

TOTALLY DISSIPATIVE DYNAMICAL PROCESSES AND THEIR UNIFORM GLOBAL ATTRACTORS

VLADIMIR V. CHEPYZHOV

Institute for Information Transmission Problems - Russian Academy of Sciences
Bolshoy Karetniy 19, Moscow 101447, Russia
National Research University Higher School of Economics
Myasnitckaya Street 20, Moscow 101000, Russia

MONICA CONTI AND VITTORINO PATA

Politecnico di Milano - Dipartimento di Matematica "F. Brioschi"
Via Bonardi 9, 20133 Milano, Italy

(Dedicated to the Memory of Professor Mark Iosifovich Vishik)

Received February 2013; revised April 2013.

E-mail address: `chep@iitp.ru` (V.V. Chepyzhov) *E-mail address:* `monica.conti@polimi.it` (M. Conti) *E-mail address:* `vittorino.pata@polimi.it` (V. Pata)

1. Introduction. Let (X, d) be a metric space, not necessarily complete. A family of maps

$$U(t, \tau) : X \rightarrow X$$

depending on two real parameters $t \geq \tau$ is said to be a *dynamical process*, or more simply a *process*, on X whenever

- $U(\tau, \tau) = \text{id}_X$ (the identity map in X) for all $\tau \in \mathbb{R}$;
- $U(t, \tau) = U(t, s)U(s, \tau)$ for all $t \geq s \geq \tau$.

Such a notion is particularly useful to describe the solutions to nonautonomous differential equations in normed spaces. Indeed, assume to have the equation

$$\frac{d}{dt}u(t) = \mathcal{A}(t, u(t)), \tag{1}$$

where, for every fixed $t \in \mathbb{R}$, $\mathcal{A}(t, \cdot)$ is a (possibly nonlinear) densely defined operator on a normed space X . If the Cauchy problem for (1) is well posed, in some weak sense, for all times $t \geq \tau$ and all initial data $u_0 \in X$ taken at the initial time $\tau \in \mathbb{R}$,

Work partially supported by the Russian Foundation of Basic Researches (projects 11-01-00339 and 12-01-00203) and the Russian Ministry of Education and Science (agreement no. 8502).

then the corresponding solution $u(t)$ on the time-interval $[\tau, \infty)$ with $u(\tau) = u_0$ reads

$$u(t) = U(t, \tau)u_0,$$

where $U(t, \tau)$ is uniquely determined by the equation, and it is easily seen to satisfy the two properties above. Within this picture, autonomous systems are just a particular case, occurring when the operator $\mathcal{A}(t, \cdot)$ is constant in time. In that situation, the evolution depends only on the difference $t - \tau$. In other words, the equality

$$U(t, \tau) = U(t - \tau, 0)$$

holds for every $t \geq \tau$, and the one-parameter family of maps

$$S(h) = U(h, 0), \quad h \geq 0,$$

fulfills the semigroup axioms, i.e.

- $S(0) = \text{id}_X$;
- $S(h + r) = S(h)S(r)$ for all $h, r \geq 0$.

Summarizing, we may say that dynamical processes extend the concept of dynamical semigroups for the evolution of open models where time-dependent external excitations are present.

When dealing with differential problems arising from concrete evolutionary phenomena, we are usually in presence of some dissipation mechanism. Adopting a global-geometrical point of view, the theory of dissipative dynamical systems describes this situation in terms of small sets of the phase space able to attract in a suitable sense the trajectories arising from bounded regions of initial data. In particular, it is interesting to locate the smallest set where the whole asymptotic dynamics is eventually confined. For autonomous systems, such a set is called the global attractor (we address the reader to the classical books [1, 9, 10, 13] for more details). A similar concept can be used in connection with nonautonomous systems. Namely, it is possible to extend the notion of global attractor for dynamical processes, or even families of dynamical processes (see [3, 4, 7, 10, 12]). In this note, paralleling what done in [2] for the semigroup case, we aim to reconsider the theory of global attractors for families of dynamical processes by defining the basic objects (e.g. the attractor) only in term of their attraction properties, without appealing to any continuity-like notion (see Sections 2-4). In fact, imposing further conditions on the processes, but still much weaker than continuity, it is possible to recover the classical characterization given in [7] of the attractor in terms of kernel sections of complete bounded trajectories (see Sections 5-6). In the final Sections 7-8, we discuss an application to nonautonomous differential equations.

Notation. For every $\varepsilon > 0$, the ε -neighborhood of a set $B \subset X$ is defined as

$$\mathcal{O}_\varepsilon(B) = \bigcup_{x \in B} \{\xi \in X : d(x, \xi) < \varepsilon\}.$$

We denote the standard Hausdorff semidistance of two (nonempty) sets $B, C \subset X$ by

$$\delta_X(B, C) = \sup_{x \in B} d(x, C) = \sup_{x \in B} \inf_{\xi \in C} d(x, \xi).$$

We have also the equivalent formula

$$\delta_X(B, C) = \inf \{\varepsilon > 0 : B \subset \mathcal{O}_\varepsilon(C)\}.$$

2. Families of dissipative processes. Rather than a single process, given another metric space Σ we will consider a family of processes

$$\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma}.$$

The parameter σ is called the *symbol* of $U_\sigma(t, \tau)$, whereas Σ is said to be the *symbol space*. A single process $U(t, \tau)$ can be clearly viewed as a family of processes with a symbol space made of one element. Usually, in connection with nonautonomous differential problems, the symbol is the collection of all explicitly time-dependent terms appearing in the equations (see [3, 4, 7]). Besides, the symbol itself may evolve in time in such a way that, in combination with the process evolution, gives rise to an autonomous dynamical system acting on the extended phase space $X \times \Sigma$, called the *skew-product flow* or *skew-product semigroup* (see Section 4). The study of this semigroup gives essential information on the evolution of the original family of processes.

Analogously to the autonomous case, we introduce a number of definitions that extend the concept of dissipation to the more general nonautonomous situation. In what follows, the word *uniform* will be always understood with respect to $\sigma \in \Sigma$.

Definition 1. A set $B \subset X$ is *uniformly absorbing* for $U_\sigma(t, \tau)$ if for every bounded set $C \subset X$ there exists a (uniform) entering time $t_e = t_e(C)$ such that

$$U_\sigma(t, \tau)C \subset B, \quad \forall \sigma \in \Sigma,$$

whenever $t - \tau \geq t_e$.

The existence of bounded absorbing sets translates in mathematical terms the fact that a given system is dissipative.

Definition 2. The family of processes $U_\sigma(t, \tau)$ is said to be *uniformly dissipative* if there is a bounded uniformly absorbing set.

Uniform dissipativity is however a rather poor notion of dissipation for a process, unless one can prove the existence of reasonably small (e.g. compact or even of finite fractal dimension) uniformly absorbing sets. In concrete differential problems, this is out of reach if, for instance, the system exhibits some hyperbolicity, which prevents the regularization of initial data. Hence, a weaker object, albeit more effective, than a uniformly absorbing set should be considered in order to depict the confinement of the longterm dynamics.

Definition 3. A set $K \subset X$ is called a *uniformly ε -absorbing set*, for some $\varepsilon > 0$, if its ε -neighborhood $\mathcal{O}_\varepsilon(K)$ is a uniformly absorbing set. If K is a uniformly ε -absorbing set for all $\varepsilon > 0$, then it is called a *uniformly attracting set*.

The latter definition can be more conveniently given in terms of Hausdorff semidistance: a set K is uniformly attracting if for any bounded set $C \subset X$ we have the limit relation

$$\lim_{t-\tau \rightarrow \infty} \left[\sup_{\sigma \in \Sigma} \delta_X(U_\sigma(t, \tau)C, K) \right] = 0. \quad (2)$$

Remark. Actually, the definition of a uniformly attracting set given in [7] (as well as the one of a uniformly absorbing set) is a little different: indeed, the limit relation (2) is there replaced by

$$\lim_{t \rightarrow \infty} \left[\sup_{\sigma \in \Sigma} \delta_X(U_\sigma(t, \tau)C, K) \right] = 0,$$

for every *fixed* $\tau \in \mathbb{R}$. Compared to this one, the definition (2) adopted in the present paper is uniform with respect to $\tau \in \mathbb{R}$, which renders the notion of attraction slightly stronger, and more closely related to the concrete examples arising from partial differential equations. Nonetheless, all the results proved in this paper remain valid (with the same proofs) in the framework of [7].

Let now \mathfrak{C}_Σ denote the collection of all possible sequences in X of the form

$$y_n = U_{\sigma_n}(t_n, \tau_n)x_n,$$

where $x_n \in X$ is a bounded sequence, $\sigma_n \in \Sigma$ and $t_n - \tau_n \rightarrow \infty$. For any $y_n \in \mathfrak{C}_\Sigma$, we consider the set

$$L_\Sigma(y_n) = \{x \in X : y_n \rightarrow x \text{ up to a subsequence}\},$$

which in fact can be empty for some y_n , or y_n may contain a subsequence y_{n_i} such that $L_\Sigma(y_{n_i}) = \emptyset$. Accordingly, we can rephrase the attraction property (2) as follows.

Lemma 4. *A set $K \subset X$ is uniformly attracting for the family $U_\sigma(t, \tau)$ if and only if*

$$d(y_n, K) \rightarrow 0, \quad \forall y_n \in \mathfrak{C}_\Sigma.$$

Next, we denote the union of all $L_\Sigma(y_n)$ by

$$A_\Sigma^* = \{x \in X : y_n \rightarrow x \text{ up to a subsequence, for some } y_n \in \mathfrak{C}_\Sigma\},$$

and, for any given bounded set $C \subset X$, we define the *uniform ω -limit* of C by

$$\omega_\Sigma(C) = \bigcap_{h \geq 0} \overline{\bigcup_{\sigma \in \Sigma} \bigcup_{t - \tau \geq h} U_\sigma(t, \tau)C}, \quad (3)$$

the *bar* standing for closure in X . Note that, without additional assumptions, both the sets A_Σ^* and $\omega_\Sigma(C)$ might be empty.

Lemma 5. *The following assertions hold:*

- (i) A_Σ^* is contained in any closed uniformly attracting set.
- (ii) For any bounded set $C \subset X$ we have the inclusion $\omega_\Sigma(C) \subset A_\Sigma^*$. Besides,

$$A_\Sigma^* = \bigcup \omega_\Sigma(C),$$

where the union is taken over all bounded sets $C \subset X$.

- (iii) If $U_\sigma(t, \tau)$ is uniformly dissipative, then for any bounded uniformly absorbing set B and any bounded set C we have the relation

$$\omega_\Sigma(C) \subset \omega_\Sigma(B) = A_\Sigma^*.$$

The latter equality in particular implies that A_Σ^* is closed in X .

Proof. If $x \in A_\Sigma^*$, then $y_n \rightarrow x$ for some $y_n \in \mathfrak{C}_\Sigma$, and we readily obtain from Lemma 4 that

$$d(y_n, K) \rightarrow 0$$

for any uniformly attracting set K . If K is also closed, then $x \in K$, which proves (i). Concerning (ii), the inclusion $\omega_\Sigma(C) \subset A_\Sigma^*$ is straightforward from (3), whereas the subsequent equality comes from the very definition of $L_\Sigma(y_n)$. Indeed,

$$x \in L_\Sigma(y_n) \quad \Rightarrow \quad x \in \omega_\Sigma(\{x_n\}),$$

where x_n is a bounded sequence in X and

$$y_n = U_{\sigma_n}(t_n, \tau_n)x_n \in \mathfrak{C}_\Sigma.$$

Finally, in light of (ii), the equality $\omega_\Sigma(B) = A_\Sigma^*$ in (iii) clearly follows from the inclusion $\omega_\Sigma(C) \subset \omega_\Sigma(B)$. Let then B and C be a bounded uniformly absorbing set and a bounded set, respectively. Then there is $t_e > 0$ such that

$$U_\sigma(\tau + t_e, \tau)C \subset B, \quad \forall \tau \in \mathbb{R}, \forall \sigma \in \Sigma.$$

Therefore, for $t - \tau \geq t_e$ we get

$$U_\sigma(t, \tau)C = U_\sigma(t, \tau + t_e)U_\sigma(\tau + t_e, \tau)C \subset U_\sigma(t, \tau + t_e)B.$$

Accordingly,

$$\bigcup_{t-\tau \geq h+t_e} U_\sigma(t, \tau)C \subset \bigcup_{t-\tau \geq h} U_\sigma(t, \tau)B,$$

and taking the union over all $\sigma \in \Sigma$ and the intersection in $h \geq 0$, from (3) we arrive at the desired inclusion. \square

Among uniformly attracting sets, of particular interest are the compact ones. Hence, following [2], we consider the collection of sets

$$\mathbb{K}_\Sigma = \{K \subset X : K \text{ is compact and uniformly attracting}\}.$$

Using the results above, we establish a necessary and sufficient condition in order for a compact set to be uniformly attracting.

Proposition 6. *Let $K \subset X$ be a compact set. Then $K \in \mathbb{K}_\Sigma$ if and only if*

$$\emptyset \neq L_\Sigma(y_n) \subset K, \quad \forall y_n \in \mathfrak{C}_\Sigma.$$

Proof. If K is uniformly attracting and $y_n \in \mathfrak{C}_\Sigma$, point (i) of Lemma 5 implies that $L_\Sigma(y_n) \subset K$. Besides, by Lemma 4,

$$d(y_n, \xi_n) \rightarrow 0,$$

for some $\xi_n \in K$. Since K is compact, there is $\xi \in K$ such that (up to a subsequence)

$$\xi_n \rightarrow \xi \in K \quad \Rightarrow \quad y_n \rightarrow \xi \quad \Rightarrow \quad L_\Sigma(y_n) \neq \emptyset.$$

Conversely, if K is not attracting,

$$d(y_n, K) > \varepsilon,$$

for some $\varepsilon > 0$ and $y_n \in \mathfrak{C}_\Sigma$. Therefore, $L_\Sigma(y_n) \cap K = \emptyset$. \square

As a straightforward consequence, we deduce a corollary.

Corollary 7. *If $K_1, K_2 \in \mathbb{K}_\Sigma$ then $K_1 \cap K_2 \in \mathbb{K}_\Sigma$.*

As it will be clear in the subsequent section, the collection \mathbb{K}_Σ plays a crucial role in the asymptotic analysis of the process. This motivates the following definition.

Definition 8. The family $U_\sigma(t, \tau)$ is called *uniformly asymptotically compact* if it has a compact uniformly attracting set, i.e. if the collection \mathbb{K}_Σ is nonempty.

Remark. It is apparent that any uniformly asymptotically compact family is in particular uniformly dissipative.

Proposition 9. *If $U_\sigma(t, \tau)$ is uniformly asymptotically compact, then $A_\Sigma^* \in \mathbb{K}_\Sigma$.*

Proof. By assumption, there exists $K \in \mathbb{K}_\Sigma$. Due to Proposition 6, $L_\Sigma(y_n) \neq \emptyset$ for all $y_n \in \mathfrak{C}_\Sigma$ and A_Σ^* is not empty, being the union of all $L_\Sigma(y_n)$. Besides, $A_\Sigma^* \subset K$ by (i) of Lemma 5. Since the family $U_\sigma(t, \tau)$ is uniformly dissipative, by (iii) of Lemma 5 we learn that A_Σ^* is closed, and since K is compact, A_Σ^* is compact as well. Finally, invoking again Proposition 6, we conclude that A_Σ^* is uniformly attracting.

3. Uniform global attractors. Like in the autonomous case, we are interested in finding the minimal compact attracting set. In fact, dealing with nonautonomous systems, the property of minimality turns out to be the natural one to define the (unique) global attractor, since we cannot rely any longer on the invariance property, typical of semigroups. Hence, the following definition sounds even more reasonable in the nonautonomous framework.

Definition 10. A compact set $A_\Sigma \subset X$ is said to be the *uniform global attractor* of the family of processes $U_\sigma(t, \tau)$ if it is uniformly attracting and is contained in any compact uniformly attracting set.

According to the previous discussion, the attractor A_Σ is also uniform with respect to the choice of the initial time $\tau \in \mathbb{R}$.

Remark. It is actually possible to develop a theory of global attractors for locally asymptotically compact semigroups (or processes), which cannot be dissipative in the traditional sense (i.e. existence of a bounded absorbing set). Still, one can prove the existence of a unique locally compact global attractor. In this case, the global attractor is defined to be the smallest closed (instead of compact) attracting set (see [3, 4, 7]).

Proposition 11. *The family $U_\sigma(t, \tau)$ possesses at most one uniform global attractor.*

Proof. By contradiction, suppose not. Then, by virtue of Corollary 7, the intersection of two different uniform global attractors belongs to \mathbb{K}_Σ , which contradicts the minimality property. \square

Remark. For any $\Sigma_0 \subset \Sigma$ we have the inclusion $A_{\Sigma_0} \subset A_\Sigma$ where Σ_0 is the symbol space of the subfamily $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma_0}$ and A_{Σ_0} is the uniform global attractor of this subfamily. In particular, $A_{\{\sigma\}} \subset A_\Sigma$ for any fixed $\sigma \in \Sigma$.

The main existence result for the uniform global attractor reads as follows.

Theorem 12. *If the family $U_\sigma(t, \tau)$ is uniformly asymptotically compact, then it possesses the global attractor A_Σ which coincides with the set A_Σ^* .*

Proof. By Proposition 9, we already know that $A_\Sigma^* \in \mathbb{K}_\Sigma$. Then, we infer from point (i) of Lemma 5 that A_Σ^* is contained in any compact uniformly attracting set, and hence it is the uniform global attractor. \square

Therefore, having a concrete family of processes, the main problem is to construct at least one compact uniformly attracting set. Such a task can be, in general, extremely difficult. However, if the underlying metric space X is complete, there is a more effective way to express asymptotic compactness. We need first a definition.

Definition 13. The family $U_\sigma(t, \tau)$ is called *uniformly ε -dissipative* if there exists a finite uniformly ε -absorbing set. If the family is uniformly ε -dissipative for all $\varepsilon > 0$, then it is called *totally uniformly dissipative*.

Remark. It is readily seen that the family $U_\sigma(t, \tau)$ is totally uniformly dissipative if and only if there is a bounded uniformly absorbing set B for which

$$\lim_{t-\tau \rightarrow \infty} \left[\sup_{\sigma \in \Sigma} \alpha(U_\sigma(t, \tau)B) \right] = 0,$$

where

$$\alpha(C) = \inf \{d : C \text{ has a finite cover of balls of } X \text{ of diameter less than } d\}$$

denotes the *Kuratowski measure of noncompactness* of a bounded set $C \subset X$ (see [9] for more details on α).

Theorem 14. *Let X be a complete metric space. Then the family of processes $U_\sigma(t, \tau)$ is uniformly asymptotically compact if and only if it is totally uniformly dissipative.*

Proof. If a uniformly attracting set K is compact, then, for any $\varepsilon > 0$, it has an ε -net $M_\varepsilon = \{x_1, \dots, x_{N_\varepsilon}\}$ and, therefore, the finite set M_ε is uniformly 2ε -absorbing. Thus, the family $U_\sigma(t, \tau)$ is totally uniformly dissipative. To show the converse implication, for every $\varepsilon > 0$, let M_ε be a finite set such that $\mathcal{O}_\varepsilon(M_\varepsilon)$ is absorbing. We denote

$$K = \bigcap_{\varepsilon > 0} B_\varepsilon \quad \text{where} \quad B_\varepsilon = \overline{\mathcal{O}_\varepsilon(M_\varepsilon)}.$$

The set K is clearly compact since it is closed and each M_ε is a finite ε -net of K . Consider an arbitrary $y_n \in \mathfrak{C}_\Sigma$. The sequence y_n is totally bounded since, for every $\varepsilon > 0$, the set B_ε is uniformly absorbing and therefore $y_n \in B_\varepsilon$ for sufficiently large n (depending on ε). Hence, y_n is precompact and, since X is complete, the set $L_\Sigma(y_n)$ is nonempty. Moreover, $L_\Sigma(y_n) \subset B_\varepsilon$ for each $\varepsilon > 0$, hence,

$$L_\Sigma(y_n) \subset \bigcap_{\varepsilon > 0} B_\varepsilon = K \quad \Rightarrow \quad K \neq \emptyset.$$

By Proposition 6 we conclude that the compact set K is uniformly attracting, i.e. $K \in \mathbb{K}_\Sigma$ and $U_\sigma(t, \tau)$ is uniformly asymptotically compact. \square

In conclusion, having a family of processes on a complete metric space, in order to construct global attractors we only need to prove the total uniform dissipation property. No continuity assumptions on the processes are required.

Remark. Let X be a Banach space, and let the family of processes $U_\sigma(t, \tau)$ be uniformly dissipative, with a bounded uniformly absorbing set B . Then a sufficient condition for $U_\sigma(t, \tau)$ to be totally uniformly dissipative is the following: for every fixed $\varepsilon > 0$ there exist a decomposition $X = Y \oplus Z$ with $\dim(Y) < \infty$ and a time $t_* > 0$ such that

$$\sup_{\sigma \in \Sigma} \sup_{x \in B} \|U_\sigma(t, \tau)x - \Pi_Y U_\sigma(t, \tau)x\| < \varepsilon$$

whenever $t - \tau \geq t_*$, where Π_Y is the canonical projection of X onto Y . In concrete situations, this condition can be verified by means of a standard Galerkin approximation scheme.

We conclude the section by discussing the following problem. Assume we are given another metric space $\Sigma_0 \subset \Sigma$. Assume also that the subfamily of processes $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma_0}$ has a uniformly (with respect to $\sigma \in \Sigma_0$) attracting set K . The question is now which conditions guarantee that K is uniformly attracting for the whole family $U_\sigma(t, \tau)$.

Proposition 15. *Let the embedding $\Sigma_0 \subset \Sigma$ be dense, and suppose that, for every bounded set $C \subset X$, there exists $t_C \geq 0$ such that the map*

$$\sigma \mapsto U_\sigma(t, \tau)x : \Sigma \rightarrow X$$

is continuous for any fixed $x \in C$ and every $t - \tau \geq t_C$. Then any uniformly attracting set K for the subfamily $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma_0}$ is uniformly attracting for $U_\sigma(t, \tau)$ as well.

Proof. Let $C \subset X$ be a bounded set, and let $\varepsilon > 0$ be arbitrarily fixed. Since K is uniformly attracting for $\{U_\sigma(t, \tau)\}_{\sigma \in \Sigma_0}$, there is an entering time $t_e = t_e(\varepsilon, C) > 0$ such that

$$U_\sigma(t, \tau)C \subset \mathcal{O}_\varepsilon(K), \quad \forall \sigma \in \Sigma_0,$$

whenever $t - \tau \geq t_e$. Since Σ_0 is dense in Σ , given $\sigma_* \in \Sigma$ there is a sequence $\sigma_n \in \Sigma_0$ such that $\sigma_n \rightarrow \sigma_*$. In turn, this yields the convergence

$$U_{\sigma_n}(t, \tau)x \rightarrow U_{\sigma_*}(t, \tau)x$$

for any fixed $x \in C$ and $t - \tau \geq t_C$. Consequently,

$$U_{\sigma_*}(t, \tau)C \subset \mathcal{O}_{2\varepsilon}(K),$$

for every $t - \tau \geq t_*$, where $t_* = \max\{t_e, t_C\}$. This tells that K is actually uniformly attracting for the whole family. \square

Corollary 16. *Let the hypotheses of Proposition 15 hold. Then the global attractors of both families of processes with symbol spaces Σ_0 and Σ coincide.*

4. The skew-product semigroup. Throughout the end of the paper, we will consider a particular but at the same time very typical situation.

4.1. General assumptions. Let Σ be a compact metric space, and let

$$T(h) : \Sigma \rightarrow \Sigma, \quad h \geq 0,$$

be a semigroup under whose action Σ is fully invariant, i.e.

$$T(h)\Sigma = \Sigma, \quad \forall h \geq 0.$$

Besides, let the translation property

$$U_\sigma(h + t, h + \tau) = U_{T(h)\sigma}(t, \tau) \tag{4}$$

hold for every $\sigma \in \Sigma$ and every $h \geq 0$ and $t \geq \tau$. In which case (see [3, 4, 7]), it is easy to verify that the map

$$S(h)(x, \sigma) = (U_\sigma(h, 0)x, T(h)\sigma), \quad h \geq 0, \tag{5}$$

defines a (skew-product) semigroup acting on the metric space

$$\mathbf{X} = X \times \Sigma.$$

4.2. Global attractors of semigroups. Before stating the main result of the section, we recall some facts on abstract semigroups. Let

$$V(h) : Y \rightarrow Y, \quad h \geq 0,$$

be a semigroup acting on a (not necessarily complete) metric space Y .

Definition 17. The semigroup $V(h)$ is said to be *asymptotically compact* if there exists a compact attracting set, namely, a compact set $K \subset Y$ such that

$$\lim_{h \rightarrow \infty} \delta_Y(V(h)C, K) = 0,$$

for every bounded set $C \subset Y$, where δ_Y denotes the Hausdorff semidistance in Y .

The main theorem in [2] reads as follows.

Theorem 18. *If the semigroup $V(h)$ is asymptotically compact, then there exists the minimal (i.e. smallest) compact attracting set A , called the global attractor of $V(h)$.*

It is worth observing that such a notion of global attractor is based only on the minimality with respect to the attraction property, and does not require any continuity on the semigroup. Indeed, contrary to the classical notion of attractor (see e.g. [1, 9, 10, 12, 13]), A may fail to be fully invariant under the action of the semigroup (see examples in [2]).

4.3. The theorem. Let the family $U_\sigma(t, \tau)$ be uniformly asymptotically compact.¹ Then, by Theorem 12, we know that $U_\sigma(t, \tau)$ has the uniform global attractor $A_\Sigma \subset X$. It is also clear from Theorem 18 that the semigroup $T(h)$ possesses the global attractor which coincides with the whole phase space Σ .

Theorem 19. *Within the assumptions above, the skew-product semigroup $S(h)$ on \mathbf{X} has a (unique) global attractor \mathbf{A} . Besides, we have the equalities*

$$\Pi_1 \mathbf{A} = A_\Sigma \quad \text{and} \quad \Pi_2 \mathbf{A} = \Sigma,$$

where Π_1 and Π_2 denote the canonical projections of \mathbf{X} onto its components X and Σ , respectively.

Proof. It is apparent from the definition (5) of skew-product semigroup that $A_\Sigma \times \Sigma$ is a (compact) attracting set for $S(h)$. On account of Theorem 18, this implies that $S(h)$ possesses the global attractor \mathbf{A} . Therefore, appealing to the minimality of A_Σ and \mathbf{A} , it is enough showing that

$$\Pi_1 \mathbf{A} \in \mathbb{K}_\Sigma \quad \text{and} \quad \Pi_2 \mathbf{A} = \Sigma. \quad (6)$$

Indeed, if $\Pi_1 \mathbf{A} \in \mathbb{K}_\Sigma$ then $\Pi_1 \mathbf{A} \supset A_\Sigma$. On the other hand, being $A_\Sigma \times \Sigma$ compact attracting for $S(h)$, we also get

$$\mathbf{A} \subset A_\Sigma \times \Sigma \quad \Rightarrow \quad \Pi_1 \mathbf{A} \subset \Pi_1(A_\Sigma \times \Sigma) = A_\Sigma.$$

To see (6), the compactness of $\Pi_1 \mathbf{A}$ being obvious, let $C \subset X$ be bounded. Then

$$\lim_{h \rightarrow \infty} \delta_{\mathbf{X}}(S(h)(C \times \Sigma), \mathbf{A}) = 0.$$

Equivalently, we can write

$$\sup_{\sigma \in \Sigma} \delta_X(U_\sigma(h, 0)C, \Pi_1 \mathbf{A}) \rightarrow 0 \quad \text{and} \quad \delta_\Sigma(T(h)\Sigma, \Pi_2 \mathbf{A}) \rightarrow 0.$$

¹Since in concrete cases the underlying space X is usually complete, this is the same as uniformly totally dissipative.

The second convergence and the full invariance of Σ readily yield the equality $\Pi_2 \mathbf{A} = \Sigma$. We are left to prove the attraction property for $\Pi_1 \mathbf{A}$. Since Σ is fully invariant for $T(h)$, for $h_\star > 0$ to be chosen later we know that, for any fixed $\sigma \in \Sigma$,

$$\sigma = T(h_\star)\sigma_\star \quad \text{for some } \sigma_\star \in \Sigma.$$

Hence, exploiting (4),

$$U_\sigma(t, \tau)C = U_{T(h_\star)\sigma_\star}(t, \tau)C = U_{\sigma_\star}(h_\star + t, h_\star + \tau)C = U_{T(h_\star + \tau)\sigma_\star}(t - \tau, 0)C,$$

upon choosing $h_\star \geq -\tau$. In light of the first convergence above, we conclude that

$$\sup_{\sigma \in \Sigma} \delta_X(U_\sigma(t, \tau)C, \Pi_1 \mathbf{A}) \leq \sup_{\sigma \in \Sigma} \delta_X(U_\sigma(t - \tau, 0)C, \Pi_1 \mathbf{A}) \rightarrow 0$$

as $t - \tau \rightarrow \infty$, proving that $\Pi_1 \mathbf{A}$ is uniformly attracting for $U_\sigma(t, \tau)$. \square

5. Structure of the Attractor. We now proceed to analyze the structure of the uniform global attractor. In some sense, this amounts to extend the notion of invariance, typical of semigroups, to dynamical processes. We begin with two definitions.

Definition 20. Let $\sigma \in \Sigma$ be fixed. A function $s \mapsto x(s) : \mathbb{R} \rightarrow X$ is a *complete bounded trajectory* (CBT) of $U_\sigma(t, \tau)$ if and only if the set $\{x(s)\}_{s \in \mathbb{R}}$ is bounded in X and

$$x(s) = U_\sigma(s, \tau)x(\tau), \quad \forall s \geq \tau, \forall \tau \in \mathbb{R}.$$

Definition 21. For a fixed $\sigma \in \Sigma$, we call *kernel* of the single process $U_\sigma(t, \tau)$ with symbol σ the collection of all its CBT. The set

$$\mathcal{K}_\sigma(t) = \{x(t) : x(s) \text{ is a CBT for } U_\sigma(t, \tau)\}$$

is called the *kernel section* at time $t \in \mathbb{R}$.

Within the framework of the previous section, the following theorem holds.

Theorem 22. *Assume that there exists $h_\star > 0$ such that the maps*

$$(x, \sigma) \mapsto U_\sigma(h_\star, 0)x : \mathbf{X} \rightarrow X \quad \text{and} \quad \sigma \mapsto T(h_\star)\sigma : \Sigma \rightarrow \Sigma$$

are closed.² Then the uniform global attractor A_Σ of the family $U_\sigma(t, \tau)$ coincides with the set

$$\mathcal{K}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0).$$

In fact, the sets $\mathcal{K}_\sigma(0)$ in the statement can be replaced by $\mathcal{K}_\sigma(t)$ for any fixed $t \in \mathbb{R}$.

Remark. Since Σ is compact, it is easy to see that $\sigma \mapsto T(h_\star)\sigma$ closed actually implies that $T(h_\star) \in \mathcal{C}(\Sigma, \Sigma)$.

Proof. We preliminary observe that the closedness assumptions of the theorem imply that the semigroup $S(h)$ defined by (5) is also a closed map on \mathbf{X} for $h = h_\star$. This fact, due to a general result from [2], is enough to ensure that the global attractor \mathbf{A} is fully invariant for $S(h)$. In which case, it is well known (see e.g. [10]) that \mathbf{A} is characterized as

$$\mathbf{A} = \{\mathbf{x}(0) : \mathbf{x}(s) \text{ is a CBT for } S(h)\},$$

²Recall that a map $f : Y \rightarrow Z$ is closed if $f(y) = z$ whenever $y_n \rightarrow y$ and $f(y_n) \rightarrow z$.

where a CBT for $S(h)$ is a bounded function $s \mapsto \mathbf{x}(s) : \mathbb{R} \rightarrow \mathbf{X}$ such that

$$\mathbf{x}(h + s) = S(h)\mathbf{x}(s), \quad \forall h \geq 0, \forall s \in \mathbb{R}.$$

The same characterization clearly applies for the global attractor Σ of $T(h)$. The proof now proceeds along the lines of Theorem IV.5.1 in [7]. For completeness, we report the details.

• $\Pi_1 \mathbf{A} \subset \mathcal{K}_\Sigma$. Indeed, let

$$\mathbf{x}(s) = (x(s), \sigma(s))$$

be a CBT of $S(h)$. By the very definition of $S(h)$, this is the same as saying that $\sigma(s)$ is a CBT of $T(h)$ (in particular, $\sigma(0) \in \Sigma$), and

$$x(s) = U_{\sigma(\tau)}(s - \tau, 0)x(\tau), \quad \forall s \geq \tau, \tau \in \mathbb{R}.$$

If $\tau \geq 0$, setting $\sigma_0 = \sigma(0)$ and using (4), we have the chain of equalities

$$U_{\sigma(\tau)}(s - \tau, 0)x(\tau) = U_{T(\tau)\sigma_0}(s - \tau, 0)x(\tau) = U_{\sigma_0}(s, \tau)x(\tau).$$

If $\tau < 0$, then $T(-\tau)\sigma(\tau) = \sigma_0$ and using (4) in the other direction we end up with

$$U_{\sigma(\tau)}(s - \tau, 0)x(\tau) = U_{T(-\tau)\sigma(\tau)}(s, \tau)x(\tau) = U_{\sigma_0}(s, \tau)x(\tau).$$

This proves that $x(s)$ is a CBT of $U_{\sigma_0}(t, \tau)$.

• $\Pi_1 \mathbf{A} \supset \mathcal{K}_\Sigma$. Let $x_0 \in \mathcal{K}_\Sigma$. Then, there exist $\sigma_0 \in \Sigma$ and a CBT $x(s)$ of the process $U_{\sigma_0}(t, \tau)$ such that $x(0) = x_0$. Since Σ is fully invariant, there is a CBT $\sigma(s)$ of $T(h)$ such that $\sigma(0) = \sigma_0$. We must show that $(x(s), \sigma(s))$ is a CBT of $S(h)$. Indeed, leaning again on the translation property (4), for $s \geq 0$ we get

$$\begin{aligned} S(h)(x(s), \sigma(s)) &= (U_{\sigma(s)}(h, 0)x(s), T(h)\sigma(s)) \\ &= (U_{T(s)\sigma_0}(h, 0)x(s), \sigma(h + s)) \\ &= (U_{\sigma_0}(h + s, s)x(s), \sigma(h + s)) = (x(h + s), \sigma(h + s)). \end{aligned}$$

The case $s < 0$ is similar and left to the reader.

Since by Theorem 19 we know that $\Pi_1 \mathbf{A} = A_\Sigma$, the proof is finished.

6. Asymptotically closed processes. The aim of this section is to extend the characterization Theorem 22 to a more general class of processes.

6.1. Asymptotically closed semigroups. We first need a definition and a theorem from [2] about dynamical semigroups.

Definition 23. A semigroup $V(h)$ acting on a metric space Y is said to be *asymptotically closed* if there exists a sequence of times $0 = h_0 < h_1 < h_2 < h_3 \dots$ with the following property: whenever the convergence $V(h_k)y_n \rightarrow \eta^k \in X$ occurs as $n \rightarrow \infty$ for every $k \in \mathbb{N}$, we have the equalities

$$V(h_k)\eta^0 = \eta^k, \quad \forall k \in \mathbb{N}.$$

The sequence h_k in the definition may be finite (but of at least two elements). In fact, if it is made exactly of two elements $h_0 = 0$ and $h_1 > 0$, then we recover the closedness of the map $V(h_1)$. On the other hand, if $V(h_*)$ is closed for some $h_* > 0$, it follows that $V(h)$ is asymptotically closed with respect to the sequence $h_k = kh_*$. This shows that asymptotic closedness is a weaker property than closedness in one point.

Remark. When the metric space Y is compact, by applying a standard diagonalization method is immediate to verify that, if $V(h)$ is asymptotically closed with respect to some sequence h_k , then

$$V(h_k) \in \mathcal{C}(Y, Y), \quad \forall k \in \mathbb{N}.$$

The following theorem holds [2].

Theorem 24. *Let $V(h)$ have the global attractor A . If $V(h)$ is asymptotically closed, then A is fully invariant under the action of the semigroup.*

6.2. The theorem. Hereafter, let the general assumptions 4.1 hold. Firstly, we extend Definition 23 to the case of a family of processes.

Definition 25. The family $U_\sigma(t, \tau)$ is said to be *asymptotically closed* if there exists a sequence of times $0 = h_0 < h_1 < h_2 < h_3 \dots$ with the following property: if

$$\sigma_n \rightarrow \sigma \in \Sigma \quad \text{and} \quad U_{\sigma_n}(h_k, 0)x_n \rightarrow \xi^k \in X$$

as $n \rightarrow \infty$ for every $k \in \mathbb{N}$, then we have the chain of equalities

$$U_\sigma(h_k, 0)\xi^0 = \xi^k, \quad \forall k \in \mathbb{N}.$$

Proposition 26. *Let $U_\sigma(t, \tau)$ be asymptotically closed with respect to some sequence h_k complying with Definition 25, and let $T(h)$ be a continuous map³ for all $h = h_k$. Then the skew-product semigroup $S(h)$ is also asymptotically closed with respect to h_k .*

Proof. Assume that, for some sequence $(x_n, \sigma_n) \in \mathbf{X}$, the convergence

$$S(h_k)(x_n, \sigma_n) \rightarrow (\xi^k, \omega^k) \in \mathbf{X}$$

holds for every $k \in \mathbb{N}$. By (5), this translates into

$$U_{\sigma_n}(h_k, 0)x_n \rightarrow \xi^k \in X \quad \text{and} \quad T(h_k)\sigma_n \rightarrow \omega^k \in \Sigma.$$

In particular,

$$\sigma_n \rightarrow \omega^0,$$

and from the continuity of $T(h_k)$ we readily obtain

$$T(h_k)\omega^0 = \omega^k,$$

for every $k \in \mathbb{N}$. Besides, appealing to the asymptotic closedness of $U_\sigma(t, \tau)$, we also deduce the chain of equalities

$$U_{\omega^0}(h_k, 0)\xi^0 = \xi^k.$$

Hence, using (5) the other way around, we conclude that

$$S(h_k)(\xi^0, \omega^0) = (\xi^k, \omega^k).$$

This proves the asymptotic closedness of $S(h)$. □

We are now ready to state the following generalized version of Theorem 22.

Theorem 27. *Let the family $U_\sigma(t, \tau)$ be uniformly asymptotically compact (or uniformly totally dissipative if X is complete). If $U_\sigma(t, \tau)$ is asymptotically closed with respect to some sequence h_k and $T(h_k)$ is continuous, then*

$$A_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0).$$

³Since Σ is compact, we could equivalently ask $T(h)$ asymptotically closed with respect to h_k .

Proof. Indeed, we learn from Proposition 26 that the skew-product semigroup $S(h)$ on \mathbf{X} is asymptotically closed with respect to h_k , hence Theorem 24 guarantees the full invariance of its global attractor \mathbf{A} . At this point, the argument is the same as in the proof of Theorem 22. \square

7. Differential equations with translation compact symbols. We finally apply the results to the study of a particular (although quite general) class of nonautonomous differential problems. More precisely, we focus on a single process $U_g(t, \tau)$ generated by a nonautonomous differential equation on a Banach space X of the form

$$\frac{d}{dt}u(t) = \mathcal{A}(u(t)) + g(t), \quad (7)$$

where $\mathcal{A}(\cdot)$ is a densely defined operator on X , and g (the symbol) is a function defined on \mathbb{R} with values in some other normed space. The problem is supposed to be well posed for every initial data $u_0 \in X$ taken at any initial time $\tau \in \mathbb{R}$.

We assume that g is *translation compact* as an element of a given metric space \mathfrak{L} . By definition, this means that the set of translates

$$\mathcal{T}(g) = \{g(\cdot + h) : h \in \mathbb{R}\}$$

is precompact in \mathfrak{L} . The closure of $\mathcal{T}(g)$ in the space \mathfrak{L} is called the *hull* of g , and is denoted by $\mathcal{H}(g)$.

Example. Given a domain $\Omega \subset \mathbb{R}^N$, we consider the space

$$\mathfrak{L} = L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)).$$

Here, f belongs to $\mathcal{H}(g)$ if there exists a sequence $h_n \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \int_a^b \|g(t + h_n) - f(t)\|_{L^2(\Omega)}^2 dt = 0, \quad \forall a > b.$$

Several translation compactness criteria can be found in [7], also for different choices of the space \mathfrak{L} , such as $L^p_{\text{loc}}(\mathbb{R}; L^q(\Omega))$ or $\mathcal{C}_b(\mathbb{R}, L^q(\Omega))$.

Then, for every $h \in \mathbb{R}$, we define the translation operator acting on a vector-valued function f on \mathbb{R} as

$$[T(h)f](t) = f(h + t).$$

It is clear that the family $\{T(h)\}_{h \in \mathbb{R}}$ satisfies the axioms of a *group* of operators on the compact space $\mathcal{H}(g)$. We also assume that $T(h)$ is strongly continuous, i.e.⁴

$$T(h) \in \mathcal{C}(\mathcal{H}(g), \mathcal{H}(g)), \quad \forall h \in \mathbb{R}.$$

In which case, it is apparent that

$$T(h)\mathcal{H}(g) = \mathcal{H}(g), \quad \forall h \in \mathbb{R}.$$

Along with the process $U_g(t, \tau)$ generated by (7), we also consider the family of processes

$$\{U_f(t, \tau)\}_{f \in \mathcal{H}(g)},$$

generated by the family of equations

$$\frac{d}{dt}u(t) = \mathcal{A}(u(t)) + f(t), \quad f \in \mathcal{H}(g). \quad (8)$$

⁴For most general concrete spaces \mathfrak{L} , like those mentioned in the example, the strong continuity of $T(h)$ is straightforward.

Again, for any choice of the symbol $f \in \mathcal{H}(g)$, the problem is supposed to be well posed for every initial data $u_0 \in X$ taken at any initial time $\tau \in \mathbb{R}$. We note that the translation property (4), namely,

$$U_f(h+t, h+\tau) = U_{T(h)f}(t, \tau), \quad \forall f \in \mathcal{H}(g), \quad (9)$$

actually holds for every $h \in \mathbb{R}$.

Remark. Such a property reflects the obvious fact that shifting the time in the initial data is the same as shifting the time in the symbol.

Hence, Theorem 27 tailored for this particular framework reads as follows.

Theorem 28. *Let the family $U_f(t, \tau)$ generated by (8) be uniformly totally dissipative. If it is also asymptotically closed, then*

$$A_{\mathcal{H}(g)} = \bigcup_{f \in \mathcal{H}(g)} \{u(0) : u(s) \text{ is a CBT for } U_f(t, \tau)\}.$$

In fact, requiring a further continuity assumption, we can also provide a description of the global attractor $A_{\{g\}}$ of the single process $U_g(t, \tau)$ generated by (7).

Theorem 29. *Let the hypotheses of Theorem 28 hold. If in addition the map*

$$f \mapsto U_f(t, \tau)u_0 : \mathcal{H}(g) \rightarrow X$$

is continuous for every fixed $t \geq \tau$ and $u_0 \in X$, then we have the equality

$$A_{\{g\}} = A_{\mathcal{H}(g)}.$$

Proof. The existence of $A_{\mathcal{H}(g)}$ implies that $A_{\mathcal{T}(g)}$ and $A_{\{g\}}$ exist too, and

$$A_{\{g\}} \subset A_{\mathcal{T}(g)} \subset A_{\mathcal{H}(g)}.$$

In light of the additional continuity, we can apply Corollary 16 to get $A_{\mathcal{T}(g)} = A_{\mathcal{H}(g)}$. So, we are left to prove the equality $A_{\{g\}} = A_{\mathcal{H}(g)}$. Indeed, for an arbitrary bounded set $B \subset X$, we infer from (9) that

$$\delta_X(U_{T(h)g}(t, \tau)B, A_{\{g\}}) = \delta_X(U_g(h+t, h+\tau)B, A_{\{g\}}).$$

This tells that the compact set $A_{\{g\}}$, in principle only contained in $A_{\mathcal{T}(g)}$, is actually uniformly attracting for the family $\{U_f(t, \tau)\}_{f \in \mathcal{T}(g)}$, hence coincides with its uniform global attractor $A_{\mathcal{T}(g)}$. \square

Remark. An interesting open question is whether or not Theorem 29 remains valid *without* the continuity hypothesis, lying only on the fact that $f \mapsto U_f(t, \tau)u_0$ is a closed map.

8. A concrete application. Given a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 1, 2$) with smooth boundary $\partial\Omega$ (for $N = 2$), let g be a translation compact function in the space

$$\mathfrak{L} = L_{\text{loc}}^2(\mathbb{R}; L^2(\Omega)).$$

In this case, it is certainly true that

$$T(h) \in \mathcal{C}(\mathcal{H}(g), \mathcal{H}(g)), \quad \forall h \in \mathbb{R}.$$

For any given initial time $\tau \in \mathbb{R}$, we consider the family of nonautonomous Cauchy problems on the time-interval $[\tau, \infty)$ in the unknown $u = u(t)$ depending on the external source $f \in \mathcal{H}(g)$

$$\begin{cases} u_{tt} + (1 + u^2)u_t - \Delta u + u^3 - u = f(t), \\ u|_{\partial\Omega} = 0, \\ u(\tau) = u_0, \quad u_t(\tau) = v_0, \end{cases}$$

which can be viewed as a model of a vibrating string ($N = 1$) or membrane ($N = 2$) in a stratified viscous medium. Arguing as in [8, 11], dealing with the same model for a time-independent f , for every $\tau' > \tau$ and every initial data $x = (u_0, v_0)$ in the weak energy space

$$X = H_0^1(\Omega) \times L^2(\Omega),$$

there is a unique variational solution

$$u \in \mathcal{C}([\tau', \tau], H_0^1(\Omega)) \cap \mathcal{C}^1([\tau', \tau], L^2(\Omega)).$$

Accordingly, the equation generates a dynamical process

$$U_f(t, \tau) : X \rightarrow X,$$

depending on the symbol $f \in \mathcal{H}(g)$. Repeating the proofs of [8, 11], we can also find a compact uniformly (with respect to $f \in \mathcal{H}(g)$) attracting set. Hence the family of processes is uniformly asymptotically compact, and by Theorem 12 we infer the existence of the uniform global attractor $A_{\mathcal{H}(g)}$. In order to understand the structure of the attractor, we shall distinguish two cases.

- If $N = 1$, repeating the proofs of [8] one can show that the map

$$(x, f) \mapsto U_f(t, \tau)x$$

is continuous from $X \times \mathcal{H}(g)$ into X . Thus both Theorem 28 and Theorem 29 apply, yielding

$$A_{\mathcal{H}(g)} = \bigcup_{f \in \mathcal{H}(g)} \{(u(0), u_t(0)) : (u(s), u_t(s)) \text{ is a CBT for } U_f(t, \tau)\} \quad (10)$$

along with the identity

$$A_{\{g\}} = A_{\mathcal{H}(g)}.$$

- If $N = 2$, the process is not strongly continuous. Nonetheless, introducing the weaker space

$$W = L^2(\Omega) \times H^{-1}(\Omega),$$

one can prove the following continuous dependence result, analogous to Proposition 2.5 of [11].

Proposition 30. *For every $t \geq \tau$, every $f_1, f_2 \in \mathcal{H}(g)$ and every $R \geq 0$, we have the estimate*

$$\|U_{f_1}(t, \tau)x_1 - U_{f_2}(t, \tau)x_2\|_W \leq Ce^{C(t-\tau)} [\|x_1 - x_2\|_X + \|f_1 - f_2\|_{L^2(t, \tau; L^2(\Omega))}],$$

for some $C = C(R) \geq 0$ and all initial data $x_1, x_2 \in X$ of norm not exceeding R .

In other words, for every fixed $t \geq \tau$, we have the weaker continuity

$$(x, f) \mapsto U_f(t, \tau)x \in \mathcal{C}(X \times \mathcal{H}(g), W).$$

This is enough to infer that the map

$$(x, f) \mapsto U_f(h, 0)x : X \times \mathcal{H}(g) \rightarrow X$$

is closed for every $h \geq 0$. We conclude from Theorem 28 that $A_{\mathcal{H}(g)}$ fulfills the same characterization (10) of the case $N = 1$.

Remark. If the function g is periodic, i.e.

$$g(\cdot + p) = g(\cdot) \quad \text{for some } p > 0,$$

then we have the trivial equality

$$\mathcal{H}(g) = \mathcal{T}(g) = \{g(\cdot + h) : 0 \leq h < p\},$$

providing at once the identity $A_{\{g\}} = A_{\mathcal{H}(g)}$. Moreover, it is known that the uniform global attractor of a periodic process coincides with the nonuniform (with respect to the initial time $\tau \in \mathbb{R}$) one. More details can be found in [5, 6].

REFERENCES

- [1] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.
- [2] V. V. Chepyzhov, M. Conti and V. Pata, A minimal approach to the theory of global attractors, *Discrete Contin. Dynam. Systems*, **32** (2012), 2079–2088.
- [3] V. V. Chepyzhov and M. I. Vishik, Nonautonomous evolution equations and their attractors, *Russian J. Math. Phys.*, **1** (1993), 165–190.
- [4] V. V. Chepyzhov and M. I. Vishik, Attractors of nonautonomous dynamical systems and their dimension, *J. Math. Pures Appl.*, **73** (1994), 279–333.
- [5] V. V. Chepyzhov and M. I. Vishik, Periodic processes and non-autonomous evolution equations with time-periodic terms, *Topol. Methods Nonlinear Anal.*, **4** (1994), 1–17.
- [6] V. V. Chepyzhov and M. I. Vishik, Attractors of periodic processes and estimates of their dimensions, *Math. Notes*, **57** (1995), 127–140.
- [7] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, 2002.
- [8] S. Gatti and V. Pata, A one-dimensional wave equation with nonlinear damping, *Glasg. Math. J.*, **48** (2006), 419–430.
- [9] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, Providence, 1988.
- [10] A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Coll. RMA no.17, Masson, Paris, 1991.
- [11] V. Pata and S. Zelik, Attractors and their regularity for 2-D wave equation with nonlinear damping, *Adv. Math. Sci. Appl.*, **17** (2007), 225–237.
- [12] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Springer-Verlag, New York, 2002.
- [13] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer, New York, 1997.