A well posedness result for nonlinear viscoelastic equations with memory

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Received 29 May 2013
Accepted 13 August 2013
Communicated by Enzo Mitidieri

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, we denote by

$$A = -\Delta$$

the Dirichlet operator with domain

$$\mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) \Subset L^2(\Omega).$$

Let $\rho \in [0, 4]$, $\theta \in [0, 1]$, $\gamma \geq 0$ and $\alpha > 0$ be fixed parameters. For $t \in \mathbb{R}^+ = (0, \infty)$, we consider the equation

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + \alpha A u - \int_0^\infty \mu(s) A u(t-s)\, \mathrm{d}s + f(u) = h \tag{1.1}$$

in the unknown variable $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ subject to the Dirichlet boundary condition

$$u(\boldsymbol{x}, t)_{|\boldsymbol{x}\in\partial\Omega} = 0.$$

The model is supplemented with the initial conditions (the dependence on $\boldsymbol{x}$ is omitted)

$$u(0) = u_0, \qquad \partial_t u(0) = v_0, \qquad u(-s)_{|s\in\mathbb{R}^+} = \psi_0(s), \tag{1.2}$$

where $u_0, v_0 : \Omega \to \mathbb{R}$ and $\psi_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are given functions, the latter accounting for the initial past history of u.

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Here, the time-independent external force h belongs to the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$, while the locally Lipschitz nonlinearity f, with $f(0) = 0$, fulfills the critical growth restriction

$$|f(u) - f(v)| \le c|u - v|(1 + |u|^4 + |v|^4), \tag{1.3}$$

along with the dissipation condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{1.4}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Finally, the convolution (or memory) kernel μ is a nonnegative, nonincreasing, piecewise absolutely continuous function on $\mathbb{R}^+$ of total mass

$$\int_0^\infty \mu(s)\,\mathrm{d}s = \kappa \in [0, \alpha).$$

Without loss of generality, we may take

$$\alpha - \kappa = 1. \tag{1.5}$$

In particular, μ is allowed to exhibit (even infinitely many) jumps, and can be unbounded about the origin.

Remark 1.1. The degenerate case $\mu \equiv 0$, corresponding to the partial differential equation

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + Au + f(u) = h,$$

is included in our analysis.

Problem (1.1), featuring the nonlinear term

$$|\partial_t u|^\rho \partial_{tt} u,$$

provides a generalization, accounting for memory effects in the material, of equations of the form

$$\varrho(\partial_t u)\partial_{tt} u + A\partial_{tt} u + Au = 0.$$

Such PDEs arise in mechanics, in the description of the vibrations of thin rods whose material density $\varrho(\partial_t u)$ is not constant (see e.g. [1]). The model under consideration has been the object of intensive investigations in the last decade, mainly in its simplified Volterra version

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + \alpha Au - \int_0^t \mu(s) Au(t - s)\,\mathrm{d}s = 0, \tag{1.6}$$

which turns out to be a particular instance of (1.1), as shown in the next Section 3.

The first result concerning (1.6) appears in [2], where the global existence of weak solutions is established for $\theta = 1$ and $\gamma \ge 0$, provided that

$$\rho \le 2.$$

Besides, assuming $\gamma > 0$ and an exponentially decaying memory kernel μ, the authors demonstrate the exponential decay of solutions. However, since no uniqueness is proved, such a result holds only for those trajectories that can be obtained as limits in the Galerkin approximation scheme.

After [2], the study of the longterm properties of the (Galerkin) solutions to (1.6) has been tackled in several works (e.g. [3–12]), with various decay hypotheses on μ. Still, in the above-mentioned papers, the restriction $\rho \le 2$ is always assumed. Indeed, all of them refer to [2] for the existence result, with the only exception of [4], which actually recasts the argument of [2].

On the contrary, the uniqueness issue has never been addressed until the very recent article [13], dealing with the more general model (1.1) with $\theta = 1$, in presence of a nonlinearity $f(u)$ of cubic growth satisfying $uf(u) \ge 0$. There, leaning on [2], the authors show the existence of solutions for $\rho \le 2$, and they prove a continuous dependence (whence uniqueness) result under the additional request that the map

$$v \mapsto |v|^\rho$$

be differentiable at zero, which introduces the further restriction

$$\rho > 1.$$

The aim of our paper is to establish the ultimate well-posedness result for (1.1)–(1.2), ensuring existence (see Section 4) and continuous dependence from the initial data (see Section 5) for the most general admissible nonlinearity $f(u)$, as well as for ρ belonging to the whole meaningful range $[0, 4]$.

Once well-posedness is attained, the study of the asymptotic properties of the solution semigroup becomes a meaningful and interesting question, that will be possibly addressed in forthcoming papers.

2. Functional setting

For $r \in \mathbb{R}$, we define the scale of compactly nested Hilbert spaces

$$\mathrm{H}^r = \mathrm{dom}(A^{r/2})$$

with inner products and norms given by

$$\langle u, v\rangle_r = \langle A^{r/2}u, A^{r/2}v\rangle_{L^2(\Omega)} \quad \text{and} \quad \|u\|_r = \|A^{r/2}u\|_{L^2(\Omega)}.$$

We will always omit the index r whenever $r = 0$. Thus, $\mathrm{H} = L^2(\Omega)$. The symbol $\langle\cdot,\cdot\rangle$ will also stand for the duality product between $\mathrm{H}^1 = H_0^1(\Omega)$ and $\mathrm{H}^{-1} = H^{-1}(\Omega)$. Then, we introduce the so-called history space

$$\mathcal{M} = L^2_\mu(\mathbb{R}^+; \mathrm{H}^1)$$

endowed with the inner product

$$\langle \eta, \xi\rangle_{\mathcal{M}} = \int_0^\infty \mu(s)\langle \eta(s), \xi(s)\rangle_1 \,\mathrm{d}s.$$

We will also consider the right-translation semigroup $\Sigma(t)$ on $\mathcal{M}$ acting as

$$[\Sigma(t)\eta](s) = \begin{cases} 0 & 0 < s \le t, \\ \eta(s-t) & s > t, \end{cases}$$

whose infinitesimal generator is the linear operator

$$T\eta = -\eta', \qquad \mathrm{dom}(T) = \left\{\eta \in \mathcal{M} : \eta' \in \mathcal{M}, \eta(0) = 0\right\},$$

the *prime* standing for weak derivative. The following inequality holds (see e.g. [14])

$$\langle T\eta, \eta\rangle_{\mathcal{M}} \le 0, \quad \forall \eta \in \mathrm{dom}(T). \tag{2.1}$$

Finally, we introduce the extended history space

$$\mathcal{H} = \mathrm{H}^1 \times \mathrm{H}^1 \times \mathcal{M}.$$

Notation. Throughout the paper, $c \ge 0$ and $\mathcal{Q}(\cdot)$ will stand for a generic constant and a generic increasing positive function, respectively. We will use, often without explicit mention, the usual Sobolev embeddings, as well as the Young, Hölder and Poincaré inequalities.

3. The equation in the history framework

Following the approach of Dafermos [15], the original problem (1.1) translates into the system in the unknown variables $u = u(t)$ and $\eta = \eta^t(s)$

$$\begin{cases} |\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + Au + \displaystyle\int_0^\infty \mu(s) A\eta(s)\,\mathrm{d}s + f(u) = h, \\ \partial_t \eta = T\eta + \partial_t u, \end{cases} \tag{3.1}$$

where the initial conditions (1.2) become

$$u(0) = u_0, \qquad \partial_t u(0) = v_0, \quad \text{and} \quad \eta_0(s) = u_0 - \psi_0(s).$$

At a formal level, this is obtained by defining the auxiliary variable

$$\eta = \eta^t(\boldsymbol{x}, s) : \Omega \times [0,\infty) \times \mathbb{R}^+ \to \mathbb{R},$$

accounting for the past history of u, as

$$\eta^t(s) = u(t) - u(t-s).$$

First, we give the definition of weak solution.

Definition 3.1. Given $\tau > 0$, a pair (u, η) with

$$u \in W^{2,\infty}(0,\tau; \mathrm{H}^1) \quad \text{and} \quad \eta \in \mathcal{C}([0,\tau], \mathcal{M})$$

is said to be a (weak) solution to (3.1) on the time-interval $[0, \tau]$ with initial data

$$(u(0), \partial_t u(0), \eta^0) = (u_0, v_0, \eta_0) \in \mathcal{H}$$

if for every test function $\phi \in \mathrm{H}^1$ the equality

$$\langle |\partial_t u|^\rho \partial_{tt} u, \phi\rangle + \langle \partial_{tt} u, \phi\rangle_1 + \gamma\langle \partial_t u, \phi\rangle_\theta + \langle u, \phi\rangle_1 + \int_0^\infty \mu(s)\langle \eta(s), \phi\rangle_1 \,\mathrm{d}s + \langle f(u), \phi\rangle = \langle h, \phi\rangle$$

holds for a.e. $t \in [0, \tau]$, and η is a mild solution on $[0, \tau]$ in the sense of Pazy [16] to the nonhomogeneous linear equation in the Hilbert space $\mathcal{M}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta = T\eta + \partial_t u. \tag{3.2}$$

Remark 3.2. Note that the condition $u \in W^{2,\infty}(0, \tau; \mathrm{H}^1)$ implies

$$u \in \mathcal{C}^1([0, \tau], \mathrm{H}^1).$$

Therefore, $\partial_t u$ can be viewed as an element of the space $L^1(0, \tau; \mathcal{M})$. Accordingly, the mild solution η to (3.2) belongs to $\mathcal{C}([0, \tau], \mathcal{M})$ (see [16, Chapter 4]).

Remark 3.3. By definition, for η to be a mild solution to (3.2) with initial datum $\eta^0 = \eta_0 \in \mathcal{M}$ is the same as saying that

$$\eta^t = \Sigma(t)\eta_0 + \int_0^t \Sigma(t-y)\partial_t u(y)\,\mathrm{d}y.$$

This entails the explicit representation formula for η

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ \eta_0(s-t) + u(t) - u_0 & s > t. \end{cases} \tag{3.3}$$

As a particular case of the problem above, we recover the Volterra equation

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + \alpha A u - \int_0^t \mu(s) A u(t-s)\,\mathrm{d}s + f(u) = h, \tag{3.4}$$

where α fulfills (1.5). Indeed (see [17]), by choosing

$$\eta_0(s) = u_0,$$

formula (3.3) yields

$$\eta^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ u(t) & s > t. \end{cases}$$

Accordingly,

$$Au(t) + \int_0^\infty \mu(s) A\eta^t(s)\,\mathrm{d}s = \alpha A u(t) - \int_0^t \mu(s) A u(t-s)\,\mathrm{d}s.$$

Therefore, Definition 3.1 for the Volterra equation (3.4) becomes

Definition 3.4. A function $u \in W^{2,\infty}(0, \tau; \mathrm{H}^1)$ is said to be a (weak) solution to (3.4) on $[0, \tau]$ with initial data

$$(u(0), \partial_t u(0)) = (u_0, v_0) \in \mathrm{H}^1 \times \mathrm{H}^1$$

if for every test function $\phi \in \mathrm{H}^1$ the equality

$$\langle |\partial_t u|^\rho \partial_{tt} u, \phi\rangle + \langle \partial_{tt} u, \phi\rangle_1 + \gamma\langle \partial_t u, \phi\rangle_\theta + \alpha\langle u, \phi\rangle_1 - \int_0^t \mu(s)\langle u(t-s), \phi\rangle_1 \,\mathrm{d}s + \langle f(u), \phi\rangle = \langle h, \phi\rangle$$

holds for a.e. $t > 0$.

4. Existence of solutions

Along the section, let $\tau > 0$ be arbitrarily fixed.

Theorem 4.1. *For any initial data $z = (u_0, v_0, \eta_0) \in \mathcal{H}$, system (3.1) admits at least a solution $(u(t), \eta^t)$ on $[0, \tau]$ satisfying the initial conditions*

$$(u(0), \partial_t u(0), \eta^0) = z.$$

In particular, defining the corresponding energy at time t as

$$E(t) = \frac{1}{2}\left[\|u(t)\|_1^2 + \|\partial_t u(t)\|_1^2 + \|\eta^t\|_{\mathcal{M}}^2\right],$$

the following proposition holds.

Proposition 4.2. *For every initial data z with $\|z\|_{\mathcal{H}} \le R$ and every $t \in [0, \tau]$, we have the uniform estimate*

$$E(t) + \|\partial_{tt} u(t)\|_1 \le \mathcal{Q}(R),$$

where the increasing function $\mathcal{Q}$ is independent of $\tau > 0$.

The existence result follows from a standard Galerkin approximation scheme, the key point being the validity of some a priori estimates.

4.1. Energy estimates

Assume for the moment to have a sufficiently regular solution (u, η) to (3.1) with initial data $\|z\|_{\mathcal{H}} \le R$, and define the energy functional

$$\mathfrak{L}(t) = \frac{1}{\rho+2}\int_\Omega |\partial_t u(t)|^{\rho+2}\,\mathrm{d}\boldsymbol{x} + E(t) + \langle \hat{f}(u(t)), 1\rangle - \langle h, u(t)\rangle,$$

where

$$\hat{f}(u) = \int_0^u f(y)\,\mathrm{d}y.$$

In light of assumptions (1.3)–(1.4), it is readily seen that

$$\nu E - c \le \mathfrak{L} \le cE^3 + c \tag{4.1}$$

for some $\nu > 0$, possibly very small, depending only on the value of the limit in (1.4). Testing system (3.1) with $(\partial_t u, \eta)$ in H $\times$ $\mathcal{M}$ and recalling (2.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L} + \gamma\|\partial_t u\|_\theta^2 = \langle T\eta, \eta\rangle_{\mathcal{M}} \le 0. \tag{4.2}$$

Hence,

$$\mathfrak{L}(t) \le \mathfrak{L}(0) \le \mathcal{Q}(R).$$

In light of (4.1), this yields

$$E(t) \le \mathcal{Q}(R). \tag{4.3}$$

Furthermore, a multiplication of the first equation of (3.1) by $\partial_{tt} u$ gives

$$\langle|\partial_t u|^\rho \partial_{tt} u, \partial_{tt} u\rangle + \|\partial_{tt} u\|_1^2 = -\gamma\langle\partial_t u, \partial_{tt} u\rangle_\theta - \langle u, \partial_{tt} u\rangle_1 - \int_0^\infty \mu(s)\langle\eta(s), \partial_{tt} u\rangle_1\,\mathrm{d}s - \langle f(u), \partial_{tt} u\rangle + \langle h, \partial_{tt} u\rangle.$$

The first term on the left-hand side is positive and, by (1.3),

$$-\langle f(u), \partial_{tt} u\rangle \le \|f(u)\|_{L^{6/5}}\|\partial_{tt} u\|_{L^6} \le c\left(1 + \|u\|_1^5\right)\|\partial_{tt} u\|_1.$$

Moreover,

$$-\int_0^\infty \mu(s)\langle\eta(s), \partial_{tt} u\rangle_1\,\mathrm{d}s \le \|\partial_{tt} u\|_1 \int_0^\infty \mu(s)\|\eta(s)\|_1\,\mathrm{d}s,$$

where

$$\int_0^\infty \mu(s)\|\eta(s)\|_1\,\mathrm{d}s \le \left(\int_0^\infty \mu(s)\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^\infty \mu(s)\|\eta(s)\|_1^2\,\mathrm{d}s\right)^{\frac{1}{2}} = \sqrt{\kappa}\,\|\eta\|_{\mathcal{M}}.$$

Thus, we infer from (4.3) that

$$\begin{aligned}\|\partial_{tt} u\|_1^2 &\le \left(\gamma\|\partial_t u\|_{2\theta-1} + \|u\|_1 + \sqrt{\kappa}\|\eta\|_{\mathcal{M}} + c + c\|u\|_1^5 + \|h\|_{-1}\right)\|\partial_{tt} u\|_1 \\ &\le \frac{1}{2}\|\partial_{tt} u\|_1^2 + \mathcal{Q}(R),\end{aligned}$$

implying the further bound

$$\|\partial_{tt} u(t)\|_1 \le \mathcal{Q}(R). \tag{4.4}$$

Incidentally, once existence will be proved, (4.3)–(4.4) establish Proposition 4.2.

4.2. Galerkin scheme

We follow a standard Galerkin method based on the choice of a suitable orthonormal basis of $\mathrm{H}^1 \times \mathcal{M}$ (see [18] for more details). We then consider the smooth solutions (u_n, η_n) on $[0, \tau]$ to the corresponding n-dimensional Cauchy approximating problems with initial conditions

$$z_n = (u_{0n}, v_{0n}, \eta_{0n}) \to z = (u_0, v_0, \eta_0) \quad \text{in } \mathcal{H}.$$

Arguing as in the previous subsection, we deduce the bounds

$$\|u_n(t)\|_1 + \|\partial_t u_n(t)\|_1 + \|\partial_{tt} u_n(t)\|_1 + \|\eta_n^t\|_{\mathcal{M}} \le \mathcal{Q}(R).$$

Therefore, we find a pair (u, η) such that, up to subsequences,

$$u_n \rightharpoonup u \quad \text{weakly-}* \text{ in } L^\infty(0, \tau; \mathrm{H}^1), \tag{4.5}$$

$$\partial_t u_n \rightharpoonup \partial_t u \quad \text{weakly-}* \text{ in } L^\infty(0, \tau; \mathrm{H}^1), \tag{4.6}$$

$$\partial_{tt} u_n \rightharpoonup \partial_{tt} u \quad \text{weakly-}* \text{ in } L^\infty(0, \tau; \mathrm{H}^1), \tag{4.7}$$

$$\eta_n \rightharpoonup \eta \quad \text{weakly-}* \text{ in } L^\infty(0, \tau; \mathcal{M}). \tag{4.8}$$

4.3. Passage to the limit

We aim to show that (u, η) is a weak solution to (3.1), passing to the limit in the n-dimensional approximated problem solved by (u_n, η_n). On account of (4.5)–(4.8), the only difficulty here is handling the nonlinear terms. To this end, calling $\Omega_\tau = \Omega \times (0, \tau)$, we begin to observe that

$$f(u_n) \text{ and } |\partial_t u_n|^\rho \partial_t u_n$$

are uniformly bounded in $L^{6/5}(\Omega_\tau)$. Indeed, since $\rho \in [0, 4]$, exploiting the H^1-bound for $\partial_t u_n$ we have

$$\begin{aligned}\int_0^\tau \int_\Omega |\partial_t u_n|^{\frac{6}{5}(\rho+1)} \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t &\le c \int_0^\tau \left(\int_\Omega |\partial_t u_n|^6 \,\mathrm{d}\boldsymbol{x} \right)^{\frac{\rho+1}{5}} \mathrm{d}t \\ &\le c \int_0^\tau \|\partial_t u_n\|_1^{\frac{6(\rho+1)}{5}} \,\mathrm{d}t \\ &\le \tau \mathcal{Q}(R),\end{aligned}$$

while the corresponding estimate for $f(u_n)$ follows in a similar fashion making use of the growth condition (1.3). Therefore, the compact embedding

$$W^{1,\infty}(0, \tau; \mathrm{H}^1) \Subset \mathcal{C}([0, \tau], \mathrm{H}),$$

jointly with the H^1-bounds for $\partial_t u_n$ and $\partial_{tt} u_n$, provide the convergence (up to a subsequence)

$$\partial_t u_n \to \partial_t u \quad \text{in } \mathcal{C}([0, \tau], \mathrm{H}).$$

In particular,

$$|\partial_t u_n|^\rho \partial_t u_n \to |\partial_t u|^\rho \partial_t u \quad \text{a.e. in } \Omega_\tau.$$

Hence, exploiting the $L^{6/5}$-bound, the weak dominated convergence theorem entails the limit

$$-\frac{1}{\rho+1} \int_0^\tau \langle |\partial_t u_n|^\rho \partial_t u_n, \partial_t \zeta \rangle \,\mathrm{d}t \to \int_0^\tau \langle |\partial_t u|^\rho \partial_{tt} u, \zeta \rangle \,\mathrm{d}t$$

for every fixed $\zeta \in \mathcal{D}([0, \tau], \mathrm{H}^1)$, having used the identity

$$\int_0^\tau \langle |\partial_t u|^\rho \partial_{tt} u, \zeta \rangle \,\mathrm{d}t = -\frac{1}{\rho+1} \int_0^\tau \langle |\partial_t u|^\rho \partial_t u, \partial_t \zeta \rangle \,\mathrm{d}t.$$

On the other hand, as

$$\int_0^\tau \langle |\partial_t u_n|^\rho \partial_{tt} u_n, \zeta \rangle \,\mathrm{d}t = -\frac{1}{\rho+1} \int_0^\tau \langle |\partial_t u_n|^\rho \partial_t u_n, \partial_t \zeta \rangle \,\mathrm{d}t,$$

we also get

$$\int_0^\tau \langle |\partial_t u_n|^\rho \partial_{tt} u_n, \zeta \rangle \,\mathrm{d}t \to \int_0^\tau \langle |\partial_t u|^\rho \partial_{tt} u, \zeta \rangle \,\mathrm{d}t.$$

Due to the arbitrariness of ζ, we conclude that

$$\langle |\partial_t u_n|^\rho \partial_{tt} u_n, \phi\rangle \to \langle |\partial_t u|^\rho \partial_{tt} u, \phi\rangle$$

for every $\phi \in \mathrm{H}^1$ and a.e. $t \in [0, \tau]$. Analogously, we can pass to the limit in the second nonlinearity. Namely, by the same compact embedding above, the H^1-bounds for u_n and $\partial_t u_n$ entail the convergence (up to a subsequence)

$$u_n \to u \quad \text{a.e. in } \Omega_\tau,$$

and, by the continuity of f,

$$f(u_n) \to f(u) \quad \text{a.e. in } \Omega_\tau.$$

At this point, the limit

$$\int_0^\tau \langle f(u_n), \zeta\rangle \, \mathrm{d}t \to \int_0^\tau \langle f(u), \zeta\rangle \, \mathrm{d}t$$

follows by virtue of the uniform $L^{6/5}$-bound for $f(u_n)$.

Summarizing, we proved that (u, η) solve in the weak sense the first equation of (3.1) on the time-interval $[0, \tau]$. Finally, we turn our attention to η. Since in the approximation scheme $\eta_n \in \mathrm{dom}(T)$, it follows that the representation formula (3.3) for η_n holds, namely,

$$\eta_n^t(s) = \begin{cases} u_n(t) - u_n(t-s) & 0 < s \le t, \\ \eta_{0n}(s-t) + u_n(t) - u_{0n} & s > t. \end{cases}$$

At this point, knowing that $u_{0n} \to u_0$ and $\eta_{0n} \to \eta_0$ strongly, and recalling (4.5), we easily conclude that $\eta_n \rightharpoonup \tilde{\eta}$ in some weak topology, where

$$\tilde{\eta}^t(s) = \begin{cases} u(t) - u(t-s) & 0 < s \le t, \\ \eta_0(s-t) + u(t) - u_0 & s > t, \end{cases}$$

and the convergence (4.8) forces the equality $\tilde{\eta} = \eta$. Thus, as η fulfills (3.3), it is a mild solution to the second equation of (3.1).

5. Continuous dependence and uniqueness

Again, $\tau > 0$ is arbitrarily fixed. Given $z_1, z_2 \in \mathcal{H}$, let $(u_\imath, \eta_\imath)$, with $\imath = 1, 2$, be any two solutions to (3.1) on the time-interval $[0, \tau]$ with initial data

$$(u_\imath(0), \partial_t u_\imath(0), \eta_\imath^0) = z_\imath.$$

Defining the difference of solutions

$$\bar{u} = u_1 - u_2 \quad \text{and} \quad \bar{\eta} = \eta_1 - \eta_2,$$

we have the following continuous dependence estimate.

Theorem 5.1. *For every $R \ge 0$, the estimate*

$$\|\bar{u}(t)\|_1^2 + \|\partial_t \bar{u}(t)\|_1^2 + \|\bar{\eta}^t\|_{\mathcal{M}}^2 \le (1+\tau^3)\mathcal{Q}(R) e^{\tau(1+\tau)\mathcal{Q}(R)} \|z_1 - z_2\|_{\mathcal{H}}^2$$

holds for every $t \in [0, \tau]$ whenever $\|z_\imath\|_{\mathcal{H}} \le R$. The increasing function $\mathcal{Q}$ is independent of $\tau > 0$.

Choosing $z_1 = z_2$, we readily draw the next corollary.

Corollary 5.2. *The solution to (3.1) is unique.*

Proof of Theorem 5.1. We introduce the new variables

$$w(t) = \int_0^t u(y)\,\mathrm{d}y \quad \text{and} \quad \xi^t(s) = \int_0^t \eta^y(s)\,\mathrm{d}y.$$

Defining the function

$$\sigma(v) = \frac{1}{1+\rho} v|v|^\rho,$$

and integrating the first equation of (3.1) on $(0, t)$, we obtain

$$\sigma(\partial_t u) + A\partial_{tt} w + \gamma A^\theta \partial_t w + Aw + \int_0^\infty \mu(s) A\xi(s)\,\mathrm{d}s + \int_0^t f(u(y))\,\mathrm{d}y = th + g,$$

where

$$g = A\partial_t u(0) + \gamma A^\theta u(0) + \sigma(\partial_t u(0)).$$

Given any two solutions $(u_\imath, \eta_\imath)$ to (3.1), denote the corresponding differences by

$$\bar{w} = w_1 - w_2 \quad \text{and} \quad \bar{\xi} = \xi_1 - \xi_2.$$

Then $(\bar{w}, \bar{\xi})$ satisfies the system

$$\sigma(\partial_t u_1) - \sigma(\partial_t u_2) + A\partial_{tt}\bar{w} + \gamma A^\theta \partial_t \bar{w} + A\bar{w} + \int_0^\infty \mu(s) A\bar{\xi}(s)\, \mathrm{d}s + F = G, \tag{5.1}$$

$$\partial_t \bar{\xi} = T\bar{\xi} + \partial_t \bar{w} - \bar{u}(0) + \bar{\eta}^0 \tag{5.2}$$

having set

$$F(t) = \int_0^t [f(u_1(y)) - f(u_2(y))]\, \mathrm{d}y$$

and

$$G = A\partial_t \bar{u}(0) + \gamma A^\theta \bar{u}(0) + \sigma(\partial_t u_1(0)) - \sigma(\partial_t u_2(0)).$$

Multiplying (5.1) by $\partial_{tt}\bar{w}$, and observing that by the monotonicity of σ

$$\langle \sigma(\partial_t u_1) - \sigma(\partial_t u_2), \partial_{tt}\bar{w} \rangle \geq 0,$$

we obtain

$$\|\partial_{tt}\bar{w}\|_1^2 \leq -\gamma \langle \partial_t \bar{w}, \partial_{tt}\bar{w} \rangle_\theta - \langle \bar{w}, \partial_{tt}\bar{w} \rangle_1 - \int_0^\infty \mu(s) \langle \bar{\xi}(s), \partial_{tt}\bar{w} \rangle_1\, \mathrm{d}s - \langle F, \partial_{tt}\bar{w} \rangle + \langle G, \partial_{tt}\bar{w} \rangle. \tag{5.3}$$

We now define the energy corresponding to the pair $(\bar{w}, \bar{\xi})$ as

$$\Lambda(t) = \frac{1}{2}\left[\|\bar{w}(t)\|_1^2 + \|\partial_t \bar{w}(t)\|_1^2 + \|\bar{\xi}^t\|_{\mathcal{M}}^2\right].$$

A multiplication of (5.1) by $\partial_t \bar{w}$ and (5.2) by $\bar{\xi}$ in $\mathcal{M}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \gamma \|\partial_t \bar{w}\|_\theta^2 = \langle T\bar{\xi}, \bar{\xi} \rangle_{\mathcal{M}} - \langle \sigma(\partial_t u_1) - \sigma(\partial_t u_2), \partial_t \bar{w} \rangle - \langle F, \partial_t \bar{w} \rangle + \langle G, \partial_t \bar{w} \rangle + \langle \bar{\eta}^0 - \bar{u}(0), \bar{\xi} \rangle_{\mathcal{M}}. \tag{5.4}$$

Adding (5.3)–(5.4), with the aid of (2.1), we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Lambda + \gamma \|\partial_t \bar{w}\|_\theta^2 + \|\partial_{tt}\bar{w}\|_1^2 \leq{} & -\langle \sigma(\partial_t u_1) - \sigma(\partial_t u_2), \partial_t \bar{w} \rangle \\ & - \frac{\mathrm{d}}{\mathrm{d}t}\langle F, \bar{w} + \partial_t \bar{w} \rangle + \langle \partial_t F, \bar{w} + \partial_t \bar{w} \rangle + \frac{\mathrm{d}}{\mathrm{d}t}\langle G, \bar{w} + \partial_t \bar{w} \rangle \\ & + \langle \bar{\eta}^0 - \bar{u}(0), \bar{\xi} \rangle_{\mathcal{M}} - \gamma \langle \partial_t \bar{w}, \partial_{tt}\bar{w} \rangle_\theta - \langle \bar{w}, \partial_{tt}\bar{w} \rangle_1 - \int_0^\infty \mu(s) \langle \bar{\xi}(s), \partial_{tt}\bar{w} \rangle_1\, \mathrm{d}s. \end{aligned}$$

In order to control the terms on the right-hand side, we exploit the bounds

$$\|u_\imath\|_1 + \|\partial_t u_\imath\|_1 + \|\eta_\imath\|_{\mathcal{M}} \leq \mathcal{Q}(R)$$

given by Proposition 4.2. First of all, as

$$|\sigma(\partial_t u_1) - \sigma(\partial_t u_2)| \leq c\left(1 + |\partial_t u_1|^4 + |\partial_t u_2|^4\right)|\partial_t \bar{u}|, \tag{5.5}$$

we deduce

$$\begin{aligned} -\langle \sigma(\partial_t u_1) - \sigma(\partial_t u_2), \partial_t \bar{w} \rangle & \leq c\left(1 + \|\partial_t u_1\|_{L^6}^4 + \|\partial_t u_2\|_{L^6}^4\right)\|\partial_{tt}\bar{w}\|_{L^6}\|\partial_t \bar{w}\|_{L^6} \\ & \leq c\left(1 + \|\partial_t u_1\|_1^4 + \|\partial_t u_2\|_1^4\right)\|\partial_{tt}\bar{w}\|_1\|\partial_t \bar{w}\|_1 \\ & \leq \mathcal{Q}(R)\|\partial_t \bar{w}\|_1^2 + \frac{1}{2}\|\partial_{tt}\bar{w}\|_1^2. \end{aligned}$$

Besides, we have

$$
\begin{aligned}
-\gamma\langle\partial_t\bar{w},\partial_{tt}\bar{w}\rangle_\theta-\langle\bar{w},\partial_{tt}\bar{w}\rangle_1-\int_0^\infty\mu(s)\langle\bar{\xi}(s),\partial_{tt}\bar{w}\rangle_1\,\mathrm{d}s &\leq \left(\|\bar{w}\|_1+\gamma\|\partial_t\bar{w}\|_{2\theta-1}+\sqrt{\kappa}\|\bar{\xi}\|_{\mathcal{M}}\right)\|\partial_{tt}\bar{w}\|_1\\
&\leq c\left(\|\bar{w}\|_1^2+\|\partial_t\bar{w}\|_1^2+\|\bar{\xi}\|_{\mathcal{M}}^2\right)+\frac{1}{2}\|\partial_{tt}\bar{w}\|_1^2. \qquad (5.6)
\end{aligned}
$$

Hence, owing to the estimate

$$
\langle\bar{\eta}^0-\bar{u}(0),\bar{\xi}\rangle_{\mathcal{M}}\leq\|\bar{u}(0)\|_1^2+\|\bar{\eta}^0\|_{\mathcal{M}}^2+c\|\bar{\xi}\|_{\mathcal{M}}^2,
$$

the differential inequality above becomes

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Lambda\leq\mathcal{Q}(R)\Lambda-\frac{\mathrm{d}}{\mathrm{d}t}\langle F,\bar{w}+\partial_t\bar{w}\rangle+\frac{\mathrm{d}}{\mathrm{d}t}\langle G,\bar{w}+\partial_t\bar{w}\rangle+\langle\partial_tF,\bar{w}+\partial_t\bar{w}\rangle+\|\bar{u}(0)\|_1^2+\|\bar{\eta}^0\|_{\mathcal{M}}^2.
$$

At this point, we choose an arbitrary $x\in[0,\tau]$, and we integrate the latter inequality on $(0,x)$. This yields

$$
\begin{aligned}
\Lambda(x)\leq{}&\mathcal{Q}(R)\int_0^x\Lambda(t)\,\mathrm{d}t-\langle F(x),\bar{w}(x)+\partial_t\bar{w}(x)\rangle+\langle G,\bar{w}(x)+\partial_t\bar{w}(x)\rangle-\langle G,\bar{u}(0)\rangle\\
&+\int_0^x\langle\partial_tF(t),\bar{w}(t)+\partial_t\bar{w}(t)\rangle\,\mathrm{d}t+(1+x)\|\bar{u}(0)\|_1^2+x\|\bar{\eta}^0\|_{\mathcal{M}}^2.
\end{aligned}
$$

Since

$$
-\langle F(x),\bar{w}(x)+\partial_t\bar{w}(x)\rangle+\langle G,\bar{w}(x)+\partial_t\bar{w}(x)\rangle-\langle G,\bar{u}(0)\rangle\leq c\|F(x)\|_{-1}^2+c\|G\|_{-1}^2+\frac{1}{2}\Lambda(x)+c\|\bar{u}(0)\|_1^2,
$$

we get

$$
\Lambda(x)\leq\mathcal{Q}(R)\int_0^x\Lambda(t)\,\mathrm{d}t+c\|F(x)\|_{-1}^2+c\|G\|_{-1}^2+\int_0^x\langle\partial_tF(t),\bar{w}(t)+\partial_t\bar{w}(t)\rangle\,\mathrm{d}t+c(1+x)\|z_1-z_2\|_{\mathcal{H}}^2. \qquad (5.7)
$$

Due to (1.3),

$$
\begin{aligned}
\|F(x)\|_{-1}^{\frac{6}{5}}&\leq c\|F(x)\|_{L^{6/5}}^{\frac{6}{5}}\\
&\leq cx^{\frac{1}{5}}\int_0^x\int_\Omega\left(1+|u_1(t)|^4+|u_2(t)|^4\right)^{\frac{6}{5}}|\bar{u}(t)|^{\frac{6}{5}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\
&\leq cx^{\frac{1}{5}}\int_0^x\left(\int_\Omega\left(1+|u_1(t)|^4+|u_2(t)|^4\right)^{\frac{3}{2}}\,\mathrm{d}\boldsymbol{x}\right)^{\frac{4}{5}}\|\bar{u}(t)\|_{L^6}^{\frac{6}{5}}\,\mathrm{d}t\\
&\leq x^{\frac{1}{5}}\mathcal{Q}(R)\int_0^x\|\bar{u}(t)\|_1^{\frac{6}{5}}\,\mathrm{d}t,
\end{aligned}
$$

implying

$$
\|F(x)\|_{-1}^2\leq x^{\frac{1}{3}}\mathcal{Q}(R)\left(\int_0^x\|\bar{u}(t)\|_1^{\frac{6}{5}}\,\mathrm{d}t\right)^{\frac{5}{3}}\leq x\mathcal{Q}(R)\int_0^x\|\bar{u}(t)\|_1^2\,\mathrm{d}t\leq x\mathcal{Q}(R)\int_0^x\Lambda(t)\,\mathrm{d}t.
$$

Analogously, by (5.5),

$$
\begin{aligned}
\|\sigma(\partial_tu_1(0))-\sigma(\partial_tu_2(0))\|_{L^{6/5}}^{\frac{6}{5}}&\leq c\int_\Omega\left(1+|\partial_tu_1(0)|^4+|\partial_tu_2(0)|^4\right)^{\frac{6}{5}}|\partial_t\bar{u}(0)|^{\frac{6}{5}}\,\mathrm{d}\boldsymbol{x}\\
&\leq\mathcal{Q}(R)\|\partial_t\bar{u}(0)\|_1^{\frac{6}{5}},
\end{aligned}
$$

which yields

$$
\|G\|_{-1}^2\leq c\|\bar{u}(0)\|_1^2+\mathcal{Q}(R)\|\partial_t\bar{u}(0)\|_1^2.
$$

Reasoning as before, we find

$$
\|\partial_tF\|_{L^{6/5}}^{\frac{6}{5}}\leq c\int_\Omega\left(1+|u_1|^4+|u_2|^4\right)^{\frac{6}{5}}|\bar{u}|^{\frac{6}{5}}\,\mathrm{d}\boldsymbol{x}\leq\mathcal{Q}(R)\|\bar{u}\|_1^{\frac{6}{5}},
$$

whence

$$\int_0^x \langle \partial_t F(t), \bar{w}(t) + \partial_t \bar{w}(t)\rangle \,\mathrm{d}t \le \int_0^x \|\partial_t F(t)\|_{L^{6/5}} \left(\|\bar{w}(t)\|_{L^6} + \|\partial_t \bar{w}(t)\|_{L^6}\right) \mathrm{d}t$$
$$\le \mathcal{Q}(R) \int_0^x \Lambda(t)\,\mathrm{d}t.$$

In light of the above computations, (5.7) improves to

$$\Lambda(x) \le (1+x)\mathcal{Q}(R) \int_0^x \Lambda(t)\,\mathrm{d}t + (1+x)\mathcal{Q}(R)\|z_1 - z_2\|^2_{\mathcal{H}}$$

for every $x \in [0, \tau]$, and an application of the Gronwall lemma entails

$$\Lambda(x) \le (1+\tau)\mathcal{Q}(R)e^{\tau(1+\tau)\mathcal{Q}(R)}\|z_1 - z_2\|^2_{\mathcal{H}}, \quad \forall x \in [0, \tau]. \tag{5.8}$$

In particular we learn that

$$\|\bar{u}(x)\|_1^2 \le (1+\tau)\mathcal{Q}(R)e^{\tau(1+\tau)\mathcal{Q}(R)}\|z_1 - z_2\|^2_{\mathcal{H}}, \quad \forall x \in [0, \tau].$$

The analogous estimate for $\partial_t\bar{u}$ comes directly from (5.3). Indeed, taking into account (5.6), (5.8) and the controls for $\|F\|_{-1}$ and $\|G\|_{-1}$ obtained before,

$$\|\partial_t\bar{u}\|_1^2 \le \frac{1}{2}\left(\gamma\|\partial_t\bar{w}\|_{2\theta-1} + \|\bar{w}\|_1 + \sqrt{\kappa}\|\bar{\xi}\|_{\mathcal{M}} + \|F\|_{-1} + \|G\|_{-1}\right)^2 + \frac{1}{2}\|\partial_t\bar{u}\|_1^2,$$

and we draw the estimate

$$\|\partial_t\bar{u}(x)\|_1^2 \le (1+\tau^3)\mathcal{Q}(R)e^{\tau(1+\tau)\mathcal{Q}(R)}\|z_1 - z_2\|^2_{\mathcal{H}}, \quad \forall x \in [0, \tau].$$

Finally, to control $\|\bar{\eta}\|_{\mathcal{M}}$, we multiply the equation for $\bar{\eta}$, i.e.

$$\partial_t\bar{\eta} = T\bar{\eta} + \partial_t\bar{u},$$

by $\bar{\eta}$ in $\mathcal{M}$, to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\eta}\|^2_{\mathcal{M}} = \langle T\bar{\eta}, \bar{\eta}\rangle_{\mathcal{M}} + \langle \partial_t\bar{u}, \bar{\eta}\rangle_{\mathcal{M}} \le \sqrt{\kappa}\|\partial_t\bar{u}\|_1\|\bar{\eta}\|_{\mathcal{M}} \le \|\bar{\eta}\|^2_{\mathcal{M}} + c\|\partial_t\bar{u}\|_1^2.$$

By the Gronwall lemma, exploiting the estimate for $\|\partial_t\bar{u}\|_1$ just proved, we are led to

$$\|\bar{\eta}^x\|^2_{\mathcal{M}} \le (1+\tau^3)\mathcal{Q}(R)e^{\tau(1+\tau)\mathcal{Q}(R)}\|z_1 - z_2\|^2_{\mathcal{H}}, \quad \forall x \in [0, \tau].$$

The proof is finished. □

6. The solution semigroup

For any given initial datum $z \in \mathcal{H}$, we now write the unique solution $(u(t), \eta^t)$ to (3.1) at time $t > 0$ with initial conditions

$$(u(0), \partial_t u(0), \eta^0) = z$$

in the form

$$(u(t), \partial_t u(t), \eta^t) = S(t)z,$$

where the map $S(t) : \mathcal{H} \to \mathcal{H}$ is a semigroup of operators, namely,

- $S(0) = \mathrm{id}_{\mathcal{H}}$ (the identity map in $\mathcal{H}$),
- $S(t+\tau) = S(t)S(\tau)$ for all $t, \tau \ge 0$.

Collecting Theorems 4.1 and 5.1, we can formulate the following result.

Theorem 6.1. *The semigroup $S(t)$ satisfies the strong continuity property*

$$z \mapsto S(t)z \in \mathcal{C}(\mathcal{H}, \mathcal{H}), \quad \forall t \ge 0.$$

Besides, $S(t)$ fulfills the further continuity in time

$$t \mapsto S(t)z \in \mathcal{C}([0, \infty), \mathcal{H}), \quad \forall z \in \mathcal{H}.$$

Remark 6.2. Actually, it is easily seen that $S(t)$ is jointly continuous, that is,

$$(t, z) \mapsto S(t)z \in \mathcal{C}([0, \infty) \times \mathcal{H}, \mathcal{H}).$$

In addition, by the a priori estimates of Proposition 4.2 we also learn that the solution fulfills

$$u \in W^{2,\infty}(0, \infty; \mathrm{H}^1), \qquad \eta \in L^\infty(0, \infty; \mathcal{M})$$

and, if $\gamma > 0$,

$$\partial_t u \in L^2(0, \infty; \mathrm{H}^\theta).$$

The latter relation is an immediate consequence of (4.1)–(4.2).

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