# CORRIGENDUM TO "MULTIPLETS OF REPRESENTATIONS, TWISTED DIRAC OPERATORS AND VOGAN'S CONJECTURE IN AFFINE SETTING".

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We have recently realized that part (2) of Lemma 8.6 of [3] is incorrect. This invalidates the proof of Theorem 8.1 (also appearing on the Introduction as Theorem 1.1). We can however prove the following weaker result (notation is as in [3]; $\phi_{\mathfrak{a}}$ is defined in (12)).

**Theorem 1.** *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{a}$ a reductive quadratic equal rank subalgebra such that assumption (8) below holds. Fix $\Lambda \in \widehat{\mathfrak{h}}^*$ such that $\Lambda + \widehat{\rho}$ belongs to the Tits cone $C_{\mathfrak{g}}$ and let $M$ be a highest weight module for $\widehat{\mathfrak{g}}$ with highest weight $\Lambda$. Let $f$ be a holomorphic $\widehat{W}$-invariant function on $C_{\mathfrak{g}}$. Suppose that a highest weight $\widehat{L}(\mathfrak{a})$-module of highest weight $\mu$ occurs in the Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0, M)$. Then $f_{|\widehat{\mathfrak{h}}^*_{\mathfrak{a}}}(\mu + \widehat{\rho}_{\mathfrak{a}}) = f(\Lambda + \widehat{\rho})$. In particular, there is $w \in \widehat{W}$ such that $(\phi^*_{\mathfrak{a}})^{-1}(\mu + \widehat{\rho}_{\mathfrak{a}}) = w(\Lambda + \widehat{\rho})$.*

As an example where the hypothesis of Theorem 1 hold, we consider in Proposition 7 the case where $\mathfrak{a} = \mathfrak{h}$.

We specialize the setting of [3] to the case $\sigma = Id$. In particular, we may simplify the notation of [3] letting $\mathfrak{h}$ (rather than $\mathfrak{h}_0$) denote a Cartan subalgebra of $\mathfrak{g}$ and $\rho$ (rather than $\rho_0$) denote the Weyl vector.

Let $\mathcal{W}^k = (U(L'(\mathfrak{g}))/(K-k)) \otimes F^-(\overline{\mathfrak{g}})$ and $\overline{\mathcal{A}}$ be the algebras defined in Sections 8.2, 8.3 of [3]. Then the map $t^r \otimes x \mapsto \tilde{x}_r$, $t^{s-\frac{1}{2}} \otimes y \mapsto \overline{y}_s$ extends to an homomorphism $\Xi : \mathcal{W}^k \to \overline{\mathcal{A}}$. Set $U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}})_p = \{x \in U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}}) \mid \deg(x) = p\}$, where in our context deg can be defined, for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{p}_\beta$, by $\deg(\tilde{x}_r) = ht(r\delta + \alpha)$, $\deg(\overline{y}_s) = ht(s\delta + \beta)$. Recall that $\mathcal{F}$ denotes the algebra of holomorphic functions on $(\mathfrak{h}^* \oplus \mathbb{C}\delta) \times (\mathfrak{h}^* \oplus \mathbb{C}\delta_{\mathfrak{a}})$. Set

$$C_{diag} = \{(\Lambda + c\delta, \Lambda + \rho - \rho_{\mathfrak{a}} + c\delta_{\mathfrak{a}}) \mid \Lambda \in \mathfrak{h}^*,\ c \in \mathbb{C}\},$$

let $\mathcal{I}_{diag} \subset \mathcal{F}$ be the set of functions that are zero when restricted to $C_{diag}$ and set $\mathcal{F}_{|C_{diag}} = \mathcal{F}/\mathcal{I}_{diag}$.

By Lemma 8.5 of [3] the map $x^- \otimes f \otimes x^+ \mapsto \Xi(x^-) f \Xi(x^+) + \overline{\mathcal{A}}^{p+1}$ is an onto map of vector spaces from $U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_{\mathfrak{p}})_-) \otimes \mathcal{F} \otimes U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}})_p$ to $\overline{\mathcal{A}}^p/\overline{\mathcal{A}}^{p+1}$ whose kernel is $U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_{\mathfrak{p}})_-) \otimes \mathcal{I}_{diag} \otimes U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}})_p$. Then we have an isomorphism (of vector spaces) between $U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_{\mathfrak{p}})_-) \otimes \mathcal{F}_{|C_{diag}} \otimes U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}})_p$ and $\overline{\mathcal{A}}^p/\overline{\mathcal{A}}^{p+1}$.

Let $\Phi$ denote the set of holomorphic functions on $\mathfrak{h}^* \oplus \mathbb{C}\delta_{\mathfrak{a}}$. We can embed $\Phi$ in $\mathcal{F}$ by mapping $f \in \Phi$ to $F_f \in \mathcal{F}$, where $F_f$ is defined by setting $F_f(\lambda, \mu) = f(\mu)$. The map $f \mapsto F_f + \mathcal{I}_{diag}$ is an isomorphism from $\Phi$ to $\mathcal{F}_{|C_{diag}}$. It follows that the map from $U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-) \otimes \Phi \otimes U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_\mathfrak{p})_p$ to $\overline{\mathcal{A}}^p/\overline{\mathcal{A}}^{p+1}$ defined by

$$x^- \otimes f \otimes x^+ \mapsto \Xi(x^-)F_f\Xi(x^+) + \overline{\mathcal{A}}^{p+1} \tag{1}$$

is an isomorphism of vector spaces. Since, as a vector space, $\overline{\mathcal{A}}/\overline{\mathcal{A}}^{p+1} = \oplus_{i\le p}\overline{\mathcal{A}}^i/\overline{\mathcal{A}}^{i+1}$, we obtain an isomorphism

$$\overline{\mathcal{A}}/\overline{\mathcal{A}}^{p+1} \simeq U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-) \otimes \mathbf{\Phi} \otimes (\oplus_{n\le p}U((\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_\mathfrak{p})_n)). \tag{2}$$

Since $\overline{\mathcal{A}}^{p+1}$ is homogeneous with respect to deg, we can induce a grading on $\overline{\mathcal{A}}/\overline{\mathcal{A}}^{p+1}$. Note that, by (1), $\deg(x) \le p$ for any homogeneous element $x \in \overline{\mathcal{A}}/\overline{\mathcal{A}}^{p+1}$.

Denote by $\mathcal{A}(\mathfrak{a})$ the subalgebra of $\overline{\mathcal{A}}$ generated by $\{F_f \mid f \in \Phi\}$ and $(\tilde{x}_\mathfrak{a})_r$ with $x \in \mathfrak{a}$, $r \in \mathbb{Z}$. Denote by $\overline{\mathcal{A}}(\mathfrak{a})$ its closure. Let $\mathcal{A}(\mathfrak{n}'_\mathfrak{a})$ be the subalgebra of $\mathcal{A}(\mathfrak{a})$ generated by $(\tilde{x}_\mathfrak{a})_r$ with $t^r \otimes x \in \mathfrak{n}'_\mathfrak{a}$ and let $\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a})$ be the ideal in $\mathcal{A}(\mathfrak{n}'_\mathfrak{a})$ generated by $(\tilde{x}_\mathfrak{a})_r$ with $t^r \otimes x \in \mathfrak{n}'_\mathfrak{a}$.

Set $B(p) = \overline{\mathcal{A}}^{p+1} + \overline{\mathcal{A}}\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a})$ and $\mathcal{B}_p = \overline{\mathcal{A}}/B(p)$. Using PBW theorem, we decompose $U(\mathfrak{n}'_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-)$ and $U(\mathfrak{n}' \oplus \overline{\mathfrak{n}}'_\mathfrak{p})$ as vector spaces as $U((\mathfrak{n}'_\mathfrak{a})_-) \otimes S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-)$, $U(\mathfrak{n}'_\mathfrak{a}) \otimes S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})$ respectively. Set $S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})^p = \oplus_{i\le p}S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})_i$. Using a multi-index notation similar to the one introduced in [3, (8.17)], we set, for $I = \{i_1, i_2, \dots\}$, $\tilde{x}^I = (\tilde{x}_1)^{i_1}(\tilde{x}_2)^{i_2}\dots$ and similarly for $\bar{x}^I$, $\tilde{x}^I_\mathfrak{a}$, $\theta(x)^I$.

**Lemma 2.** *If $v \in \overline{\mathcal{A}}\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a})$, then $v + \overline{\mathcal{A}}^{p+1} = \sum_{I\ne 0} q_I\tilde{x}^I_\mathfrak{a} + \overline{\mathcal{A}}^{p+1}$, where $q_I$ is a sum of elements of type $\Xi(a^-)\Xi(x^-)F_f\Xi(x^+), a^- \in U({\mathfrak{n}'_\mathfrak{a}}^-)$, $x^- \in S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-)$, $f \in \mathbf{\Phi}$, $x^+ \in S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})^p$.*

*Proof.* Write $v = \sum_i P_iQ_i$ with $P_i \in \overline{\mathcal{A}}$ and $Q_i \in \mathcal{A}^+(\mathfrak{n}'_\mathfrak{a})$. Using PBW theorem for $U(\mathfrak{n}'_\mathfrak{a})$ we have that $Q_i = \sum_{I\ne 0} c^i_I\tilde{x}^I_\mathfrak{a}$ with $x^I \in U(\mathfrak{n}'_\mathfrak{a})$ and $c^i_I \in \mathbb{C}$. Thus $v + \overline{\mathcal{A}}^{p+1} = \sum_{I\ne 0}(\sum_i c^i_IP_i)\tilde{x}^I_\mathfrak{a} + \overline{\mathcal{A}}^{p+1}$. Since $\deg(\tilde{x}^I_\mathfrak{a}) > 0$ we have that $\overline{\mathcal{A}}^{p+1}\tilde{x}^I_\mathfrak{a} \subset \overline{\mathcal{A}}^{p+1}$. Using (1) we write $\sum_i c^i_IP_i + \overline{\mathcal{A}}^{p+1} = \sum_j u^I_j + \overline{\mathcal{A}}^{p+1}$, where $u^I_j$ are terms of type $\Xi(a^-)\Xi(x^-)F_f\Xi(x^+)\Xi(a^+)$ and $a^- \in U({\mathfrak{n}'_\mathfrak{a}}^-)$, $x^- \in S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-)$, $a^+ \in U(\mathfrak{n}'_\mathfrak{a})$, $x^+ \in S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})$, $f \in \mathbf{\Phi}$. Hence

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{I\ne 0}(\sum_j u^I_j)\tilde{x}^I_\mathfrak{a} + \overline{\mathcal{A}}^{p+1}.$$

Since $\Xi(a^+)$ is a linear combination of $\tilde{x}^R$ with $x^R \in U(\mathfrak{n}'_\mathfrak{a})$, we write $\sum_j u^I_j = \sum_R q_{R,I}\tilde{x}^R$ with $q_{R,I}$ a finite sum of terms of type $\Xi(a^-)\Xi(x^-)F_f\Xi(x^+)$, $a^- \in U({\mathfrak{n}'_\mathfrak{a}}^-)$, $x^- \in S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-)$, $x^+ \in S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})$, $f \in \mathbf{\Phi}$.

Using the fact that $[\tilde{x}_r, \bar{y}_s] = 0$, we have that

$$\tilde{x}^R = \sum_{M\le R} c_M\sigma(\theta(x)^{R-M})\tilde{x}^M_\mathfrak{a}, \tag{3}$$

where $\sigma(\theta(x^1)^{n_1}\cdots\theta(x^k)^{n_k}) = \theta(x^k)^{n_k}\cdots\theta(x^1)^{n_1}$. Moreover $c_R = 1$. Thus

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{I\ne 0}\sum_R q_{R,I}\tilde{x}^R\tilde{x}^I_\mathfrak{a} = \sum_{I\ne 0}\sum_R\sum_{M\le R} q_{R,I}\sigma(\theta(x)^{R-M})\tilde{x}^M_\mathfrak{a}\tilde{x}^I_\mathfrak{a} + \overline{\mathcal{A}}^{p+1}.$$

Since $\deg(\tilde{x}^M_\mathfrak{a}\tilde{x}^I_\mathfrak{a}) > 0$ we have that $\overline{\mathcal{A}}^{p+1}\tilde{x}^M_\mathfrak{a}\tilde{x}^I_\mathfrak{a} \subset \overline{\mathcal{A}}^{p+1}$. Write

$$\sigma(\theta(x)^{R-M}) + \overline{\mathcal{A}}^{p+1} = \sum_j w^{R,M}_j + \overline{\mathcal{A}}^{p+1},$$

with $w^{R,M}_j$ a sum of terms of type $\Xi(x^-)\Xi(x^+)$ with $x^- \in S({\overline{\mathfrak{n}}'_\mathfrak{p}}^-)$, $x^+ \in S(\overline{\mathfrak{n}}'_\mathfrak{p})$, so we can write $v + \overline{\mathcal{A}}^{p+1} = \sum_{I\ne 0}\sum_R\sum_{M\le R} q_{R,I}\sum_j w^{R,M}_j\tilde{x}^M_\mathfrak{a}\tilde{x}^I_\mathfrak{a} + \overline{\mathcal{A}}^{p+1}$. We note that, if $x \in \mathfrak{p}_\alpha$, then

$$F_f \bar{x}_r = \bar{x}_r F_{f_{\alpha+r\delta_{\mathfrak{a}}}}. \tag{4}$$

Thus, by the defining relations in $\mathcal{A}$, setting $r_{I,M} = \sum_{j,R\geq M} q_{R,I} w_j^{R,M}$, we have that $r_{I,M}$ is a sum of terms of type $\Xi(a^-)\Xi(x^-)F_f\Xi(x^+), a^- \in U(\mathfrak{n}_{\mathfrak{a}}'^-)$, $x^- \in S((\mathfrak{n}'_{\mathfrak{p}})_-\oplus(\overline{\mathfrak{n}}'_{\mathfrak{p}})_-)$, $x^+ \in S(\mathfrak{n}'_{\mathfrak{p}}\oplus\overline{\mathfrak{n}}'_{\mathfrak{p}})$, $f \in \mathbf{\Phi}$. Hence $v+\overline{\mathcal{A}}^{p+1} = \sum_{I\neq 0}\sum_M r_{I,M}\tilde{x}_{\mathfrak{a}}^M\tilde{x}_{\mathfrak{a}}^I+\overline{\mathcal{A}}^{p+1}$. By PBW theorem, we have $\tilde{x}_{\mathfrak{a}}^M\tilde{x}_{\mathfrak{a}}^I = \sum_T c_{T,M,I}\tilde{x}_{\mathfrak{a}}^T$. with $c_{0,M,I} = 0$ (since $I \neq 0$), hence, as wished

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{T\neq 0}(\sum_{M,I} c_{T,M,I} r_{I,M})\tilde{x}_{\mathfrak{a}}^T + \overline{\mathcal{A}}^{p+1}.$$

□

**Lemma 3.** *The map* $\Theta : U(\mathfrak{n}_{\mathfrak{a}}'^-)\otimes S((\mathfrak{n}'_{\mathfrak{p}})_-\oplus(\overline{\mathfrak{n}}'_{\mathfrak{p}})_-)\otimes\mathbf{\Phi}\otimes S(\mathfrak{n}'_{\mathfrak{p}}\oplus\overline{\mathfrak{n}}'_{\mathfrak{p}})^p \to \mathcal{B}_p$ *defined by* $a^-\otimes x^-\otimes f\otimes x^+ \mapsto \Xi(a^-)\Xi(x^-)F_f\Xi(x^+) + B(p)$ *is a linear isomorphism.*

*Proof.* First we check that $\Theta$ is onto. By (1), if $v \in \mathcal{B}_p$ then $v$ is a sum of terms of type

$$\Xi(a^-)\Xi(x^-)F_f\Xi(x^+)\Xi(a^+) + B(p), \tag{5}$$

with $a^- \in U(\mathfrak{n}_{\mathfrak{a}}'^-)$, $x^- \in S((\mathfrak{n}'_{\mathfrak{p}})_-\oplus(\overline{\mathfrak{n}}'_{\mathfrak{p}})_-)$, $a^+ \in U(\mathfrak{n}'_{\mathfrak{a}})$, $x^+ \in S(\mathfrak{n}'_{\mathfrak{p}}\oplus\overline{\mathfrak{n}}'_{\mathfrak{p}})$, $f \in \mathbf{\Phi}$.

We can clearly assume that $a^+$ is a monomial, thus $\Xi(a^+) = \tilde{x}^I$. By (3), there is a constant $c$ such that $\Xi(a^+) = \tilde{x}^I = c\sigma(\theta(x)^I) \mod \overline{\mathcal{A}}\mathcal{A}^+(\mathfrak{n}'_{\mathfrak{a}})$

Remark that, setting $l = \deg(\theta(x)_r)$, then $\theta(x)_r \in \prod_{q\geq l}\mathcal{W}_{l-q}\mathcal{W}_q^+$, thus $\theta(x)_r \in \sum_{q=l}^p \mathcal{W}_{l-q}\mathcal{W}_q^+ \mod \overline{\mathcal{A}}^{p+1}$. Since $\overline{\mathcal{A}}_i^n\overline{\mathcal{A}}_j^m \subset \overline{\mathcal{A}}^t$ with $t = \max(n, m+j)$, we see that there is an element of $u \in S(L(\bar{\mathfrak{p}}))$ such that $\sigma(\theta(x)^I) \equiv \Xi(u) \mod \overline{\mathcal{A}}^{p+1}$. Substituting in (5), we see that $v \in \mathcal{B}_p$ is a sum of terms of type $\Xi(a^-)\Xi(x^-)F_f\Xi(x^+)\Xi(u)+B(p)$ with $u \in S(L(\bar{\mathfrak{p}}))$. It is now clear using the defining relations of $\mathcal{A}$ and (4) that $\Xi(x^-)F_f\Xi(x^+)\Xi(u)$ can be rewritten as a sum of terms of type $\Xi(x'^-)F_{f'}\Xi(x'^+)$ with $x' \in S((\mathfrak{n}'_{\mathfrak{p}})_-\oplus(\overline{\mathfrak{n}}'_{\mathfrak{p}})_-)$, $f' \in \mathbf{\Phi}$, $x'^+ \in S(\mathfrak{n}'_{\mathfrak{p}}\oplus\overline{\mathfrak{n}}'_{\mathfrak{p}})^p$, thus $v$ is in the image of $\Theta$.

We now check that the map $\Theta$ is injective. Assume that $v = \Theta(\sum_i u_i)$ with $u_i$ of type $a^-\otimes x^-\otimes f\otimes x^+$ and that $v \in \overline{\mathcal{A}}^{p+1} + \overline{\mathcal{A}}\mathcal{A}^+(\mathfrak{n}'_{\mathfrak{a}})$. This means that, as in the statement of Lemma 2, $v + \overline{\mathcal{A}}^{p+1} = \sum_{I\neq 0} q_I\tilde{x}_{\mathfrak{a}}^I + \overline{\mathcal{A}}^{p+1}$. Since

$$\tilde{x}_{\mathfrak{a}}^I = \sum_{M\leq I} c_{M,I}\theta(x)^{I-M}\tilde{x}^M, \tag{6}$$

with $c_{I,I} = 1$, we have that

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{I\neq 0} q_I\tilde{x}^I + \sum_M(\sum_{I>M} c_{M,I}q_I\theta(x)^{I-M})\tilde{x}^M + \overline{\mathcal{A}}^{p+1}.$$

Arguing as in the first part of the proof we can show that $q_I\theta(x)^{I-M} + \overline{\mathcal{A}}^{p+1} = q'_{I,M} + \overline{\mathcal{A}}^{p+1}$ with $q'_{I,M}$ a sum of terms of type $\Xi(x'^-)F_{f'}\Xi(x'^+)$ with $x' \in S((\mathfrak{n}'_{\mathfrak{p}})_-\oplus(\overline{\mathfrak{n}}'_{\mathfrak{p}})_-)$, $f' \in \mathbf{\Phi}$, $x'^+ \in S(\mathfrak{n}'_{\mathfrak{p}}\oplus\overline{\mathfrak{n}}'_{\mathfrak{p}})^p$, hence

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{I\neq 0} q_I\tilde{x}^I + \sum_M\sum_{I>M} c_{M,I}q'_{I,M}\tilde{x}^M + \overline{\mathcal{A}}^{p+1}. \tag{7}$$

Let $m = \max\{|I| \mid q_I\tilde{x}_{\mathfrak{a}}^I + \overline{\mathcal{A}}^{p+1} \neq \overline{\mathcal{A}}^{p+1}\}$, then

$$v + \overline{\mathcal{A}}^{p+1} = \sum_{|I|=m} q_I\tilde{x}^I + \sum_{|I|<m} q''_I\tilde{x}^I + \overline{\mathcal{A}}^{p+1},$$

where the coefficients $q''_I$ are obtained from (7) in the obvious way. It follows that

$$\sum_{|I|=m} q_I\tilde{x}^I + \overline{\mathcal{A}}^{p+1} = \Theta(\sum_i u_i) - \sum_{|I|<m} q''_I\tilde{x}^I + \overline{\mathcal{A}}^{p+1}.$$

If $m > 0$, this contradicts (1). Thus $v \in \overline{\mathcal{A}}^{p+1}$. Since $v = \Theta(\sum_i u_i)$ with $u_i$ of type $a^- \otimes x^- \otimes \bar{h} \otimes f \otimes x^+$ this implies, by (1) again, that $\sum_i u_i = 0$ as wished. □

Obviously $\overline{\mathcal{A}}$ acts on $B(p) = \overline{\mathcal{A}}^{p+1} + \overline{\mathcal{A}}\mathcal{A}^+(\mathfrak{n}'_{\mathfrak{a}})$ by left multiplication. In particular we can define an action of $L'(\mathfrak{a})$ on $\mathcal{B}_p = \overline{\mathcal{A}}/B(p)$ by letting $t^r \otimes x$ act by left multiplication by $(\tilde{x}_{\mathfrak{a}})_r$ and $K_S$ by $(k + g - g_S)I$.

Set $\Omega_{\mathfrak{a}} = (\tilde{L}^{\mathfrak{a}})_0 + (k+g)d_{\mathfrak{a}}$. Since $\deg((\tilde{L}^{\mathfrak{a}})_0) = \deg(\Omega_{\mathfrak{a}}) = 0$, bracketing with $(\tilde{L}^{\mathfrak{a}})_0$ and $\Omega_{\mathfrak{a}}$ leaves $\overline{\mathcal{A}}^p$ stable. This implies that that $[\Omega_{\mathfrak{a}}, B(p)] \subset B(p)$. Indeed, if $v \in B(p)$, then, by Lemma 2, $v + \overline{\mathcal{A}}^{p+1} = \sum_{I\neq 0} q_I\tilde{x}^I_{\mathfrak{a}} + \overline{\mathcal{A}}^{p+1}$, hence

$$[\Omega_{\mathfrak{a}}, v] + B(p) = [\Omega_{\mathfrak{a}}, \sum_{I\neq 0} q_I\tilde{x}^I_{\mathfrak{a}}] + B(p) = (\sum_{I\neq 0} q_I\tilde{x}^I_{\mathfrak{a}})q + B(p)$$

where $q = \sum_i \tilde{h}^i_{\mathfrak{a}0}\tilde{h}^i_{\mathfrak{a}0} + (\tilde{h}_{2\rho_{\mathfrak{a}}})_{\mathfrak{a}0} + (k+g)d_{\mathfrak{a}}$. We can assume that $\tilde{x}^I_{\mathfrak{a}}$ are weight vectors of weight $\mu_I$ under the action of $\{\tilde{h}_{\mathfrak{a}0} \mid h \in \mathfrak{h}\} \cup \mathbb{C}d_{\mathfrak{a}}$. Therefore

$$[\Omega_{\mathfrak{a}}, v] + B(p) = (\sum_{I\neq 0} q_I q_{-\mu_I}\tilde{x}^I_{\mathfrak{a}}) + B(p) = B(p).$$

Thus bracketing with $\Omega_{\mathfrak{a}}$ defines an operator $\Omega^p$ on $\mathcal{B}_p$.

Since $\deg(F_f) = 0$ for $f \in \Phi$, we have that $\overline{\mathcal{A}}^{p+1}F_f \subset \overline{\mathcal{A}}^{p+1}$. The argument above shows that $B(p)F_f \subset B(p)$, hence we can define a right action of $\Phi$ on $\mathcal{B}_p$. In particular, we might consider $h \in \mathfrak{h} \oplus \mathbb{C}d_{\mathfrak{a}}$ as a function in $\Phi$, so we have a well defined action $ad$ of $\mathfrak{h} + \mathbb{C}d_{\mathfrak{a}}$ on $\mathcal{B}_p$ given by $ad(h)(x) = hx - x\cdot h$, $x \in \mathcal{B}_p$.

For $f \in \Phi$, set

$$\mathcal{B}_p[f] = \{v \in \mathcal{B}_p \mid \Omega^p(v) = v \cdot f\}.$$

**Proposition 4.** *Assume that for all $p$*

$$\mathcal{B}_p = U(L'(\mathfrak{a}))(\mathcal{B}_p^{\mathfrak{n}'_{\mathfrak{a}}}). \tag{8}$$

*Then*

$$\mathcal{B}_p = \bigoplus_{f\in\Phi} \mathcal{B}_p[f].$$

*Proof.* We first show that the spaces $\mathcal{B}_p[f]$ generate $\mathcal{B}_p$. Let $x \in \mathcal{B}_p^{\mathfrak{n}'_{\mathfrak{a}}}$. We can assume that $x$ is a weight vector for the action $ad$ of $\mathfrak{h} + \mathbb{C}d_{\mathfrak{a}}$ on $\mathcal{B}_p$ of weight $\mu \in \mathfrak{h}^* \oplus \mathbb{C}\delta_{\mathfrak{a}}$. If $x = v + B(p)$, then $\Omega^p(v + B(p)) = [\Omega_{\mathfrak{a}}, v] + B(p)$, hence

$$\Omega^p(x) = [\tilde{L}^{\mathfrak{a}} + (k+g)d_{\mathfrak{a}}, v] + B(p) = F_q x - xF_q + B(p).$$

where $q = \sum_i \tilde{h}^i_{\mathfrak{a}0}\tilde{h}^i_{\mathfrak{a}0} + (\tilde{h}_{2\rho_{\mathfrak{a}}})_{\mathfrak{a}0} + (k+g)d_{\mathfrak{a}}$. Hence

$$\Omega^p(x) = v(F_{q_\mu} - F_q) + B(p) = x \cdot (q_\mu - q).$$

Since $[\Omega_{\mathfrak{a}}, U(L'(\mathfrak{a}))] = 0$ we obtain the our claim.

We now check that the sum is direct. Assume that $v \cdot f = 0$ for a nonzero $f \in \Phi$ and $v \in \mathcal{B}_p$. Lemma 3 identifies $\mathcal{B}_p$ with $U(\mathfrak{n}'^-_{\mathfrak{a}}) \otimes S((\mathfrak{n}'_{\mathfrak{p}})_- \oplus (\overline{\mathfrak{n}}'_{\mathfrak{p}})_-) \otimes \boldsymbol{\Phi} \otimes S(\mathfrak{n}'_{\mathfrak{p}} \oplus \overline{\mathfrak{n}}'_{\mathfrak{p}})^p$ so we can write accordingly $v = \sum_i \Theta(a^-_i \otimes x^-_i \otimes f_i \otimes x^+_i)$ and assume that the weight of $x^+_i$ under the action of $\mathfrak{h} \oplus \mathbb{C}d_{\mathfrak{a}}$ is $\mu_i$. Here $h \in \mathfrak{h}$ acts by the adjoint action of $\tilde{h}_{\mathfrak{a}0}$. Then $v \cdot f = \sum_i \Theta(a^-_i \otimes x^-_i \otimes \bar{h}_i \otimes f_i f_{-\mu_i} \otimes x^+_i)$ By Lemma 3, $v \cdot f = 0$ implies that $f_i f_{-\mu_i} = 0$ for all $i$. Since $f$ is nonzero, then clearly $f_{-\mu_i}$ is nonzero. Since $f, f_i$ are holomorphic functions $f_i f_{-\mu_i} = 0$ implies $f_i = 0$ for all $i$. Thus, if

$v \cdot f = 0$ for a nonzero $f \in \Phi$, then $v = 0$. A standard argument of linear algebra proves by induction on $n$ that, if $\sum_{i=1}^{n} v_i = 0$ with $v_i \in \mathcal{B}_p[f_i]$ and $f_i \neq f_j$ for $i \neq j$ then $v_i = 0$ for all $i$. □

Set $G_0 = (G_{\mathfrak{g},\mathfrak{a}})_0$. Since $\deg(G_0) = 0$, $[G_0, \overline{\mathcal{A}}^p] \subset \overline{\mathcal{A}}^p$. Moreover

$$[G_0, \overline{\mathcal{A}}\,\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a})] \subseteq [G_0, \overline{\mathcal{A}}]\,\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a}) \subseteq \overline{\mathcal{A}}\,\mathcal{A}^+(\mathfrak{n}'_\mathfrak{a}).$$

It follows that bracketing with $G_0$ stabilizes $B(p)$, hence induces a map $\bar{d}_p$ on $\mathcal{B}_p$. We set

$$\mathcal{B}_p^{inv} = Ker(\Omega^p).$$

It follows from Proposition 4 that we can write

$$\mathcal{B}_p = \mathcal{B}_p^{inv} \oplus W,$$

where $W = \bigoplus_{f \neq 0} \mathcal{B}_p[f]$. Clearly $W$ is stable under the action of $\Omega^p$.

**Lemma 5.**

(1) *$\mathcal{B}_p^{inv}$ and $W$ are $\bar{d}_p$-stable.*
(2) *$\bar{d}_p^2 = 0$ on $\mathcal{B}_p^{inv}$.*

*Proof.* The first statement is clear since $[G_0, \Omega^p] = 0$.

For the second statement, we start by observing that (3.12) in [3] says that $[\tilde{L}_0^{\mathfrak{g}}, \tilde{x}_r] = -(k+g)r\tilde{x}_r$ and $[\tilde{L}_0^{\mathfrak{g}}, \overline{x}_r] = 0$. By the definition of the product in $\overline{\mathcal{A}}$ and formula (3.10) of [3] (with $s_i = 0$), we see that $[\tilde{L}_0^{\mathfrak{g}}, f] = 0$ for $f$ in $\mathcal{F}$. On the other hand $[d, \tilde{x}_r] = r\tilde{x}_r$, $[d, \overline{x}_r] = 0$ and $[d, f] = 0$ for $f$ in $\mathcal{F}$. Thus bracketing with $d$ and $\tilde{L}_0^{\mathfrak{g}}$ stabilizes the subalgebra of $\overline{\mathcal{A}}$ generated by $\tilde{x}_r, \overline{x}_r, f$ and if $x$ is in this subalgebra, then $[\tilde{L}_0^{\mathfrak{g}}, x] = -(k+g)[d, x]$. By [3, Lemma 8.5] this subalgebra is dense in $\overline{\mathcal{A}}$, hence $[\tilde{L}_0^{\mathfrak{g}}, x] = -(k+g)[d, x]$ for all $x \in \overline{\mathcal{A}}$.

Now notice that, if $x + B(p) \in \mathcal{B}_p^{inv}$, then $\bar{d}_p^2(x + B(p)) = [G_0^2, x] + B(p)$, $G_0$ being odd. It follows from [3, (4.18)] that

$$[G_0^2, x] = [\tilde{L}_0^{\mathfrak{g}} - \tilde{L}_0^{\mathfrak{a}} - (k+g)L_0^{\overline{\mathfrak{p}}}, x] = -[\tilde{L}_0^{\mathfrak{a}} + (k+g)(d + L_0^{\overline{\mathfrak{p}}}), x] = -[\Omega_\mathfrak{a}, x] \in B(p)$$

as desired. □

Identify $S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-) \otimes S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})$ with $S(L(\mathfrak{p})) \otimes \wedge L(\overline{\mathfrak{p}})$ (as vector spaces). We introduce an increasing filtration $\mathcal{B}_p[0] \subset \mathcal{B}_p[1] \subset \cdots \subset \mathcal{B}_p[n] \subset \ldots$ on $\mathcal{B}_p$ by setting $\mathcal{B}_p[n] = \Theta(U(\mathfrak{n}_\mathfrak{a}^-) \otimes \mathbf{\Phi} \otimes K^p[n])$ where

$$K^p[n] = (\oplus_{m \leq n} S^m(L(\mathfrak{p})) \otimes S(L(\overline{\mathfrak{p}}))) \cap (S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-) \otimes S(\mathfrak{n}'_\mathfrak{p} \oplus (\overline{\mathfrak{n}}'_\mathfrak{p}))^p).$$

Observe that the corresponding graded space is $Gr(\mathcal{B}_p) = \oplus_n Gr_n(\mathcal{B}_p)$ where

$$Gr_n(\mathcal{B}_p) = U((\mathfrak{n}_\mathfrak{a})_-) \otimes \mathbf{\Phi} \otimes K_n^p,$$
$$K_n^p = (S^n(L(\mathfrak{p})) \otimes S(L(\overline{\mathfrak{p}}))) \cap (S((\mathfrak{n}'_\mathfrak{p})_- \oplus (\overline{\mathfrak{n}}'_\mathfrak{p})_-) \otimes S(\mathfrak{n}'_\mathfrak{p} \oplus \overline{\mathfrak{n}}'_\mathfrak{p})^p).$$

This filtration can be defined in a more natural way as follows: introduce a filtration $\mathcal{W}[0] \subset \mathcal{W}[1] \subset \cdots \subset \mathcal{W}[n] \subset \ldots$ on $\mathcal{W}$ by giving degree 1 to $\tilde{x}_r$ if $x \in \mathfrak{p}$ and degree 0 to $\tilde{x}_r$ if $x \in \mathfrak{a}$ and to $\bar{x}_r$ if $x \in \mathfrak{p}$. This filtration induces a filtration on $\mathcal{B}_p$ and, since $L'(\mathfrak{a})$ is a subalgebra of $L'(\mathfrak{g})$, the two filtrations coincide. As a consequence we have that

$$\Omega^p(\mathcal{B}_p[n]) \subset \mathcal{B}_p[n]. \tag{9}$$

This is due to the fact that, if $x \in \mathcal{W}$, then $\Omega^p(x + B(p)) = [\Omega_\mathfrak{a}, x] + B(p) = [y, x] + B(p)$ for some $y \in \mathcal{W}[0]$ (indeed $y = \Omega_\mathfrak{a} \mod \mathcal{A}^q$, $q \gg 0$).

Let $\delta_\mathfrak{p} : S(L(\mathfrak{p})) \otimes \wedge L(\overline{\mathfrak{p}}) \to S(L(\mathfrak{p})) \otimes \wedge L(\overline{\mathfrak{p}})$ be the Koszul differential. Since $[G_0, F_f] = 0$ for $f \in \mathbf{\Phi}$, as in the proof of Lemma 8.7 of [3] we can show that the map induced by $\bar d_p$ on $Gr(\mathcal{B}_p)$ is the restriction to $\oplus_n(U((\mathfrak{n}'_\mathfrak{a})_-)\otimes\mathbf{\Phi}\otimes K^p_n)$ of $Id\otimes Id\otimes\delta_\mathfrak{p}$. It follows that, if $x\in\mathcal{B}_p[n]$ is such that $\bar d_p(x)=0$, then

$$x = x_\mathfrak{a} + \bar d_p(y) + u \tag{10}$$

with $x_\mathfrak{a}\in\Theta(U(\mathfrak{n}'^-_\mathfrak{a})\otimes\mathbf{\Phi})$ and $u,y\in\mathcal{B}_p[n-1]$.

**Proposition 6.** *If $x\in\mathcal{B}_p^{inv}$ and $\bar d_p(x)=0$ then there are $a\in\overline{\mathcal{A}}(\mathfrak{a})$ and $y\in\mathcal{B}_p^{inv}$ such that*

$$x = a + B(p) + \bar d_p(y).$$

*Proof.* Assume $x\in\mathcal{B}_p[n]$. The proof will be by induction on $n$.

First assume $x\in\mathcal{B}_p[0]$. Then, by (10), we have that $x=x_\mathfrak{a}$. Write $x_\mathfrak{a}=x'_\mathfrak{a}+B(p)$ where $x'_\mathfrak{a}=\sum_I\tilde x^I F_{q_I}$ with $\tilde x^I\in U(\mathfrak{n}'^-_\mathfrak{a})$ and $q_I\in\mathbf{\Phi}$. Let $m=\max\{|I|\mid q_I\neq 0\}$. Similarly to (3), we have

$$\tilde x^I = \sum_{M\le I} a_M\tilde x^M_\mathfrak{a}\sigma(\theta(x)^{I-M}) \tag{11}$$

with $a_I=1$, hence we can rewrite $x'_\mathfrak{a}$ as $x'_\mathfrak{a}=\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+\sum_{|I|<m}\tilde x^I_\mathfrak{a}q'_I$, with $q'_I\in\overline{\Xi(S(L(\bar{\mathfrak{p}})))\mathcal{F}_\mathbf{\Phi}}$, where $\mathcal{F}_\mathbf{\Phi}=\{F_f\mid f\in\mathbf{\Phi}\}$. Applying (6) we can rewrite this as $x'_\mathfrak{a}=\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+\sum_{|I|<m}\tilde x^I q''_I$, with $q''_I\in\overline{\Xi(S(L(\bar{\mathfrak{p}})))\mathcal{F}_\mathbf{\Phi}}$. Thus $x_\mathfrak{a}=\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+\sum_{|I|<m}\tilde x^I q'''_I+B(p)$, with $q'''_I\in\Xi(S(L(\bar{\mathfrak{p}})))\mathcal{F}_\mathbf{\Phi}$. Since $x_\mathfrak{a}$ and $\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+B(p)$ are both in $\mathcal{B}_p^{inv}$, we have that $\sum_{|I|<m}\tilde x^I q'''_I+B(p)\in\mathcal{B}_p^{inv}$. Since $0=\bar d_p(x_\mathfrak{a})=\bar d_p(\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+B(p))$ we have that $\bar d_p(\sum_{|I|<m}\tilde x^I q'''_I+B(p))=0$. On $\mathcal{B}_p[0]$ the differential $\bar d_p$ is just $Id\otimes Id\otimes\delta_\mathfrak{p}$. By exactness of $\delta_p$ we find that $q'''_I\in\mathcal{F}_\mathbf{\Phi}$. By an obvious induction on $m$ we deduce that $x_\mathfrak{a}=a+B(p)$ for some $a\in\overline{\mathcal{A}}(\mathfrak{a})$.

Assume now $n>0$. Then, by (10), we have that $x=x_\mathfrak{a}+\bar d_p(y)+u$ with $y,u\in\mathcal{B}_p[n-1]$. Arguing as above we have $x_\mathfrak{a}=\sum_{|I|=m}\tilde x^I_\mathfrak{a}q_I+\sum_{|I|<m}\tilde x^I q'''_I+B(p)$. Note that $\sum_{|I|<m}\tilde x^I q'''_I+B(p)\in\mathcal{B}_p[0]$.

Setting $u'=u+\sum_{|I|<m}\tilde x^I q'''_I+B(p)$, we have that $x=a+B(p)+\bar d_p(y)+u'$ with $y,u'\in\mathcal{B}_p[n-1]$. We can write $y=y_0+y'$ and $u'=u_0+u''$ with $y_0,u_0\in\mathcal{B}_p^{inv}$ and $y',u''\in W$. Since $x\in\mathcal{B}_p^{inv}$ we can write $x=a+B(p)+\bar d_p(y_0)+u_0$ so, since $\bar d_p(x)=\bar d_p(a+B(p))=0$, we have that $\bar d_p(u_0)=0$.

By (9) we can assume $u_0\in\mathcal{B}_p[n-1]$, hence we can apply the induction hypothesis, obtaining $u_0=a'+B(p)+\bar d_p(z)$ and proving that $x=a+a'+B(p)+\bar d_p(y_0+z)$. with $a+a'\in\overline{\mathcal{A}}(\mathfrak{a})$. □

We now come to the proof of the main result. Let $C_\mathfrak{g}$ be the Tits cone of $\widehat{L}(\mathfrak{g})$. Let $\phi_\mathfrak{a}:\widehat{\mathfrak{h}}_\mathfrak{a}\to\widehat{\mathfrak{h}}$ be the map defined by

$$\phi_{\mathfrak{a}|\mathfrak{h}} = Id_\mathfrak{h},\quad \phi_\mathfrak{a}(d_\mathfrak{a})=d\quad d_\mathfrak{a}(K_S)=K\ \text{ for all}\, S. \tag{12}$$

If $f$ is a function on $C_\mathfrak{g}$ we denote by $f_{|\widehat{\mathfrak{h}}^*_\mathfrak{a}}$ the function on $\phi^*_\mathfrak{a}(C_\mathfrak{g})\cap\widehat{\mathfrak{h}}^*_\mathfrak{a}$ defined by $f_{|\widehat{\mathfrak{h}}^*_\mathfrak{a}}(\lambda)=(f\circ(\phi^*_\mathfrak{a})^{-1})(\lambda)$.

*Proof of Theorem 1.* Recall that the Dirac cohomology $H((G_{\mathfrak{g},\mathfrak{a}})_0,M)$ is the $\widehat{L}(\mathfrak{a})$-module $Ker\,G_0/Ker\,G_0\cap Im\,G_0$, where $G_0$ is seen as an operator on $M\otimes F(\bar{\mathfrak{p}})$.

If $v\in M\otimes F(\bar{\mathfrak{p}})$ is a weight vector of weight $\sum k_S\Lambda_0^S+\nu$ and $q\in\mathbf{\Phi}$, we define an action of $F_q$ on $v$ by setting $F_q\cdot v=q(\nu)v$. This extends the action of $\widehat{L}(\mathfrak{a})$ on $M\otimes F(\bar{\mathfrak{p}})$ to an action of $\mathcal{A}(\mathfrak{a})$. As in § 8.5 of [3] we get the existence of a central element $z_f$ of $\overline{\mathcal{A}}$ such that $z_f\cdot v=f(\Lambda+\widehat{\rho})v$ for any $v\in M\otimes F(\bar{\mathfrak{p}})$. Let $v_0+Ker\,G_0^M\cap Im\,G_0^M$ be the highest vector of a $\widehat{L}(\mathfrak{a})$-submodule of $H((G_{\mathfrak{g},\mathfrak{h}})_0,M)$ with highest weight $\mu=\sum_S(k+g-g_S)\Lambda_0^S+\bar\mu$ with $\bar\mu\in\mathfrak{h}^*+\mathbb{C}\delta_\mathfrak{a}$. Choose $p$ big enough so that $\overline{\mathcal{A}}^{p+1}v_0=0$. Since $z_f$ is central we have that $z_f+B(p)\in\mathcal{B}_p^{inv}$

and $\bar{d}_p(z_f) = 0$. Applying Proposition 6 we can write $z_f = a + [G_0, y] + u$ with $a \in \overline{\mathcal{A}}(\mathfrak{a})$ and $u \in B(p)$. It follows that $f(\Lambda + \widehat{\rho})v_0 = z_f v_0 = av_0 + G_0 y v_0 + uv_0 = av_0 + G_0 y v_0$. Since both $v_0$ and $av_0$ are in $KerG_0$ we see that $G_0 y v_0 \in Ker\, G_0$ so $f(\Lambda + \widehat{\rho})(v_0 + Ker\, G_0^M \cap Im\, G_0^M) = a \cdot (v_0 + Ker\, G_0^M \cap Im\, G_0^M)$.

On the other hand, since $\deg(z_f) = 0$ we can assume $\deg(a) = 0$ so $a = F_q + a'$ with $q \in \mathbf{\Phi}$ and $a' \in \overline{\mathcal{A}}(\mathfrak{a})\mathcal{A}^+(\mathfrak{n}'_{\mathfrak{a}})$. It follows that $f(\Lambda + \widehat{\rho}) = q(\bar{\mu})$.

By Corollary 7.2 of [3], $v_\Lambda \otimes 1 + Ker\, G_0^M \cap Im\, G_0^M$ is the highest vector for a nonzero $\widehat{L}(\mathfrak{a})$-submodule of $H((G_{\mathfrak{g},\mathfrak{h}})_0, M)$ having highest weight $\Lambda + \widehat{\rho} - \widehat{\rho}_{\mathfrak{a}}$. It follows that $f(\Lambda+\widehat{\rho}) = q(\bar{\Lambda}+\rho-\rho_{\mathfrak{a}})$ for any $\Lambda \in -\widehat{\rho}+C_{\mathfrak{g}}$. Since $\mu+\widehat{\rho}_{\mathfrak{a}} \in C_{\mathfrak{g}}$, it follows that $f(\mu + \widehat{\rho}_{\mathfrak{a}}) = q(\bar{\mu}) = f(\Lambda + \widehat{\rho})$. and the first statement is proven. The second statement follows from a theorem of Looijenga [1], asserting that holomorphic $\widehat{W}$-invariant functions separate the orbits of the action of $\widehat{W}$ on $C_{\mathfrak{g}}$. □

**Proposition 7.** *If $k + g \neq 0$ then, as a $L'(\mathfrak{h})$-module, $\mathcal{B}_p$ is generated by $\mathcal{B}_p^{\mathfrak{n}'_{\mathfrak{h}}}$.*

*Proof.* Write for simplicity $h_r$ for $(\tilde{h}_{\mathfrak{h}})_r$, $h \in \mathfrak{h}$. Consider the infinite Heisenberg subalgebra $\mathfrak{s} = \sum_{r\neq 0} t^r \otimes \mathfrak{h} \oplus \mathbb{C}K_{\mathfrak{h}}$ of $L'(\mathfrak{h})$. Recall that $K_{\mathfrak{h}}$ acts as on $\mathcal{B}_p$ as $(k + g)Id$. Note that $\mathcal{B}_p^{\mathfrak{n}'_{\mathfrak{h}}}$ is the set $x \in \mathcal{B}_p$ such that $(t^r \otimes h)x = 0$ fon any $h \in \mathfrak{h}$ and $r > 0$. By Lemma 9.13 of [2] it suffices to check that given $x \in \mathcal{B}_p$ then there is $N$ such that $(h^1)_{i_1} \dots (h^n)_{i_n} \cdot x = 0$ whenever $i_j > 0$ for all $j$ and $n > N$ $(h^i \in \mathfrak{h})$. We can clearly assume that $x$ is homogeneous with respect to deg. Then it is enough to choose $N = -\deg(x)+p$, for, if $n > N$, then $\deg((h^1)_{i_1} \dots (h^n)_{i_n}) = (i_1 + \dots + i_n) \geq n > N$. It follows that $\deg((h^1)_{i_1} \dots (h^n)_{i_n} x) > N + \deg(x) = p$ so $(h^1)_{i_1} \dots (h^n)_{i_n} x = 0$. □

## References


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