ON TWO PHASE FREE BOUNDARY PROBLEMS GOVERNED BY ELLIPTIC EQUATIONS WITH DISTRIBUTED SOURCES

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Abstract. We present some recent progress on the analysis of two-phase free boundary problems governed by elliptic operators, with non-zero right hand side. We also discuss on several open questions, object of future investigations.

1. **Introductory examples.** In the last few years, a significant progress has been achieved in the analysis of free boundary problems (f.b.p.) governed by elliptic equations with forcing terms, in particular on the regularity issues. In this brief survey we describe the new results, ideas and techniques introduced in the paper [12] by De Silva and subsequently refined in [16], [13], [14] to cover a broad spectrum of applications. In absence of distributed sources, this theory has been developpped by a number of authors along the ideas of Caffarelli in the seminal papers [5, 6], with a substantially different approach that we shall briefly recall and comment later on.

Before introducing the precise setting we are going to work in, we exhibit a few motivating examples.

The first one comes from classical hydrodynamics and concerns travelling gravity waves on the surface of an ideal fluid (that is with constant density, inviscid, no surface tension). The motion is assumed to be two dimensional in a vertical (x, y) −plane. In a reference frame moving with the waves, the motion is steady and the fluid occupies a fixed region Ω which lies above a flat bottom $y = 0$ and below an unknown free surface F.

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If the flow is incompressible with constant speed, one is reduced to seek for a stream function ψ that solves the following one phase f.b. problem

$$\begin{cases} \Delta\psi = -\gamma(\psi) & \text{in } \Omega \\ 0 < \quad (x,y) < B & \text{in } \Omega \\ = B & \text{on } \{y=0\} \\ \psi = 0, & \text{on } F(\psi) \equiv \{\psi = 0\} \\ |\nabla\psi|^2 = -2gy + q & \text{on } F(\psi)\,. \end{cases} \tag{1}$$

The function $\gamma : [0, B] \to \mathbb{R}$ is the so called *vorticity function* and the first equation expresses a functional dependence between ψ and the vorticity $\omega = \Delta\psi$.

The last equation is the Bernoulli law at the free surface $F(\psi)$; B and q are given constants. Of particular interest are the periodic waves in which $F(\psi)$ is periodic along the horizontal direction.

A classical conjecture of Stokes in the irrotational case $\gamma = 0$ refers to the existence of the so called *extreme waves,* exhibiting at sharp crests a corner with included angle of $120°$. At these points, called *stagnation points*, the velocity of the fluid relative to the reference frame must be zero, that is $q - 2gy = 0$. Also, between sharp crests the wave profile is conjectured to be strictly convex (hence locally Lipschitz). The Stokes conjecture has been proved in [2] and in [22].

The case $\gamma \neq 0$, γ smooth, has been recently investigated by [24] and [25]: a symmetric wave, locally monotone at either sides of a stagnation point, at these points has either a $120°$ corner or an horizontal tangent.

Among the questions left open in the above papers there is the regularity of the wave profile away from stagnation points. From [11] we know that in these regions $F(\psi)$ is locally Lipschitz. In [12] is proved that indeed $F(\psi)$ is smooth.

Back in 1956 Batchelor proposed a model for large Reynolds number limits of the steady Navier-Stokes equations that leads to the following free boundary problem for a two dimensional flow with closed streamlines.

Let Ω be a bounded domain in the plane, whose boundary is a simple closed curve Γ. Let γ be another simple closed curve contained in Ω, a priori unknown (the free boundary). Call Ω_1 the annular domain bounded by Γ and γ and Ω_2 the domain bounded by γ.

Given two constants $\omega > 0$ and $\mu < 0$, we seek for two functions ψ_1, ψ_2 solutions of the following so called Prandtl-Batchelor system:

$$\begin{cases} \Delta\psi_1 = 0 \text{ in } \Omega_1 & \psi_1 = 0 \text{ on } \gamma,\ \psi_1 = \mu \text{ on } \Gamma \\ \Delta\psi_2 = \omega \text{ in } \Omega_2 & \psi_2 = 0 \text{ on } \gamma. \end{cases}$$

Moreover, the following jump condition on the tangential velocities holds on γ :

$$|\nabla\psi_1|^2 - |\nabla\psi_2|^2 = \sigma$$

for some constant $\sigma \geq 0$.

Thus, ψ_1 is a stream function for an irrotational flow in Ω_2, and ψ_2 is a stream function for a $\omega-$vorticity flow in Ω_2.

The Prandtl-Batchelor problem is somehow connected with the minimization problem, arising in flow of two fluids in jets and cavities[1],

$$J(u)=\int_{\Omega}\left\{\sum_{i,j=1}^{n}a_{ij}(x)\,u_{x_i}u_{x_j}-f(x)\,u+q(x)\,\lambda(u)\right\}\to\min$$

over $u\in g+H_0^1(\Omega)$, with $q\geq c>0$ a.e. and

$$\lambda(u)=\begin{cases}\lambda_1 & u<0\\ \lambda_2 & u>0\end{cases}\qquad \lambda_1,\lambda_2\geq\Lambda\equiv\lambda_1-\lambda_2>0,0\leq\lambda(0)\leq\lambda_2.$$

Here the free boundary condition takes the form

$$\left(u_{\nu^*}^+\right)^2-\left(u_{\nu^*}^-\right)^2=\Lambda q(x)$$

where $u_{\nu^*}^+$ and $u_{\nu^*}^-$ denote the conormal derivatives of u, in the inward directions to the positive and negative phase, respectively.

When $n=2$, $a_{ij}=\delta_{ij}$ and $\lambda(u)=-(2\omega u+\sigma)\,\chi_{\{u<0\}}$, J can be written in the form

$$J(u)=\int_{\Omega}|\nabla u|^2-\omega\int_{\{u<0\}}u+\sigma\,|\{u<0\}|$$

and in [17] it is shown that a minimizer of J solves *formally* the Prandtl-Batchelor system, but the author could not derive the free boundary condition.

Thus, there is no satisfactory theory for this problem. Viscosity solutions are Lipschitz across γ as shown in [8], but neither existence nor regularity is known (uniqueness fails already in the radial case, where two explicit solution can be found).

Here we shall prove that flat or Lipschitz free boundaries are smooth (see [13]).

Other examples come from limits of singular perturbation problems with forcing term as in [20], where the authors analyze solutions arising in the study of flame propagation with nonlocal effects.

2. **Main definitions and results.** All the model examples in Section 1 fit in the following general framework. In a bounded domain $\Omega\subset\mathbb{R}^n$, consider the problem

$$\begin{cases}\mathcal{L}u=f & \text{in } \Omega^+(u)\cup\Omega^-(u)\\ \\ u_\nu^+=G(u_\nu^-,x) & \text{on } F(u)=\partial\Omega^+(u)\cap\Omega,\end{cases}\tag{2}$$

where

$$\Omega^+(u)=\{x\in\Omega:u(x)>0\},\quad \Omega^-(u)=\{x\in\Omega:u(x)\leq 0\}^\circ.$$

Here f is bounded on Ω and continuous in $\Omega^+(u)\cup\Omega^-(u)$, while u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively. $F(u)$ is called the *free boundary.*

$\mathcal{L}$ is a uniformly elliptic operator with ellipticity constants $\lambda,\Lambda>0$, in nondivergence form

$$\mathcal{L}u=\mathrm{Tr}\left(A(x)\,D^2u\right)+b(x)\cdot\nabla u\tag{3}$$

with Hölder continuous coefficients, or a fully nonlinear operator

$$\mathcal{L}u=\mathcal{F}\left(D^2u\right)\qquad(\mathcal{F}(O)=0)\tag{4}$$

where D^2u is the Hessian matrix of u.

[1]See [ACF] for the case $f=0,a_{ij}=\delta_{ij}$.

Expressely note that, in the fully nonlinear case, we assume for $\mathcal{F}$ neither concavity nor homogeneity of degree one. Also observe that if $\mathcal{F}$ is an operator in our class then for every $r > 0$

$$\mathcal{F}_r(M) = \frac{1}{r}\mathcal{F}(rM)$$

is still an operator in the same class.

The function

$$G(\eta, x) : [0, \infty) \times \Omega \to (0, \infty)$$

satisfies the following assumptions:

(H1) $G(\eta, \cdot) \in C^{0,\bar{\gamma}}(\Omega)$ uniformly in η; $\quad G(\cdot, x) \in C^{1,\bar{\gamma}}([0, L])$ for every $x \in \Omega$.

(H2) $G_\eta(\cdot, x) > 0$ with $G(0, x) \geq \gamma_0 > 0$ uniformly in x.

(H3) There exists $N > 0$ such that $\eta^{-N} G(\eta, x)$ is strictly decreasing in η, uniformly in x.

Classical comparison sub/super solutions are defined as follows.

Definition 2.1. We say that $v \in C(\Omega)$ is a C^2 strict (comparison) subsolution (resp. supersolution) to our f.b.p. in Ω, if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

1. $\mathcal{L}v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;
2. If $x_0 \in F(v)$, then

$$v_\nu^+(x_0) > G(v_\nu^-(x_0), x_0) \quad (\text{resp. } v_\nu^+(x_0) < G(v_\nu^-(x_0), x_0),\ v_\nu^+(x_0) \neq 0.)$$

Observe that the free boundary of a strict comparison sub/supersolution is C^2.

Viscosity sub/super solutions are defined in the usual way. Given $u, \varphi \in C(\Omega)$, we say that φ *touches u by below* (resp. *above*) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly by below (resp. above).

Definition 2.2. Let u be a continuous function in Ω. We say that u is a viscosity solution to our f.b.p. in Ω, if the following conditions are satisfied:

1. $\mathcal{L}u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense;
2. Let $x_0 \in F(u)$ and $v \in C^2(\overline{B^+(v)}) \cap C^2(\overline{B^-(v)})$ ($B = B_\delta(x_0)$) with $F(v) \in C^2$. If v touches u by below (resp.above) at $x_0 \in F(v)$, then

$$v_\nu^+(x_0)) \leq G(v_\nu^-(x_0)) \quad (\text{resp. } \geq).$$

When $f = 0$ and $\mathcal{L}$ can be put into divergence form or $\mathcal{L}u = \mathcal{F}\left(D^2 u\right)$ is concave, homogeneous of degree one, the existence of viscosity solutions has been settled by Caffarelli in [7] and by Wang in [26], respectively. In particular, the positivity set of u has finite perimeter and, with respect to $(n-1)-$dimensional Hausdorff measure H^{n-1}, a.e. points on $F(u)$ have a normal in the measure theoretical sense.

We shall assume optimal (Lipschitz) regularity of our solution. Indeed, in our generality, the existence of Lipschitz viscosity solutions with proper measure theoretical properties of the free boundary is an open problem and it will be object of future investigations.

When $\mathcal{L} = \Delta$, under the assumption $G(\eta, x) \to \infty,$ as $\eta \to \infty$, the Lipschitz continuity of the solution in the nonhomogeneous case has been proven in [8], Theorem 4.5, as a consequence of the following monotonicity formula:

Theorem 2.3. *Let u, v be nonnegative, continuous functions in B_1, with*

$$\Delta u \geq -1, \Delta v \geq -1 \text{ in the sense of distributions}$$

and $u(0) = v(0) = 0$, $u(x) v(x) = 0$ in B_1. Then, for $r \leq 1/2$,

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla v|^2}{|x|^{n-2}} \leq c(n) \left(1 + \|u\|^2_{L^2(B_1)}\right) \left(1 + \|v\|^2_{L^2(B_1)}\right). \quad (5)$$

Observe that if the supports of u and v were separated by a smooth surface with normal ν at $x = 0$ then, by taking the limit as $r \to 0$, we could deduce that

$$(u_\nu(0))^2 (v_\nu(0))^2 \leq \Phi(1/2).$$

Hence $\Phi(r)$ "morally" gives a control in average of the product of the normal derivatives of u at the origin.

If $\mathcal{L}$ is linear and can be written in divergence form an estimate like (5) is available (see [21]) and one can reproduce the proof of Theorem 4.5 in [8], to recover the Lipschitz continuity of a viscosity solution. Observe that then $f = f(x, u, \nabla u)$ is allowed, with $f(x, \cdot, \cdot)$ locally bounded.

As we have said, we are mainly interested in the regularity properties of the free boundary, in particular in proving that *flat* or *Lipschitz* free boundaries are smooth ($C^{1,\gamma}$).

A way to express the flatness of the free boundary is to assume that $F(u)$ or the zero set of u^+ is trapped between two parallel hyperplanes at $\delta-$distance from each other, for a small δ ($\delta-$flatness). While this looks like a somewhat strong assumption, it is indeed a natural one since it is satisfied for example by rescaling a solution around a point of the free boundary where there is a normal in some weak sense (*regular points*), for instance in the measure theoretical one. We have seen that in the homogeneous case $H^{n-1}-$a.e. points on $F(u)$ are of this kind, when $\mathcal{L}$ is a divergence form operator or $\mathcal{L}u = \mathcal{F}\left(D^2 u\right)$. Moreover, starting form a Lipschitz free boundary, $H^{n-1}-$a.e. points on $F(u)$ are regular, by Rademacher Theorem.

The following results are proved in [13], [14]. A constant depending only on (some of) the parameters n, $\mathrm{Lip}(u), \lambda, \Lambda, [a_{ij}]_{C^{0,\bar{\gamma}}}, \|\mathbf{b}\|_{L^\infty}, \|f\|_{L^\infty}, [G(\eta, \cdot)]_{C^{0,\bar{\gamma}}}$, γ_0 and N is called universal. The $C^{1,\bar{\gamma}}$ norm of $G(\cdot, x)$ may depend on x and enters in a qualitative way only. We will always assume that

$$0 \in F(u).$$

Theorem 2.4 (Flatness implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to (2) in B_1, with $Lip(u) \leq L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \leq L$ and G satisfies ($H1$)-($H3$).*

There exists a universal constant $\bar{\delta} > 0$ such that, if

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \qquad (\delta - \textit{flatness}) \quad (6)$$

with $0 \leq \delta \leq \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

When $\mathcal{L}$ is linear or if $\mathcal{F}$ (positively) homogeneous of degree one (or when $\mathcal{F}_r(M)$ has a limit $\mathcal{F}^*(M)$, as $r \to 0$, which is always homogeneous of degree one) we also have:

Theorem 2.5. *(Lipschitz implies $C^{1,\gamma}$) Let u be a Lipschitz viscosity solution to (2) in B_1, with $Lip(u) \leq L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \leq L$ and G satisfies ($H1$)-($H3$). If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

Here some of the ideas are presented for the model case $G(\beta, x) = \sqrt{1+\beta^2}$.

Theorem 2.5 follows from Theorem 2.4 and the main result in [5] or [18], via a blow-up argument.

The flatness conditions present in the literature (see, for instance [6]), are often stated in terms of "ε- monotonicity" along a large cone of directions $\Gamma(\theta_0, e)$ of axis e and opening θ_0. Precisely, a function u is said to be ε-monotone ($\varepsilon > 0$ small) along the direction τ in the cone $\Gamma(\theta_0, e)$ if for every $\varepsilon' \geq \varepsilon$,

$$u(x + \varepsilon'\tau) \geq u(x).$$

A variant of Theorem 2.4 states the following (for simplicity we take $\mathcal{L} = \Delta$).

Theorem 2.6. *([13]) Let u be a solution to our f.b.p in B_1, $0 \in F(u)$. Suppose that u^+ is non-degenerate. Then there exist $\theta_0 < \pi/2$ and $\varepsilon_0 > 0$ such that if u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then u^+ is fully monotone in $B_{1/2}$ along any direction in $\Gamma(\theta_1, e_n)$ for some θ_1 depending on θ_0, ε_0. In particular $F(u)$ is Lipschitz and therefore $C^{1,\gamma}$.*

Geometrically, the ε-monotonicity of u^+ can be interpreted as ε-closeness of $F(u)$ to the graph of a Lipschitz function. Our flatness assumption requires ε-closeness of $F(u)$ to a hyperplane. If $\|f\|_\infty$ is small enough, depending on ε, it is not hard to check that ε-flatness of $F(u)$ implies $c\varepsilon$-monotonicity of u^+ along the directions of a flat cone, for a c depending on its opening.

The proof of Theorem 2.6 follows immediately from the following elementary lemma:

Lemma 2.7. *Let u be a solution to to our f.b.p in B_1, $0 \in F(u)$. Suppose that u^+ is non-degenerate. Assume that u^+ is ε-monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then there exist a radius $r_0 > 0$ and $\delta_0 > 0$ depending on ε_0, θ_0 such that u^+ is δ_0-flat in B_{r_0}, that is*

$$\{x_n \leq -\delta_0\} \subset B_{r_0} \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta_0\}.$$

3. Reduction of Theorem 2.4 to a localized form.

The proof of Theorem 2.4 is based on an iterative procedure that "squeezes" our solution around an optimal configuration $U_\beta\,(x \cdot \nu)$ at a geometric rate in dyadically decreasing balls. Here $U_\beta = U_\beta\,(t)$ is given by

$$U_\beta\,(t) = \alpha t^+ - \beta t^- \qquad \beta \geq 0, \alpha = G_0\,(\beta) \equiv G\,(\beta, 0)$$

and ν is a unit vector, which plays the role of the normal vector at the origin. $U_\beta\,(x \cdot \nu)$ is a so-called *two plane solution* when $f = 0$.

The above plan of flatness improvement works nicely in the one phase case (then $\beta = 0$) or as long as the two phases u^+, u^- are, say, comparable (*nondegenerate case*). The difficulties arise when the negative phase becomes very small but at the same time not negligible (*degenerate case).* In this case the flatness assumption in Theorem 2.4 gives a control of the positive phase only, through the closeness to a *one plane solution* $U_0\,(x_n) = x_n^+$.

As we shall see, this requires to face a dychotomy in the final iteration. On the other hand a similar situation is already present in the homogeneous case $f = 0$ (see e.g. [6]).

The first step is to check that the flatness condition (6) implies that u is close to U_β for some β. Indeed we prove that

Lemma 3.1. *Given any $\eta > 0$ there exist $\bar{\delta}, \bar{\rho} > 0$ depending only on η, n, and L such that if $\delta \leq \bar{\delta}$, then*

$$\|u - U_\beta\|_{L^\infty(B_{\bar{\rho}})} \leq \eta \bar{\rho} \tag{7}$$

for some $0 \leq \beta \leq L$.

The proof, by contradiction, follows from the following compactness result, where

$$\mathcal{L}^k u = \mathrm{Tr}\left(A^k(x)\, D^2 u\right) \quad \text{or} \quad \mathcal{L}^k u = \mathcal{F}^k\left(D^2 u\right).$$

Lemma 3.2. *Let u_k be a sequence of (Lipschitz) viscosity solutions to*

$$\begin{cases} |\mathcal{L}^k u_k| \leq M, & \text{in } \Omega^+(u_k) \cup \Omega^-(u_k), \\ \\ (u_k^+)_\nu = G_k((u_k^-)_\nu, x), & \text{on } F(u_k). \end{cases}$$

Assume that:

1. *$u_k \to u^*$ uniformly on compact sets of Ω*
2. *$A^k \to A^*$ uniformly on compact sets of Ω, or $\mathcal{F}^k \to \mathcal{F}^*$ uniformly on compact sets of matrices,*
3. *$G_k(\eta, \cdot) \to G(\eta, \cdot)$ on compact sets of Ω, uniformly on $0 \leq \eta \leq L = Lip(u_k)$,*
4. *$\{u_k^+ = 0\} \to \{(u^*)^+ = 0\}$ in the Hausdorff distance*[2]

Then

$$|\mathcal{L}^* u^*| \leq M, \quad \text{in} \quad \Omega^+(u^*) \cup \Omega^-(u^*)$$

and u^* satisfies the free boundary condition

$$(u^*)_\nu^+ = G((u^*)_\nu^-, x) \quad \text{on } F(u^*),$$

both in the viscosity sense.

In view of Lemma 3.1, after proper rescaling, Theorem 2.4 follows from the following result.

Lemma 3.3. *Let u be a (Lipschitz) viscosity solution to our f.b.p. in B_1, with $Lip(u) \leq L$. There exists a universal constant $\bar{\eta} > 0$ such that, if*

$$\|u - U_\beta\|_{L^\infty(B_1)} \leq \bar{\eta} \quad \text{for some } 0 \leq \beta \leq L, \tag{8}$$

$$\{x_n \leq -\bar{\eta}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \bar{\eta}\}, \tag{9}$$

$$\|f\|_{L^\infty(B_1)} \leq \bar{\eta},\ [G(\eta, \cdot)]_{C^{0,\bar{\gamma}}(B_1)} \leq \bar{\eta}, \quad \forall 0 \leq \eta \leq L,$$

and, when $\mathcal{L} = Tr\left(AD^2\right) + b \cdot \nabla$,

$$[A]_{C^{0,\bar{\gamma}}(B_1)} \leq \bar{\eta}, \quad \|\mathbf{b}\|_{L^\infty(B_1)} \leq \bar{\eta},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

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[2] If K_1, K_2 are two compact sets, their Hausdorff distance is defined by

$$d^H(K_1, K_2) = \inf\{\alpha > 0, K_1 \subset N_\alpha(K_2) \text{ and } K_2 \subset N_\alpha(K_1)\}$$

where

$$N_\alpha(K) = \{x \in \mathbb{R}^n; d(x, K) \leq \alpha\}.$$

Equivalently,

$$d^H(K_1, K_2) = \|d(x, K_1) - d(x, K_2)\|_{L^\infty(\mathbb{R}^n)}.$$

We are almost ready to start the *improvement of flatness procedure.* This means that from (8) and (9) we should be able to squeeze more the graph of u (and therefore $F(u)$) around a possibly rotated new two plane solution in a neighborhood of the origin. A closer look to (8) reveals that, when α and β are comparable, a nice control on the location of $F(u)$ is available but when $\beta \ll \alpha$ only a one side control of $F(u)$ is possible. This dichotomy is well reflected in the following elementary lemma, that we give for a general continuous function and that translates the "vertical" closeness between the graphs of u and U_β given by (8) into "horizontal" closeness, which is much more confortable for our purposes.

Lemma 3.4. *Let u be a continuous function. If, for a small $\eta > 0$,*

$$\|u - U_\beta\|_{L^\infty(B_1)} \leq \eta$$

and

$$\{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\},$$

then:

If $\beta \geq \eta^{1/3}$,

$$U_\beta \left(x_n - \eta^{1/3}\right) \leq u(x) \leq U_\beta \left(x_n + \eta^{1/3}\right) \qquad \text{in } B_{3/4}$$

If $\beta < \eta^{1/3}$,

$$U_0 \left(x_n - \eta^{1/3}\right) \leq u^+(x) \leq U_0 \left(x_n + \eta^{1/3}\right) \qquad \text{in } B_{3/4}.$$

Set $\bar{\eta} = \tilde{\varepsilon}^3$ in the Main Lemma. Then, according to Lemma 3.4, the dichotomy *nondegenerate versus degenerate* translates quantitatively into the two cases:

$$\beta \geq \tilde{\varepsilon} : \textbf{nondegenerate}, \qquad \beta < \tilde{\varepsilon} : \textbf{degenerate.}$$

The parameter $\tilde{\varepsilon}$ will be chosen later in the final iteration, as shown in the next section.

4. The nondegenerate case.

4.1. Improvement of flatness.

Assume that for some $\varepsilon > 0$, small, we have

$$U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1, \tag{10}$$

with $0 < \beta \leq L$, $\alpha = G(\beta, 0) \equiv G_0(\beta)$. One would like to get in a smaller ball an improvement of (10). It is convenient to consider first the *nondegenerate case*, that at this stage reads $\beta \geq \varepsilon$. After a rescaling we may assume that f is small compared to β, in particular,

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta. \tag{11}$$

Then the basic step in the improvement of flatness reads as follows.

Lemma 4.1. *If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then*

$$U_{\beta'}(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \leq u(x) \leq U_{\beta'}(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r, \tag{12}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\varepsilon$, and $|\beta - \beta'| \leq \tilde{C}\beta\varepsilon$ for a universal constant $\tilde{C}$.

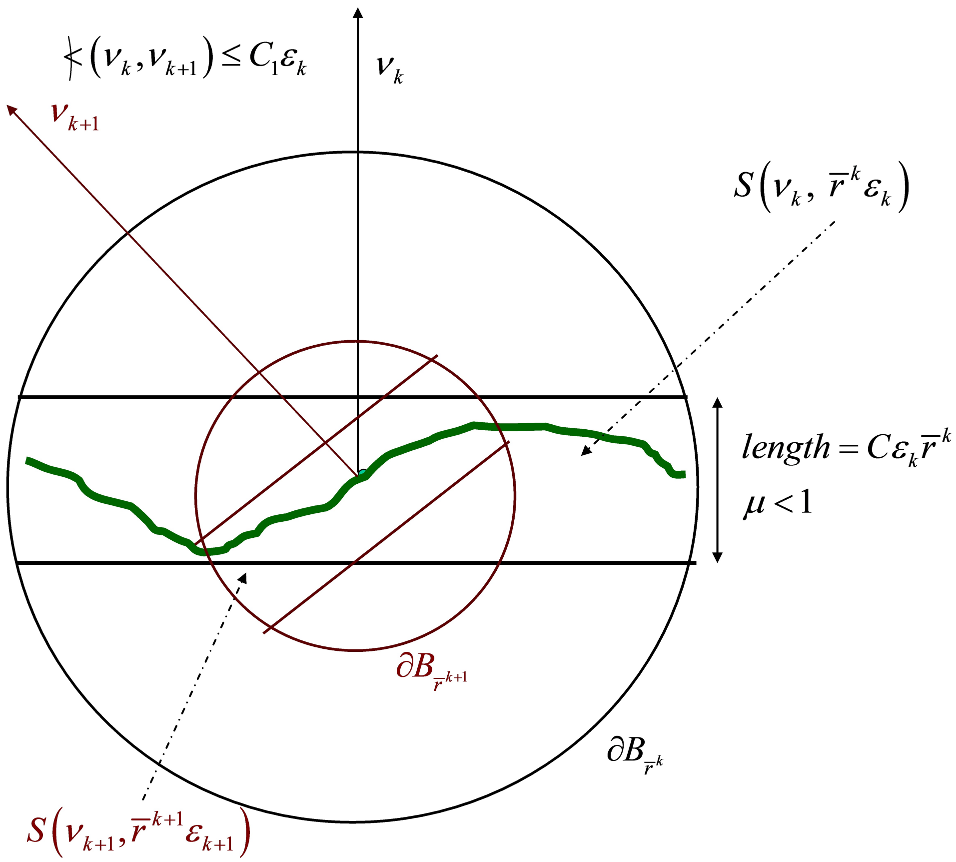

FIGURE 1. Improvement of flatness

To prove Lemma 3.3 we rescale considering a blow up sequence

$$u_k(x) = \frac{u(\bar{r}^k x)}{\bar{r}^k} \qquad x \in B_1 \tag{13}$$

for suitable $\bar{r} \le \min\left\{r_0, \frac{1}{16}\right\}$, $\tilde{\varepsilon} \le \varepsilon_0(\bar{r})$, as in Lemma 4.1, and iterate to get, at the kth step,

$$U_{\beta_k}(x \cdot \nu_k - \bar{r}^k \varepsilon_k) \le u(x) \le U_{\beta_k}(x \cdot \nu_k + \bar{r}^k \varepsilon_k) \quad \text{in } B_{\bar{r}^k},$$

with $\varepsilon_k = 2^{-k}\tilde{\varepsilon}$, $|\nu_k| = 1$, $|\nu_k - \nu_{k-1}| \le \tilde{C}\varepsilon_{k-1}$,

$$|\beta_k - \beta_{k-1}| \le \tilde{C}\beta_{k-1}\varepsilon_{k-1}, \quad \varepsilon_k \le \beta_k \le L.$$

Note that at each step we have the correct inductive hypotheses. For instance, starting with $\beta = \beta_0 \ge \varepsilon_0 = \tilde{\varepsilon}$, if $k \ge 1$ and $\beta_{k-1} \ge \varepsilon_{k-1}$, then

$$\begin{aligned} \beta_k &\ge \beta_{k-1}(1 - \tilde{C}\varepsilon_{k-1}) \ge 2^{-k+1}\tilde{\varepsilon}\left(1 - \tilde{C}2^{-k+1}\tilde{\varepsilon}\right) \\ &\ge 2^{-k}\tilde{\varepsilon} = \varepsilon_k. \end{aligned}$$

Moreover, since $f_k(x) = \rho_k f(\rho_k x)$, $x \in B_1$ and (recall that $\bar{\eta} = \tilde{\varepsilon}^3$)

$$\|f_k\|_{L^\infty(B_1)} \le \rho_k \tilde{\varepsilon}^3 \le \varepsilon_k^2 \beta_k.$$

The Figure 1 describes the step from k to $k+1$.

This implies that $F(u)$ is $C^{1,\alpha}$ at the origin. Repeating the procedure for points in a neighborhood of $x=0$, since all estimates are universal, we conclude that there exists a unit vector $\nu_\infty = \lim \nu_k$ and $C>0$, $\gamma \in (0,1]$, both universal, such that, in the coordinate system $e_1,...,e_{n-1},\nu_\infty$, $\nu_\infty \perp e_j$, $e_j \cdot e_k = \delta_{jk}$, $F(u)$ is $C^{1,\gamma}$ graph, say $x_n = f(x')$, with $f(0')=0$ and

$$|f(x') - \nu_\infty \cdot x'| \le C|x'|^{1+\gamma}$$

in a neighborhood of $x=0$.

The main question is: *where is it hidden the information allowing one to realize the step from (10) to (12)*?

Let us examine briefly Caffarelli's technique in the case $f=0$, $\mathcal{L}=\Delta$. As we have already mentioned, the starting point is $\varepsilon-$monotonicity along the directions of a large cone of directions $\Gamma(\theta_0, e_n)$, of axis e_n and opening θ_0. To get $C^{1,\gamma}$ Caffarelli proves first that $F(u)$ is Lipschitz and then that Lipschitz free boundaries are smooth[3]. The first part amounts to prove that actually u is fully monotone along a possibly rotated cone with a smaller opening. Let us focus more on the part Lipschitz implies $C^{1,\gamma}$. Here are the main steps:

1. To prove that the level sets of u are all Lipschitz graphs with a common Lipschitz constant in the direction e_n, say, in B_1. This follows from full monotonicity along the directions of a cone, say, $\Gamma(\theta_0, e_n)$.
2. To improve the flatness of the level sets in some ball away from the free boundary. This amounts to enlarge the cone of monotonicity in this ball.
3. To carry this gain (giving up a little amount of it) up to the free boundary in $B_{1/2}$. This step is the most crucial and we describe it below.
4. Rescale and iterate the steps 2 and 3.

The final situation of the process is shown in Figure 2.

Here there are two main questions: first, where is it stored the key information that allows to enlarge the cone away from the free boundary? second, how can we transport the gain from inside to the free boundary?

Concerning the first question, one looks at the direction of ∇u at the point $p = e_n/2$. Then $\Gamma(\theta_0, e_n)$ is contained in the half space

$$H^+ = \{\tau : D_\tau u(p) = \langle \nabla u(p), \tau \rangle \ge 0\}.$$

From pure geometric considerations there is a cone $\Gamma(\theta_1, \nu_1) \subset H^+$, containing $\Gamma(\theta_0, e_n)$, such that $\pi/2 - \theta_1 \le \mu(\pi/2 - \theta_0)$, for some universal μ. This means a geometric interior decay of the Lipschitz constant of the level sets of u. In particular, $D_\tau u(p) \ge 0$ if $\tau \in \Gamma(\theta_1, \nu_1)$ and, since the directional derivatives are harmonic, by Harnack inequality, the positivity of $D_\tau u$, $\tau \in \Gamma(\theta_1, \nu_1)$,[4] propagates to a ball $B(p) \subset\subset \Omega^+(u)$.

To carry this gain up to $F(u)$, Caffarelli uses a continuity method based on the construction of a family of continuous deformations (sort of *supconvolutions*) of the type

$$v_{\varepsilon\varphi_t}(x) = \sup_{B_{\varepsilon\varphi_t(x)}(x)} u(x-\tau)$$

with $\varepsilon > 0, \tau \in \Gamma(\theta_0/2, e_n)$ and the parameter $t \in [0,1]$. The variable radius φ has to be chosen such that $\varphi_t \sim 1 + t(\pi/2 - \theta_1)$ in $B(p)$, $\varphi_t \sim 1 + \sigma t(\pi/2 - \theta_1)$ in $B_{1/2}$ for some positive σ, and $\varphi_t \sim 1 + t(\pi/2 - \theta_0)$ everywhere else in B_1.

[3]See however [3], [19] where one goes directly from flatness to $C^{1,\gamma}$.

[4]Actually for a slightly smaller θ_1.

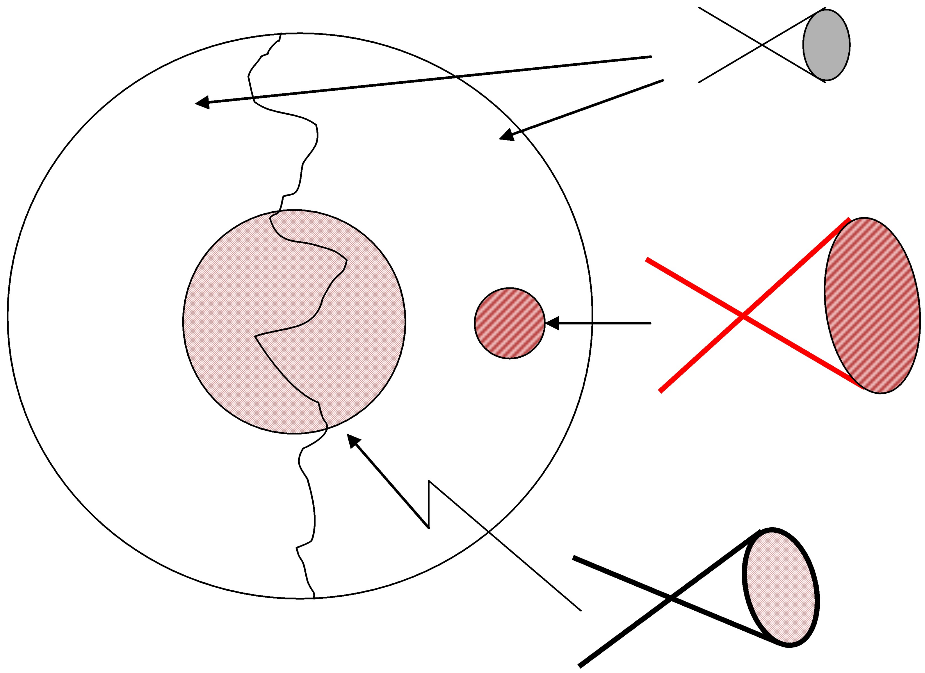

FIGURE 2. Enlargement of the monotonicity cone

Then, the inequality

$$v_{\varepsilon\varphi_0}(x) \le u(x) \quad \text{ in } B_1$$

for each $\varepsilon > 0$ and $\tau \in \Gamma(\theta_0/2, e_n)$ simply means that u is monotone along these directions. One would like to show that the same holds for $v_{\varepsilon\varphi_t}(x) \le u(x)$ and for every $t \in [0,1]$. Then

$$v_{\varepsilon\varphi_t}(x) \le u(x) \quad \text{ in } B_1$$

realizes a geometric improvement of the flatness of $F(u)$ in $B_{1/2}$.

To implement the program, the deformations $v_{\varepsilon\varphi_t}(x)$ must act as comparison subharmonic functions on their support and the main problem is *to find under which condition on φ this happens.*

The answer is the following differential inequality (see [5]):

$$\varphi\Delta\varphi \ge C(n)\left|\nabla\varphi\right|^2. \tag{14}$$

At this point, the construction of a smooth function φ with the desired properties it is not difficult.

The situation for more general operators is much more involved. For instance, in the variable coefficient case (see [19]), if we have a nonnegative function u such that $\mathcal{L}u = \mathrm{Tr}\big(A(x)\,D^2u\big) + b(x)\cdot\nabla u = 0$ on its support, the condition that $\varphi = \varphi_0$ has to satisfy in order to make v_φ an $\mathcal{L}-$subsolution on its support takes the following form:

$$\mathcal{L}\varphi \ge C(n,\lambda,\Lambda)\left\{\frac{\left|\nabla\varphi\right|^2+\omega^2}{\varphi} + \|b\|_{L^\infty}\right\} \tag{15}$$

where ω is the modulus of continuity of A computed at $\max\varphi/\Lambda$.

Let us now return to the case when distributed sources are present. The above deformation method seems to be quite complicated to implement and indeed, using

De Silva technique, one avoids the use of supconvolutions and works directly on u rather than on its derivatives. This is another big advantage, since there is no need to differentiate the equation or to use perturbation methods. With respect to the homogeneous case, the only disadvantage is that we have to assume the Lipschitz continuity of u while, using the deformation method, this comes out as a consequence of the process.

We are back to our basic question: *where is it hidden the information allowing to go from (10) to (12)*?

Here a *linearized problem* comes into play.

4.2. **The linearized problem.** Let us first consider the one-phase case (see [12]) where $u \geq 0$ in B_1,

$$\mathcal{L}u = \operatorname{Tr}\left(A(x) D^2 u\right) = f \qquad \text{in } B_1^+(u)$$

and $u_\nu = |\nabla u| = g(x)$ on the free boundary. Assume that

$$|f| \leq \varepsilon^2, \;\; |a_{ij}(x) - \delta_{ij}| \leq \varepsilon, \;\; |g(x) - 1| \leq \varepsilon^2.$$

The flatness condition writes ($U_0(x) = x_n^+$)

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \text{in } B_1. \tag{16}$$

Renormalize by setting

$$\tilde{u}_\varepsilon(x) = \frac{u(x) - x_n}{\varepsilon} \qquad \text{in } B_1^+(u) \cup F(u)$$

or

$$u(x) = x_n + \varepsilon \tilde{u}_\varepsilon(x) \qquad \text{in } B_1^+(u) \cup F(u). \tag{17}$$

In (17), u appears as a first order perturbation of the hyperplane x_n.

The idea is that the key information we are looking for is **stored precisely in the "coefficient"** $\tilde{u}_\varepsilon$. To extract it we look at what happens to $\tilde{u}_\varepsilon$, asymptotically as $\varepsilon \to 0$. Note that, as $\varepsilon \to 0$, $B_1^+(u) \to \{x_n > 0\}$ and $F(u)$ goes to $\{x_n = 0\}$, both in Hausdorff distance.

We have:

$$\mathcal{L}\tilde{u}_\varepsilon = \frac{f}{\varepsilon} \sim \varepsilon \quad \text{in } B_1^+(u)$$

and on $F(u)$,

$$|\nabla u|^2 = |e_n + \varepsilon \nabla \tilde{u}_\varepsilon|^2 = g^2 \sim 1 + \varepsilon^2$$

that is, after simplifying by ε,

$$2\tilde{u}_{x_n} + \varepsilon |\nabla \tilde{u}_\varepsilon|^2 \sim \varepsilon.$$

Thus, formally, letting $\varepsilon \to 0$, we get "for the limit" $\tilde{u} = \tilde{u}_0$ the following problem:

$$\Delta \tilde{u} = 0, \;\; \text{in } B_{1/2}^+ = B_{1/2} \cap \{x_n > 0\} \tag{18}$$

and the Neumann condition (linearization of the free boundary condition)

$$\tilde{u}_{x_n} = 0 \text{ on } B_{1/2} \cap \{x_n = 0\}. \tag{19}$$

We call (18), (19) the *linearized problem.*

Let us see how the general condition

$$\left|\nabla u^+\right| = G\left(\left|\nabla u^-\right|, x\right)$$

linearizes in the *nondegenerate* two phase problem.

First let

$$\mathcal{L}u = \operatorname{Tr}\left(A(x) D^2 u\right) + \mathbf{b}(x) \cdot \nabla u = f \qquad \text{in } B_1.$$

Assume that

$$|a_{ij}(x) - \delta_{ij}| \le \varepsilon, \ \ |b_j(x)| \le \varepsilon^2, \ \ |f| \le \varepsilon^2 \min\{\alpha, \beta\}$$

and

$$|G(\eta, \cdot) - G_0(\eta)| \le \varepsilon^2 \qquad \forall \eta \in [0, L].$$

The flatness condition

$$\alpha(x_n - \varepsilon)^+ - \beta(x_n - \varepsilon)^- \le u(x) \le \alpha(x_n + \varepsilon)^+ - \beta(x_n + \varepsilon)^- \quad \text{in } B_1, \tag{20}$$

with $0 < \beta \le L$, $\alpha = G_0(\beta)$, suggests the renormalization

$$\tilde{u}_\varepsilon(x) = \begin{cases} \dfrac{u(x) - \alpha x_n}{\alpha \varepsilon}, & x \in B_1^+(u) \cup F(u) \\ \\ \dfrac{u(x) - \beta x_n}{\beta \varepsilon}, & x \in B_1^-(u) \end{cases}$$

or

$$u(x) = \begin{cases} \alpha x_n + \varepsilon \alpha \tilde{u}_\varepsilon(x), & x \in B_1^+(u) \cup F(u) \\ \\ \beta x_n + \varepsilon \beta \tilde{u}_\varepsilon(x), & x \in B_1^-(u). \end{cases} \tag{21}$$

We have

$$\mathcal{L}\tilde{u}_\varepsilon \sim \varepsilon \qquad \text{in } B_1^+(u) \cup B_1^-(u).$$

On $F(u)$,

$$\left|\nabla u^+\right| = \alpha \left|e_n + \varepsilon \nabla \tilde{u}_\varepsilon(x)\right| \sim \alpha \left(1 + \varepsilon (\tilde{u}_\varepsilon)_{x_n} + \frac{1}{2}\varepsilon^2 \left|\nabla \tilde{u}_\varepsilon\right|^2\right)$$

and

$$\begin{aligned} G\left(\left|\nabla u^-\right|, x\right) &= G\left(\left|\beta e_n + \varepsilon \beta \nabla \tilde{u}_\varepsilon\right|, x\right) \sim G\left(\beta\left(1 + \varepsilon (\tilde{u}_\varepsilon)_{x_n} + \frac{1}{2}\varepsilon^2 \left|\nabla \tilde{u}_\varepsilon\right|^2\right), x\right) \\ &\sim G_0(\beta) + \varepsilon G_0'(\beta)\left(\beta (\tilde{u}_\varepsilon)_{x_n} + \frac{1}{2}\varepsilon \beta \left|\nabla \tilde{u}_\varepsilon\right|^2\right) + \varepsilon^2. \end{aligned}$$

As before, letting $\varepsilon \to 0$, we get formally for "the limit" $\tilde{u} = \tilde{u}_0$ the following problem:

$$\Delta \tilde{u} = 0, \ \text{ in } B_{1/2}^+ \cup B_{1/2}^- \tag{22}$$

and ($\alpha = G_0(\beta)$) the transmission condition (linearization of the free boundary condition)

$$\alpha (\tilde{u}_{x_n})^+ - \beta G_0'(\beta)(\tilde{u}_{x_n})^- = 0 \ \text{ on } B_{1/2} \cap \{x_n = 0\} \tag{23}$$

where $(\tilde{u}_{x_n})^+$ and $(\tilde{u}_{x_n})^-$ denote the e_n-derivatives of $\tilde{u}$ restricted to $\{x_n > 0\}$ and $\{x_n < 0\}$, respectively.

When $\mathcal{L}u = \mathcal{F}\left(D^2 u\right)$, the Laplace equation in (22) must be replaced by the fully nonlinear equation

$$\mathcal{F}^{\pm}\left(D^2 \tilde{u}\right) = 0 \quad \text{ in } B_{1/2}^{\pm} \tag{24}$$

where $\mathcal{F}^+(M), \mathcal{F}^-(M)$ are limits (of sequences) of operators of the form

$$\mathcal{F}^+(M) = \frac{1}{\alpha \varepsilon}\mathcal{F}^+(\alpha \varepsilon M) \quad \text{and } \mathcal{F}^+(M) = \frac{1}{\beta \varepsilon}\mathcal{F}^+(\beta \varepsilon M)$$

respectively.

Thus, at least formally, we have found an asymptotic problem for the limits of the renormalizations $\tilde{u}_\varepsilon$. The crucial information we were mentioning before is contained in the following regularity result. Consider the transmission problem, ($\tilde{\alpha} \neq 0$)

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_1 \cap \{x_n \neq 0\}, \\ \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{25}$$

Theorem 4.2. *Let $\tilde{u}$ be a viscosity solution to (25) in B_1 such that $\|\tilde{u}\|_\infty \leq 1$. Then $\tilde{u} \in C^\infty\left(\bar{B}^{\pm}_{1/2}\right)$ and in particular, there exists a universal constant $\bar{C}$ such that*

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'} \tilde{u}(0) \cdot x' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq \bar{C} r^2, \quad \textit{in } B_r \tag{26}$$

for all $r \leq 1/2$ and with $\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0$.

If the Laplace equation in (25) is replaced by $\mathcal{F}^{\pm}\left(D^2 \tilde{u}\right) = 0$ in $B^{\pm}_{1/2}$, $\tilde{u} \in C^{1+\gamma'}\left(\bar{B}^{\pm}_{1/2}\right)$ and the right hand side of (26) must be replaced by $\bar{C} r^{1+\gamma'}, 0 < \gamma' < 1$.

The question is now to transfer the estimate (26) to $\tilde{u}_\varepsilon$ and then read it in terms of flatness for u through formulas (21).

The right way is to proceed by contradiction.

Fix $r \leq r_0$, to be chosen suitably. Assume that for a sequence $\varepsilon_k \to 0$ there is a sequence u_k of solutions of our free boundary problem in B_1, with right hand side f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2 \beta_k$ and

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1,\, 0 \in F(u_k), \tag{27}$$

with $0 \leq \beta_k \leq L$, $\alpha_k = G_k(\beta_k, 0)$, but the conclusion of Lemma 4.1 does not hold for every $k \geq 1$.

Construct the corresponding sequence of renormalized functions

$$\tilde{u}_k(x) = \begin{cases} \dfrac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\ \\ \dfrac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

At this point we need compactness to show that $\tilde{u}_k$ converges uniformly (up to a subsequence) to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$. Also $\alpha_k = G_k(\beta_k, 0)$ converges to $\tilde{\alpha} = \tilde{G}_0\left(\tilde{\beta}\right)$. The compactness is provided by the Harnack inequality stated in Theorem 4.3 and its corollary, as we shall see later, and is inspired by the work of Savin, see [23].

It turns out that the limit function $\tilde{u}$ satisfies the linearized problem (22) or (24) and (23) with $\tilde{\beta}^2$ replaced by $\tilde{\beta} \tilde{G}_0'(\tilde{\beta})$, in the viscosity sense. Hence, from (26), having $\tilde{u}(0) = 0$,

$$|\tilde{u}(x) - (x' \cdot \nu' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq C r^2, \quad x \in B_r, \tag{28}$$

for all $r \leq 1/4$ (say), with

$$\tilde{\alpha} p - \tilde{\beta} \tilde{G}_0'(\tilde{\beta}) q = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \leq C.$$

Since $\tilde{u}_k$ converges uniformly to $\tilde{u}$ in $B_{1/2}$, (28) transfers to $\tilde{u}_k$:

$$|\tilde{u}_k(x) - (x' \cdot \nu' + \tilde{p} x_n^+ - \tilde{q} x_n^-)| \leq C' r^2, \quad x \in B_r. \tag{29}$$

Set

$$\beta_k' = \beta_k\left(1+\varepsilon_k\tilde{q}\right) \qquad \nu_k = \frac{1}{\sqrt{1+\varepsilon_k^2|\nu'|^2}}\left(e_n+\varepsilon_k\left(\nu',0\right)\right).$$

Then,

$$\begin{aligned}\alpha_k' &= G_k(\beta_k(1+\varepsilon_k q),0) = G_k(\beta_k,0)+\beta_k G_k'(\beta_k,0)\varepsilon_k q + O(\varepsilon_k^2)\\ &= \alpha_k(1+\beta_k\frac{G_k'(\beta_k,0)}{\alpha_k}q\varepsilon_k)+O(\varepsilon_k^2) = \alpha_k(1+\varepsilon_k p)+O(\varepsilon_k^2)\end{aligned}$$

where to obtain the first equality we used that $\tilde{\alpha}p - \tilde{\beta}\tilde{G}_0'(\tilde{\beta})q = 0$ and hence

$$\beta_k\frac{G_k'(\beta_k,0)}{\alpha_k}q = p + O(\varepsilon_k).$$

Moreover

$$\nu_k = e_n + \varepsilon_k(\nu',0)+\varepsilon_k^2\tau, \quad |\tau|\le C.$$

With these choices we can now show that (for k large and $r \le r_0$)

$$\widetilde{U}_{\beta_k'}(x\cdot\nu_k - \varepsilon_k\frac{r}{2}) \le \tilde{u}_k(x) \le \widetilde{U}_{\beta_k'}(x\cdot\nu_k + \varepsilon_k\frac{r}{2}), \quad \text{in } B_r$$

where again we are using the notation:

$$\widetilde{U}_{\beta_k'}(x) = \begin{cases}\dfrac{U_{\beta_k'}(x)-\alpha_k x_n}{\alpha_k\varepsilon_k}, & x\in B_1^+(U_{\beta_k'})\cup F(U_{\beta_k'})\\[2ex] \dfrac{U_{\beta_k'}(x)-\beta_k x_n}{\beta_k\varepsilon_k}, & x\in B_1^-(U_{\beta_k'}).\end{cases}$$

This will clearly imply that

$$U_{\beta_k'}(x\cdot\nu_k - \varepsilon_k\frac{r}{2}) \le u_k(x) \le U_{\beta_k'}(x\cdot\nu_k + \varepsilon_k\frac{r}{2}), \quad \text{in } B_r$$

leading to a contradiction.

In view of (29) we need to show that in B_r

$$\widetilde{U}_{\beta_k'}(x\cdot\nu_k - \varepsilon_k\frac{r}{2}) \le (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2$$

and

$$\widetilde{U}_{\beta_k'}(x\cdot\nu_k + \varepsilon_k\frac{r}{2}) \ge (x'\cdot\nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

This can be shown after some elementary calculations as long as $r \le r_0$, r_0 universal, and $\varepsilon \le \varepsilon_0(r)$.

We are left with compactness. The Harnack inequality takes the following form.

Theorem 4.3. *Let u be a solution of our f.b.p. in B_1 with Lipschitz constant L. There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_1$ and u satisfies the following condition:*

$$U_\beta(x_n+a_0) \le u(x) \le U_\beta(x_n+b_0) \qquad \text{in } B_r(x_0)\subset B_1 \tag{30}$$

with

$$\|f\|_{L^\infty(B_2)} \le \varepsilon^2\beta, \qquad 0<\beta\le L$$

and

$$0 < b_0 - a_0 \le \varepsilon r$$

for some $0<\varepsilon\le\tilde{\varepsilon}$, then

$$U_\beta(x_n+a_1) \le u(x) \le U_\beta(x_n+b_1) \qquad \text{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0 \quad \text{and} \quad b_1 - a_1 \leq (1-c)\,\varepsilon r$$

and $0 < c < 1$ *universal.*

If u satisfies (30) with, say $r = 1$, then we can apply Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \leq u(x) \leq U_\beta(x_n + b_m) \quad \text{in } B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \leq (1-c)^m \varepsilon$$

for all m's such that

$$(1-c)^m 20^m \varepsilon \leq \bar{\varepsilon}.$$

This implies that for all such m's, the oscillation of the renormalized functions $\tilde{u}_k$ in $B_r(x_0), r = 20^{-m}$, is less than $(1-c)^m = 20^{-\gamma m} = r^\gamma$. Thus, the following corollary holds.

Corollary 1. *Let $r = 1$ in Theorem 5.1. Then*

$$|\tilde{u}_k\,(x) - \tilde{u}_k\,(x_0)| \leq C\,|x - x_0|^\gamma$$

for all $x \in B_1\,(x_0)$ such that $|x - x_0| \geq \varepsilon_k/\tilde{\varepsilon}$.

Note now that

$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1 \tag{31}$$

and $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. These facts, together with Ascoli-Arzela Theorem give that as $\varepsilon_k \to 0$ the graphs of the $\tilde{u}_k$ converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function $\tilde{u}$ over $B_{1/2}$.

Thus the improvement of flatness process in the nondegenerate case can be concluded.

5. The degenerate case.

5.1. Improvement of flatness.

In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0$). Thus, assume that for some $\varepsilon > 0$, small, we have

$$U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1. \tag{32}$$

Again one would like to get in a smaller ball an improvement of (32). At this stage nondegeneracy reads $\beta < \varepsilon$. This time the key lemma is:

Lemma 5.1. *Let u satisfies (32). Assume that*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4,$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2. \tag{33}$$

There exists a universal r_1, such that if $0 < r \leq r_1$ and $0 < \varepsilon \leq \varepsilon_1$ for some ε_1 depending on r, then

$$U_0(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r, \tag{34}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ for a universal constant C.

The proof follows the same pattern of the nondegenerate case.

Fix $r \leq r_1$, to be chosen suitably. By contradiction assume that, for some sequences $\varepsilon_k \to 0$ and u_k, solutions of our f.b.p. in B_1 with r.h.s. f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^4$ and

$$\|u_k^-\|_{L^\infty(B_1)} \leq \varepsilon_k^2,$$

$$U_0(x_n - \varepsilon_k) \leq u_k(x) \leq U_0(x_n + \varepsilon_k) \quad \text{in } B_1, 0 \in F(u_k)$$

but the conclusion of the lemma does not hold.

Then one proves via a Harnack inequality (see below), that the sequence of normalized functions

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k} \quad x \in B_1^+(u_k) \cup F(u_k)$$

converges to a limit function $\tilde{u}$, Hölder continuous in $B_{1/2}$.

The limit function $\tilde{u}$ is a viscosity solution of the linearized problem

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \\ \tilde{u}_{x_n} = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{35}$$

The regularity of $\tilde{u}$ is not a problem and the contradiction argument proceeds as before with obvious changes.

The Harnack inequality takes the following form.

Theorem 5.2. *Let u be a solution of our f.b.p. in B_1 with Lipschitz constant L. There exists a universal $\tilde{\varepsilon} > 0$ such that, if $x_0 \in B_1$ and u satisfies the following condition*

$$(x_n + a_0)^+ \leq u^+(x) \leq (x_n + b_0)^+ \qquad \textit{in } B_r(x_0) \subset B_1 \tag{36}$$

with

$$\|f\|_{L^\infty(B_2)} \leq \varepsilon^4, \quad \left\|u^-\right\|_{L^\infty(B_2)} \leq \varepsilon^2$$

and

$$0 < b_0 - a_0 \leq \varepsilon r$$

for some $0 < \varepsilon \leq \tilde{\varepsilon}$, then

$$(x_n + a_1)^+ \leq u^+(x) \leq (x_n + b_1)^+ \qquad \textit{in } B_{r/20}(x_0)$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0 \quad \textit{and} \quad b_1 - a_1 \leq (1 - c)\,\varepsilon r$$

and $0 < c < 1$ universal.

Lemma 5.1 provides the first step in the flatness improvement. Notice that this improvement is obtained through the closeness of the positive phase to a *one plane solution*, as long as inequality (33) holds. This inequality expresses in another quantitative way the degeneracy of the negative phase and should be kept valid at each step of the iteration of Lemma 5.1. However, it could happen that this is not the case and in some step of the iteration, say at the level ε_k of flatness, the norm $\|u^-\|_{L^\infty(B_1)}$ becomes of order ε_k^2. When this occurs, a suitable rescaling restores a nondegenerate situation. This give rise in the final iteration to the dychotomy we have mentioned in Section 2.

The situation is precisely described in the following lemma.

Lemma 5.3. *Let u be a solution in B_1 satisfying*

$$U_0(x_n - \varepsilon) \le u^+(x) \le U_0(x_n + \varepsilon) \quad \text{in } B_1 \tag{37}$$

with

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^4,$$

and for $\tilde{C}$ universal,

$$\|u^-\|_{L^\infty(B_2)} \le \tilde{C}\varepsilon^2, \ \|u^-\|_{L^\infty(B_1)} > \varepsilon^2. \tag{38}$$

There exists (universal) ε_1 such that, if $0 < \varepsilon \le \varepsilon_1$, the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2} u\left(\varepsilon^{1/2} x\right)$$

satisfies, in $B_{2/3}$:

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \le u_\varepsilon(x) \le U_{\beta'}(x_n + C'\varepsilon^{1/2})$$

with $\beta' \sim \varepsilon^2$ and C' depending on $\tilde{C}$.

Let us see how the dychotomy arises. To prove Lemma 3.3 in the degenerate case, choose $\bar{r} \le \min\{r_0, r_1, 1/16\}$, and $\tilde{\varepsilon} \le \min\left\{\varepsilon_0(\bar{r}), \varepsilon_1(\bar{r})/2, 1/\left(2\tilde{C}\right)\right\}$ and assume $\beta < \tilde{\varepsilon}$. In view of our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the relation

$$U_0(x_n - \tilde{\varepsilon}) \le u^+(x) \le U_0(x_n + \tilde{\varepsilon}) \quad \text{in } B_1.$$

Since

$$\|u - U_\beta\|_{L^\infty(B_1)} \le \bar{\eta} = \tilde{\varepsilon}^3$$

we infer

$$\left\|u^-\right\|_{L^\infty(B_1)} \le \beta + \tilde{\varepsilon}^3 \le 2\tilde{\varepsilon}.$$

Call $\varepsilon' = \sqrt{2\tilde{\varepsilon}}$. Then

$$U_0(x_n - \varepsilon') \le u^+(x) \le U_0(x_n + \varepsilon') \quad \text{in } B_1$$

and

$$\|f\|_{L^\infty(B_1)} \le (\varepsilon')^4, \|u^-\|_{L^\infty(B_1)} \le (\varepsilon')^2.$$

From Lemma 5.1, we get

$$U_0(x \cdot \nu_1 - \bar{r}\frac{\varepsilon'}{2}) \le u^+(x) \le U_0(x \cdot \nu_1 + \bar{r}\frac{\varepsilon'}{2}) \quad \text{in } B_{\bar{r}}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le C\varepsilon'$ for a universal constant C.

We now rescale considering a blow up sequence

$$u_k(x) = \frac{u\left(\bar{r}^k x\right)}{\bar{r}^k} \qquad x \in B_1 \tag{39}$$

and set $\varepsilon_k = 2^{-k}\varepsilon'$

$$f_k(x) = \rho_k f(\rho_k x) \qquad x \in B_1.$$

Note that

$$\|f_k\|_{L^\infty(B_1)} \le \rho_k (\varepsilon')^4 \le \frac{1}{16}(\varepsilon')^4 = \varepsilon_k^4.$$

We can iterate Lemma 5.1 and obtain

$$U_0(x \cdot \nu_k - \varepsilon_k) \le u_k^+(x) \le U_0(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1$$

with $|\nu_k - \nu_{k-1}| \le C\varepsilon_{k-1}$, **as long as**

$$\|u_k^-\|_{L^\infty(B_1)} \le \varepsilon_k^2.$$

Let $k^* > 1$ be the first integer for which this fails:

$$\|u_{k^*}^-\|_{L^\infty(B_1)} > \varepsilon_{k^*}^2$$

and

$$\|u_{k^*-1}^-\|_{L^\infty(B_1)} \le \varepsilon_{k^*-1}^2.$$

We also have

$$U_0(x \cdot \nu_{k^*-1} - \varepsilon_{k^*-1}) \le u_{k^*-1}^+(x) \le U_0(x \cdot \nu_{k^*-1} + \varepsilon_{k^*-1}) \quad \text{in } B_1.$$

By usual comparison arguments we can write

$$u_{k^*-1}^+(x) \le C\,|x_n - \varepsilon_{k^*-1}|\,\varepsilon_{k^*-1}^2 \qquad \text{in } B_{19/20}$$

for C universal. Rescaling, we have

$$\|u_{k^*}^-\|_{L^\infty(B_1)} \le C_1\varepsilon_{k^*}^2$$

where C_1 universal (C_1 depends on $\bar{r}$). Then u_{k^*} satisfies the assumptions of Lemma 5.3 and therefore the rescaling

$$v\,(x) = \varepsilon_{k^*}^{-1/2} u_{k^*}(\varepsilon_{k^*}^{1/2} x)$$

satisfies in $B_{2/3}$:

$$U_{\beta'}(x \cdot \nu_{k^*} - C'\varepsilon_{k^*}^{1/2}) \le v(x) \le U_{\beta'}(x \cdot \nu_{k^*} + C'\varepsilon_{k^*}^{1/2})$$

with $\beta' \sim \varepsilon_{k^*}^2$. Call $\hat{\varepsilon} = C'\varepsilon_{k^*}^{1/2}$. Then v is a solution of our f.b.p. in $B_{2/3}$ with r.h.s.

$$g\,(x) = \varepsilon_{k^*}^{1/2} f_{k^*}(\varepsilon_{k^*}^{1/2} x)$$

and the flatness assumption

$$U_{\beta'}(x \cdot \nu_{k^*} - \hat{\varepsilon}) \le v(x) \le U_{\beta'}(x \cdot \nu_{k^*} + \hat{\varepsilon}).$$

Since $\beta' \sim \varepsilon_{k^*}^2$, we have

$$\|g\|_{L^\infty(B_1)} \le \varepsilon_{k^*}^{1/2}\varepsilon_{k^*}^4 \le \hat{\varepsilon}^2\beta'$$

as long as $\hat{\varepsilon} \le \min\left\{\varepsilon_0\,(\bar{r})\,, \frac{1}{2\tilde{C}}\right\}$, which is true if $C'\,(2\tilde{\varepsilon})^{1/4} \le \min\left\{\varepsilon_0\,(\bar{r})\,, \frac{1}{2\tilde{C}}\right\}$ or

$$\tilde{\varepsilon} \le \frac{1}{2C'^4}\min\left\{\varepsilon_0\,(\bar{r})\,, \frac{1}{2\tilde{C}}\right\}^4.$$

Under these restrictions, v satisfies the assumptions of the nondegenerate case and we can proceed accordingly.

This concludes the proof of the main Lemma.

6. **Further developments.** With the two Theorems 2.4 and 2.5 the regularity theory for two phase problems with forcing terms has reached a reasonably satisfactory level. However many open questions remain open, object of future investigations.

The first one is to provide an existence results for viscosity solutions satisfying a Dirichlet boundary condition, extending for instance the results in the homogeneous case in [7].

Another question is the $C^\infty-$smoothness (resp. analyticity) of the free boundary in presence of C^∞ (resp. analytic) coefficients and data.

We shall deal with these two questions in forthcoming papers.

Also of great importance, we believe, is to have information on the Hausdorff measure or dimension of the *singular* (*nonflat*) points of the free boundary. For instance, in 3 dimensions, the free boundary for local energy minimizer in the variational problem

$$\int_{\Omega} \left\{ |\nabla u|^2 + \chi_{\{u>0\}} \right\} \to \min$$

is a smooth surface (see [9]). In dimension $n = 7$, De Silva and Jerison in [15] provided an example of a minimizer with singular free boundary. Thus the conjecture is that energy minimizing free boundaries should be smooth for $n < 7$.

Nothing is known in the nonhomogeneous case.

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