Optimal investment strategies with a minimum performance constraint

Emilio Barucci[*], Daniele Marazzina[†] and Elisa Mastrogiacomo[‡]

Abstract

We consider the optimal investment problem of a fund manager in the presence of a minimum guarantee constraint on the fund performance. The manager receives a fee which is proportional to the liquidation value of the portfolio or of the surplus over the guarantee in case it is positive and zero otherwise, eventually augmented by a constant fee. Her remuneration is reduced through the application of a penalty if the value of the fund at maturity is below a specified-in-advance threshold (minimum guarantee). We deal with two different settings: a continuous time economy with constant instantaneous interest rate and the case where the short-term interest rate evolves as the Vasicek model. Explicit formulas for the optimal investment strategy are presented. We compare our portfolio strategies to the Merton portfolio and to the Option Based Portfolio Insurance strategy.

JEL Classification: C61, G11, G23

Keywords: Asset management; Minimum guarantee; Martingale approach; Management Fee; Performance Fee.

1 Introduction

Life insurance products are often characterized by a minimum guarantee: the insurance company manages funds guaranteeing a minimum return to policyholders. In this paper we investigate how the presence of a minimum guarantee affects the asset manager's strategy assuming that the liability of the insurance company is partially charged to the asset manager.

Usually, the funds of policyholders are pooled together in a segregated fund, the insurance company manages it in order to refund claims and lapses of policyholders. The company is remunerated through

[*] Politecnico di Milano, Dipartimento di Matematica, Milano, Italy. emilio.barucci@polimi.it
[†] Politecnico di Milano, Dipartimento di Matematica, Milano, Italy. daniele.marazzina@polimi.it
[‡] Università degli Studi dell'Insubria, Dipartimento di Economia, Varese, Italy. elisa.mastrogiacomo@uninsubria.it

a constant fee, and a fee that depends on the assets under management (AUM) of the fund (asset management fee) or through a share of the surplus over the guarantee in case it is positive and zero otherwise (performance fee). In some cases a combination of the two schemes is at work. Life insurance products with a minimum guarantee establish that the insurance company is endowed with a liability in case the fund goes below it. If this is the case, then the insurance company has to refund the performance gap to policyholders and, therefore, the company is short of a put option written on the AUM of the fund. This type of contract affects the management of the segregated fund by the insurance company. In what follows, we investigate the asset management problem in a dynamic setting assuming that the payoff of the asset manager is made up of a constant fee, an asset management/performance fee and the liability in case the performance target is not reached. The guarantee is defined as a threshold on the AUM.

The problem has been analyzed in a stochastic/constant interest rate setting in a way that doesn't fully fit the features of life insurance contracts. In [3, 6, 7, 8, 9, 10] the optimal investment problem has been analyzed assuming that the goal of the company is to maximize the expected utility of a fraction of the AUM or of the positive surplus over the guarantee in case it is positive and zero otherwise under the constraint that at maturity the guarantee is reached.

The assumption that at maturity the AUM should be greater than a threshold is not motivated by life insurance policies. In [6] the rationale for imposing the constraint is traced back to the fact that the regulator imposes it to the company. In this setting, the management of a segregated fund is reduced to an asset management problem under the constraint that AUM cannot go below a certain threshold. This interpretation refers to the literature on Option Based Portfolio Insurance (OBPI) strategies, see [8, 13, 18]. Actually, this assumption doesn't fit life insurance contracts because the insurance company has some degrees of freedom in a sense that the AUM of the segregated fund can go below the threshold. If this is the case, then the insurance company is obliged to refund the performance gap to policyholders at maturity.

In a recent article [14], the problem has been analyzed in a constant interest rate setting assuming that the AUM can go below the guarantee threshold, and considering the perspective of the asset manager of the insurance company, see also [11] for an analysis of a manager of a hedge fund. In these two articles, authors assume that the asset manager takes the whole loss in case the guarantee is not reached, if this is the case then her remuneration may even become negative. An assumption that doesn't fully match the remuneration schemes adopted in the asset management industry, as a matter of fact the remuneration cannot be negative. Allowing for a negative remuneration, they consider an S-shaped utility function.

On the opposite, in this article we assume that the manager's remuneration decreases in case the AUM is below the guarantee threshold, concurring to the loss of the insurance company, but it cannot become negative. Therefore, it is the insurance company, with its revenues from other activities or its capital, that ensures the payment of the minimum return to policyholders, while the manager only concurs to the loss in the sense that her remuneration is negatively affected if the minimum guarantee is not reached. We deal with a stochastic and a constant risk interest rate. Our analysis is strictly related to the optimization problems when the remuneration is a piecewise linear function of wealth, see [1, 2, 4, 5, 16].

We show that an asset management fee or a performance fee lead to a similar investment strategy with the latter yielding a lower level of risk exposure (investment in the risky asset). Differently from what is observed when the AUM at maturity is constrained to be above a threshold, we show that the manager may invest in the risky asset even if the put option is in the money, i.e., when AUM are below the threshold of the guarantee. In that region, the investment is hump shaped: when the put option is deep in the money the manager doesn't invest in the risky asset; as AUM increase, the investment in the risky asset increases and then decreases just below the guarantee with a kink and then the investment in the risky asset increases again converging towards the solution obtained without constraint in the region where the guarantee is satisfied (Merton solution). If the company is remunerated also through a constant fee, then the investment strategy may be hump shaped also above the threshold yielding excess risk taking with respect to the Merton solution. Differently from the case where the asset manager's remuneration can become negative, we observe a no-investment (non-reachable) region when the AUM are deep out of the money.

The paper is organized as follows. Section 2 presents the general framework of our analysis. In Section 3 we describe the optimal investment problem and we provide our main results. Section 4 deals with numerical results concerning the optimal dynamic trading strategies. We compare our results with the strategy obtained in the standard investment problem (i.e., the Merton problem) and with the OBPI optimal strategy.

2 The model

In this section, we present the general framework of our analysis, which is based on [6]. In what follows, $(z_r(t))_{t\geq 0}$ and $(z(t))_{t\geq 0}$ are two independent Wiener processes defined on a complete probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$, the classical conditions are satisfied.

We assume that the risk-free asset evolves as

$$\frac{\mathrm{d}S_0(t)}{S_0(t)} = r(t)\mathrm{d}t, \quad S_0(0) = 1,$$

and the short rate process $r(t)$ evolves according to the following equation (Vasicek model)

$$\mathrm{d}r(t) = (a - br(t))\mathrm{d}t - \sqrt{\eta}\mathrm{d}z_r(t), \quad r(0) = r_0 \tag{1}$$

for r_0, a, b, η positive constants in the stochastic interest rate framework, $a = b = \eta = 0$ in the constant interest rate case, yielding $r(t) = r_0$ for any $t \in [0, T]$.

The evolution of the risky asset is described by a Geometric Brownian motion, i.e.,

$$\frac{\mathrm{d}S(t)}{S(t)} = r(t)\mathrm{d}t + \sigma_1(\mathrm{d}z(t) + \lambda_1\mathrm{d}t) + \sigma_2\sqrt{\eta}(\mathrm{d}z_r(t) + \lambda_2\sqrt{\eta}\mathrm{d}t)$$

with $S(0) = 1$, λ_1, λ_2 constants and σ_1, σ_2 positive constants. Notice that in the case of a constant interest rate the evolution of the risky asset price reduces to

$$\frac{\mathrm{d}S(t)}{S(t)} = (r_0 + \sigma_1\lambda_1)\mathrm{d}t + \sigma_1\mathrm{d}z(t)$$

where λ_1 is the Sharpe ratio.

Coherently with Equation (1), we have that the no arbitrage price of the zero coupon-bond evolves as follows:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = r(t)\mathrm{d}t + \sigma_B(T-t, r(t))(\mathrm{d}z_r(t) + \lambda_2\sqrt{\eta}\mathrm{d}t), \quad B(T,T) = 1,$$

where

$$\sigma_B(T-t, r(t)) = \frac{1 - e^{-b(T-t)}}{b}\sqrt{\eta}.$$

In what follows, we assume that the maturity of the bond coincides with that of the contract.

We define the following process for the AUM of the segregated fund: for any $t < T$

$$\mathrm{d}X(t) = X(t)(1 - \pi^B(t) - \pi^S(t))\frac{\mathrm{d}S_0(t)}{S_0(t)} + X(t)\pi^B(t)\frac{\mathrm{d}B(t,T)}{B(t,T)} + X(t)\pi^S(t)\frac{\mathrm{d}S(t)}{S(t)},$$

$$X(0) = X_0 > 0,$$

where $1 - \pi^B(t) - \pi^S(t), \pi^B(t), \pi^S(t)$ denote, respectively, the proportion of AUM invested into the risk free asset, the bond and the stock. In the constant risk-free case, $\pi_B \equiv 0$.

The evolution of AUM can be rewritten as

$$\frac{\mathrm{d}X(t)}{X(t)} = r(t)\mathrm{d}t + \sigma_1 \pi^S(t)(\mathrm{d}z(t) + \lambda_1 \mathrm{d}t) + \pi^B(t)\sigma_B(T-t, r(t))(\mathrm{d}z_r(t) + \lambda_2\sqrt{\eta}\mathrm{d}t) \\ + \pi^S(t)\sigma_2\sqrt{\eta}(\mathrm{d}z_r(t) + \lambda_2\sqrt{\eta}\mathrm{d}t).$$

Set $\boldsymbol{\lambda} = (\lambda_1, \lambda_2\sqrt{\eta})^\top$, $\mathbf{z}(t) = (z(t), z_r(t))^\top$, and

$$\boldsymbol{\pi}(t) = (\pi^S(t), \pi^B(t))^\top, \qquad \Sigma(t) = \begin{pmatrix} \sigma_1 & \sigma_2\sqrt{\eta} \\ 0 & \sigma_B \end{pmatrix} \tag{2}$$

the above equation can be rewritten as

$$\frac{\mathrm{d}X(t)}{X(t)} = r(t)\mathrm{d}t + \boldsymbol{\lambda}^\top(t)\Sigma^\top(t)\boldsymbol{\pi}(t)\mathrm{d}t + [\Sigma^\top(t)\boldsymbol{\pi}(t)]^\top \mathrm{d}\mathbf{z}(t), \qquad X(0) = X_0.$$

As already pointed out in the Introduction, the optimal investment strategy of a fund manager in case of a minimum guarantee has been already studied in literature. We refer to [6, 8] for the analysis of the performance fee and of the asset management fee, respectively. Authors investigate a finite horizon optimal investment problem when the constraint is imposed on the terminal date (European guarantee) or on every intermediate date (American guarantee).

More precisely, in [8], in the constant risk-free case, the following utility maximization problem is solved

$$\max_{\pi^S} \mathbb{E}[u(X(T))] \quad \text{s.t.} \quad X(T) \geq K \tag{3}$$

in the European guarantee case, and

$$\max_{\pi^S} \mathbb{E}[u(X(T))] \quad \text{s.t.} \quad X(t) \geq K \quad \text{for any } t \in [0, T]$$

in the American case. u is a power utility function, K the minimum guarantee.

Instead, in [6], it is assumed that the asset manager takes a share of the positive surplus of the segregated fund over a minimum under the constraint that the segregated fund at maturity is above it.

As a consequence, the manager solves the following maximization problem:

$$\max_{(\pi^S,\pi^B)} \mathbb{E}[u(X(T)-K)] \quad \text{s.t.} \quad X(T) \geq K.$$

As already discussed in the Introduction, imposing the condition $X(T) \geq K$ is not borne out by the insurance contract. To cope with life insurance peculiarities, we introduce a liability in case a guarantee is not reached. In what follows, we start from considering an asset management fee and we analyze the following problem

$$\max_{(\pi^S,\pi^B)} \mathbb{E}[u(\alpha_1 X(T) + \widehat{H} - \alpha_2 (K - X(T))^+)]$$

where $(\cdot)^+ = \max\{\cdot, 0\}$, and α_1, α_2 and $\widehat{H}$ are positive constants. α_1 is the percentage of AUM at the terminal date that the manager receives as a fee, $\widehat{H}$ is a constant fee received by the manager independently of the performance of the fund, and $\alpha_2(K - X(T))^+$ is the liability due to the need to refund the policyholder: if $X(T) < K$, then the remuneration of the asset manager is reduced by $\alpha_2(K - X(T)) > 0$. Therefore, the remuneration is $(\alpha_1 + \alpha_2)X(T) + \widehat{H} - \alpha_2 K$ if $X(T) < K$ and $\alpha_1 X(T) + \widehat{H}$ otherwise, $(\alpha_1 + \alpha_2)X(T) + \widehat{H} - \alpha_2 K \geq 0$, where the inequality holds true due to the definition of admissible strategy we will provide in the next section. Notice that, if $\alpha_2 = 1$, then the manager's fee is reduced by the money necessary to refund the minimum guarantee, while if $0 < \alpha_2 < 1$ the manager only shares a part of the performance gap.

In what follows, we deal with the mathematical objective function $\mathbb{E}[u(\alpha X(T) + H - (K - X(T))^+)]$, where α is the ratio between α_1 and α_2, and $H = \widehat{H}/\alpha_2$.

Considering a remuneration scheme based on a share of the positive surplus of the segregated fund over the threshold guaranteed by the contract, the optimization problem becomes:

$$\max_{(\pi^S,\pi^B)} \mathbb{E}[u(\widehat{H} + \alpha_1 (X(T) - K)^+ - \alpha_2 (K - X(T))^+)].$$

A similar problem has been already analyzed in constant risk free context in [1, 4]. We assume $0 < \alpha_1 < \alpha_2$ and $\widehat{H} > 0$: the first assumption results in a concave utility function. If $\alpha_2 < \alpha_1$, then the solution can still be computed via concavification techniques [1, 4, 5, 15, 16]. As above, in what follows we deal with the corresponding optimization function $\mathbb{E}[u(H + \alpha(X(T) - K)^+ - (K - X(T))^+)]$ with $\alpha = \alpha_1/\alpha_2 < 1$.

3 The optimal investment problem

We assume that the manager's preferences are described by a power utility function:

$$u(x) = \frac{x^\gamma}{\gamma}, \qquad 0 < \gamma < 1 \;\text{ or }\; \gamma < 0.$$

The aim of the manager is then to maximize the expected utility of the final payoff

$$\max_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}\left[u(\alpha X(T) + H - (K - X(T))^+)\right] \tag{4}$$

(for α, H, K positive constants) over a set of admissible investment strategies

$$\begin{aligned} \mathcal{A} \;\; = \;\; & \{(\boldsymbol{\pi}(t))_{t\geq 0} \text{ s.t. } \boldsymbol{\pi}(t) \in \mathcal{F}_t,\; X(t)\boldsymbol{\pi}(t) \text{ is square integrable, and} \\ & \alpha X(T) + H - (K - X(T))^+ \geq 0 \text{ a.e.}\}. \end{aligned} \tag{5}$$

The condition that the remuneration scheme cannot become negative is imposed because the asset manager cannot be called to refund the return gap with its own resources, the insurance company is the ultimate stakeholder and will do it. The manager refunds a share of the performance gap up to a non negative remuneration.

Our problem can be solved via the martingale technique, obtaining a closed form solution. To this end, we introduce the following notation:

$$U(x) := u(\alpha x + H - (K - x)^+); \tag{6}$$

thus the maximization problem (4) becomes

$$\max_{\boldsymbol{\pi} \in \mathcal{A}} \mathbb{E}\left[U(X(T))\right].$$

Notice that $U(x)$ is well-defined for $x \geq \frac{K-H}{\alpha+1}$ ($x > \frac{K-H}{\alpha+1}$ if γ is negative) and it is not differentiable in $x = K$. In what follows, we assume $K - H > 0$, i.e., the minimum guarantee is greater than the fixed component of the manager's remuneration, which is the case in real applications. Furthermore, the first order derivative of U is not uniquely defined in $x = K$, since we have

$$U'_+(K) := \lim_{x \to K^+} U'(x) = \alpha u'(\alpha K + H),$$
$$U'_-(K) := \lim_{x \to K^-} U'(x) = (\alpha + 1) u'(\alpha K + H).$$

In what follows, we denote by U' its set-valued derivative, i.e., the set-function

$$U'(x) = \begin{cases} (\alpha+1)u'((\alpha+1)x + H - K), & \frac{K}{\alpha+1} < x < K, \\ [U'_+(K), U'_-(K)], & x = K, \\ \alpha u'(\alpha x + H), & x > K. \end{cases} \tag{7}$$

Following [5], the optimal strategy can be computed for any $t \in [0, T]$ through the condition

$$X^*(t) = X^*(t, r(t), \zeta(t)) = \mathbb{E}_t \left[\frac{\zeta(T)}{\zeta(t)} I(\lambda \zeta(T)) \right].$$

where $\zeta(t)$ is the state price density and is given by

$$\zeta(t) = \exp \left\{ -\int_0^t r(s) \mathrm{d}s - \int_0^t \boldsymbol{\lambda}^\top(s) \mathrm{d}\mathbf{z}(s) - \frac{1}{2} \int_0^t |\boldsymbol{\lambda}(s)|^2 \mathrm{d}s \right\}$$

and I is the (well-defined) inverse function of U' and λ solves $\mathbb{E}[\zeta(T) I(\lambda \zeta(T))] = X_0$.

We would like to stress that the optimal policy displays three distinct patterns, depending on the state of the world, represented by the state price density $\zeta(T)$, with low $\zeta(T)$ representing good states and high $\zeta(T)$ bad states. In fact, the optimal terminal wealth is

$$X^*(T) = I(\lambda \zeta(T)),$$

with I having different expressions in the intervals $\big(0, U'_+(K)\big)$, $\big[U'_+(K), U'_-(K)\big]$ and $(U'_-(K), +\infty)$. If $\zeta(T)$ is lower than $U'_+(K)/\lambda$, then the terminal wealth is above the guarantee and consequently the manager remuneration is not affected by the penalty; if $\zeta(T)$ is high, then the fund performs below the guarantee K and thus a penalty is applied to the manager's remuneration. In addition, there is an intermediate region, namely $\zeta(T) \in [U'_+(K)/\lambda, U'_-(K)/\lambda]$, in which the fund is performing exactly the guarantee.

Our first result concerns the optimal investment policy in case of a constant risk-free rate, i.e., the case $a = b = \eta = \lambda_2 = 0$. Notice that in this case AUM evolve as:

$$\mathrm{d}X(t) = [r + \sigma_1 \lambda_1 \pi^S(t)] X(t)\mathrm{d}t + \pi^S(t)\sigma_1 X(t)\mathrm{d}z(t), \qquad X(0) = X_0.$$

The following Theorem characterizes the optimal investment solution in the constant interest rate framework.

Theorem 1 *Let $a = b = \eta = \lambda_2 = 0$, then the optimal portfolio is*

$$\begin{aligned}(\pi^S)^*(t) = {} & \frac{\lambda_1}{\sigma_1(1-\gamma)}\left(1 + \frac{K}{X^*(t)}e^{-r(T-t)}\left(N(d^+ - \sigma) - \frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1}N(d^- - \sigma)\right)\right. \\ & \left. + \frac{H}{X^*(t)}e^{-r(T-t)}\left(\frac{1}{\alpha}N(d^+ - \sigma) + \frac{1}{\alpha+1}(1 - N(d^- - \sigma))\right)\right).\end{aligned} \tag{8}$$

where the optimal AUM $X^*(t)$ is given by

$$\begin{aligned}X^*(t) = e^{-r(T-t)} & \left(\frac{\alpha K + H}{\alpha}\frac{N'(d^+ - \sigma)}{N'\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right)}N\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right) + K\left(N(d^- - \sigma) - N(d^+ - \sigma)\right)\right. \\ & + \frac{\alpha K + H}{\alpha+1}\frac{N'(d^- - \sigma)}{N'\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)}\left(1 - N\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)\right) + \frac{K}{\alpha+1}(1 - N(d^- - \sigma)) \\ & \left. - H\left(\frac{1}{\alpha}N(d^+ - \sigma) + \frac{1}{\alpha+1}(1 - N(d^- - \sigma))\right)\right),\end{aligned}$$

while $\sigma = \lambda_1\sqrt{T-t}$, and $d^\pm = \frac{\ln\frac{U_\pm(K)}{\lambda\zeta(t)} + r(T-t) + \frac{\sigma^2}{2}}{\sigma}$.

Remark 2 *In case of a constant risk-free rate, the investment strategy in the risky asset (Merton strategy) is $\pi^M(t) = \frac{\lambda_1}{\sigma_1(1-\gamma)}$, and it is recovered by (8) if $K, H \to 0$.*

The second result concerns the optimal solution in case the risk free rate evolves according to the Vasicek model.

Theorem 3 *The optimal portfolio is*

$$\boldsymbol{\pi}^*(t) = \frac{1}{X^*(t)}(\Sigma^\top)^{-1}\begin{pmatrix} 0 \\ -X^*_r(t, r(t)\zeta(t))\sqrt{\eta}\end{pmatrix} - (\Sigma^\top)^{-1}\zeta(t)\frac{X^*_\zeta(t, r(t), \zeta(t))}{X^*(t)}\boldsymbol{\lambda},$$

where the optimal AUM $X^*(t)$ is given by

$$
\begin{aligned}
X^*(t) = f(t)e^{-\mu+\frac{\sigma^2}{2}} &\left(\frac{\alpha K + H}{\alpha} \frac{N'(d^+ - \sigma)}{N'\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right)} N\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right) + K\left(N(d^- - \sigma) - N(d^+ - \sigma)\right) \right. \\
&+ \frac{\alpha K + H}{\alpha+1} \frac{N'(d^- - \sigma)}{N'\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)} \left(1 - N\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)\right) + \frac{K}{\alpha+1}(1 - N(d^- - \sigma)) \\
&\left. - H\left(\frac{1}{\alpha}N(d^+ - \sigma) + \frac{1}{\alpha+1}(1 - N(d^- - \sigma))\right)\right),
\end{aligned}
$$

$f(t), \mu, \sigma, d^+, d^-$ being defined as:

$$
\begin{aligned}
f(t) &= \exp\left\{-\left(\frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\eta\right) + \lambda_2 a\right)(T-t) - \lambda_2 r(t)\right\}, \\
\mu &= \frac{1 - \lambda_2 b - e^{-b(T-t)}}{b} r(t) + \frac{a}{b}(T-t)(1-\lambda_2 b) - \frac{a}{b^2}(1 - e^{-b(T-t)}), \\
\sigma^2 &= (1 - b\lambda_2)^2 \eta \int_t^T \left(\frac{1 - e^{-b(T-s)}}{b}\right)^2 \mathrm{d}s + \lambda_1^2(T-t) + \lambda_2^2 \frac{\eta}{2b}\left(1 - e^{-2b(T-t)}\right), \\
&- 2\lambda_2(1 - b\lambda_2)\eta \int_t^T e^{-b(T-s)} \left(\frac{1 - e^{-b(T-s)}}{b}\right) \mathrm{d}s, \\
d^\pm &= \frac{\ln \frac{U_\pm(K)}{\lambda\zeta(t)} + \mu - \ln f(t)}{\sigma}.
\end{aligned}
$$

We would like to point out that Theorem 3 implies the following optimal strategy for the risky asset:

$$
\begin{aligned}
(\pi^S)^*(t) = \frac{\lambda_1}{\sigma_1(1-\gamma)} &\left(1 + \frac{K}{X^*(t)} f(t) e^{-\mu+\frac{\sigma^2}{2}} \left(N(d^+ - \sigma) - \frac{1}{\alpha+1} - \frac{\alpha}{\alpha+1} N(d^- - \sigma)\right)\right. \\
&\left. + \frac{H}{X^*(t)} f(t) e^{-\mu+\frac{\sigma^2}{2}} \left(\frac{1}{\alpha}N(d^+ - \sigma) + \frac{1}{\alpha+1}(1 - N(d^- - \sigma))\right)\right),
\end{aligned}
\tag{9}
$$

see Remark 5. Note the similarities between Theorem 1 (in the constant interest rate case) and Theorem 3 (in the Vasicek interest rate case). In the Appendix we only provide the proof of Theorem 3.

Considering a remuneration scheme based on a share of the positive surplus over the minimum guarantee, the maximization problem becomes

$$
\max_{\boldsymbol{\pi}\in\mathcal{A}} \mathbb{E}[u(H + \alpha(X(T) - K)^+ - (K - X(T))^+)]. \tag{10}
$$

with

$$\begin{aligned}\mathcal{A} \quad = \quad & \{(\boldsymbol{\pi}(t))_{t\geq 0} \text{ s.t. } \boldsymbol{\pi}(t) \in \mathcal{F}_t,\ X(t)\boldsymbol{\pi}(t) \text{ is square integrable, and} \\ & H + \alpha(X(T)-K)^+ - (K - X(T))^+ \geq 0 \text{ a.e.}\}.\end{aligned}$$

The optimal strategy in the Vasicek model is provided by the following Theorem.

Theorem 4 *The optimal portfolio is*

$$\boldsymbol{\pi}^*(t) = \frac{1}{X^*(t)}(\Sigma^\top)^{-1}\begin{pmatrix} 0 \\ -X_r^*(t, r(t)\zeta(t))\sqrt{\eta}\end{pmatrix} - (\Sigma^\top)^{-1}\zeta(t)\frac{X_\zeta^*(t, r(t), \zeta(t))}{X^*(t)}\boldsymbol{\lambda},$$

where the optimal AUM $X^*(t)$ is given by

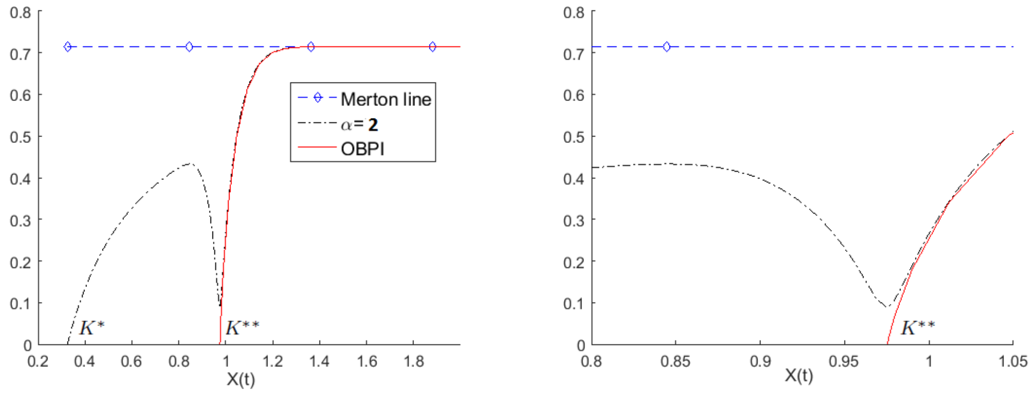
$$\begin{aligned} X^*(t) = f(t)e^{-\mu+\frac{\sigma^2}{2}} & \left(\frac{H}{\alpha}\frac{N'(d^+ - \sigma)}{N'\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right)} N\left(d^+ - \frac{\gamma}{\gamma - 1}\sigma\right)\right. \\ & + H\frac{N'(d^- - \sigma)}{N'\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)}\left(1 - N\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)\right) \\ & \left. + K - H - \frac{H}{\alpha}N(d^+ - \sigma) + HN(d^- - \sigma)\right), \end{aligned}$$

$f(t), \mu, \sigma, d^+, d^-$ being defined as in Theorem 3.

The proof closely follows the one of Theorem 3, and therefore it is not reported in the Appendix.

4 Numerical analysis

In this section we present some numerical results concerning the manager's optimal trading strategy for the risky asset. We compare our results with the strategy obtained in the standard investment problem, i.e., the Merton problem. In the Merton problem, the optimal portfolio (in what follows "Merton strategy") is characterized by a constant proportion $\pi^M(t) = \frac{\lambda_1}{\sigma_1(1-\gamma)}$ invested in the risky asset. When r is constant, we also compare our strategy to the one obtained in [8], the so-called OBPI strategy. By construction, this strategy satisfies the constraint $X(T) \geq K$.



(a) Influence of α

(b) Zoom

Figure 1: **The manager's optimal investment in the risky asset under constant interest rate**. The figure displays the profile of the optimal investment in the risky asset against AUM with $\alpha = 2$ in $t = 0.5$. Constant interest rate, i.e., $a = b = \eta = \lambda_2 = 0$. The remaining parameter values are those of the baseline scenario, see (11). $K^*(t) = 0.3251$ and $K^{**}(t) = 0.9753$.

The reference parameters of our analysis are

$$T = 1;\ t = 0.5;\ \alpha = 0.3;\ K = 1;\ H = 0;\ \gamma = -0.4;\ X_0 = 1;\ b = 0.2399;\ a = 0.01b;$$
$$\eta = 0.007;\ r_0 = 0.05;\ \sigma_1 = 0.2;\ \lambda_1 = 0.2;\ \sigma_2 = 0.02;\ \lambda_2 = 1;\ S_0 = 1. \tag{11}$$

We develop our analysis considering the asset management fee, i.e., the objective function (4), then we will consider the remuneration scheme derived considering a performance fee, i.e., the objective function (10). The analysis of the two different settings yields similar results.

We start with a constant risk free rate ($a = b = \eta = \lambda_2 = 0$). In this setting, the optimal strategy in case of the asset management fee is the one described in Theorem 1. In Figure 1 we plot the manager's optimal risk exposure (investment in the risky asset) and the optimal strategy OBPI, proposed in [8]. The strategy is fully described by two critical values $K^*(t)$ and $K^{**}(t)$, where $K^*(t) = e^{-r(T-t)}\frac{K-H}{1+\alpha}$, $K^{**}(t) = e^{-r(T-t)}K$ $(0 < K^*(t) < \frac{K-H}{1+\alpha} < K^{**}(t) < K)$. Notice that with a (positive) constant risk free rate it may be possible to reach a certain target starting just below it by simply investing all the AUM at the risk free rate. It is interesting to observe that $\frac{K-H}{1+\alpha}$ is the threshold of the AUM at maturity that allows the asset manager to obtain a positive remuneration (above it), instead K is the strike price of the put option describing the liability at maturity. $K^*(t)$ and $K^{**}(t)$ are the discounted values of these two thresholds, that means that for $X(t) > K^{**}(t)$ the asset manager can always beat the minimum guarantee by investing all the AUM at the risk free rate, instead for $X(t) > K^*(t)$ the asset manager can always reach a positive wealth. The strategy is similar to the one derived in [14], the main difference is that as they allow for a negative remuneration the no-investment region is not observed, i.e., in their setting $K^*(t) = 0$, and the hump shaped part starts at $X(t) = 0$.

First of all, we notice that the presence of a liability in case of underperformance provides an incentive for the manager to deviate from the Merton strategy. In particular, the optimal proportion of AUM invested in the risky asset is always lower than the Merton's one. We can conclude that the presence of a penalty in the remuneration scheme leads the asset manager to decrease the risk exposure with respect to the solution obtained maximizing the expected utility of terminal wealth.

The picture shows that the manager invests in the risky asset also when the fund is below the minimum guarantee. Note that the sensitivity of the remuneration to AUM is $\alpha+1$ for $X(T) \leq K$ and α for $X(T) \geq K$. For a low performance of the fund ($X(t) < K^*(t)$) there is no admissible strategy because for that level of AUM it is impossible to construct a strategy that allows for a positive remuneration of the manager at time T with certainty. However, if $X_0 > K^*(0)$, then the optimal strategy implies $X^*(t) \geq K^*(t)$ for any $t \in [0, T]$. For a performance of the fund close to $K^*(t)$ the asset manager invests a limited amount of AUM in the risky asset because the marginal utility explodes when the remuneration approaches zero (at $\frac{K-H}{1+\alpha}$) and therefore the asset manager takes a limited risk exposure fearing for the downside risk. As the AUM increase, the manager becomes less sensitive to the downside risk of a null remuneration and therefore she increases the risk exposure of her strategy. The increase is bounded by the fact that as AUM reach K the sensitivity of the remuneration decreases from $\alpha+1$ to α. This feature drives a reduction in the risk exposure yielding a maximum and then a minimum/kink in $K^{**}(t)$. At that point the risk exposure is limited because the manager may get rid of the put option by investing in the risk free asset. As a result, the optimal exposure in the risky asset is hump shaped: it is nihil for $X(t) = K^*(t)$, then becomes positive and increases with the AUM, reaches a maximum and then decreases with a minimum at the kink $K^{**}(t)$. Finally, as the AUM are above K the optimal stock strategy increases smoothly converging (from below) towards the Merton strategy. The optimal strategy doesn't entail excessive risk taking with respect to the Merton strategy.

We can compare our optimal strategy with that obtained imposing that $X(T) \geq K$ (OBPI strategy). Imposing an additional constraint, the OBPI strategy yields a lower risk exposure than the one obtained in our setting for any level of AUM, see Figure 1. Both strategies converge to the Merton strategy as AUM are above the guarantee. This is due to the fact that the probability of performing less than K decreases to zero when the AUM of the fund are far ahead of the minimum guarantee and in this scenario the two remuneration schemes look almost the same. The two strategies differ for AUM in a neighborhood of the guarantee and below it. In case of the OBPI strategy, because of the constraint $X^*(T) \geq K$, the

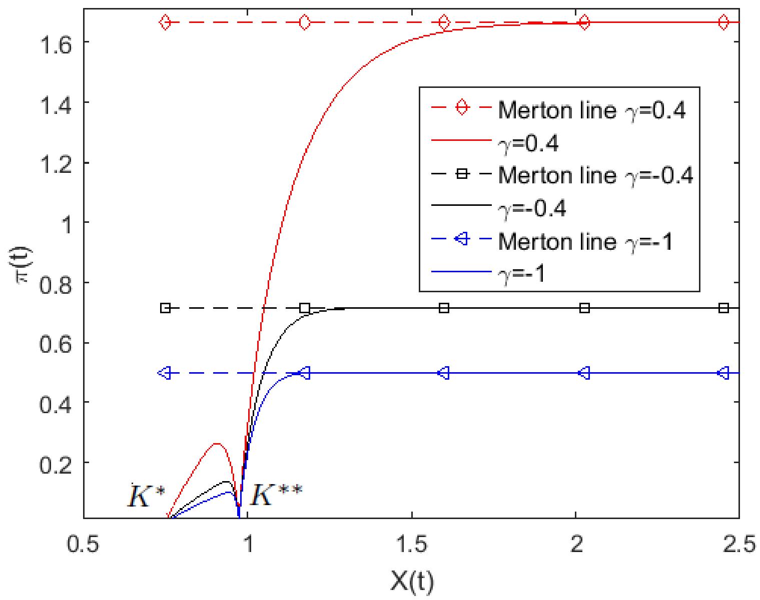
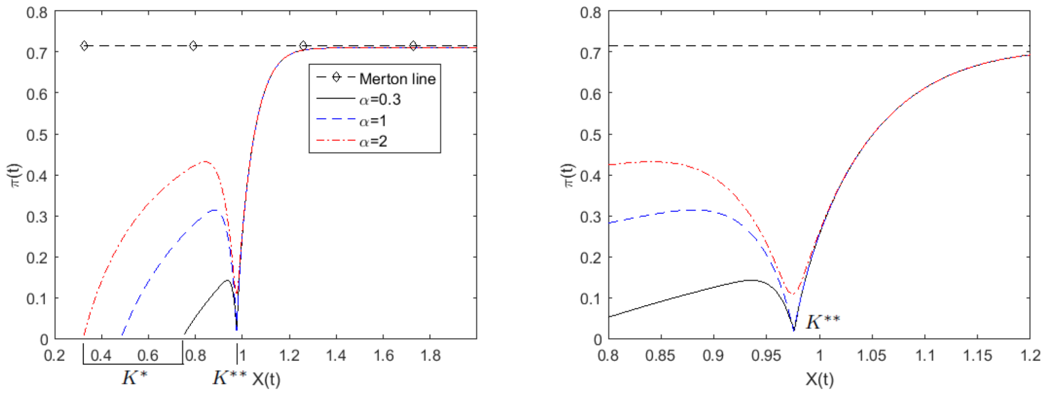


Figure 2: **The manager's optimal investment in the risky asset under stochastic interest rate, varying** γ. The figure displays the optimal investment against AUM as γ assumes three different values $-1, -0.4, 0.4$. The remaining parameters are those of the baseline scenario, see (11).

AUM cannot go below the discounted value of K at any $t \in [0, T)$. In fact, if at any time $t < T$ the AUM touches $Ke^{-r(T-t)}$, then the optimal allocation strategy requires to invest all the wealth in the risk-free asset: as a matter of fact this is the unique strategy yielding $X^*(T) \geq K$ with certainty. So according to the OBPI strategy, the strategy is not defined for $X(t) < Ke^{-r(T-t)} = K^{**}(t)$ and it is equal to zero for $X(t) = Ke^{-r(T-t)}$. This is not anymore true in our setting: the manager can invest in the risky asset also in case the put option is in the money and stops to do it only in case it is deep in the money.

Let us consider now the stochastic interest rate setting. The optimal strategy is described in Theorem 3. In Figure 2 we plot the optimal investment strategy in the risky asset $\pi^S(t)$ for the baseline parameters as a function of the AUM $X(t)$ at the middle of the investment period ($t = 0.5$, $T = 1$), assuming $r(t) = r_0$ for different degrees of risk aversion. The picture shows that also in this case the manager invests in the risky asset also when the fund is below the minimum guarantee. The shape of the optimal strategy as a function of AUM is similar to the one observed in the constant interest rate case. The investment function is characterized by two cutoff points: $K^*(t)$ and $K^{**}(t)$ such that $0 < K^*(t) < \frac{K-H}{1+\alpha} < K^{**}(t) < K$. The interpretation of these two values is similar to the constant interest rate case, since we can verify that $K^*(t) = B(t,T)\frac{K-H}{1+\alpha}$, and $K^{**}(t) = B(t,T)K$.

In $K^*(t)$ we have $(\pi^S)^* = 0$ and $(\pi^B)^* = 1$, i.e., the manager invests all the AUM in the bond: this is the only strategy yielding a terminal remuneration equal to zero ($X^*(T) = \frac{K-H}{1+\alpha}$) and thus not



(a) Influence of α (b) Zoom

Figure 3: **Sensitivity analysis of the optimal investment in the risky asset with respect to the management fee parameter**. The figure displays the optimal investment against AUM for different values of α. The remaining parameters are those of the baseline scenario, see (11).

negative with certainty. This is in line with the definition of admissible strategy in (5). The optimal allocation strategy in the risky asset is not defined for $X(t) < K^*(t)$, it is equal to zero at $K^*(t)$, then it increases reaching a maximum and then decreases until a kink at a point $K^{**}(t)$. When the AUM are between $K^*(t)$ and $K^{**}(t)$, we thus observe a hump in the manager's optimal risk exposure. As expected, the optimal stock strategy increases smoothly towards the Merton strategy, as $X(t)$ increases above the guarantee. Notice that, increasing γ, and thus decreasing the manager risk-aversion, the optimal strategy π^S increases.

It is possible to perform a sensitivity analysis of the risk exposure of the strategy with respect to the management fee parameter α (see Figure 3) and to the constant fee H (Figure 4). Increasing α, and thus decreasing the importance of the minimum guarantee penalty in the payoff or increasing the management fee, we observe that the asset manager takes more risk. As far as the constant fee H is concerned, in spite of the common wisdom that a constant salary prevents manager to take risk in excess, a constant floor induces the manager to take excessive risk as claimed in [17] and proved in [1]. In Figure 4, it is shown that the investment in the risky asset increases in H, moreover, when a constant fee is inserted ($H > 0$) the investment strategy is not anymore monotonic when the put option is out of the money. It becomes hump shaped above the guarantee yielding excessive risk taking with respect to the Merton strategy.

Finally, in Figure 5 we compare the optimal strategy obtained in Theorem 3 in case of a fee proportional to AUM (management fee) with the one obtained in Theorem 4 when the manager is remunerated through a share of the positive surplus with respect to the minimum guarantee (performance fee) as in [6]. We observe that a linear remuneration scheme renders a higher risk exposure than an asymmetric

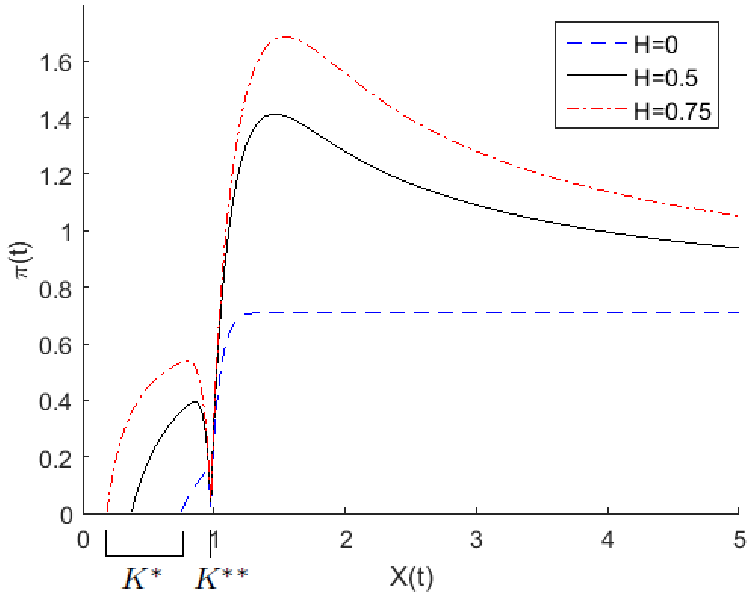


Figure 4: **Sensitivity analysis of the optimal investment in the risky asset with respect to the constant fee**. The figure displays the optimal investment against AUM for different values of H. The remaining parameters are those of the baseline scenario, see (11).

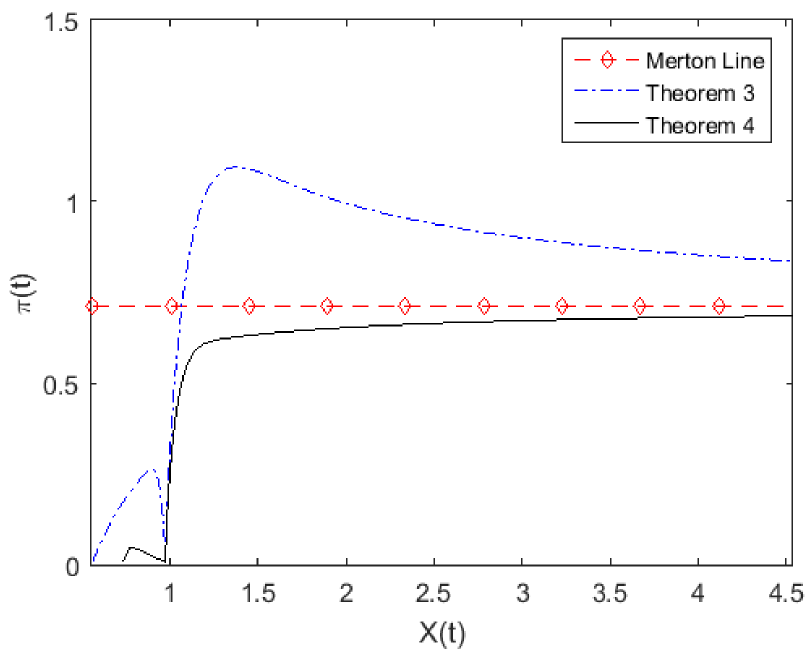


Figure 5: **Optimal investment in the risky asset in case of a performance fee**. This figure displays the profile of the optimal investment in the risky asset against AUM as described in Theorem 3 (asset management fee) and in Theorem 4 (performance fee). H is set equal to 0.25. The remaining parameters are those of the baseline scenario, see (11).

scheme which is based on a performance fee. This result agrees with what is observed in Figure 4. In fact, assuming $X(T) > K$, the two remuneration schemes render $\alpha X(T) + H$ in the framework of Theorem 3 and $\alpha X(T) + H - \alpha K$ in the framework of Theorem 4. Comparing the two payoffs we observe that the fixed fee H in the first case is replaced by a lower value $H - \alpha K$ in the second case, as a consequence the optimal strategy associated with an asset management fee takes more risk than the second one. A similar argument holds true also in the case $X(T) < K$.

Notice that differently from what is observed in case of an asset management fee, in Figure 5 the strategy associated to the performance fee doesn't induce the manager to take risk in excess with respect to the Merton strategy, i.e., the investment strategy in Theorem 4 is bounded from above by the Merton strategy. This is due to the choice of the parameters, since in this case $H - \alpha K < 0$. Numerical results not reported show that, if $H - \alpha K > 0$, then even the investment strategy associated to the performance fee is not monotonic when the put option is out of the money.

We can conclude that when the manager is remunerated through a constant fee, she takes a larger risk exposure when the remuneration is based on a management fee (linear scheme) than in case of a performance fee (option like scheme).

5 Conclusions

A minimum guarantee is a peculiarity of traditional life insurance policies. Often the contracts establish a performance target and the insurance company is committed to refund the performance gap to the policyholder at maturity.

The asset management problem considering a contract like this was not properly analyzed in the literature. In our analysis, we have modeled the guarantee assuming the insurance company is short of a put option and we have shown four main results: a) if there isn't a constant fee, the investment strategy in the risky asset is always below the one obtained without the guarantee (Merton strategy); b) the manager may invest in the risky asset even when AUM are below the strike price of the guarantee, in that case the investment is hump shaped: when the put option is deep in the money the manager doesn't invest in the risky asset; as AUM increase, the investment in the risky assets increases, reaches a maximum and then decreases just below the guarantee with a kink and then the investment in the risky asset increases again converging towards the solution obtained without constraint (Merton strategy); c) if the company is also remunerated through a (high) constant fee then the investment strategy is characterized by excessive risk taking with respect to the Merton's strategy; d) a remuneration scheme based on a proportion of the

AUM yields a higher exposure on the risky asset with respect to the case in which the remuneration of the manager is provided by a share of the positive performance surplus over the guarantee.

Acknowledgments

This paper has been presented at the 29th European Conference on Operational Research, EURO 2018. We thank all the participants for their helpful feedback. Usual caveat applies.

References

[1] Emilio Barucci, Gaetano La Bua and Daniele Marazzina (2018) On relative performance, remuneration and risk taking of asset managers, forthcoming in *Annals of Finance*.

[2] Suleyman Basak, Anna Pavlova, and Alexander Shapiro (2007) Optimal asset allocation and risk shifting in money management, *The Review of Financial Studies*, 20: 1583–1621

[3] Jean-Francois Boulier, ShaoJuan Huang, and Gregory Taillard (2001) Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund, *Insurance: Mathematics and Economics*, 28: 137-189.

[4] Andrea Buraschi, Robert Kosowski, and Worrawat Sritrakul (2014) Incentives and endogenous risk taking: A structural view on hedge fund alphas, *The Journal of Finance*, 69(6): 2819–2870.

[5] Jennifer Carpenter (2000) Does option compensation increase managerial risk appetite?, *The Journal of Finance*, 55: 2311–2331.

[6] Griselda Deelstra, Martino Grasselli, and Pierre Francois Koehl (2003) Optimal investment strategies in the presence of a minimum guarantee, *Insurance: Mathematics and Economics*, 33: 189-207.

[7] Griselda Deelstra, Martino Grasselli, and Pierre Francois Koehl (2004) Optimal design of the guarantee for defined contribution funds, *Journal of Economic Dynamics and Control*, 28: 2239-2260.

[8] Nicole El Karoui, Monique Jeanblanc, and Vincent Lacoste (2005) Optimal portfolio management with American capital guarantee, *Journal of Economic Dynamics and Control*, 29(3): 449-468.

[9] Guohui Guan, and Zongxia Liang (2014) Optimal management of DC pension plan in a stochastic interest rate and stochastic volatility framework, *Insurance: Mathematics and Economics*, 57: 58-66.

[10] Nen-wei Han, and Mao-wei Hung (2012) Optimal asset allocation for DC pension plans under inflation, *Insurance: Mathematics and Economics*, 51: 172-181.

[11] Xue Dong He, and Steven Kou (2018) Profit sharing in hedge funds, *Mathematical Finance*, 28: 50-81.

[12] Ioannis Karatzas, and Steve Shreve (1998) *Methods of Mathematical Finance*, volume 39. Springer, Science & Business Media.

[13] Hayne Leland, and Mark Rubinstein (1976) The evolution of portfolio insurance, in Luskin, D.L. (Ed.) *Portfolio Insurance: A Guide to Dynamic Hedging*. Wiley, New York.

[14] Hogcan Lin, David Saunders, and Chengguo Weng (2017) Optimal investment strategies for participating contracts, *Insurance: Mathematics and Economics*, 73: 137-155.

[15] Marco Nicolosi (2018) Optimal strategy for a fund manager with option compensation, *Decisions in Economics and Finance*, 41(1): 1–17.

[16] Marco Nicolosi, Flavio Angelini, and Stefano Herzel (2018) Portfolio management with benchmark related incentives under mean reverting processes, *Annals of Operations Research*, 266(1-2): 373–394.

[17] Stephen Ross (2004) Compensation, incentives, and the duality of risk aversion and riskiness, *The Journal of Finance*, 59: 207-225.

[18] Lucie Tepla (2001) Optimal investment with minimum performance constraints, *Journal of Economic Dynamics & Control*, 25: 1629-1645.

A Proofs

A.1 The optimal process

Following [5], the optimal strategy can be computed for any $t \in [0, T]$ as

$$X^*(t) = X^*(t, r(t), \zeta(t)) = \mathbb{E}_t \left[\frac{\zeta(T)}{\zeta(t)} I(\lambda \zeta(T)) \right].$$

where I is the inverse function of U' (U being the function introduced in (6)) and λ solves $\mathbb{E}[\zeta(T) I(\lambda \zeta(T))] = X_0$. Since U is not differentiable in $x = K$, U' denotes the set-valued first derivative of U given in Equation

(7). Let us also use the notation

$$i(z) := (u')^{-1}(z) = z^{\frac{1}{\gamma-1}}.$$

Consequently,

$$I(z) = \begin{cases} \frac{1}{\alpha} i\left(\frac{z}{\alpha}\right) - \frac{H}{\alpha}, & z < U'_+(K) \\ K, & U'_+(K) \le z \le U'_-(K) \\ \frac{1}{\alpha+1} i\left(\frac{z}{\alpha+1}\right) + \frac{K-H}{\alpha+1}, & z > U'_-(K). \end{cases} \tag{12}$$

Taking into account (12), the optimal process then becomes

$$\begin{aligned} X^*(t) &= \frac{1}{\alpha}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha}\right)\mathbf{1}_{\lambda\zeta(T)<U'_+(K)}\right] - \frac{H}{\alpha}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)<U'_+(K)}\right] \\ &+ K\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{U'_+(K)<\lambda\zeta(T)<U'_-(K)}\right] + \frac{1}{\alpha+1}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha+1}\right)\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right] \\ &+ \frac{K-H}{\alpha+1}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right]. \end{aligned} \tag{13}$$

Concerning the first term, we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha}\right)\mathbf{1}_{\lambda\zeta(T)<U'_+(K)}\right] = \left(\frac{\lambda\zeta(t)}{\alpha}\right)^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_+(K)}{\lambda\zeta(t)}}\right]. \tag{14}$$

For the second term, we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)<U'_+(K)}\right] = \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_+(K)}{\lambda\zeta(t)}}\right] \tag{15}$$

For the third term, we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{U'_+(K)<\lambda\zeta(T)<U'_-(K)}\right] = \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right] - \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_+(K)}{\lambda\zeta(t)}}\right]. \tag{16}$$

Concerning the fourth term, we have

$$\begin{aligned} \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha+1}\right)\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right] &= \left(\frac{\lambda\zeta(t)}{\alpha+1}\right)^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}>\frac{U'_-(K)}{\lambda\zeta(t)}}\right] \\ = \left(\frac{\lambda\zeta(t)}{\alpha+1}\right)^{\frac{1}{\gamma-1}}&\left(\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\right] - \mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right]\right). \end{aligned} \tag{17}$$

Concerning the last term we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right]=\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\right]-\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right]. \tag{18}$$

We then proceed through the estimation of

$$\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\Gamma}\right] \quad \text{and} \quad \mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\Gamma}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\Lambda}\right],$$

for $\Gamma,\Lambda\in\mathbb{R}$ (in our case Γ will be 1 or $\gamma/(\gamma-1)$ and Λ will be $\frac{U'_-(K)}{\lambda\zeta(t)}$ or $\frac{U'_+(K)}{\lambda\zeta(t)}$).

A.1.1 Stochastic interest rate

Notice that r is stochastic and it is given by the Vasicek dynamics the state price density becomes

$$\frac{\zeta(T)}{\zeta(t)}=\exp\left\{-\frac{1}{2}\left(\lambda_1^2+\lambda_2^2\eta\right)(T-t)-\int_t^T r(s)\mathrm{d}s-\int_t^T\lambda_1\mathrm{d}z(s)-\int_t^T\lambda_2\sqrt{\eta}\mathrm{d}z_r(s)\right\} \tag{19}$$

where $r(t)$ satisfies the SDE

$$\mathrm{d}r(t)=(a-br(t))\mathrm{d}t-\sqrt{\eta}\mathrm{d}z_r(t).$$

Following the lines of [6, Proof of Lemma 5], we substitute $\sqrt{\eta}\mathrm{d}z_r(t)$ in equation (19) by

$$\sqrt{\eta}\mathrm{d}z_r(t)=(a-br(t))\mathrm{d}t-\mathrm{d}r(t)$$

to obtain

$$\begin{aligned}\frac{\zeta(T)}{\zeta(t)}&=\exp\left\{-\frac{1}{2}\left(\lambda_1^2+\lambda_2^2\eta\right)(T-t)\right\}\exp\Big\{-\int_t^T a\lambda_2\mathrm{d}s-\int_t^T(1-b\lambda_2)r(s)\mathrm{d}s\\&\quad-\lambda_1(z(T)-z(t))+\lambda_2(r(T)-r(t))\Big\}\\&=\exp\left\{-\left(\frac{1}{2}\left(\lambda_1^2+\lambda_2^2\eta\right)+\lambda_2 a\right)(T-t)-\lambda_2 r(t)\right\}\\&\quad\exp\left\{-\int_t^T(1-b\lambda_2)r(s)\mathrm{d}s-\lambda_1(z(T)-z(t))+\lambda_2 r(T)\right\}\\&=f(t)\exp\left\{-V(t)\right\},\end{aligned}$$

where

$$f(t)=\exp\left\{-\left(\frac{1}{2}\left(\lambda_1^2+\lambda_2^2\eta\right)+\lambda_2 a\right)(T-t)-\lambda_2 r(t)\right\},$$

and

$$V(t) = \int_t^T (1 - b\lambda_2) r(s) \mathrm{d}s + \lambda_1 (z(T) - z(t)) - \lambda_2 r(T).$$

We thus have

$$\mathbb{E}_t \left[\left(\frac{\zeta(T)}{\zeta(t)} \right)^{\Gamma} \right] = f(t)^{\Gamma} \mathbb{E}_t \left[e^{-\Gamma V(t)} \right],$$
$$\mathbb{E}_t \left[\left(\frac{\zeta(T)}{\zeta(t)} \right)^{\Gamma} \mathbf{1}_{\frac{\zeta(T)}{\zeta(t)} < \Lambda} \right] = f(t)^{\Gamma} \mathbb{E}_t \left[e^{-\Gamma V(t)} \mathbf{1}_{e^{-V(t)} < \frac{\Lambda}{f(t)}} \right],$$

for any $\Gamma, \Lambda \in \mathbb{R}$.

We notice, as in [6, Proof of Lemma 2], that $V(t)$ is a Gaussian with mean

$$\mu = \mathbb{E}_t \left[V(t) \right] = (1 - \lambda_2 b) \left[\left(\frac{1 - e^{-b(T-t)}}{b} \right) r(t) + \frac{a}{b}(T - t) - \frac{a}{b^2}(1 - e^{-b(T-t)}) \right]$$
$$- \lambda_2 e^{-b(T-t)} r(t) - \lambda_2 \frac{a}{b}(1 - e^{-b(T-t)}),$$

that is

$$\mu = \frac{1 - \lambda_2 b - e^{-b(T-t)}}{b} r(t) + \frac{a}{b}(T - t)(1 - \lambda_2 b) - \frac{a}{b^2}(1 - e^{-b(T-t)}) \tag{20}$$

and variance

$$\begin{aligned}
\sigma^2 = VAR(V(t)) &= (1 - b\lambda_2)^2 VAR \left(\int_t^T r(s) \mathrm{d}s \right) + \lambda_1^2 (T - t) + \lambda_2^2 VAR(r(T)) \\
&- 2(1 - b\lambda_2)\lambda_2 COV \left(\int_t^T r(s) \mathrm{d}s, r(T) \right) \\
&= (1 - b\lambda_2)^2 \eta \int_t^T \left(\frac{1 - e^{-b(T-s)}}{b} \right)^2 \mathrm{d}s + \lambda_1^2 (T - t) + \lambda_2^2 \frac{\eta}{2b} \left(1 - e^{-2b(T-t)} \right) \\
&- 2\lambda_2 (1 - b\lambda_2) \eta \int_t^T e^{-b(T-s)} \left(\frac{1 - e^{-b(T-s)}}{b} \right) \mathrm{d}s.
\end{aligned} \tag{21}$$

Consequently, we can write

$$\mathbb{E}_t \left[\left(\frac{\zeta(T)}{\zeta(t)} \right)^{\Gamma} \right] = f(t)^{\Gamma} \mathbb{E}_t \left[e^{-\Gamma V(t)} \right] = f(t)^{\Gamma} e^{-\Gamma \mu + \frac{\Gamma^2 \sigma^2}{2}}.$$

Now we calculate

$$\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\Gamma}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\Lambda}\right]=f(t)^{\Gamma}\mathbb{E}_t\left[e^{-\Gamma V(t)}\mathbf{1}_{e^{-V(t)}<\frac{\Lambda}{f(t)}}\right].$$

With an abuse of notation we replace $\Lambda/f(t)$ with Λ and we proceed by estimating

$$\mathbb{E}_t\left[e^{-\Gamma V(t)}\mathbf{1}_{e^{-V(t)}<\Lambda}\right].$$

Taking into account that $V(t)$ is Gaussian with mean μ and variance σ as in (20) and (21), we have

$$\begin{aligned}\mathbb{E}_t\left[e^{-\Gamma V(t)}\mathbf{1}_{e^{-V(t)}<\Lambda}\right]&=\mathbb{E}_t\left[e^{-\Gamma V(t)}\mathbf{1}_{-V(t)<\ln(\Lambda)}\right]=\mathbb{E}_t\left[e^{-\Gamma V(t)}\mathbf{1}_{V(t)>-\ln(\Lambda)}\right]\\&=\int_{-\ln(\Lambda)}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\Gamma v}e^{-\frac{1}{2}\left(\frac{v-\mu}{\sigma}\right)^2}\mathrm{d}v=e^{-\Gamma\mu+\frac{\Gamma^2\sigma^2}{2}}\int_{-\ln(\Lambda)}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{v+(\Gamma\sigma^2-\mu)}{\sigma}\right)^2}\mathrm{d}v\\&=e^{-\Gamma\mu+\frac{\Gamma^2\sigma^2}{2}}\int_{\frac{-\ln(\Lambda)-\mu}{\sigma}+\Gamma\sigma}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\mathrm{d}v=e^{-\Gamma\mu+\frac{\Gamma^2\sigma^2}{2}}N(d-\Gamma\sigma),\end{aligned}$$

where $d=\frac{\ln(\Lambda)+\mu}{\sigma}$.

We are now able to determine the expectations (12)-(18) in the case Vasicek setting. Thanks to the calculation above, we have

$$\begin{aligned}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}i\left(\frac{\lambda\zeta(T)}{\alpha}\right)\mathbf{1}_{\lambda\zeta(T)<U_+'(K)}\right]&=\left(\frac{\lambda\zeta(t)}{\alpha}\right)^{\frac{1}{\gamma-1}}\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U_+'(K)}{\lambda\zeta(t)}}\right]\\&=\left(\frac{\lambda\zeta(t)f(t)}{\alpha}\right)^{\frac{1}{\gamma-1}}f(t)e^{-\frac{\gamma}{\gamma-1}\mu+\frac{\gamma^2}{2(\gamma-1)^2}\sigma^2}N\left(d^+-\frac{\gamma}{\gamma-1}\sigma\right),\end{aligned}$$

where

$$d^+:=\frac{\ln\frac{U_+'(K)}{\lambda\zeta(t)f(t)}+\mu}{\sigma}.$$

Concerning (15) we have

$$\begin{aligned}\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)<U_+'(K)}\right]&=\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U_+'(K)}{\lambda\zeta(t)}}\right]\\&=f(t)e^{-\mu+\frac{\sigma^2}{2}}N\left(d^+-\sigma\right).\end{aligned}$$

For (16) we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{U'_+(K)<\lambda\zeta(T)<U'_-(K)}\right] = \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right] - \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_+(K)}{\lambda\zeta(t)}}\right]$$
$$= f(t)e^{-\mu+\frac{\sigma^2}{2}}\left(N(d^- - \sigma) - N(d^+ - \sigma)\right),$$

with d^+ as above and

$$d^- = \frac{\ln\frac{U'_-(K)}{\lambda\zeta(t)f(t)} + \mu}{\sigma}.$$

Concerning (17), we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}i\left(\frac{\lambda\zeta(T)}{\alpha+1}\right)\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right]$$
$$= \left(\frac{\lambda\zeta(t)}{\alpha+1}\right)^{\frac{1}{\gamma-1}}\left(\mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\right] - \mathbb{E}_t\left[\left(\frac{\zeta(T)}{\zeta(t)}\right)^{\frac{\gamma}{\gamma-1}}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right]\right)$$
$$= \left(\frac{\lambda\zeta(t)f(t)}{\alpha+1}\right)^{\frac{1}{\gamma-1}} f(t)e^{-\frac{\gamma}{\gamma-1}\mu+\frac{\gamma^2}{2(\gamma-1)^2}\sigma^2}\left(1 - N\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)\right)$$

Concerning (18) we have

$$\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\lambda\zeta(T)>U'_-(K)}\right] = \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\right] - \mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)}\mathbf{1}_{\frac{\zeta(T)}{\zeta(t)}<\frac{U'_-(K)}{\lambda\zeta(t)}}\right]$$
$$= f(t)e^{-\mu+\frac{\sigma^2}{2}}\left(1 - N\left(d^- - \sigma\right)\right).$$

Finally, we notice that

$$\frac{N'(d^+ - \sigma)}{N'\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right)} = \left(\frac{\lambda\zeta(t)f(t)}{U'_+(K)}\right)^{\frac{1}{\gamma-1}} e^{-\frac{\gamma}{\gamma-1}\mu+\frac{\gamma^2}{2(\gamma-1)^2}\sigma^2}e^{\mu-\frac{\sigma^2}{2}},$$

or, equivalently,

$$(U'_+(K))^{\frac{1}{\gamma-1}}e^{-\mu+\frac{\sigma^2}{2}}\frac{N'(d^+ - \sigma)}{N'\left(d^+ - \frac{\gamma}{\gamma-1}\sigma\right)} = (\lambda\zeta(t)f(t))^{\frac{1}{\gamma-1}}\, e^{-\frac{\gamma}{\gamma-1}\mu+\frac{\gamma^2}{2(\gamma-1)^2}\sigma^2},$$

and, analogously,

$$(U'_-(K))^{\frac{1}{\gamma-1}}e^{-\mu+\frac{\sigma^2}{2}}\frac{N'(d^- - \sigma)}{N'\left(d^- - \frac{\gamma}{\gamma-1}\sigma\right)} = (\lambda\zeta(t)f(t))^{\frac{1}{\gamma-1}}\, e^{-\frac{\gamma}{\gamma-1}\mu+\frac{\gamma^2}{2(\gamma-1)^2}\sigma^2}.$$

Hence, (14) and (17) become

$$\begin{aligned}
&\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha}\right)\mathbf{1}_{\lambda\zeta(T)<U_+'(K)}\right] \\
&\qquad = \left(\frac{U_+'(K))}{\alpha}\right)^{\frac{1}{\gamma-1}} f(t)e^{-\mu+\frac{\sigma^2}{2}}\frac{N'(d^+-\sigma)}{N'\left(d^+-\frac{\gamma}{\gamma-1}\sigma\right)}N\left(d^+-\frac{\gamma}{\gamma-1}\sigma\right), \\
&\mathbb{E}_t\left[\frac{\zeta(T)}{\zeta(t)} i\left(\frac{\lambda\zeta(T)}{\alpha+1}\right)\mathbf{1}_{\lambda\zeta(T)>U_-'(K)}\right] \\
&\qquad = \left(\frac{U_-'(K)}{\alpha+1}\right)^{\frac{1}{\gamma-1}} f(t)e^{-\mu+\frac{\sigma^2}{2}}\frac{N'(d^--\sigma)}{N'\left(d^--\frac{\gamma}{\gamma-1}\sigma\right)}\left(1-N\left(d^--\frac{\gamma}{\gamma-1}\sigma\right)\right).
\end{aligned}$$

Summing up, the optimal process is given by

$$\begin{aligned}
X^*(t) = f(t)e^{-\mu+\frac{\sigma^2}{2}}&\left(\frac{\alpha K+H}{\alpha}\frac{N'(d^+-\sigma)}{N'\left(d^+-\frac{\gamma}{\gamma-1}\sigma\right)}N\left(d^+-\frac{\gamma}{\gamma-1}\sigma\right)+K\left(N(d^--\sigma)-N(d^+-\sigma)\right)\right. \\
&+\frac{\alpha K+H}{\alpha+1}\frac{N'(d^--\sigma)}{N'\left(d^--\frac{\gamma}{\gamma-1}\sigma\right)}\left(1-N\left(d^--\frac{\gamma}{\gamma-1}\sigma\right)\right)+\frac{K}{\alpha+1}(1-N(d^--\sigma)) \\
&\left.-H\left(\frac{1}{\alpha}N(d^+-\sigma)+\frac{1}{\alpha+1}(1-N(d^--\sigma))\right)\right).
\end{aligned} \tag{22}$$

A.2 The optimal portfolio

Following [12, Theorem 3.7.3], the optimal portfolio $\boldsymbol{\pi}=(\pi^S,\pi^B)^\top$ is given by

$$\Sigma^\top\boldsymbol{\pi}(t)=\frac{1}{X^*(t)\zeta(t)}\boldsymbol{\psi}(t)+\boldsymbol{\lambda}, \tag{23}$$

where Σ has been introduced in (2) and $\boldsymbol{\psi}(\cdot)$ is such that

$$dM(t)=\boldsymbol{\psi}^\top(t)\mathrm{d}\mathbf{z}(t), \tag{24}$$

M being the martingale defined as

$$M(t):=\zeta(t)X^*(t).$$

In order to apply this result we need to find the explicit expression of $\boldsymbol{\psi}$. By Itô formula, we have

$$\mathrm{d}M(t)=\mathrm{d}\zeta(t)X^*(t)+\zeta(t)\mathrm{d}X^*(t)+\mathrm{d}\zeta(t)\mathrm{d}X^*(t). \tag{25}$$

Moreover, notice that $\zeta(t)$ is of the form $\zeta(t) = \exp\left[-A(t) - \boldsymbol{\lambda}^\top \mathbf{z}(t)\right]$, while $X^*(t)$ is of the form $X^*(t) = X^*(t, r(t), \zeta(t))$. The dependence of X^* on r enters in the terms μ and f, while the dependence on ζ enters in $d^+ = d^+(\zeta, r), d^- = d^-(\zeta, r)$. Hence

$$\mathrm{d}\zeta(t) = [\cdots]\mathrm{d}t - \zeta(t)\boldsymbol{\lambda}^\top(t)\mathrm{d}\mathbf{z}(t),$$

while

$$\mathrm{d}X^*(t) = [\cdots]\mathrm{d}t + X^*_r(t, r(t), \zeta(t))\mathrm{d}r(t) - \zeta(t)X^*_\zeta(t, r(t), \zeta(t))\boldsymbol{\lambda}^\top(t)\mathrm{d}\mathbf{z}(t).$$

Here X^*_r, X^*_ζ denote the first order derivative of X^* wrt r, ζ. Finally, we notice that

$$\mathrm{d}\zeta(t)\mathrm{d}X^*(t) = [\cdots]\mathrm{d}t.$$

We recall that, being M a martingale, we did not study the drift terms in the above equations, since the sum of all the contributions in Equation (25) will result in a null drift.

We thus have

$$\begin{aligned}\mathrm{d}M(t) = [\cdot]\mathrm{d}t - \zeta(t)X^*_r(t, r(t), \zeta(t))\sqrt{\eta}\mathrm{d}z_r(t) \\ - \zeta(t)[X^*(t) + \zeta(t)X^*_\zeta(t, r(t), \zeta(t))]\boldsymbol{\lambda}^\top(t)\mathrm{d}\mathbf{z}(t).\end{aligned} \tag{26}$$

We now compute the derivative X^*_ζ. First of all, we notice that

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}N(d^\pm - \sigma) = \frac{\mathrm{d}d^\pm(\zeta)}{\mathrm{d}\zeta}N'(d^\pm - \sigma) = -\frac{1}{\sigma\zeta}N'(d^\pm - \sigma).$$

Moreover, since $N''(d) = -dN'(d)$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\zeta}N'(d^\pm - \sigma) &= \frac{d^\pm - \sigma}{\sigma\zeta}N'(d^\pm - \sigma), \\ \frac{\mathrm{d}}{\mathrm{d}\zeta}N'\left(d^\pm - \frac{\gamma}{\gamma - 1}\sigma\right) &= \frac{d^\pm - \frac{\gamma}{\gamma-1}\sigma}{\sigma\zeta}N'\left(d^\pm - \frac{\gamma}{\gamma - 1}\sigma\right),\end{aligned}$$

while

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{N'(d^{\pm}-\sigma)}{N'\left(d^{\pm}-\frac{\gamma}{\gamma-1}\sigma\right)} = \frac{1}{\sigma\zeta}\frac{(d^{\pm}-\sigma)N'(d^{\pm}-\sigma)}{N'\left(d^{\pm}-\frac{\gamma}{\gamma-1}\sigma\right)} - \frac{1}{\sigma\zeta}\frac{\left(d^{\pm}-\frac{\gamma}{\gamma-1}\sigma\right)N'(d^{\pm}-\sigma)}{N'\left(d^{\pm}-\frac{\gamma}{\gamma-1}\sigma\right)}$$
$$= \frac{1}{\zeta}\frac{1}{\gamma-1}\frac{N'(d^{\pm}-\sigma)}{N'\left(d^{\pm}-\frac{\gamma}{\gamma-1}\sigma\right)}.$$

Hence

$$\begin{aligned}
X_{\zeta}^{*}(t,r,\zeta) &= \frac{1}{\zeta}\frac{1}{\gamma-1}f(t)e^{-\mu+\frac{\sigma^2}{2}}\left[\frac{\alpha K+H}{\alpha}\frac{N'(d^{+}-\sigma)}{N'\left(d^{+}-\frac{\gamma}{\gamma-1}\sigma\right)}N\left(d^{+}-\frac{\gamma}{\gamma-1}\sigma\right)\right.\\
&\quad\left.+\ \frac{\alpha K+H}{\alpha+1}\frac{N'(d^{-}-\sigma)}{N'\left(d^{-}-\frac{\gamma}{\gamma-1}\sigma\right)}\left(1-N\left(d^{-}-\frac{\gamma}{\gamma-1}\sigma\right)\right)\right]\\
&= \frac{1}{\zeta}\frac{1}{\gamma-1}\left[X^{*}(t,r,\zeta)+Kf(t)e^{-\mu+\frac{\sigma^2}{2}}\left(N(d^{+}-\sigma)-\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}N(d^{-}-\sigma)\right)\right.\\
&\quad\left.+\ Hf(t)e^{-\mu+\frac{\sigma^2}{2}}\left(\frac{1}{\alpha}N(d^{+}-\sigma)+\frac{1}{\alpha+1}(1-N(d^{-}-\sigma))\right)\right].
\end{aligned}$$

Similarly $X_r^{*}(t,r,\zeta)$ can be computed.

Considering now $\mathrm{d}M(t)$ as in Equations (24) and (26) we obtain:

$$\boldsymbol{\psi}(t) = \zeta(t)\begin{pmatrix}0\\ -X_r^{*}(t,r(t)\zeta(t))\sqrt{\eta}\end{pmatrix} - \zeta(t)\left[X^{*}(t)+\zeta(t)X_{\zeta}^{*}(t,r(t),\zeta(t))\right]\boldsymbol{\lambda},$$

and thus Equation (23) implies

$$\Sigma^{T}\boldsymbol{\pi}(t) = \frac{1}{X^{*}(t)}\begin{pmatrix}0\\ -X_r^{*}(t,r(t)\zeta(t))\sqrt{\eta}\end{pmatrix} - \zeta(t)\frac{X_{\zeta}^{*}(t,r(t),\zeta(t))}{X^{*}(t)}\boldsymbol{\lambda}.$$

Remark 5 *Since*

$$(\Sigma^{\top})^{-1}\boldsymbol{\lambda} = \begin{pmatrix}\frac{\lambda_1}{\sigma_1}\\ \frac{-\sigma_2\sqrt{\eta}}{\sigma_1\sigma_B}\lambda_1+\frac{\lambda_2\sqrt{\eta}}{\sigma_B}\end{pmatrix},$$

from the above calculation we notice that the proportion invested into the stock is given by

$$\begin{aligned}
(\pi^{S})^{*}(t) &= \frac{\lambda_1}{\sigma_1(1-\gamma)}\left[1+\frac{K}{X^{*}(t)}f(t)e^{-\mu+\frac{\sigma^2}{2}}\left(N(d^{+}-\sigma)-\frac{1}{\alpha+1}-\frac{\alpha}{\alpha+1}N(d^{-}-\sigma)\right)\right.\\
&\quad\left.+\ \frac{H}{X^{*}(t)}f(t)e^{-\mu+\frac{\sigma^2}{2}}\left(\frac{1}{\alpha}N(d^{+}-\sigma)+\frac{1}{\alpha+1}(1-N(d^{-}-\sigma))\right)\right].
\end{aligned}$$