Carleman Type Approximation Theorem in the Quaternionic Setting and Applications

Sorin G. Gal* Irene Sabadini

Abstract

In this paper we prove Carleman's approximation type theorems in the framework of slice regular functions of a quaternionic variable. Specifically, we show that any continuous function defined on $\mathbb{R}$ and quaternion valued, can be approximated by an entire slice regular function, uniformly on $\mathbb{R}$, with an arbitrary continuous "error" function. As a byproduct, one immediately obtains result on uniform approximation by polynomials on compact subintervals of $\mathbb{R}$. We also prove an approximation result for both a quaternion valued function and its derivative and, finally, we show some applications.

1 Introduction and Preliminaries

Carleman's approximation theorem in complex setting was proved in Carleman [2] and can be stated as follows.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{C}$ and $\varepsilon : \mathbb{R} \to (0, +\infty)$ be continuous on $\mathbb{R}$. Then there exists an entire function $G : \mathbb{C} \to \mathbb{C}$ such that*

$$|f(x) - G(x)| < \varepsilon(x), \text{ for all } x \in \mathbb{R}.$$

The Carleman's theorem is a pointwise approximation result which generalizes the Weierstrass result on uniform approximation by polynomials in compact

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intervals, since on any compact subinterval of $\mathbb{R}$, the entire function can in turn be approximated uniformly by polynomials, more exactly by the partial sums of is power series (see Remark 2.9 for the quaternionic setting).

A natural question is to ask what kind of approximation results one can obtain in the quaternionic setting. In the literature, there are approximation results obtained on balls, see [6], [7], [8], [9] and also Runge theorems, see [4], on uniform approximation for slice regular functions by using rational functions or polynomials.

The goal of the present paper is to extend Theorem 1.1 and other Carleman-type results to the case of entire functions of a quaternionic variable. The class of functions we will consider are expressed by converging power series of the quaternion variable q. This class is a subset of the class of the so-called slice regular functions, see e.g. [3] for a systematic treatment of these functions as well as their applications to the construction of a quaternionic functional calculus. To the best of our knowledge, a Carleman-type theorem has never proved neither for Cauchy-Fueter regular functions of a quaternionic variable nor for monogenic functions with values in a Clifford algebra.

In order to introduce the framework in which we will work, let us introduce some preliminary notations and definitions.

The noncommutative field $\mathbb{H}$ of quaternions consists of elements of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, $x_i \in \mathbb{R}$, $i = 0,1,2,3$, where the imaginary units i, j, k satisfy

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

The real number x_0 is called real part of q, and is denoted by $\mathrm{Re}(q)$, while $x_1 i + x_2 j + x_3 k$ is called imaginary part of q and is denoted by $\mathrm{Im}(q)$. We define the norm of a quaternion q as $\|q\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^3}$. By $\mathbb{S}$ we denote the unit sphere of purely imaginary quaternion, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^3 = 1\}.$$

Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For any fixed $I \in \mathbb{S}$ we define $\mathbb{C}_I := \{x + Iy;\ |\ x, y \in \mathbb{R}\}$, which can be can be identified with a complex plane. Obviously, the real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$. Any non real quaternion q is uniquely associated to the element $I_q \in \mathbb{S}$ defined by $I_q := (ix_1 + jx_2 + kx_3)/\|ix_1 + jx_2 + kx_3\|$ and so q belongs to the complex plane $\mathbb{C}_{I_q}$.

The functions we will consider are entire in a suitable sense of analyticity, the so called left slice regularity (or left slice hyperholomorphy) for functions of a quaternion variable, see [5].

Definition 1.2. *Let U be an open set in $\mathbb{H}$ and let $f : U \to \mathbb{H}$ be real differentiable. The function f is called left slice regular if for every $I \in \mathbb{S}$, its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ satisfies*

$$\overline{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0, \quad \text{on } U \cap \mathbb{C}_I.$$

The following result allows to look at slice regular functions as power series of the variable q with quaternionic coefficients on the right (see [5]):

Theorem 1.3. *Let $\mathbb{B}_R = \{q \in \mathbb{H} \, ; \, \|q\| < R\}$. A function $f : \mathbb{B}_R \to \mathbb{H}$ is left slice regular on $\mathbb{B}_R$ if and only if it has a series representation of the form*

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \qquad a_n \in \mathbb{H} \tag{1}$$

uniformly convergent on $\mathbb{B}_R$.

Unless otherwise stated, the *entire functions* considered in this paper will be power series of the form (1) converging for any $R > 0$.

Definition 1.4. *The functions which, on a ball $\mathbb{B}_R$, admit a series expansion of the form (1) with real coefficients a_n are called quaternionic intrinsic. They form a class denoted by $\mathcal{N}(\mathbb{B}_R)$.*

To complete the preliminary notions we note that for any slice regular function we have

$$\frac{\partial}{\partial x} f(x+Iy) = -I \frac{\partial}{\partial y} f(x+Iy) \qquad \forall I \in \mathbb{S},$$

and therefore, analogously to what happens in the complex case, for all $I \in \mathbb{S}$ the following equality holds:

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(x+Iy) = \partial_x(f)(x+Iy).$$

By setting $q = x + Iy$ we will write $f'(q)$ instead of $\partial_x(f)(q)$. For a discussion of the relation between $f'(q)$ and the so-called slice derivative of a slice regular function, we refer the interested reader to [3], p.115.

The plan of the present paper goes as follows. In Section 2 we prove the Carleman's approximation theorem i.e. a pointwise approximation for the class of slice regular functions. In Section 3 we prove a simultaneous approximation result, namely an approximation for both a quaternion valued function and its derivative. Finally, in Section 4 we discuss some applications.

2 Carleman Approximation Theorem

The first main result of this section is the following.

Theorem 2.1. *Let $f : \mathbb{R} \to \mathbb{H}$ and $\varepsilon : \mathbb{R} \to (0, +\infty)$ be continuous on $\mathbb{R}$. Then there exists an entire function $G : \mathbb{H} \to \mathbb{H}$ such that*

$$\|f(x) - G(x)\| < \varepsilon(x), \ \textit{for all } x \in \mathbb{R}.$$

The proof of Theorem 2.1 requires some auxiliary results and follows the ideas in the complex case in Hoischen's paper [10], see also Burckel's book [1], pp. 273-276.

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{H}$ be continuous on $\mathbb{R}$. There exists a zero free entire function $g : \mathbb{H} \to \mathbb{H}$ such that $g(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ and $g(x) > \|f(x)\|$, for all $x \in \mathbb{R}$.*

Proof. For $n \in \mathbb{N}$ denote $M_n = \max\{\|f(x)\|; |x| \leq n+1\}$ and choose a natural number $k_n \geq n$ such that $\left(\frac{n^2}{n+1}\right)^{k_n} > M_n$. If $q \in \mathbb{H}$ is such that $\|q\| \leq N$, then $\|q^2/(n+1)\| < 1/2$ for all $n \geq 2N^2$, which implies that the power series in quaternions $h(q) = M_0 + \sum_{n=1}^{\infty} \left(\frac{q^2}{n+1}\right)^{k_n}$ converges uniformly in any closed ball $\overline{B(0;N)}$, with arbitrary $N > 0$, which shows that h is entire on $\mathbb{H}$. Also, note that the coefficients in the series development are all real (and positive).

Evidently $h(x) \geq 0$ for all $x \in \mathbb{R}$. Then for $|x| < 1$ we have $h(x) \geq M_0 \geq \|f(x)\|$, while for $1 \leq n \leq |x| < n+1$ we have $h(x) > \left(\frac{x^2}{n+1}\right)^{k_n} \geq \left(\frac{n^2}{n+1}\right)^{k_n} > M_n \geq \|f(x)\|$, which implies $h(x) \geq \|f(x)\|$, for all $x \in \mathbb{R}$. Finally, set $g(q) = e^{h(q)}$ to get the required entire function. Here a comment is in order: in general the composition $f \circ h$ of two slice regular functions f and h is not, in general, slice regular, but it is so when h is quaternionic intrinsic, see [3]. It also worth noting that $g \in \mathcal{N}(B(0;R))$ for all $R > 0$, i.e. the coefficients in its series development are all real.

Lemma 2.3. *Let $I = [a, b]$ be an interval in $\mathbb{R}$ and let $f : I :\to \mathbb{H}$ be a continuous function. For any $k \in \mathbb{N}$ define*

$$f_k(x) = \frac{k}{C} \int_a^b e^{-k^2(x-t)^2} f(t)\, dt, \quad x \in \mathbb{R}, \tag{2}$$

where $C = \int_{-\infty}^{+\infty} e^{-x^2}\, dx$. Then for every $\varepsilon > 0$

$$\lim_{k\to+\infty} f_k(x) = \begin{cases} f(x) & \text{uniformly for } x \in [a+\varepsilon, b-\varepsilon] \\ 0 & \text{uniformly for } x \in \mathbb{R} \setminus [a+\varepsilon, b-\varepsilon] \end{cases}$$

Proof. Let us choose a basis $\{1, i, j, k\}$, with $i^2 = j^2 = k^2 = -1, ij = -ji = k$ for the (real) vector space of quaternions. Let us write $f(x) = f_0(x) + f_1(x)i + f_2(x)j + f_3(x)k = \varphi(x) + \psi(x)j$ where the functions $\varphi(x) = f_0(x) + f_1(x)i$, $\psi(x) = f_2(x) + f_3(x)i$ have values in the complex plane $z = x + iy$. Since the result holds true for complex valued functions, see e.g. [1, Exercise 8.26 (ii)], we can define, for each $k \in \mathbb{N}$, the functions $\varphi_k(x)$ and $\psi_k(x)$ as in formula (2) by writing $\varphi(t)$, $\psi(t)$ instead of $f(t)$ in the integrand. Then for every $\varepsilon > 0$ we have that, uniformly, $\lim_{k\to+\infty} \varphi_k(x)$ is $\varphi(x)$ in $[a+\varepsilon, b-\varepsilon]$ and is 0 outside. In an analogous way, we have that, uniformly, $\lim_{k\to+\infty} \psi_k(x)$ is $\psi(x)$ in $[a+\varepsilon, b-\varepsilon]$ and is 0 outside. By setting $f_k(x) = \varphi_k(x) + \psi_k(x)$ we obtain the statement.

Lemma 2.4. *Let $f : \mathbb{R} \to \mathbb{H}$ be continuous on $\mathbb{R}$. Then for each $n \in \mathbb{Z}$, there exists a continuous function $f_n : \mathbb{R} \to \mathbb{H}$ with support in $[-1, 1]$, such that for all $x \in \mathbb{R}$ we have $f(x) = \sum_{n=-\infty}^{+\infty} f_n(x-n)$.*

Proof. We write the function $f(x)$ as $\varphi(x) + \psi(x)j$ as we have done in the proof of Lemma 2.3. Since the result is true for complex valued functions, see e. g.

[1, Exercise 8.28 (i)], we have

$$\varphi(x) = \sum_{n=-\infty}^{+\infty} \varphi_n(x-n)$$
$$\psi(x) = \sum_{n=-\infty}^{+\infty} \psi_n(x-n),$$

and by setting $f_n(x-n) = \varphi_n(x-n) + \psi_n(x-n)j$ the result follows.

Remark 2.5. It is interesting for the sequel to explicitly construct the functions $\varphi_n(x-n)$, $\psi_n(x-n)$ following [1]. Let $\sigma(x)$ be a piecewise linear function which is equal 1 on $(-1/2, 1/2)$ and is 0 outside $(-1,1)$ and let

$$\Sigma(x) = \sum_{n=-\infty}^{\infty} \sigma(x-n), \quad x \in \mathbb{R}$$

Then the functions $\varphi_n(x)$ can be constructed as

$$\varphi_n(x) = \frac{\sigma(n)\varphi(x+n)}{\Sigma(x+n)}, \; n \in \mathbb{N}.$$

and similarly we can construct $\psi_n(x)$.

Lemma 2.6. *Let $f : \mathbb{R} \to \mathbb{H}$ be continuous on $\mathbb{R}$ and having compact support in $[-1,1]$. Set*

$$T = \{q \in \mathbb{H} \; : \; |\mathrm{Re}(q)| > 3 \text{ and } |\mathrm{Re}(q)| > 2\|\mathrm{Im}(q)\|\}.$$

For any number $\varepsilon > 0$, there exists an entire function $F : \mathbb{H} \to \mathbb{H}$, such that $\|f(x) - F(x)\| < \varepsilon$ for all $x \in \mathbb{R}$ and $\|F(q)\| < \varepsilon$ for all $q \in T$.

Proof. For any $k \in \mathbb{N}$ let us define

$$f_k(q) = \frac{k}{C} \int_{-1}^{1} e^{-k^2(q-t)^2} f(t)dt, \quad q \in \mathbb{H},$$

where $C = \int_{-\infty}^{+\infty} e^{-x^2} dx$. First of all note that the function $e^{-k^2(q-t)^2}$ is slice regular and when we multiply it on the right by the quaternion valued function $f(t)$ it remains slice regular, since slice regular functions form a right vector space over $\mathbb{H}$, and with compact support in $[-1,1]$, since so is f. If we expand the exponential in power series, by the uniform convergence we can exchange the series and the integral, thus $f_k(q)$ can be written as power series and so it is an entire slice regular function for all $k \in \mathbb{N}$. If we apply Lemma 2.3 to the function $f(x)$ by choosing $a = -2, b = 2$ we obtain that $f_k \to f$ uniformly in $[-3/2, 3/2]$ while, by choosing $a = -1, b = 1$ we have that $f \to 0$ uniformly in $\mathbb{R} \setminus [-3/2, 3/2]$ and so $f_k \to f$ uniformly on $\mathbb{R}$. Let $q \in T$ and $t \in [-1,1]$ and write $q = x_0 + \mathrm{Im}(q)$. Easy computations show that

$$\mathrm{Re}(k^2(q-t)^2) = k^2((x_0-t)^2 - \|\mathrm{Im}(q)\|^2) > \frac{3}{4}k^2.$$

On each interval $[a,b]$, the function $f_k(x)$, that we can write in real components as $f_k = f_{k0} + f_{k1}i + f_{k2}j + f_{k3}k$, is such that

$$\| \int_a^b f_k(x)\, dx\| \le \sqrt{\sum_{n=0}^{3} \left(\int_a^b f_{kn}(x) dx \right)^2} \le \sum_{n=0}^{3} \int_a^b \|f_{kn}(x)\| dx \le 4 \int_a^b \|f_k(x)\|\, dx$$

Then, for all $q \in T$, we have

$$\begin{aligned} \|f_k(q)\| &\le 4\frac{k}{C}\int_{-1}^{1} \|e^{-k^2(q-t)^2} f(t)\| dt \\ &\le 4\frac{k}{C}\int_{-1}^{1} e^{-\mathrm{Re}(-k^2(q-t)^2)}\|f(t)\| dt \\ &\le 4\frac{k}{C} e^{-\frac{3}{4}k^2}\int_{-1}^{1} \|f(t)\| dt \le \frac{k}{C}\frac{16}{3k^2} M \end{aligned} \tag{3}$$

where $M = \int_{-1}^{1} \|f(t)\| dt$. If we choose $F(q) = f_k(q)$ for k large we have that $\|f(x) - F(x)\| < \varepsilon$ for $x \in \mathbb{R}$ since $f_k \to f$ uniformly on $\mathbb{R}$, moreover $\|F(q)\| < \varepsilon$ for $q \in T$ by the estimate (3).

Lemma 2.7. *Let $f : \mathbb{R} \to \mathbb{H}$ be continuous on $\mathbb{R}$. There exists an entire function $F : \mathbb{H} \to \mathbb{H}$, such that $\|f(x) - F(x)\| < 1$ for all $x \in \mathbb{R}$.*

Proof. Let f_n be as in Lemma 2.4, for $n \in \mathbb{Z}$. By Lemma 2.6 we can associate to each f_n an entire function F_n such that $\|f_n(x) - F_n(x)\| < 2^{-|n|-2}$, $\|F_n(x)\| < 2^{-|n|}$. Let $N \in \mathbb{N}$, then choose q such that $\|q\| \le N$ and $n \in \mathbb{Z}$ such that $|n| > 3N + 3$. We have

$$|\mathrm{Re}(q-n)| \ge |n| - |\mathrm{Re}(q)| > 3$$

and

$$\|\mathrm{Im}(q-n)\| = \|\mathrm{Im}(q)\| \le N < \frac{1}{3}(|n| - N) \le \frac{1}{2}|\mathrm{Re}(q-n)|.$$

The above inequalities allows to conclude that $q - n$ belongs to the set T defined in Lemma 2.6. Our assumption allows to obtain

$$\|F_n(q-n)\| < 2^{-|n|} \qquad \text{for } \|q\| \le N,\ |n| > 3N+3.$$

The estimate implies that the series $\sum_{n=-\infty}^{+\infty} F_n(q-n)$ converges uniformly for any q such that $\|q\| \le N$, for any $N \in \mathbb{N}$. Thus the sequence $s_m(q) = \sum_{n=-m}^{m} F_n(q-n)$ converges uniformly to a function F, as well as its restrictions to any complex plane $\mathbb{C}_I$, for all $I \in \mathbb{S}$. Thus we have that

$$(\partial_x + I\partial_y)F(x+Iy) = (\partial_x + I\partial_y)\lim_{m\to\infty} s_m(x+Iy) = \lim_{m\to\infty}(\partial_x + I\partial_y)s_m(x+Iy) = 0,$$

for any q such that $\|q\| \le N$, for any $N \in \mathbb{N}$ and and so F is an entire function. Moreover for any $x \in \mathbb{R}$ we have

$$\begin{aligned} \|F(x) - f(x)\| &\le \| \sum_{n=-\infty}^{+\infty} F_n(x-n) - f_n(x-n)\| \\ &\le \sum_{n=-\infty}^{+\infty} \|F_n(x) - f_n(x)\| < \sum_{n=-\infty}^{+\infty} 2^{-|n|-2} < 1 \end{aligned}$$

and this concludes the proof.

Proof of Theorem 2.1. By Lemma 2.2 there exists a zero free entire function $h : \mathbb{H} \to \mathbb{H}$, with all the coefficients in its series development being real numbers, such that $h(x) > \frac{1}{\varepsilon(x)}$, for all $x \in \mathbb{R}$. Then, Lemma 2.7 gives an entire function $F : \mathbb{H} \to \mathbb{H}$ such that $\|h(x)f(x) - F(x)\| < 1$, for all $x \in \mathbb{R}$. Since $h(x)$ is real valued, this implies

$$\left\| f(x) - \frac{F(x)}{h(x)} \right\| < \frac{1}{h(x)} < \varepsilon(x), \text{ for all } x \in \mathbb{R}.$$

Then the proof follows by choosing $G(q) = [h(q)]^{-1} \cdot F(q)$.

Remark 2.8. Note that if the function $h(q) \notin \mathcal{N}(\mathbb{H})$ then one would have chosen $G(q) = [h(q)]^{-*} * F(q)$ where $*$ denotes the star multiplication, see [3], i.e. a multiplication which preserves slice regularity.

Remark 2.9. The Weierstrass result on uniform approximation by polynomials on compact subintervals of $\mathbb{R}$ easily follows from Theorem 2.1. Indeed, choose $[A, B] \subset \mathbb{R}$ and an arbitrary small constant $\varepsilon(x) := \varepsilon/2 > 0$, for all $x \in \mathbb{R}$. By Theorem 2.1, there exists an entire function $G(q) = \sum_{k=0}^{\infty} q^k a_k$, such that $\|f(x) - G(x)\| < \varepsilon/2$, for all $x \in [A, B]$. But from the uniform convergence of the series $G(q)$ in a closed ball $\overline{B(0; R)}$ that includes $[A, B]$, clearly there exists n_0 such that for all $n \geq n_0$ we have $\|G(q) - \sum_{k=0}^{n} q^k a_k\| < \varepsilon/2$, for all $q \in \overline{B(0; R)}$, which implies

$$\|f(x) - \sum_{k=0}^{n} x^k a_k\| \leq \|f(x) - G(x)\| + \|G(x) - \sum_{k=0}^{n} x^k a_k\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for all $x \in [A, B]$ and all $n \geq n_0$.

3 Carleman-Type Theorem on Simultaneous Approximation

In this section we derive the following Carleman-type result on simultaneous approximation generalizing those obtained in Kaplan [11] in the complex case.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{H}$ having a continuous derivative on $\mathbb{R}$ and $E : \mathbb{R} \to (0, +\infty)$ be continuous on $\mathbb{R}$. Then there exists an entire function $G : \mathbb{H} \to \mathbb{H}$ such that simultaneously we have*

$$\|f(x) - G(x)\| < E(x), \ \|f'(x) - G'(x)\| < E(x), \ \textit{for all } x \in \mathbb{R}.$$

We will adapt the proof of [11, Theorem 3] which holds in the case of a complex variable to our setting. That proof is based on Lemma 1 and Lemma 2 in the same paper. Since Lemma 1 refers only to real valued functions of real variable, it will remain unchanged. Therefore we have to deal just with the analogue of Lemma 2 in the quaternionic setting. We have:

Lemma 3.2. *Let $E_1 : \mathbb{R} \to \mathbb{R}_+$ be continuous, satisfying $E_1(x) = E_1(-x)$, for all $x \in \mathbb{R}$ and such that $k = \int_{-\infty}^{+\infty} E_1(t)dt$ is finite. Let $A, B \in \mathbb{H}$ be satisfying $\|A - B\| < 2k$. Then there exists an entire function $h : \mathbb{H} \to \mathbb{H}$, such that*

$$\|h'(x)\| < E_1(x), \ \textit{for all } x \in \mathbb{R}, \ \textit{and} \ \lim_{x \to -\infty} h(x) = A, \ \lim_{x \to +\infty} h(x) = B.$$

Proof. If $A = B$, then clearly we can choose $h(q) = A$, for all $q \in \mathbb{H}$. If $A \neq B$, denote $r = \|A - B\|/(2k)$ and $s = (1-r)/(2(1+r))$. By Theorem 2.1, there exists an entire function $G : \mathbb{H} \to \mathbb{H}$, such that for all $x \in \mathbb{R}$ we have $\|G(x) - E_1(x)\| < sE_1(x)$.

Now, if $G(q) = \sum_{n=0}^{\infty} q^n a_n$ then $h_0(q) = \sum_{n=0}^{\infty} q^{n+1} \cdot \frac{a_n}{n+1}$ remains a convergent series with the same ray of convergence as G, therefore h_0 is also entire. In addition, it is clear that $\partial_s h_0(q) = G(q)$ for all q. Therefore, we get that there exists an entire function $h_0 : \mathbb{H} \to \mathbb{H}$, such that

$$\|h_0'(x) - E_1(x)\| < sE_1(x), \text{ for all } x \in \mathbb{R}.$$

This last inequality implies $\|h'(x)\| \leq (1+s)E_1(x)$ and therefore by the Leibniz-Newton formula $h_0(x) = \int_0^x h'(t)dt + h_0(0)$, we get that the next two limits exist (in $\mathbb{H}$)

$$\lim_{x \to +\infty} h_0(x) = \int_0^{+\infty} h_0'(t)dt + h_0(0) := B_0,$$

$$\lim_{x \to -\infty} h_0(x) = \int_0^{-\infty} h_0'(t)dt + h_0(0) := A_0.$$

In addition, we easily get $\mathrm{Re}[h'(x)] > (1-s)E_1(x)$ for all $x \in \mathbb{R}$ and therefore

$$\|A_0 - B_0\| = \left\| \int_{-\infty}^{+\infty} h'(x)dx \right\| > \int_{-\infty}^{+\infty} \mathrm{Re}[h'(x)]dx > 2k(1-s).$$

Choosing now the constants $a, b \in \mathbb{H}$ such that $aA_0 + b = A$, $aB_0 + b = B$ and defining $h(q) = ah_0(q) + b$, by similar reasonings with those in the proof of Lemma 2 in [11] we get the desired conclusion.

Proof of Theorem 3.1. Without loss of generality, we may suppose that $E(x) = E(-x)$, for all $x \in \mathbb{R}$ (this is due to the simple fact for any positive function $E(x)$ on $\mathbb{R}$, we can define $E^*(x) = \min(E(x), E(-x))$, which is now an even function on $\mathbb{R}$). Let $E_1(x)$ (depending on $E(x)$ as in Lemma 1 in [11]) so that E_1 is also an even function. By Theorem 2.1, there exists an entire function G_1 such that $\|G_1(x) - f'(x)\| < E_1(x)$, for all $x \in \mathbb{R}$.

Set $g(x) = \int_0^x [G_1(t) - f'(t)]dt$. By the choice of $E_1(x)$, there exist (in $\mathbb{H}$) the limits $\lim_{x \to +\infty} g(x) = B$, $\lim_{x \to -\infty} g(x) = A$ and $\|A - B\| < \int_{-\infty}^{+\infty} E_1(x)dx := 2k$. For these A, B and $E_1(x)$, let h the entire function given by the above Lemma 3.2.

Define now $G(q) = \int_0^q G_1(t)dt + f(0) - h(q)$, $q \in \mathbb{H}$. The conclusion of the theorem follows as in the proof of Theorem 3 in [11].

4 Applications

The first application of Theorem 2.1 is the following.

Theorem 4.1. *Let $f : (-1,1) \to \mathbb{H}$ and $\varepsilon : (-1,1) \to (0,+\infty)$ be continuous on $(-1,1)$. Then there exists a power series $P(u) = \sum_{n=0}^{\infty} u^n a_n$, with $a_n \in \mathbb{H}$, such that*

$$\|f(u) - P(u)\| < \varepsilon(u), \text{ for all } u \in (-1,1).$$

In addition, if f is real-valued on $(-1,1)$ then also P can be chosen real-valued on $(-1,1)$.

Proof. It is an immediate consequence of Theorem 2.1 by using the entire function $w \in \mathcal{N}(B(0;R))$ for all $R > 0$, defined by

$$w(q) = \tan\left(\frac{\pi}{2}q\right) = \sum_{n=1}^{\infty} q^{2n-1} \cdot \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}$$

$$= q + q^3 \cdot \frac{1}{3} + q^5 \cdot \frac{2}{15} + q^7 \cdot \frac{17}{315} + ... +,$$

where B_n denotes the nth Bernoulli number.

Indeed, defining $F : \mathbb{R} \to \mathbb{H}$ by $F(x) = f((2/\pi)\arctan(x))$, clearly F is continuous on $\mathbb{R}$ and then by Theorem 2.1, for the continuous function $E : \mathbb{R} \to \mathbb{R}_+$ defined by $E(x) = \varepsilon((2/\pi)\arctan(x))$, there exists an entire function $G : \mathbb{H} \to \mathbb{H}$, such that $\|F(x) - G(x)\| < E(x)$, for all $x \in \mathbb{R}$, i.e. $\|f((2/\pi)\arctan(x)) - G(x)\| < E(x)$ for all $x \in \mathbb{R}$.

Denoting $(2/\pi)\arctan(x) = u$ and replacing in the last inequality, we obtain

$$\|f(u) - G(\tan(\pi u/2))\| < E(\tan(\pi u/2)) = \varepsilon(u), \text{ for all } u \in (-1, 1).$$

Denoting now $P(q) = G(w(q))$, since $w \in \mathcal{N}(B(0;R))$ for all $R > 0$ it follows that P is an entire function on $\mathbb{H}$ and therefore we can write $P(q) = \sum_{n=0}^{\infty} q^n a_n$, for all $q \in \mathbb{H}$ and the statement follows.

Similar to the case of complex variable of Theorem 7 in Kaplan [11], one can prove the following.

Corollary 4.2. *Let $f : \partial(B(0;1)) \to \mathbb{R}$ be real-valued and measurable. Then there exists a function $u : \overline{B(0;1)} \to \mathbb{H}$, harmonic in $B(0;1)$ (that is if $u(q) = u(x + Iy)$ then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, for all $\|q\| < 1$), such that for any $I \in \mathbb{S}$ we have $u(re^{I\varphi}) \to f(e^{I\varphi})$ as $r \nearrow 1$, for almost everywhere φ.*

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University of Oradea
Department of Mathematics and Computer Science
Str. Universitatii Nr. 1
410087 Oradea, Romania
galso@uoradea.ro

Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9
20133 Milano, Italy
irene.sabadini@polimi.it