.1].

THEOREM 7.1. *Assume* $(\mathbf{A.1}--\mathbf{A.6})$ *and* $(\mathbf{E.1})$. *Then the following holds:*

(i) For arbitrary control γ we have $J(x,\gamma) \geq \lambda$, and equality holds if and only if the following holds $\mathbb{P}$- a.s. for a.e. $t \geq 0$:

$$L(X_t^x,\gamma_t) + \bar{\zeta}_1(X_t^x)G^{-1}(X_t^x)R_1(\gamma_t) + \bar{\zeta}_2(X_t^x)R_2(\gamma_t) = \psi(X_t^x, \bar{\zeta}_1(X_t^x)G^{-1}(X_t^x), \bar{\zeta}_2(X_t^x)).$$

(ii) *If the infimum is attained in (7.1) and $\rho : \Xi^* \times H^* \to \Gamma$ is any measurable function realizing the minimum (that always exists by Filippov selection theorem, see [22]) then the control $\bar{\gamma}_t = \rho(X_t^x, \bar{\zeta}_1(X_t^x), \bar{\zeta}_2(X_t^x))$ is optimal, that is $J(x, \bar{\gamma}) = \lambda$.*

(iii) *$\bar{v}$ admits a generalized directional gradient and $(\bar{v}, \lambda)$ is the mild solution of the equation (5.1), in the sense of definition (5.2) and $\bar{\zeta}_1, \bar{\zeta}_2 \in \tilde{\nabla}^{(QG,D)}$.*

(iv) *Finally if $\bar{v}$ is in class $\mathcal{G}^1$ then $(\bar{v}, \lambda)$ is a mild solution of equation (5.1), in the sense of definition (5.3) and $\bar{\zeta}_1 = \nabla \bar{v} QG$ and $\bar{\zeta}_2 = \nabla \bar{v} D$.*

7.1. Examples.

EXAMPLE 7.1. We consider an ergodic control problem for a stochastic heat equation controlled through the boundary

$$\left\{\begin{array}{ll} d_t x(t,\xi) = \frac{\partial}{\partial \xi^2} x(t,\xi)\,dt + d(\xi)\dot{\mathcal{W}}(t,\xi)\,dt, & t \geq 0,\ \xi \in (0,\pi), \\ x(t,0) = y(t), \qquad x(t,\pi) = 0, & \\ x(0,\xi) = x_0(\xi), & \xi \in (0,\pi) \\ dy(t) = b(y(t))\,dt + \sigma(y(t))\rho(\gamma(t))dt + \sigma(y(t))\,dB_t, & t \geq 0, \\ y(0) = x \in \mathbb{R}. & \end{array}\right. \tag{7.4}$$

where $\mathcal{W}$ is the space-time white noise on $[0, +\infty) \times [0, \pi]$ and B is a Brownian motion. An admissible control γ is a predictable process $\gamma : \Omega \times [0, +\infty) \to \mathbb{R}$. The cost functional is

$$J(x_0, \gamma) = \liminf_{T \to +\infty} \frac{1}{T}\,\mathbb{E} \int_0^T \int_0^\pi \ell(x(t,\xi), \gamma(t))\,d\xi\,dt. \tag{7.5}$$

We assume that

1. $b : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$|b(y) - b(y')| \leq L_b |y - y'|,$$

for a suitable positive constant L_b, for every $y, y \in \mathbb{R}$.

2. $\sigma : \mathbb{R} \to \mathbb{R}$ is a measurable and bounded function, such that

$$|\sigma(y) - \sigma(y')| \leq L_\sigma |y - y|,$$

for suitable positive constants L_σ and there exists a suitable positive δ such that:

$$|\sigma(y))| \geq \delta > 0,$$

for every $y \in \mathbb{R}$.

3. there exists $\mu > 0$ such that for all $y, y' \in \mathbb{R}$:

$$2\langle b(y) - b(y'), y - y' \rangle + |\sigma(y) - \sigma(y')|^2 \ \leq\ -\mu |y - y'|^2, \tag{7.6}$$

4. $d : [0, \pi] \to \mathbb{R}$, $\rho : \mathbb{R} \to \mathbb{R}$ are bounded and measurable functions.
5. $\ell : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a measurable and bounded function such that

$$|\ell(x, \gamma) - \ell(x', \gamma)| \leq L |x - x'|,$$

for a suitable positive constant L, for every $x, x', \gamma \in \mathbb{R}$.

Under these hypotheses, see [18], the above equation can be reformulated in an infinite dimensional space as:

$$
(7.7)\qquad \begin{cases} d_t\mathcal{X}_t = \Delta \mathcal{X}_t\, dt - \Delta \mathfrak{r} y(t) dt + \tilde{D} d\tilde{W}_t\,, & t\geq 0,\ \xi\in[0,\pi],\\ \mathcal{X}_0 = x_0(\cdot), & \xi\in(0,\pi)\\ dy(t) = b(y(t))\, dt + \sigma(y(t))\rho(u(t))dt + \sigma(y(t))\, dB(t), & t\geq 0,\\ y(0)=y_0\in\mathbb{R}. \end{cases}
$$

where $\mathcal{X}_t := x(\cdot)$ is in $L^2(0,\pi)$, $\tilde{W}$ is a cylindrical Wiener process in $L^2(0,\pi)$, $\tilde{D}$ is the bounded operator in $L^2(0,\pi)$ corresponding to multiplication by a bounded function d, Δ is the realisation of the Laplace operator with Dirichlet boundary conditions in $L^2(0,\pi)$, that is (denoting by $\mathcal{D}(\Delta)$ the domain of the operator)

$$
\mathcal{D}(\Delta) = H^2(0,\pi)\cap H^1_0(0,\pi), \qquad \Delta f = \frac{\partial^2 f}{\partial \xi^2},\ \ \forall f \in \mathcal{D}(\Delta)
$$

Finally $\mathfrak{r}(\xi) = 1-\frac{\xi}{\pi}$, $\xi\in[0,\pi]$ is the solution to

$$
(7.8)\qquad \begin{cases} \frac{\partial^2 \mathfrak{r}}{\partial \xi^2}(\xi) = 0, & \xi\in(0,\pi),\\ \mathfrak{r}(0)=1, \qquad \mathfrak{r}(\pi)=0. \end{cases}
$$

It is well known that Δ generates an analytic semigroup of contractions (of negative type -1) moreover, for any $\delta>0$, $\mathfrak{r}\in\mathcal{D}((-\Delta)^{1/2-\delta})$ (where $(-\Delta)^\alpha$ denotes the fractional power). Standard results on analytic semigroups then yield:

$$
(7.9)\qquad |(-\Delta)e^{t\Delta}\mathfrak{r}|_{L^2(0,\pi)} \leq c_{\mathfrak{r}} e^{-t} t^{-(\frac{1}{2}+\delta)}, \qquad t>0.
$$

We are now in a position to rephrase the problem according to our general framework. Indeed setting $H = L^2(0,\pi)\times\mathbb{R}$, $\Xi=\mathbb{R}$ and $X_t = \big(\mathcal{X}_t, y(t)\big)$ equation (7.7) becomes

$$
(7.10)\qquad \begin{cases} dX^x_t = AX^x_t dt + F(X^x_t)dt + QG(X^x_t)\rho(\gamma_t)dt + QG(X^x_t)dW^1_t + DdW^2_t, & t\geq 0,\\ X^x_0 = x. \end{cases}
$$

where:

1. $A = \begin{pmatrix} -\Delta & -\Delta R\\ 0 & 0\end{pmatrix}$ where $R:\mathbb{R}\to D((-\Delta)^{\frac{1}{2}-\delta})$, is defined as $Ry = \mathfrak{r}(\cdot)y$, $y\in\mathbb{R}$
   It is easy to verify that A generates a C_0-semigroup in H.
2. $F: H\to H$, is defined as: $F\begin{pmatrix}\mathcal{X}\\ y\end{pmatrix} = \begin{pmatrix}0\\ b(y)\end{pmatrix}$,
   $Q:\Xi\to H$ is defined as: $Qy = \begin{pmatrix}0\\ y\end{pmatrix}$,
   $G:\Xi\to\Xi$, is defined as: $G(y)=\sigma(y)$
   $D: H\to H$ is defined as: $D\begin{pmatrix}\mathcal{X}\\ y\end{pmatrix} = \begin{pmatrix}\tilde{D}\mathcal{X}\\ 0\end{pmatrix}$.
3. $W^1(t) = B(t)$ and (W^2) is a cylindrical Wiener process in H.

Hypotheses ($\mathbf{A.1}--\mathbf{A.5}$) are immediately verified, we have to check ($\mathbf{A.6}$). We come back to the formulation (7.7) and start with the second component y (that only depends on y_0). By (7.6), Proposition 3.2 gives:

$$
(7.11)\qquad \mathbb{E}|y^{y_0}(t) - y^{y_0'}(t)|^2 \leq e^{-2\mu t}|y_0 - y_0'|^2.
$$

Coming now to the first component we have that it fullfills in $L^2(0,\pi)$ the following mild formulation:

$$\mathcal{X}_t^{x_0,y_0} = e^{t\Delta}x_0 - \int_0^t \left[\Delta e^{(t-s)\Delta}\mathfrak{r}\right] y^{y_0}(s)\,ds + \int_0^t e^{(t-s)\Delta} D\,dW_s.$$

Thus considering two different initial data

$$\mathcal{X}_t^{x_0,y_0} - \mathcal{X}_t^{x_0',y_0'} = e^{t\Delta}(x_0 - x_0') - \int_0^t \Delta e^{(t-s)\Delta}(\mathfrak{r}y^{y_0}(s) - \mathfrak{r}y^{y_0'}(s))\,ds.$$

By (7.9) and (7.11) choosing $\mu_0 \in (0, 1\wedge\mu)$

$$\begin{aligned}\mathbb{E}|\mathcal{X}_t^{x_0,y_0} - \mathcal{X}_t^{x_0',y_0'}| &\le e^{-t}|x_0 - x_0'| + \int_0^t e^{-(t-s)}(t-s)^{-(\frac{1}{2}+\delta)}e^{-\mu s}|y_0 - y_0'|\,ds \\ &\le e^{-t}|x_0 - x_0'| + e^{-\mu_0 t}\left[\int_0^t e^{-(1-\mu_0)(t-s)}(t-s)^{-(\frac{1}{2}+\delta)}\,ds\right]|y_0 - y_0'|.\end{aligned}$$

That implies that (3.5) holds. In the same way one gets the proof of (3.4).

We notice that it is not at all obvious that the stronger versions (3.7), (3.8) holds in this case.

As far as the control functional is concerned it is enough to set $L(X,\gamma) = \int_0^\pi \ell(\xi, \mathcal{X}(\xi), \gamma)d\xi$ and to verify in a straightforward way that **(E.1)** holds (in this case $R_1 = \rho$, $R_2 = 0$, $\Gamma = \mathbb{R}$).

Thus all the hypotheses of Theorem 7.1 hold and points (i) and (ii) in its thesis give the optimal ergodic cost and strategy in terms of the solution to the ergodic BSDE in (7.2). Moreover by point (iii) of Theorem 7.1 we have that $(\bar{v}, \lambda)$ is the mild solution of the equation (5.1), in the sense of definition (5.2) and the optimal feedback law can be characterized in terms of the generalized directional gradient of $\bar{v}$.

Example 7.2. We consider an ergodic control problem for a stochastic heat equation with Dirichlet boundary conditions with nonlinearity controlled through a one dimensional process y.

$$(7.12)\qquad \begin{cases} d_t x(t,\xi) = \frac{\partial}{\partial\xi^2}x(t,\xi)\,dt + f(x(t,\xi), y(t)) + d(\xi)\dot{\mathcal{W}}(t,\xi)\,dt, & t \ge 0,\ \xi \in (0,1), \\ x(t,0) = x(t,1) = 0, & \\ x(0,\xi) = x_0(\xi), & \xi \in (0,1) \\ dy(t) = b(y(t))\,dt + \sigma(y(t))\gamma(t)dt + \sigma(y(t))\,dB_t, & t \ge 0, \\ y(0) = y_0 \in [-1,1]. & \end{cases}$$

where $\mathcal{W}$ is the space-time white noise on $[0,+\infty)\times[0,1]$ and B is a Brownian motion. An admissible control γ is a predictable process $\gamma : \Omega \times [0,+\infty) \to [-1,1]$. The cost functional is

$$(7.13)\qquad J(x_0,\gamma) = \liminf_{T\to+\infty} \frac{1}{T}\,\mathbb{E}\int_0^T \left[\int_0^1 (\ell(x(t,\xi), y(t))d\xi + \gamma^2(t)\right] dt.$$

We assume:

1. $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz map. We fix two constants $L_f > 0$ and $\mu_f \in \mathbb{R}$ such that

$$|f(x,y) - f(x',y)| \le L_f(|x - x'| + |y - y'|), \quad \langle f(x,y) - f(x,y'), x - x'\rangle \quad \le -\mu_f|x - x'|^2,$$

for every $x, x', y, y \in \mathbb{R}$.

2. $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz. We fix a constant $\mu_b \in \mathbb{R}$ such that:

$$\langle b(y) - b(y'), y - y' \rangle \le -\mu_b |y - y'|^2, \qquad \forall y, y' \in \mathbb{R}$$

3. $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz and bounded. We fix L_σ such that

$$|\sigma(y) - \sigma(y')| \le L_\sigma |y - y'|, \qquad \forall y, y' \in \mathbb{R},$$

We also assume that there exists a suitable positive δ such that:

$$|\sigma(y))| \ge \delta > 0, \qquad \forall y \in \mathbb{R}.$$

4. $d : [0,1] \to \mathbb{R}$ is a bounded and measurable function.
5. $\ell : \mathbb{R}^2 \to \mathbb{R}$ is bounded and Lipschitz

As in the previous example the above equation can be reformulated in an infinite dimensional space as:

$$\begin{cases} d_t \mathcal{X}_t = \Delta \mathcal{X}_t \, dt + f(\mathcal{X}_t, y(t))dt + \tilde{D} d\tilde{W}_t \,, & t \ge 0, \ \xi \in [0,1], \\ \mathcal{X}_0 = x_0(\cdot), & \xi \in [0,1] \\ dy(t) = b(y(t)) \, dt + \sigma(y(t))\gamma(t)dt + \sigma(y(t)) \, dB(t), & t \ge 0, \\ y(0) = y_0 \in \mathbb{R}. \end{cases}$$

where $\mathcal{X}_t := x(\cdot)$ is in $L^2(0,1)$, $\tilde{W}$ is a cylindrical Wiener process in $L^2(0,1)$, Δ is the realisation of the Laplace operator with Dirichlet boundary conditions in $L^2(0,1)$, $\tilde{D}$ is the bounded operator in $L^2(0,1)$ corresponding to multiplication by a bounded function d.

Finally setting $H = L^2(0,1) \times \mathbb{R}$, $\Xi = \mathbb{R}$, $\Gamma = [-1,1]$ and $X_t = \big(\mathcal{X}_t, y(t)\big)$ equation (7.4) becomes

$$\text{(7.14)} \qquad \begin{cases} dX_t^x = & AX_t^x dt + F(X_t^x)dt + QG(X_t^x)\gamma_t dt + QG(X_t^x)dW_t^1 + DdW_t^2, \quad t \ge 0, \\ X_0^x = & x. \end{cases}$$

and the cost takes our general form:

$$J(x_0, \gamma) = \liminf_{T \to +\infty} \frac{1}{T} \mathbb{E} \int_0^T L(X(t), \gamma(t)) \, dt.$$

where

1. $A = \begin{pmatrix} -\Delta & 0 \\ 0 & 0 \end{pmatrix}$ generates a C_0-semigroup in H. We also have that

$$\langle AX, X \rangle_H = \langle \Delta \mathcal{X}, \mathcal{X} \rangle_{L^2(0,1)} \le -\mu_\Delta |\mathcal{X}|^2_{L^2(0,1)},$$

for some $\mu_\Delta > 0$.

2. $F : H \to H$, is defined as: $F \begin{pmatrix} \mathcal{X} \\ y \end{pmatrix} = \begin{pmatrix} f(\mathcal{X}, y) \\ b(y) \end{pmatrix}$,

$Q : \Xi \to H$ is defined as: $Qy = \begin{pmatrix} 0 \\ y \end{pmatrix}$,

$G : \Xi \to \Xi$, is defined as: $G(y) = \sigma(y)$,

$D : H \to H$ is defined as: $D \begin{pmatrix} \mathcal{X} \\ y \end{pmatrix} = \begin{pmatrix} \tilde{D}\mathcal{X} \\ 0 \end{pmatrix}$.

3. $W^1(t) = B(t)$ and (W^2) is a cylindrical Wiener process in H.

4. $L : H \times \Gamma \to \mathbb{R}$, $L(X,\gamma) = \int_0^1 \ell(\mathcal{X}(\xi), y) d\xi + |\gamma|^2$.

We also notice that in this case the Hamiltonian defined as in (7.1) becomes:

$$\psi\left(\begin{pmatrix}\mathcal{X}\\ y\end{pmatrix}, z\right) = -\frac{z^2}{4} I_{[-2,2]}(z) + (1-|z|) I_{[-2,2]^c}(z) + \int_0^1 \ell(\mathcal{X}(\xi), y) d\xi. \tag{7.15}$$

We also assume that there exists $\bar{\mu} > 0$ such that

$$\begin{pmatrix} -\mu_\Delta - \mu_f & \frac{1}{2} L_f \\ \frac{1}{2} L_f & -\mu_b + \frac{1}{2} L_\sigma \end{pmatrix} \leq -\bar{\mu}\, I_{\mathbb{R}^2}. \tag{7.16}$$

Hypotheses (**A.1** − −**A.5**) are immediately verified. Moreover relation (7.16) ensures that (**A.7**) holds as well. Finally **(E.1)** is straight forward (in this case $R_1 = id$, $R_2 = 0$). Thus the hypotheses of Theorem 7.1 hold and points (i), (ii) and (iii) in its thesis give the optimal ergodic cost, the strategy in terms of the solution to the ergodic BSDE in (7.2) and we have that $(\bar{v}, \lambda)$ is the mild solution of the equation (5.1), in the sense of definition (5.2) and the optimal feedback law can be characterized in terms of the generalized directional gradient of $\bar{v}$.

We finally wish to apply the differentiability result in Theorem 6.1 to this specific example. We notice that by (7.15) the Hamiltonian ψ is concave and differentiable with respect to z with $\nabla_z \psi \leq 1$. Thus **(B.1)** holds and we can choose $L_z = 1$ in (4.2). If we assume that f b σ and ℓ are of class C^1 in all their variables then **(C.1)** and **(C.2)** hold, moreover if we impose that $\bar{\mu} > 2\delta^{-2}$ (here, comparing with Theorem 6.1, $L_u = 0$, $M_{G^{-1}} = \delta^{-1}$) then all the assumptions of Theorem 6.1 are verified and we can conclude that function $\bar{v}$ in Theorem 7.1 is differentiable. Consequently point (iv) in Theorem 7.1 as well applies here and we obtain that $(\bar{v}, \lambda)$ is a mild solution of equation (5.1), in the sense of definition (5.3), and that the optimal feedback law can be characterized in terms of the gradient of $\bar{v}$.

Appendix A. Proof of Theorem 6.1.

We will need to use some results from [23, Theorem 5.21 and Section 5.6]. The first concerns finite horizon BSDEs and the estimate of their solution, while the second concerns the infinite horizon case. We restate them in our setting as follows:

Lemma A.1. *Let us consider the following equation:*

$$-d\,Y_t = (\phi(t, Z_t, U_t)\, dt - \alpha Y_t)\, dt - Z_t\, dW_t^1 - U_t\, dW_t^2, \qquad Y_T = \eta, \qquad t \in [0,T],\ \alpha \geq 0. \tag{A.1}$$

assume that:

1. $|\phi(t,z,u) - \phi(t,z',u')| \leq \ell(t)(|z-z'|^2 + |u-u'|^2)^{1/2}$, $\forall z, z' \in \Xi^*, u, u' \in H^*$, $\mathbb{P}-a.s.$ *for some* $\ell \in L^2([0,T])$*;*
2. *for* $\nu_t := \int_0^t \ell^2(s)\, ds$*, one has*

$$\mathbb{E}\left(e^{2\nu_T - 2\alpha T}|\eta|^2\right) < \infty, \qquad \mathbb{E}\left(\int_0^T e^{\nu_s - \alpha s}|\phi(s,0,0)|\, ds\right)^2 < \infty. \tag{A.2}$$

Then there exists a unique solution $(Y,Z,U) \in L^2_{\mathcal{P}}(\Omega; C([0,T];\mathbb{R})) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; \Xi^*) \times L^2_{\mathcal{P}}(\Omega \times [0,T]; H^*)$ *and it verifies for all* $0 \leq t \leq T$*:*

$$\mathbb{E}^{\mathcal{F}_t}(\sup_{s\in[t,T]} e^{2(\nu_s - \alpha s)}|Y_s|^2) + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2(\nu_s-\alpha s)}|Z_s|^2\, ds\right) + \mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2(\nu_s-\alpha s)}|U_s|^2\, ds\right) \leq$$

$$\text{(A.3)}\quad \mathbb{E}^{\mathcal{F}_t}\left(e^{2\nu_T-2\alpha T}|\eta|^2\right)+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{V_s-\alpha s}|\phi(s,0,0)|\,ds\right)^2, \qquad \mathbb{P}-a.s., \qquad t\in[0,T].$$

LEMMA A.2. *Let us consider the following equation for $\alpha \geq 0$:*

$$\text{(A.4)}\qquad -d\,Y_t=(\phi(t,Z_t,U_t)\,dt-\alpha Y_t)\,dt-Z_t\,dW_t^1-U_t\,dW_t^2, \qquad t\geq 0,\ .$$

Assume that:

1. $|\phi(t,z,u)-\phi(t,z',u')|\leq \ell(t)(|z-z'|^2+|u-u'|^2)^{1/2}$, $\forall z,z'\in\Xi^*, u,u'\in H^*$, $\mathbb{P}-a.s.$ *for some* $\ell\in L^2_{loc}([0,+\infty[)$*;*
2. *for* $\nu_t:=\int_0^t \ell^2(s)\,ds$*, one has*

$$\text{(A.5)}\qquad \mathbb{E}\left(\int_0^\infty e^{\nu_s}|\phi(s,0,0)|\,ds\right)^2<\infty.$$

Then there exists a unique triple of processes (Y,Z,U) with $Y\in L^{2,loc}_{\mathcal{P}}(\Omega;C([0,+\infty[;\mathbb{R}))$, $Z\in L^{2,loc}_{\mathcal{P}}(\Omega\times[0,+\infty[;\Xi^*)$, $U\in L^{2,loc}_{\mathcal{P}}(\Omega\times[0,+\infty[;H^*)$, such that

$$\text{(A.6)}\qquad \mathbb{E}(\sup_{t\in[0,T]}e^{2\nu_t}|Y_t|^2)<+\infty,\ \forall T\geq 0, \qquad \lim_{T\to\infty}\mathbb{E}(e^{2\nu_T}|Y_T|^2)=0.$$

Moreover

$$\text{(A.7)}\quad \mathbb{E}^{\mathcal{F}_t}(\sup_{s\geq t}e^{2\nu_s}|Y_s|^2)+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{2\nu_s}(|Z_s|^2+|U_s|^2)\,ds\right)\leq C\,\mathbb{E}^{\mathcal{F}_t}\left(\int_t^\infty e^{\nu_s}|\phi(s,0,0)|\,ds\right)^2,$$

for some positive constant C.

Proof of Theorem 6.1. The proof is split into two parts. The first deals with approximating functions v^α defined in (4.10)

Part I - Differentiability of v^α

We first have to come back to the elliptic approximations:

$$\text{(A.8)}\ Y_t^{x,\alpha}=Y_T^{x,\alpha}+\int_t^T[\psi(X_s^x,Z_s^{x,\alpha}G^{-1}(X_s^x),U_s^{x,\alpha})-\alpha Y_s^{x,\alpha}]\,ds-\int_t^T Z_s^{x,\alpha}\,dW_s^1-\int_t^T U_s^{x,\alpha}\,dW_s^2,$$

and for those equations we prove that:

PROPOSITION A.1. *Under the same assumptions of Theorem 6.1 we have that, for each $\alpha>0$, the map $x\to Y_0^{x,\alpha}$ belongs to $\mathcal{G}^1(H,\mathbb{R})$.*

Proof. We fix $n\in\mathbb{N}$ and introduce the following finite horizon approximations where $0\leq t\leq n$:

$$Y_t^{x,\alpha,n}=\int_t^n[\psi(X_s^x,Z_s^{x,\alpha,n}G^{-1}(X_s^x),U_s^{x,\alpha,n})-\alpha Y_s^{x,\alpha,n}]\,ds-\int_t^n Z_s^{x,\alpha,n}\,dW_s^1-\int_t^n U_s^{x,\alpha,n}\,dW_s^2.$$

For such equations [16, Prop. 3.2] holds true, moreover we have from [10, Propositions 5.6 and 5.7] that $x\to Y_0^{x,\alpha,n}:=v^{\alpha,n}(x)$ belongs to $\mathcal{G}^1(H,\mathbb{R})$ and $Z_t^{x,\alpha,n}=\nabla_x v^{\alpha,n}(X_t^x)G(X_t^x)$ and $U_t^{x,\alpha,n}=\nabla_x v^{\alpha,n}(X_t^x)D$.

Hence, arguing as in Proposition 4.2, we deduce that $|Z_t^{\alpha,x,n}|\leq|\nabla_x v^{\alpha,n}(X_t^x)G(X_t^x)|\leq C/\mu$ and $|U_t^{\alpha,x,n}|\leq|\nabla_x v^{\alpha,n}(X_t^x)D|\leq\dfrac{C}{\mu}$, with C independent of n and α.

Moreover, see [10, Prop 5.2], the map $x \to (Y_t^{x,\alpha,n}, Z_t^{x,\alpha,n}, U_t^{x,\alpha,n})$ is Gateaux differentiable and the equation for the derivative in the direction $h \in H$, $|h| = 1$, is the following:

$$\nabla_x Y_t^{x,\alpha,n} h = \int_t^n [\phi^{h,\alpha}(s, \nabla_x Z_s^{x,\alpha,n} h, \nabla_x U_s^{x,\alpha,n} h) - \alpha \nabla_x Y_s^{x,\alpha,n} h]\, ds - \int_t^n \nabla_x Z_s^{x,\alpha,n} h\, dW_s^1$$
$$- \int_t^n \nabla_x U_s^{x,\alpha,n} h\, dW_s^2, \qquad 0 \le t \le n.$$

where

$$\phi^{h,\alpha,n}(s,z,u) = \nabla_x \psi(X_s^x, Z_s^{x,\alpha,n} G^{-1}(X_s^x), U_s^{x,\alpha,n}) \nabla_x X_s^x h + \nabla_u \psi(X_s^x, Z_s^{x,\alpha,n} G^{-1}(X_s^x), U_s^{x,\alpha,n}) uh$$
$$+ \nabla_z \psi(X_s^x, Z_s^{x,\alpha,n} G^{-1}(X_s^x), U_s^{x,\alpha,n}) [Z_s^{x,\alpha,n} \nabla_x G^{-1}(X_s^x) \nabla_x X_s^x h + z\, h G^{-1}(X_s^x)].$$

Notice that $\phi^{h,\alpha}(t,z,u)$ is affine in z and u and :

$$|\phi^{h,\alpha,n}(s,z,u) - \phi^{h,\alpha,n}(s,0,0)| \le L_u |u| + L_z M_{G^{-1}} |z| \le (L_z^2 M_{G^{-1}}^2 + L_u^2)^{1/2} (|z|^2 + |u|^2)^{1/2}, \qquad \mathbb{P}-a.s.$$

where here and in the following the constant C may change from line to line but always independently from n, ε and from α.

We can apply Lemma A.1 with $\nu_s = (L_z^2 M_{G^{-1}}^2 + L_u^2)s =: Ks$, indeed for $\varepsilon = \frac{1}{2}(\mu - 2K)$, we have, recalling also that $U_s^{x,\alpha,n}$ and $Z_s^{x,\alpha,n}$ are bounded uniformly in s, α and n

$$\text{(A.9)} \qquad \mathbb{E}\left[\int_0^n |\phi^{h,\alpha,n}(s,0,0)| e^{(-\alpha+K)s}\, dt\right]^2 \le \frac{C}{\varepsilon} \int_0^n e^{(\varepsilon - 2\alpha + 2K)s} \mathbb{E} |\nabla_x X_s^x h|^2\, dt \le \frac{C}{\mu - 2K}.$$

Therefore the following estimate holds, arguing as before in (A.9), for all $0 \le t \le n$:

$$\text{(A.10)} \quad \mathbb{E} \sup_{s \in [t,n]} e^{2(-\alpha+K)s} |\nabla_x Y_s^{x,\alpha,n} h|^2 + \mathbb{E} \int_t^n e^{2(-\alpha+K)s} \left[|\nabla_x Z_s^{x,\alpha,n} h|^2 + |\nabla_x U_s^{x,\alpha,n} h|^2\right] dt$$
$$\le C\, \mathbb{E}\left[\int_t^n e^{(-\alpha+K)s} |\phi^{h,\alpha,n}(s,0,0)|\, ds\right]^2 \le \frac{C e^{(-2\alpha - \frac{1}{2}\mu + K)t}}{\mu - 2K}, \quad t \le s \le n.$$

In particular, we have for all $t \ge 0$:

$$\text{(A.11)} \qquad \mathbb{E}\Big(e^{2Kt} |\nabla_x Y_t^{x,\alpha,n} h|^2 \Big) \le C\, e^{(-\frac{1}{2}\mu + K)\, t}.$$

From estimate (A.10) we deduce that $(\nabla_x Y^{x,\alpha,n} h, \nabla_x Z^{x,\alpha,n} h, \nabla_x U^{x,\alpha,n} h)$ weakly converges in the Hilbert space $L^2(\Omega \times (0,T); \mathbb{R} \times \Xi^* \times H^*)$ to some $(R^{x,\alpha,h}, V^{x,\alpha,h}, M^{x,\alpha,h})$, for every $T > 0$. From (A.11) we also have that $\nabla_x Y_0^{x,\alpha,n} h$ converge in $\mathbb{R}$ to $\xi^{x,\alpha,h}$.

We define for every $t \ge 0$

$$\tilde{R}_t^{x,\alpha,h} = \xi^{x,\alpha,h} + \int_0^t \left[\phi^{h,\alpha}(s, V_s^{x,\alpha,h}, M_s^{x,\alpha,h}) - \alpha R_s^{x,\alpha,h}\right] ds - \int_0^t V_s^{x,\alpha,h} dW_s^1 - \int_0^t M_s^{x,\alpha,h} dW_s^2.$$

Compare the above with the forward equation fulfilled by $(\nabla_x Y^{x,\alpha,n} h, \nabla_x Z^{x,\alpha,n} h, \nabla_x U^{x,\alpha,n} h)$, namely:

$$\nabla_x Y_t^{x,\alpha,n} h = \nabla_x Y_0^{x,\alpha,n} h + \int_0^t \left[\phi^{h,\alpha,n}(s, \nabla_x Z_s^{x,\alpha,n}, \nabla_x U_s^{x,\alpha,n}) - \alpha \nabla_x Y_s^{x,\alpha,n} h\right] ds$$

$$- \int_0^t \nabla_x Z_s^{x,\alpha,n} h \, dW_s^1 - \int_0^t \nabla_x U_s^{x,\alpha,n} h \, dW_s^2, \qquad \mathbb{P} - a.s..$$

Since every term in the R.H.S., passing to a subsequence if necessary, weakly converges in $L^2(\Omega \times (0,T); \mathbb{R})$, see also [16, Theo. 3.1], we have that $\tilde{R}_t^{x,\alpha,h} = R_t^{x,\alpha,h}$, $\mathbb{P}-$a.s. for a.e. $t \geq 0$. Thus the triplet processes $(\tilde{R}^{x,\alpha,h}, V^{x,\alpha,h}, M^{x,\alpha,h})$ verifies for all $t > 0$, $\mathbb{P}$-a.s.:

$$\tilde{R}_t^{x,\alpha,h} = \tilde{R}_0^{x,\alpha,h} + \int_0^t \left[\phi^{h,\alpha}(s, V_s^{x,\alpha,h}, M_s^{x,\alpha,h}) - \alpha \tilde{R}_s^{x,\alpha,h} \right] ds - \int_0^t V_s^{x,\alpha,h} \, dW_s^1 - \int_0^t M_s^{x,\alpha,h} \, dW_s^2.$$

where

$$\phi^{h,\alpha}(s,z,u) = \nabla_x \psi(X_s^x, Z_s^{x,\alpha} G^{-1}(X_s^x), U_s^{x,\alpha}) \nabla_x X_s^x h + \nabla_u \psi(X_s^x, Z_s^{x,\alpha} G^{-1}(X_s^x), U_s^{x,\alpha}) uh$$
$$+ \nabla_z \psi(X_s^x, Z_s^{x,\alpha} G^{-1}(X_s^x), U_s^{x,\alpha}) [Z_s^{x,\alpha} \nabla_x G^{-1}(X_s^x) \nabla_x X_s^x h + z \, h G^{-1}(X_s^x)].$$

Moreover, thanks to (A.10) and (A.11) we have that

$$\text{(A.12)} \qquad \mathbb{E} \sup_{s \in [0,T]} e^{2Ks} |\tilde{R}_s^{x,\alpha,h}|^2 < +\infty \quad \text{and} \quad \mathbb{E} \, e^{2Ks} |\tilde{R}_s^{x,\alpha,h}|^2 \leq \tilde{C} \, e^{(-\mu + 2K) \, s},$$

therefore, $(\tilde{R}^{x,\alpha,h}, V^{x,\alpha,h}, M^{x,\alpha,h})$ is the unique solution of equation:

$$\text{(A.13)} \qquad d_s R_s = [\phi^{h,\alpha}(s, V_s, M_s) - \alpha R_s] ds - V_s dW_s^1 - M_s dW_s^2,$$

in the class of processes with the regularity imposed in Lemma A.2 veryfying:

$$\text{(A.14)} \qquad \mathbb{E} \sup_{t \in [0,T]} |\tilde{R}_t^{x,\alpha,h}|^2 < +\infty \quad \text{and} \quad \lim_{T \to +\infty} \mathbb{E} \, e^{2K2T} |\tilde{R}_T^{x,\alpha,h}|^2 = 0, \qquad \forall T > 0.$$

We then closely follow the proof of [16, Prop 3.2], indeed we get that $\lim_{n \to +\infty} \nabla_x Y_0^{\alpha,n,x} h = \tilde{R}^{\alpha,x,h}(0)$, defines a linear and bounded operator $\tilde{R}^{\alpha,x}(0)$ from H to H, by (A.11), such that $\tilde{R}^{\alpha,x}(0)h = \tilde{R}^{x,\alpha,h}(0)$, moreover for every fixed $h \in H$, $x \to \tilde{R}^{\alpha,x}(0)h$ is continuous in x, we will sketch the argument by the the end of the proof in a similar point. Therefore, by dominated convergence, we get that:

$$\text{(A.15)} \quad \lim_{\ell \downarrow 0} \frac{Y_0^{x+\ell h,\alpha} - Y_0^{x,\alpha}}{\ell} = \lim_{\ell \downarrow 0} \lim_{n \to \infty} \frac{Y_0^{x+\ell h,\alpha,n} - Y_0^{x,\alpha,n}}{\ell} = \lim_{\ell \downarrow 0} \lim_{n \to \infty} \int_0^1 \nabla_x Y_0^{x+\theta \ell h,\alpha,n} h \, d\theta$$
$$= \lim_{\ell \downarrow 0} \int_0^1 \tilde{R}^{x+\theta \ell h,\alpha}(0) h \, d\theta = \tilde{R}^{x,\alpha}(0)h.$$

Thus v^α is differentiable and since $Y_t^{x,\alpha} = v^\alpha(X_t^x)$ we have $\nabla_x Y_t^{x,\alpha} h = v^\alpha(X_t^x) \nabla_x X_t^x h$.

Fixing $T > 0$ we can see the equation satisfied by $(Y^{x,\alpha}, Z^{x,\alpha}, U^{x,\alpha})$ as a BSDE on $[0,T]$ with final condition $v^\alpha(X_T^x)$ and we can apply standard results on the differentiability of markovian, finite horizon BSDEs (see, for instance, [10]) to deduce that the map $x \to Y^{x,\alpha}$ is of class $\mathcal{G}^1$ from H to $L^2_{\mathcal{P}}(\Omega, ; C([0,T]; \mathbb{R}))$ and $x \to Z^{x,\alpha}$ is of class $\mathcal{G}^1$ from $L^2_{\mathcal{P}}([0,T] \times \Omega; \Xi^*)$. Moreover for every $h \in H$, for every $0 \leq t \leq T$ it holds that:

$$\nabla_x Y_t^{x,\alpha} h = \nabla_x Y_T^{x,\alpha} h + \int_t^T [\phi^h(s, \nabla_x Z_s^{x,\alpha} h, \nabla_x U_s^{x,\alpha} h) - \alpha \nabla_x Y_s^{x,\alpha} h] \, ds$$
$$\text{(A.16)} \qquad - \int_t^T \nabla_x Z_s^{x,\alpha} h \, dW_s^1 - \int_t^T \nabla_x U_s^{x,\alpha} h \, dW_s^2, \qquad 0 \leq t \leq n.$$

Comparing the above with (A.13) and noticing that for all $T > 0$:

$$\mathbb{E}e^{2KT}|\nabla_x Y_T^{x,\alpha}h|^2 = \mathbb{E}e^{2KT}|\nabla_x v^\alpha(X_T^x)\nabla_x X_T^x h|^2 \leq Ce^{(2K-\mu)T},$$

the uniqueness part of Lemma A.2 tells us that $(\nabla_x Y_\cdot^{x,\alpha}h, \nabla_x Z_\cdot^{x,\alpha}h, \nabla_x U_\cdot^{x,\alpha}h)$ coincides with $(\tilde{R}^{x,h,\alpha}, V^{x,h,\alpha}, M^{x,h,\alpha})$ and is the unique solution of equation (A.13) in the sense of Lemma A.2.

Part II - Differentiability of $\bar{v}$

We also introduce the following infinite horizon BSDE:

(A.17) $$-d\,R_s^{x,h} = \phi^h(s, V_s^{x,h}, M_s^{x,h})ds - V_t^{x,h}\,dW_t^1 - M_t^{x,h}\,dW_t^2 \qquad t \geq 0.$$

with

$$\begin{aligned}\phi^h(s,z,u) =& [\nabla_x\psi(X_s^x, \bar{Z}_s^x G^{-1}(X_s^x), \bar{U}_s^x) + \nabla_z\psi(X_s^x, \bar{Z}_s^x G^{-1}(X_s^x), \bar{U}_s^x)\bar{Z}_s^x\nabla_x G^{-1}(X_s^x)]\nabla_x X_s^x h \\ &+ \nabla_u\psi(X_s^x, \bar{Z}_s^x G^{-1}(X_s^x), \bar{U}_s^x)u + \nabla_z\psi(X_s^x, \bar{Z}_s^x G^{-1}(X_s^x), \bar{U}_s^x)z.\end{aligned}$$

By Lemma A.2 has a unique solution in the class of processes $R^{x,h} \in L_{\mathcal{P}}^{2,loc}(\Omega; C([0,+\infty[;\mathbb{R}))$, $V^{x,h} \in L_{\mathcal{P}}^{2,loc}(\Omega\times[0,+\infty[;\Xi^*)$, $M \in L_{\mathcal{P}}^{2,loc}(\Omega\times[0,+\infty[;H^*)$ verifying:

(A.18) $$\lim_{T\to+\infty} e^{2KT}\mathbb{E}\,|R_T^{x,h}|^2 = 0, \qquad \forall T > 0.$$

As in [19, Theorem 5.1] we claim that, along the sequence (α_m) introduced in (4.21), it holds:

(A.19) $$\nabla_x v^{\alpha_m}(x)h = \nabla_x Y_0^{\alpha_m,x}h = R_0^{x,\alpha_m,h} \to R_0^{x,h},$$

as $m \to \infty$.

Let us introduce again some parabolic approximations. For $s \in [0,n]$ consider:

$$\begin{cases} -d\,R_s^{x,\alpha,n,h} = & \phi^{h,\alpha}(s, V_s^{x,\alpha,n,h}, M^{x,\alpha,n,h})ds - \alpha R_s^{x,\alpha,n,h}\,ds - V_s^{x,\alpha,n,h}\,dW_s^1 - M_s^{x,\alpha,n,h}\,dW_s^2, \\ R_n^{x,\alpha,n,h} = & 0. \end{cases}$$

and

$$\begin{cases} -d\,R_s^{x,n,h} &= \phi^h(s, V_s^{x,h,n}, M^{x,n,h})ds - V_s^{x,h,n}\,dW_s^1 - M_s^{x,h}\,dW_s^2, \\ R_n^{x,h,n} &= 0, \end{cases}$$

Since along the sequence (α_m) selected in Section 4 we have

$$\mathbb{E}\sup_{s\in[0,n]}|\bar{Y}_s^x - Y_s^{x,\alpha_m}|^2 + \mathbb{E}\int_0^n \left[|\bar{Z}_s - Z_s^{x,\alpha_m}|^2 + |\bar{U}_s^x - U_s^{x,\alpha_m}|^2\right]ds \to 0.$$

and consequently

$$\mathbb{E}\int_0^n |\phi^{h,\alpha_m}(s,0,0) - \phi^h(s,0,0)|^2 ds \to 0 \quad \text{as } m\to\infty.$$

standard estimates on finite horizon BSDEs give:

(A.20) $$\mathbb{E}\sup_{s\in[0,n]}|R_s^{x,n,h} - R_s^{x,\alpha_m,n,h}|^2 \to 0, \quad \text{as } m\to\infty.$$

Moreover if we compare with the solution $(\tilde R^{x,\alpha,h}, V^{x,\alpha,h}, M^{x,\alpha,h})$ of equation (A.13)

(A.21)
$$\begin{cases} -d\,(R_s^{x,\alpha,n,h} - \tilde R_s^{x,\alpha,h}) = [\phi^{h,\alpha}(s, V_s^{x,\alpha,n,h} - V_s^{x,\alpha,h}, M_s^{x,\alpha,n,h} - M_s^{x,\alpha,h}) - \alpha(R_s^{x,\alpha,n,h} - \tilde R_s^{x,\alpha,h})]\,ds \\ \qquad -[V_s^{x,\alpha,n,h} - V_s^{x,\alpha,h}]\,dW_s^1 - [M_s^{x,\alpha,n,h} - M_s^{x,\alpha,h}]\,dW_s^2, \\ R_n^{x,\alpha,n,h} - \tilde R_n^{x,\alpha,h} = -\nabla_x v^\alpha(X_n^x)\nabla_x X_n^x h \end{cases}$$

Thus Lemma A.1 estimate (A.3) yields:

$$|R_0^{x,\alpha,n,h} - \tilde R_0^{x,\alpha,h}|^2 \le \mathbb{E}\left(e^{2kn}|\nabla_x v^\alpha(X_n^x)\nabla_x X_n^x h|^2\right) \le Ce^{(2K-\mu)n} \to 0, \quad \text{as } n \to +\infty. \tag{A.22}$$

Notice that the right hand side does not depend on α . Finally

$$\begin{cases} -d\,(R_s^{x,n,h} - R_s^{x,h}) = \phi^h(s, V_s^{x,n,h} - V_s^{x,h}, M_s^{x,n,h} - M_s^{x,h})ds \\ \qquad -[V_s^{x,n,h} - V_s^{x,h}]\,dW_s^1 - [M_s^{x,n,h} - M_s^{x,h}]\,dW_s^2, \\ R_n^{x,n,h} - R_n^{x,h} = -\tilde R_n^{x,h}, \end{cases} \tag{A.23}$$

and taking into account (A.18), one has, again by Lemma A.1 relation (A.3):

$$|R_0^{x,n,h} - R_0^{x,h}|^2 \le \mathbb{E}\left(e^{2Kn}|R_n^{x,h}|^2\right) \le Ce^{(2K-\mu)n} \to 0, \qquad \text{as } N \to +\infty. \tag{A.24}$$

Therefore summing up (A.22), (A.24) and (A.20) we have that:

$$R_0^{x,\alpha_m,h} \to R_0^{x,h}, \qquad \text{as } m \to +\infty.$$

Finally the continuity with respect to x of $R_0^{x,h}$ descends immediately from (A.24) and from the continuity of the map $x \to R_0^{x,n,h}$ proved in [10, Prop. 4.3].

We can now conclude as above (and as in [16, Prop 3.2]); $R^{x,h}(0)$, defines a linear and bounded operator $R^x(0)$ from H to H, such that $R^x(0)h = R^{x,h}(0)$, and we have:

$$\lim_{t\downarrow 0} \frac{\bar v(x+th) - \bar v(x)}{t} = \lim_{t\downarrow 0} \frac{\bar Y_0^{x+th} - \bar Y_0^x}{t} = \lim_{t\downarrow 0}\lim_{m\to 0} \frac{Y_0^{x+th,\alpha_m} - Y_0^{x,\alpha}}{t} =$$
$$= \lim_{t\downarrow 0}\lim_{m\to 0} \int_0^1 \nabla_x Y_0^{x+\theta th,\alpha_m} h\, d\theta = \lim_{t\downarrow 0}\lim_{m\to 0} \int_0^1 R^{x+\theta th,\alpha_m,h}(0)h\, d\theta =$$
$$= \lim_{t\downarrow 0} \int_0^1 R^{x+\theta th}(0)h\, d\theta = R^x(0)h.$$

□

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