

TWO-PHASE PROBLEMS WITH DISTRIBUTED SOURCE: REGULARITY OF THE FREE BOUNDARY

D. DE SILVA, F. FERRARI, AND S. SALSA

ABSTRACT. We investigate the regularity of the free boundary for a general class of two-phase free boundary problems with non-zero right hand side. We prove that Lipschitz or flat free boundaries are $C^{1,\gamma}$. In particular, viscosity solutions are indeed classical.

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider two phase free boundary problems governed by uniformly elliptic equations with distributed sources. Our purpose is to investigate the regularity of the free boundary under additional hypotheses such as flatness or Lipschitz continuity. A model problem we have in mind is the following:

$$(1.1) \quad \begin{cases} \Delta u = f, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1, & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases}$$

Here, as usually for any bounded domain $\Omega \subset \mathbb{R}^n$,

$$\Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \quad \Omega^-(u) := \{x \in \Omega : u(x) \leq 0\}^\circ,$$

and u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $\Omega^+(u)$ and $\Omega^-(u)$ respectively.

Typical examples are the Prandtl-Bachelor model in fluid dynamics (see e.g. [B1, EM]), where $f = \mathbf{1}_{\Omega^-(u)}$, the characteristic function of the negative phase, or the eigenvalue problem in magnetohydrodynamics (1,1) considered in [FL], where $f = \lambda u$. Other examples come from limits of singular perturbation problems with forcing term as in [LW], where the authors analyze solutions to (1.1), arising in the study of flame propagation with nonlocal effects.

The homogeneous case $f \equiv 0$ was settled in the classical works of Caffarelli [C1, C2]. A key step in these papers is the construction of a family of continuous supconvolution deformations that act as comparison subsolutions.

The results in [C1, C2] have been widely generalized to different classes of homogeneous elliptic problems. See for example [CFS, FS1, FS2] for linear operators, [AF, F1, F2, Fe1, W1, W2] for fully nonlinear operators and [LN] for the p -Laplacian. All these papers follow the guidelines of [C1, C2].

D. D. and F. F. are supported by the ERC starting grant project 2011 EPSILON (Elliptic PDEs and Symmetry of Interfaces and Layers for Odd Nonlinearities). F. F. is supported by Miur, Italy, by University of Bologna, Italy. S. S. is supported by Miur Grant, Geometric Properties of Nonlinear Diffusion Problems. F. F. wishes to thank the Department of Mathematics of Columbia University, New York, for the kind hospitality.

In [D], De Silva introduced a new strategy to investigate inhomogeneous free boundary problems, motivated by a classical one phase problem in hydrodynamic. This method has been successfully applied in [DR] to nonlocal one phase Bernoulli type problems, governed by the fractional Laplacian. For another application of the techniques in [D] see also [LT].

Here we extend the method in [D] to two phase problems to prove that flat (see below) or Lipschitz free boundaries of (1.1) are $C^{1,\gamma}$.

In order to better emphasize the ideas involved, we first develop the regularity theory for free boundaries of viscosity solutions to problem (1.1) (see Section 2 for the relevant definitions), and then we extend our results to a more general class of free boundary problems. For simplicity, in order to avoid the machinery of L^p -viscosity solution, we assume that f is bounded in Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$. Our results may be extended to the case when f is merely bounded measurable.

We remark that in view of Theorem 4.5 in [CJK], a viscosity solution to (1.1) is locally Lipschitz. In fact, as it can be easily checked, our viscosity solutions are also weak solutions in the sense of Definition 4.4 in that paper and both $\Delta u^\pm - f$ are non negative Radon measures.

We now state our first main results. Here constants depending only on $n, \|f\|_\infty$, and $Lip(u)$ will be called universal.

Theorem 1.1 (Flatness implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to (1.1) in B_1 . Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. There exists a universal constant $\bar{\delta} > 0$ such that, if*

$$(1.2) \quad \{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.1 still holds when (1.2) is replaced by other common flatness conditions (see Subsection 6.3).

Theorem 1.2 (Lipschitz implies $C^{1,\gamma}$). *Let u be a (Lipschitz) viscosity solution to (1.1) in B_1 , with $0 \in F(u)$. Assume that $f \in L^\infty(B_1)$ is continuous in $B_1^+(u) \cup B_1^-(u)$. If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

The proof of Theorem 1.1 is based on an improvement of flatness, obtained via a compactness argument which linearizes the problem into a limiting one. The key tool is a geometric Harnack inequality that localizes the free boundary well, and allows the rigorous passage to the limit.

The main difficulty in the analysis comes from the case when u^- is degenerate, that is very close to zero without being identically zero. In this case the flatness assumption does not guarantee closeness of u to an “optimal” (two-plane) configuration. Thus one needs to work only with the positive phase u^+ to balance the situation in which u^+ highly predominates over u^- and the case in which u^- is not too small with respect to u^+ .

Theorem 1.2 follows from Theorem 1.1 and the main result in [C1], via a blow-up argument.

Sections 2 through 6 are devoted to the proof of the theorems above. In particular, in Section 2 we introduce the relevant definitions and some preliminary

lemmas. In Section 3 we describe the linearized problem associated to (1.1). Section 4 is devoted to the proof of Harnack inequality both in the non-degenerate and in the degenerate setting. In Section 5, we present the proof of the improvement of flatness lemmas. Section 6 contains the proof of the Theorem 1.1 and Theorem 1.2.

From Section 7 to Section 10 we deal with more general problems of the form

$$(1.3) \quad \begin{cases} \mathcal{L}u = f, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x), & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

with f bounded on Ω and continuous in $\Omega^+(u) \cup \Omega^-(u)$, and u Lipschitz continuous with $Lip(u) \leq L$. Here

$$\mathcal{L} = \sum_{i,j=1}^n a_{ij}(x)D_{ij} + \mathbf{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar{\gamma}}(\Omega), \quad \mathbf{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic i.e. there exist $0 < \lambda \leq \Lambda$ such that, for every $\xi \in \mathbb{R}^n$ and every $x \in \Omega$,

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,$$

and

$$G(\eta, x) : [0, \infty) \times \Omega \rightarrow (0, \infty)$$

satisfies the following assumptions:

- (H1) $G(\eta, \cdot) \in C^{0,\bar{\gamma}}(\Omega)$ uniformly in η ; $G(\cdot, x) \in C^{1,\bar{\gamma}}([0, L])$ for every $x \in \Omega$.
- (H2) $G'(\cdot, x) > 0$ with $G(0, x) \geq \gamma_0 > 0$ uniformly in x .
- (H3) There exists $N > 0$ such that $\eta^{-N}G(\eta, x)$ is strictly decreasing in η , uniformly in x .

In this framework we prove the following main results. Here, a constant depending (possibly) on $n, Lip(u), \lambda, \Lambda, [a_{ij}]_{C^{0,\bar{\gamma}}}, \|\mathbf{b}\|_{L^\infty}, \|f\|_{L^\infty}, [G(\eta, \cdot)]_{C^{0,\bar{\gamma}}}, \gamma_0$ and N is called universal. The $C^{1,\bar{\gamma}}$ norm of $G(\cdot, x)$ may depend on x and enters our proofs in a qualitative way only.

Theorem 1.3 (Flatness implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to (1.3) in B_1 , with $Lip(u) \leq L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \leq L$ and G satisfies (H1)-(H3). There exists a universal constant $\bar{\delta} > 0$ such that, if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Theorem 1.4 (Lipschitz implies $C^{1,\gamma}$). *Let u be a Lipschitz viscosity solution to (1.3) in B_1 , with $0 \in F(u)$ and $Lip(u) \leq L$. Assume that f is continuous in $B_1^+(u) \cup B_1^-(u)$, $\|f\|_{L^\infty(B_1)} \leq L$ and G satisfies (H1)-(H3). If $F(u)$ is a Lipschitz graph in a neighborhood of 0, then $F(u)$ is $C^{1,\gamma}$ in a (smaller) neighborhood of 0.*

Some remarks are in order. In particular, further extensions can be achieved with small extra effort. Actually, there is no problem to extend our results to the case when \mathbf{b} and f are merely bounded measurable, however as we already mentioned in the prototype problem we wish to avoid too many technicalities.

In Theorems 1.3 and 1.4 we need to assume the Lipschitz continuity of our solution unless the operator can be put into divergence form. Indeed, in this case an almost monotonicity formula is available (see [MP]) and under the assumption $G(\eta, x) \rightarrow \infty$, as $\eta \rightarrow \infty$ one can reproduce the proof of Theorem 4.5 in [CJK], to recover the Lipschitz continuity of a viscosity solution. Observe that then $f = f(x, u, \nabla u)$ is allowed, with $f(x, \cdot, \cdot)$ locally bounded.

2. COMPACTNESS AND LOCALIZATION LEMMAS

In this section, we state basic definitions and we prove some elementary lemmas. First we need the following standard notion.

Definition 2.1. Given $u, \varphi \in C(\Omega)$, we say that φ touches u by below (resp. above) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly by below (resp. above).

We refer to the usual C -viscosity definition of subsolution, supersolution and solution of an elliptic PDE, see e.g. [CC]. Let us introduce the notion of comparison subsolution/supersolution.

Definition 2.2. We say that $v \in C(\Omega)$ is a strict (comparison) subsolution (resp. supersolution) to (1.1) in Ω , if and only if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

- (i) $\Delta v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;
- (ii) If $x_0 \in F(v)$, then

$$(v_\nu^+)^2 - (v_\nu^-)^2 > 1 \quad (\text{resp. } (v_\nu^+)^2 - (v_\nu^-)^2 < 1, v_\nu^+(x_0) \neq 0).$$

Notice that by the implicit function theorem, according to our definition the free boundary of a comparison subsolution/supersolution is C^2 .

Finally we can give the definition of viscosity solution to the problem (1.1).

Definition 2.3. Let u be a continuous function in Ω . We say that u is a viscosity solution to (1.1) in Ω , if the following conditions are satisfied:

- (i) $\Delta u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense;
- (ii) Any (strict) comparison subsolution v (resp. supersolution) cannot touch u by below (resp. by above) at a point $x_0 \in F(v)$ (resp. $F(u)$.)

The next lemma shows that “ δ -flat” viscosity solutions (in the sense of our main Theorem 1.1) enjoy non-degeneracy of the positive part δ -away from the free boundary. Precisely,

Lemma 2.4. *Let u be a solution to (1.1) in B_2 with $Lip(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. If*

$$\{x_n \leq g(x') - \delta\} \subset \{u^+ = 0\} \subset \{x_n \leq g(x') + \delta\},$$

with g a Lipschitz function, $Lip(g) \leq L, g(0) = 0$, then

$$u(x) \geq c_0(x_n - g(x')), \quad x \in \{x_n \geq g(x') + 2\delta\} \cap B_{\rho_0},$$

for some $c_0, \rho_0 > 0$ depending on n, L as long as $\delta \leq c_0$.

Proof. All constants in this proof will depend on n, L .

It suffices to show that our statement holds for $\{x_n \geq g(x') + C\delta\}$ for a possibly large constant C . Then one can apply Harnack inequality to obtain the full statement.

We prove the statement above at $x = de_n$ (recall that $g(0) = 0$). Precisely, we want to show that

$$u(de_n) \geq c_0 d, \quad d \geq C\delta.$$

After rescaling, we reduce to proving that

$$u(e_n) \geq c_0$$

as long as $\delta \leq 1/C$, and $\|f\|_\infty$ is sufficiently small. Let $\gamma > 0$ and

$$w(x) = \frac{1}{2^\gamma}(1 - |x|^{-\gamma})$$

be defined on the closure of the annulus $B_2 \setminus \bar{B}_1$ with $\|f\|_\infty$ small enough so that

$$\Delta w < -\|f\| \quad \text{on } B_2 \setminus \bar{B}_1.$$

Let

$$w_t(x) = w(x + te_n).$$

Notice that

$$|\nabla w_0| < 1 \quad \text{on } \partial B_1.$$

From our flatness assumption for $t > 0$ sufficiently large (depending on the Lipschitz constant of g), w_t is strictly above u . We decrease t and let \bar{t} be the first t such that $w_{\bar{t}}$ touches u by above. Since $w_{\bar{t}}$ is a strict supersolution to $\Delta u = f$ in $B_2 \setminus \bar{B}_1$ the touching point z can occur only on the $\eta := \frac{1}{2^\gamma}(1 - 2^{-\gamma})$ level set in the positive phase of u , and $|z| \leq C = C(L)$. Since u is Lipschitz continuous, $0 < u(z) = \eta \leq Ld(z, F(u))$, that is a full ball around z of radius η/L is contained in the positive phase of u . Thus, for $\bar{\delta}$ small depending on η, L we have that $B_{\eta/2L}(z) \subset \{x_n \geq g(x') + 2\bar{\delta}\}$. Since $x_n = g(x') + 2\bar{\delta}$ is Lipschitz we can connect e_n and z with a chain of intersecting balls included in the positive side of u with radii comparable to $\eta/2L$. The number of balls depends on L . Then we can apply Harnack inequality and obtain

$$u(e_n) \geq cu(z) = c_0,$$

as desired.

Next, we state a compactness lemma. For its proof, we refer the reader to Section 7 where the analogue of this result for a more general class of operators and free boundary conditions is stated and proved (see Lemma 7.3).

Lemma 2.5. *Let u_k be a sequence of viscosity solutions to (1.1) with right-hand-side f_k satisfying $\|f_k\|_{L^\infty} \leq L$. Assume $u_k \rightarrow u^*$ uniformly on compact sets, and $\{u_k^+ = 0\} \rightarrow \{(u^*)^+ = 0\}$ in the Hausdorff distance. Then*

$$-L \leq \Delta u^* \leq L, \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

in the viscosity sense and u^ satisfies the free boundary condition*

$$(u_\nu^{*+})^2 - (u_\nu^{*-})^2 = 1 \quad \text{on } F(u^*)$$

in the viscosity sense of Definition 2.3.

We are now ready to re-formulate our main Theorem 1.1 using the two lemmas above. First, we denote by U_β the following one-dimensional function,

$$U_\beta(t) = \alpha t^+ - \beta t^-, \quad \beta \geq 0, \quad \alpha = \sqrt{1 + \beta^2},$$

where

$$t^+ = \max\{t, 0\}, \quad t^- = -\min\{t, 0\}.$$

Then $U_\beta(x) = U_\beta(x_n)$ is the so-called two-plane solution to (1.1) when $f \equiv 0$.

Lemma 2.6. *Let u be a solution to (1.1) in B_1 with $Lip(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. For any $\varepsilon > 0$ there exist $\delta, \bar{r} > 0$ depending on ε, n , and L such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then

$$(2.1) \quad \|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and \bar{r} depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \rightarrow 0$ and a sequence of solutions u_k to the problem (1.1) with right-hand-side f_k such that $Lip(u_k), \|f_k\| \leq L$ and

$$(2.2) \quad \{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\},$$

but the u_k do not satisfy the conclusion (2.1).

Then, up to a subsequence, the u_k converge uniformly on compacts to a function u^* . In view of (2.2) and the non-degeneracy of u_k^+ $2\delta_k$ -away from the free boundary (Lemma 2.4), we can apply our compactness lemma and conclude that

$$-L \leq \Delta u^* \leq L, \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

in the viscosity sense and also

$$(2.3) \quad (u_n^{*+})^2 - (u_n^{*-})^2 = 1 \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\},$$

with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus,

$$u^* \in C^{1,\gamma}(B_{1/2} \cap \{x_n \geq 0\}) \cap C^{1,\gamma}(B_{1/2} \cap \{x_n \leq 0\})$$

for all γ and in view of (2.3) we have that (for any \bar{r} small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq C(n, L) \bar{r}^{1+\gamma}$$

with $\alpha^2 = 1 + \beta^2$. If \bar{r} is chosen depending on ε so that

$$C(n, L) \bar{r}^{1+\gamma} \leq \frac{\varepsilon}{2} \bar{r},$$

since the u_k converge uniformly to u^* on $B_{1/2}$ we obtain that for all k large

$$\|u_k - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r},$$

a contradiction.

In view of Lemma 2.6, after rescaling our main Theorem 1.1 follows from the following main Theorem 2.7.

Theorem 2.7. *Let u be a solution to (1.1) in B_1 with $\text{Lip}(u) \leq L$ and $\|f\|_{L^\infty} \leq L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$(2.4) \quad \|u - U_\beta\|_{L^\infty(B_1)} \leq \bar{\varepsilon} \quad \text{for some } 0 \leq \beta \leq L,$$

and

$$\{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\},$$

and

$$\|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon},$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

Finally, we also need the following elementary lemma which holds for any arbitrary continuous function u .

Lemma 2.8. *Let u be a continuous function. If for $\eta > 0$ small,*

$$\|u - U_\beta\|_{L^\infty(B_2)} \leq \eta, \quad 0 \leq \beta \leq L,$$

and

$$\{x_n \leq -\eta\} \subset B_2 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\},$$

then

- If $\beta \geq \eta^{1/3}$, then

$$U_\beta(x_n - \eta^{1/3}) \leq u(x) \leq U_\beta(x_n + \eta^{1/3}), \quad \text{in } B_1$$

- If $\beta < \eta^{1/3}$, then

$$U_0(x_n - \eta^{1/3}) \leq u^+(x) \leq U_0(x_n + \eta^{1/3}), \quad \text{in } B_1.$$

3. THE LINEARIZED PROBLEM

This section is devoted to the study of the linearized problem associated with our free boundary problem (1.1), that is the following boundary value problem, ($\tilde{\alpha} \neq 0$)

$$(3.1) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2(\tilde{u}_n)^+ - \tilde{\beta}^2(\tilde{u}_n)^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}. \end{cases}$$

Here $(\tilde{u}_n)^+$ (resp. $(\tilde{u}_n)^-$) denotes the derivative in the e_n direction of \tilde{u} restricted to $\{x_n > 0\}$ (resp. $\{x_n < 0\}$).

We remark that Theorem 2.7 will follow, see Section 6, via a compactness argument from the regularity properties of viscosity solutions to (3.1).

Definition 3.1. A continuous function u is a viscosity solution to (3.1) if

- (i) $\Delta \tilde{u} = 0$ in $B_\rho \cap \{x_n \neq 0\}$, in the viscosity sense;

- (ii) Let ϕ be a function of the form

$$\phi(x) = A + px_n^+ - qx_n^- + BQ(x - y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then ϕ cannot touch u strictly by below at a point $x_0 = (x'_0, 0) \in B_\rho$.

Analogously, if

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q < 0$$

then ϕ cannot touch u strictly by above at x_0 .

We wish to prove the following regularity result for viscosity solutions to the linearized problem.

Theorem 3.2. *Let \tilde{u} be a viscosity solution to (3.1) in $B_{1/2}$ such that $\|\tilde{u}\|_\infty \leq 1$. There exists a universal constant \bar{C} such that*

$$(3.2) \quad |\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'} \tilde{u}(0) \cdot x' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq \bar{C}r^2, \quad \text{in } B_r$$

for all $r \leq 1/4$ and with $\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0$.

In order to prove the above result, first we show in the following Theorem 3.3, that problem (3.1) admits a classical solution.

Theorem 3.3. *Let h be a continuous function on ∂B_1 . There exists a (unique) classical solution \tilde{v} to (3.1) with $\tilde{v} = h$ on ∂B_1 , that is $\tilde{v} \in C^\infty(B_1 \cap \{x_n \geq 0\}) \cap C^\infty(B_1 \cap \{x_n \leq 0\})$. In particular, there exists a universal constant \tilde{C} such that*

$$(3.3) \quad |\tilde{v}(x) - \tilde{v}(\bar{x}) - (\nabla_{x'} \tilde{v}(\bar{x}) \cdot (x' - \bar{x}') + \tilde{p}(\bar{x})x_n^+ - \tilde{q}(\bar{x})x_n^-)| \leq \tilde{C}\|\tilde{v}\|_{L^\infty} r^2, \quad \text{in } B_r(\bar{x})$$

for all $r \leq 1/4$, $\bar{x} = (\bar{x}', 0) \in B_{1/2}$ and with $\tilde{\alpha}^2 \tilde{p}(\bar{x}) - \tilde{\beta}^2 \tilde{q}(\bar{x}) = 0$.

Proof. Let w be the harmonic function in $B_1 \cap \{x_n > 0\}$ such that

$$w = 0 \quad \text{on } B_1 \cap \{x_n = 0\}, \quad w(x) = h(x', x_n) - h(x', -x_n) \quad \text{on } \partial B_1 \cap \{x_n > 0\}.$$

Then $w \in C^\infty(B_1 \cap \{x_n \geq 0\})$. Call

$$\phi(x') = w_n(x', 0), \quad (x', 0) \in B_1.$$

Let

$$\tilde{v}_1(x) = w(x) + \tilde{v}_2(x', -x_n) \quad \text{in } \bar{B}_1 \cap \{x_n \geq 0\}$$

where \tilde{v}_2 is the solution to the problem

$$\begin{cases} \Delta \tilde{v}_2 = 0 & \text{in } B_1 \cap \{x_n < 0\} \\ \tilde{v}_2 = h & \text{on } \partial B_1 \cap \{x_n < 0\} \\ (\tilde{v}_2)_n = \tilde{q}\phi & \text{on } B_1 \cap \{x_n = 0\} \end{cases}$$

with

$$\tilde{q} = \frac{\tilde{\alpha}^2}{\tilde{\beta}^2 + \tilde{\alpha}^2}.$$

Then it is easily verified that the function

$$\tilde{v} = \begin{cases} \tilde{v}_1 & \text{in } \bar{B}_1 \cap \{x_n \geq 0\} \\ \tilde{v}_2 & \text{in } \bar{B}_1 \cap \{x_n \leq 0\} \end{cases}$$

is the unique classical solution to our problem and hence it satisfies the estimate (3.3) with

$$\tilde{q}(\bar{x}) = \tilde{q}\phi(\bar{x}), \quad \tilde{p}(\bar{x}) = \tilde{p}\phi(\bar{x}),$$

and

$$\tilde{p} = \frac{\tilde{\beta}^2}{\tilde{\beta}^2 + \tilde{\alpha}^2}.$$

Finally, to obtain our regularity result we only need to show the following fact.

Theorem 3.4. *Let \tilde{u} be a viscosity solution to (3.1) in B_1 such that $\|\tilde{u}\|_\infty \leq 1$ and let \tilde{v} be the classical solution to (3.1) in $B_{1/2}$ with boundary data \tilde{u} . Then $\tilde{u} = \tilde{v}$.*

Proof. We prove that $\tilde{v} \leq \tilde{u}$ in $B_{1/2}$. The opposite inequality is obtained in a similar way.

Let $\varepsilon > 0, t \in \mathbb{R}$ and denote by

$$\tilde{v}_{t,\varepsilon}(x) = \tilde{v} + \varepsilon|x_n| + \varepsilon x_n^2 - \varepsilon - t, \quad x \in \bar{B}_{1/2}.$$

Since \tilde{u} is bounded, for $t > 0$ large enough

$$(3.4) \quad \tilde{v}_{t,\varepsilon} \leq \tilde{u}.$$

Let \bar{t} be the smallest t such that (3.4) holds and let \bar{x} be the first touching point. We want to show that $\bar{t} < 0$. Assume $\bar{t} \geq 0$. Since

$$\tilde{v}_{\bar{t},\varepsilon} < \tilde{u} \quad \text{on } \partial B_{1/2}$$

such touching point must belong to $B_{1/2}$. However,

$$\Delta \tilde{v}_{\bar{t},\varepsilon}(x) > 0 \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

and

$$\Delta \tilde{u} = 0 \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}.$$

Thus $\bar{x} \in B_{1/2} \cap \{x_n = 0\}$. We claim that there exists a function ϕ of the form

$$\phi(x) = A + px_n^+ - qx_n^- + BQ(x - y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

such that ϕ touches $\tilde{v}_{\bar{t},\varepsilon}(x)$ strictly by below at \bar{x} . This would contradict the definition of viscosity solutions hence $\bar{t} < 0$. In particular

$$\tilde{v} + \varepsilon|x_n| + \varepsilon x_n^2 - \varepsilon < \tilde{u} \quad \text{on } B_{1/2}$$

and for ε going to 0 we obtain as desired

$$\tilde{v} \leq \tilde{u} \quad \text{on } B_{1/2}.$$

We are left with the proof of the claim. Call

$$\nu' = \nabla_{x'} \tilde{v}(\bar{x})$$

and set

$$y' = \bar{x}' + \frac{\nu'}{B}, \quad A = \tilde{v}(\bar{x}) - \varepsilon - \bar{t} - BQ(\bar{x} - y)$$

with $B > 0$ to be chosen later. Then in view of the estimate (3.3), in order to verify that in a small neighborhood of \bar{x}

$$\phi(x) < \tilde{v}_{\bar{t},\varepsilon}(x), \quad x \neq \bar{x}$$

we need to show that we can find $B > 0, p, q$ such that for $|x - \bar{x}| \neq 0$ small enough (\tilde{C} universal)

$$\frac{B}{2}(n-1)x_n^2 - \frac{B}{2}|x' - \bar{x}'|^2 + px_n^+ - qx_n^- < (\tilde{p} + \varepsilon)x_n^+ - (\tilde{q} - \varepsilon)x_n^- - \tilde{C}|x - \bar{x}|^2$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0,$$

(for simplicity we dropped the dependence of \tilde{p}, \tilde{q} on \bar{x} .)

It is then enough to choose,

$$B = 4\tilde{C}, \quad p = \tilde{p} + \frac{\varepsilon}{2}, \quad q = \tilde{q} - \frac{\varepsilon}{2}.$$

4. HARNACK INEQUALITY

In this section we prove our main tool, that is a Harnack-type inequality for solutions to our free boundary problem. The results contained here will allow us to pass to the limit in the compactness argument of our improvement of flatness lemmas in Section 5.

Throughout this section we consider a Lipschitz solution u to (1.1) with $Lip(u) \leq L$.

We need to distinguish two cases, which we call the non-degenerate and the degenerate case.

4.1. Non-degenerate case. In this case our solution u is trapped between two translation of a “true” two-plane solution U_β that is with $\beta \neq 0$.

Theorem 4.1 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$, such that if u satisfies at some point $x_0 \in B_2$*

$$(4.1) \quad U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_2,$$

with

$$\|f\|_{L^\infty} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_\beta(x_n + a_1) \leq u(x) \leq U_\beta(x_n + b_1) \quad \text{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - c)\varepsilon r,$$

and $0 < c < 1$ universal.

Before giving the proof we deduce an important consequence.

If u satisfies (4.1) with, say $r = 1$, then we can apply Harnack inequality repeatedly and obtain

$$U_\beta(x_n + a_m) \leq u(x) \leq U_\beta(x_n + b_m) \quad \text{in } B_{20^{-m}}(x_0),$$

with

$$b_m - a_m \leq (1 - c)^m \varepsilon$$

for all m 's such that

$$(1 - c)^m 20^m \varepsilon \leq \bar{\varepsilon}.$$

This implies that for all such m 's, the oscillation of the function

$$\tilde{u}_\varepsilon(x) = \begin{cases} \frac{u(x) - \alpha x_n}{\alpha \varepsilon} & \text{in } B_2^+(u) \cup F(u) \\ \frac{u(x) - \beta x_n}{\beta \varepsilon} & \text{in } B_2^-(u) \end{cases}$$

in $B_r(x_0)$, $r = 20^{-m}$ is less than $(1 - c)^m = 20^{-\gamma m} = r^\gamma$. Thus, the following corollary holds.

Corollary 4.2. *Let u be as in Theorem 4.1 satisfying (4.1) for $r = 1$. Then in $B_1(x_0)$ \tilde{u}_ε has a Hölder modulus of continuity at x_0 , outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$*

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality relies on the following lemma.

Lemma 4.3. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u(x) \geq U_\beta(x), \quad \text{in } B_1$$

with

$$(4.2) \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta, \quad 0 < \beta \leq L,$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$(4.3) \quad u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon),$$

then

$$(4.4) \quad u(x) \geq U_\beta(x_n + c\varepsilon), \quad \text{in } \bar{B}_{1/2},$$

for some $0 < c < 1$ universal. Analogously, if

$$u(x) \leq U_\beta(x), \quad \text{in } B_1$$

and

$$u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon),$$

then

$$u(x) \leq U_\beta(x_n - c\varepsilon), \quad \text{in } \bar{B}_{1/2}.$$

Proof. We prove the first statement. For notational simplicity we drop the sub-index β from U_β .

Let

$$(4.5) \quad w = c(|x - \bar{x}|^{-\gamma} - (3/4)^{-\gamma})$$

be defined in the closure of the annulus

$$A := B_{3/4}(\bar{x}) \setminus \bar{B}_{1/20}(\bar{x}).$$

The constant c is such that w satisfies the boundary conditions

$$\begin{cases} w = 0 & \text{on } \partial B_{3/4}(\bar{x}), \\ w = 1 & \text{on } \partial B_{1/20}(\bar{x}). \end{cases}$$

Then, for a fixed $\gamma > n - 2$,

$$\Delta w \geq k(\gamma, n) = k(n) > 0, \quad 0 \leq w \leq 1 \quad \text{on } A.$$

Extend w to be equal to 1 on $B_{1/20}(\bar{x})$.

Notice that since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u \geq U$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - U \geq 0$ and solves $\Delta(u - U) = f$ in $B_{1/10}(\bar{x})$ and we can apply Harnack inequality to obtain

$$(4.6) \quad u(x) - U(x) \geq c(u(\bar{x}) - U(\bar{x})) - C\|f\|_{L^\infty} \quad \text{in } \overline{B_{1/20}(\bar{x})}.$$

From the assumptions (4.2) and (4.3) we conclude that (for ε small enough)

$$(4.7) \quad u - U \geq \alpha c \varepsilon - C \alpha \varepsilon^2 \geq \alpha c_0 \varepsilon \quad \text{in } \overline{B_{1/20}(\bar{x})}.$$

Now set $\psi = 1 - w$ and

$$v(x) = U(x_n - \varepsilon c_0 \psi(x)), \quad x \in \overline{B_{3/4}(\bar{x})},$$

and for $t \geq 0$,

$$v_t(x) = U(x_n - \varepsilon c_0 \psi(x) + t\varepsilon), \quad x \in \overline{B_{3/4}(\bar{x})}.$$

Then,

$$v_0(x) = U(x_n - \varepsilon c_0 \psi(x)) \leq U(x) \leq u(x) \quad x \in \overline{B_{3/4}(\bar{x})}.$$

Let \bar{t} be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B_{3/4}(\bar{x})}.$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed,

$$u(x) \geq v_{\bar{t}}(x) = U(x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon) \geq U(x_n + c\varepsilon) \quad \text{in } B_{1/2} \subset \subset B_{3/4}(\bar{x})$$

with c universal. In the last inequality we used that $\|\psi\|_{L^\infty(B_{1/2})} < 1$.

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \overline{B_{3/4}(\bar{x})}$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\overline{B_{1/20}(\bar{x})}$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$

$$v_{\bar{t}}(x) = U(x_n - \varepsilon c_0 \psi(x) + \bar{t}\varepsilon) < U(x) \leq u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

We now show that \tilde{x} cannot belong to the annulus A . Indeed,

$$\Delta v_{\bar{t}} \geq \beta \varepsilon c_0 k(n) > \varepsilon^2 \beta \geq \|f\|_\infty, \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough.

Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = 1 + \varepsilon^2 c_0^2 |\nabla \psi|^2 - 2\varepsilon c_0 \psi_n \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A$$

as long as

$$\psi_n < 0 \quad \text{on } F(v_{\bar{t}}) \cap A.$$

This can be easily verified from the formula for ψ (for ε small enough.)

Thus, $v_{\bar{\varepsilon}}$ is a strict subsolution to (1.1) in A which lies below u , hence by the definition of viscosity solution, \tilde{x} cannot belong to A .

Therefore, $\tilde{x} \in \overline{B}_{1/20}(\tilde{x})$ and

$$u(\tilde{x}) = v_{\bar{\varepsilon}}(\tilde{x}) = U(\tilde{x}_n + \bar{\varepsilon}) \leq U(\tilde{x}) + \alpha \bar{\varepsilon} < U(\tilde{x}) + \alpha c_0 \varepsilon$$

contradicting (4.7).

The proof of the second statement follows from a similar argument.

We can now prove our Theorem 4.1.

Proof of Theorem 4.1. Assume without loss of generality that $x_0 = 0, r = 1$. We distinguish three cases.

Case 1. $a_0 < -1/5$. In this case it follows from (4.1) that $B_{1/10} \subset \{u < 0\}$ and

$$0 \leq v(x) := \frac{u(x) - \beta(x_n + a_0)}{\beta \varepsilon} \leq 1$$

with

$$|\Delta v| \leq \varepsilon \quad \text{in } B_{1/10}.$$

The desired claim follows from standard Harnack inequality applied to the function v .

Case 2. $a_0 > 1/5$. In this case it follows from (4.1) that $B_{1/5} \subset \{u > 0\}$ and

$$0 \leq v(x) := \frac{u(x) - \alpha(x_n + a_0)}{\alpha \varepsilon} \leq 1$$

with

$$|\Delta v| \leq \varepsilon \quad \text{in } B_{1/5}.$$

Again, the desired claim follows from standard Harnack inequality for v .

Case 3. $|a_0| \leq 1/5$. Assumption (4.1) gives that

$$U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + a_0 + \varepsilon) \quad \text{in } B_1.$$

Assume that (the other case is treated similarly)

$$(4.8) \quad u(\bar{x}) \geq U_\beta(\bar{x}_n + a_0 + \frac{\varepsilon}{2}), \quad \bar{x} = \frac{1}{5}e_n.$$

Call

$$v(x) := u(x - a_0 e_n), \quad x \in B_{4/5}.$$

Then the inequality above reads

$$U_\beta(x_n) \leq v(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_{4/5}.$$

From (4.8)

$$v(\bar{x}) \geq U_\beta(\bar{x}_n + \frac{\varepsilon}{2}).$$

Then, by Lemma 4.3,

$$v(x) \geq U_\beta(x_n + c\varepsilon), \quad \text{in } B_{2/5}$$

which gives the desired improvement

$$u(x) \geq U_\beta(x + a_0 + c\varepsilon) \quad \text{in } B_{3/5}.$$

4.2. Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0$).

Theorem 4.4 (Harnack inequality). *There exists a universal constant $\bar{\varepsilon}$, such that if u satisfies at some point $x_0 \in B_2$*

$$(4.9) \quad U_0(x_n + a_0) \leq u^+(x) \leq U_0(x_n + b_0) \quad \text{in } B_r(x_0) \subset B_2,$$

with

$$\|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4,$$

and

$$b_0 - a_0 \leq \varepsilon r,$$

for some $\varepsilon \leq \bar{\varepsilon}$, then

$$U_0(x_n + a_1) \leq u^+(x) \leq U_0(x_n + b_1) \quad \text{in } B_{r/20}(x_0),$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - c)\varepsilon r,$$

and $0 < c < 1$ universal.

We can argue as in the nondegenerate case and get the following result.

Corollary 4.5. *Let u be as in Theorem 4.1 satisfying (4.9) for $r = 1$. Then in $B_1(x_0)$*

$$\tilde{u}_\varepsilon := \frac{u^+(x) - x_n}{\varepsilon}$$

has a Hölder modulus of continuity at x_0 , outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

The proof of the Harnack inequality can be deduced from the following lemma, as in the one-phase case [D].

Lemma 4.6. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \geq U_0(x), \quad \text{in } B_1$$

with

$$(4.10) \quad \|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|f\|_{L^\infty} \leq \varepsilon^4,$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$(4.11) \quad u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon),$$

then

$$(4.12) \quad u^+(x) \geq U_0(x_n + c\varepsilon), \quad \text{in } \bar{B}_{1/2},$$

for some $0 < c < 1$ universal. Analogously, if

$$u^+(x) \leq U_0(x), \quad \text{in } B_1$$

and

$$u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon),$$

then

$$u^+(x) \leq U_0(x_n - c\varepsilon), \quad \text{in } \bar{B}_{1/2}.$$

Proof. We prove the first statement. The proof follows the same line as in the nondegenerate case.

Since $x_n > 0$ in $B_{1/10}(\bar{x})$ and $u^+ \geq U_0$ in B_1 we get

$$B_{1/10}(\bar{x}) \subset B_1^+(u).$$

Thus $u - x_n \geq 0$ and solves $\Delta(u - x_n) = f$ in $B_{1/10}(\bar{x})$ and we can apply Harnack inequality and the assumptions (4.10) and (4.11) to obtain that (for ε small enough)

$$(4.13) \quad u - x_n \geq c_0 \varepsilon \quad \text{in } \overline{B_{1/20}(\bar{x})}.$$

Let w be as in the proof of Lemma 4.3 and $\psi = 1 - w$. Set

$$v(x) = (x_n - \varepsilon c_0 \psi(x))^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x))^- , \quad x \in \overline{B_{3/4}(\bar{x})},$$

and for $t \geq 0$,

$$v_t(x) = (x_n - \varepsilon c_0 \psi + t\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi + t\varepsilon)^- , \quad x \in \overline{B_{3/4}(\bar{x})}.$$

Here C_1 is a universal constant to be made precise later. We claim that

$$v_0(x) = v(x) \leq u(x) \quad x \in \overline{B_{3/4}(\bar{x})}.$$

This is readily verified in the set where u is non-negative using that $u \geq x_n^+$. To prove our claim in the set where u is negative we wish to use the following fact:

$$(4.14) \quad u^- \leq C x_n^- \varepsilon^2, \quad \text{in } B_{\frac{19}{20}}, C \text{ universal.}$$

This estimate is easily obtained using that $\{u < 0\} \subset \{x_n < 0\}$, $\|u^-\|_\infty < \varepsilon^2$ and the comparison principle with the function w satisfying

$$\Delta w = -\varepsilon^4 \quad \text{in } B_1 \cap \{x_n < 0\}, \quad w = u^- \quad \text{on } \partial(B_1 \cap \{x_n < 0\}).$$

Thus our claim immediately follows from the fact that for $x_n < 0$ and $C_1 \geq C$,

$$\varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi(x)) \leq C x_n \varepsilon^2.$$

Let \bar{t} be the largest $t \geq 0$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B_{3/4}(\bar{x})}.$$

We want to show that $\bar{t} \geq c_0$. Then we get the desired statement. Indeed, it is easy to check that if

$$u(x) \geq v_{\bar{t}}(x) = (x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 \psi + \bar{t}\varepsilon)^- \quad \text{in } B_{3/4}(\bar{x})$$

then

$$u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } B_{1/2} \subset\subset B_{3/4}(\bar{x})$$

with c universal, $c < c_0 \inf_{B_{1/2}} w$

Suppose $\bar{t} < c_0$. Then at some $\tilde{x} \in \overline{B_{3/4}(\bar{x})}$ we have

$$v_{\bar{t}}(\tilde{x}) = u(\tilde{x}).$$

We show that such touching point can only occur on $\overline{B_{1/20}(\bar{x})}$. Indeed, since $w \equiv 0$ on $\partial B_{3/4}(\bar{x})$ from the definition of v_t we get that for $\bar{t} < c_0$

$$v_{\bar{t}}(x) = (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^+ - \varepsilon^2 C_1 (x_n - \varepsilon c_0 + \bar{t}\varepsilon)^- < u(x) \quad \text{on } \partial B_{3/4}(\bar{x}).$$

In the set where $u \geq 0$ this can be seen using that $u \geq x_n^+$ while in the set where $u < 0$ again we can use the estimate (4.14).

We now show that \tilde{x} cannot belong to the annulus A . Indeed,

$$\Delta v_{\bar{t}} \geq \varepsilon^3 c_0 k(n) > \varepsilon^4 \geq \|f\|_\infty, \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough.

Also,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 = (1 - \varepsilon^4 C_1^2)(1 + \varepsilon^2 c_0^2 |\nabla \psi|^2 - 2\varepsilon c_0 \psi_n) \quad \text{on } F(v_{\bar{t}}) \cap A.$$

Thus,

$$(v_{\bar{t}}^+)_\nu^2 - (v_{\bar{t}}^-)_\nu^2 > 1 \quad \text{on } F(v_{\bar{t}}) \cap A$$

as long as ε is small enough (as in the non-degenerate case one can check that $\inf_{F(v_{\bar{t}}) \cap A} (-\psi_n) > c > 0$, c universal.) Thus, $v_{\bar{t}}$ is a strict subsolution to (1.1) in A which lies below u , hence by definition \tilde{x} cannot belong to A .

Therefore, $\tilde{x} \in \bar{B}_{1/20}(\tilde{x})$ and

$$u(\tilde{x}) = v_{\bar{t}}(\tilde{x}) = (\tilde{x}_n + \bar{t}\varepsilon) < \tilde{x}_n + c_0\varepsilon$$

contradicting (4.13).

5. IMPROVEMENT OF FLATNESS

In this section we prove our key ‘‘improvement of flatness’’ lemmas. As in Section 4, we need to distinguish two cases.

5.1. Non-degenerate case. In this case our solution u is trapped between two translations of a two-plane solution U_β with $\beta \neq 0$. We plan to show that when we restrict to smaller balls, u is trapped between closer translations of another two-plane solution (in a different system of coordinates).

Lemma 5.1 (Improvement of flatness). *Let u satisfy*

$$(5.1) \quad U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u),$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \beta.$$

If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r , then

$$(5.2) \quad U_{\beta'}(x \cdot \nu_1 - r \frac{\varepsilon}{2}) \leq u(x) \leq U_{\beta'}(x \cdot \nu_1 + r \frac{\varepsilon}{2}) \quad \text{in } B_r,$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\varepsilon$, and $|\beta - \beta'| \leq \tilde{C}\beta\varepsilon$ for a universal constant \tilde{C} .

Proof. We divide the proof of this Lemma into 3 steps.

Step 1 – Compactness. Fix $r \leq r_0$ with r_0 universal (the precise r_0 will be given in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \rightarrow 0$ and a sequence u_k of solutions to (1.1) in B_1 with right hand side f_k with L^∞ norm bounded by $\varepsilon_k^2 \beta_k$, such that

$$(5.3) \quad U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{for } x \in B_1, \quad 0 \in F(u_k),$$

with $L \geq \beta_k > 0$, but u_k does not satisfy the conclusion (5.2) of the lemma.

Set $(\alpha_k^2 = 1 + \beta_k^2)$,

$$\tilde{u}_k(x) = \begin{cases} \frac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\ \frac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

Then (5.3) gives,

$$(5.4) \quad -1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1.$$

From Corollary 4.2, it follows that the function \tilde{u}_k satisfies

$$(5.5) \quad |\tilde{u}_k(x) - \tilde{u}_k(y)| \leq C|x - y|^\gamma,$$

for C universal and

$$|x - y| \geq \varepsilon_k/\bar{\varepsilon}, \quad x, y \in B_{1/2}.$$

From (5.3) it clearly follows that $F(u_k)$ converges to $B_1 \cap \{x_n = 0\}$ in the Hausdorff distance. This fact and (5.5) together with Ascoli-Arzelà give that as $\varepsilon_k \rightarrow 0$ the graphs of the \tilde{u}_k converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \tilde{u} over $B_{1/2}$. Also, up to a subsequence

$$\beta_k \rightarrow \tilde{\beta} \geq 0$$

and hence

$$\alpha_k \rightarrow \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}.$$

Step 2 – Limiting Solution. We now show that \tilde{u} solves the following linearized problem (transmission problem)

$$(5.6) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2(\tilde{u}_n)^+ - \tilde{\beta}^2(\tilde{u}_n)^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

Since

$$|\Delta u_k| \leq \varepsilon_k^2 \beta_k \quad \text{in } B_1^+(u_k) \cup B_1^-(u_k),$$

one easily deduces that \tilde{u} is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that \tilde{u} satisfies the boundary condition in (5.6) in the viscosity sense.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x - y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0$$

and

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Then we must show that $\tilde{\phi}$ cannot touch u strictly by below at a point $x_0 = (x'_0, 0) \in B_{1/2}$ (the analogous statement by above follows with a similar argument.)

Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let

$$(5.7) \quad \Gamma(x) = \frac{1}{n-2}[(|x'|^2 + |x_n - 1|^2)^{\frac{2-n}{2}} - 1]$$

and let

$$(5.8) \quad \Gamma_k(x) = \frac{1}{B\varepsilon_k} \Gamma(B\varepsilon_k(x - y) + AB\varepsilon_k^2 e_n).$$

Now, call

$$\phi_k(x) = a_k \Gamma_k^+(x) - b_k \Gamma_k^-(x) + \alpha_k (d_k^+(x))^2 \varepsilon_k^{3/2} + \beta_k (d_k^-(x))^2 \varepsilon_k^{3/2}$$

where

$$a_k = \alpha_k(1 + \varepsilon_k p), \quad b_k = \beta_k(1 + \varepsilon_k q)$$

and $d_k(x)$ is the signed distance from x to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k))$.

Finally, let

$$\tilde{\phi}_k(x) = \begin{cases} \frac{\phi_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(\phi_k) \cup F(\phi_k) \\ \frac{\phi_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(\phi_k). \end{cases}$$

By Taylor's theorem

$$\Gamma(x) = x_n + Q(x) + O(|x|^3) \quad x \in B_1,$$

thus it is easy to verify that

$$\Gamma_k(x) = A\varepsilon_k + x_n + B\varepsilon_k Q(x - y) + O(\varepsilon_k^2) \quad x \in B_1,$$

with the constant in $O(\varepsilon_k^2)$ depending on A, B , and $|y|$ (later this constant will depend also on p, q).

It follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ ($Q^y(x) = Q(x - y)$)

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k)$$

and analogously in $B_1^-(\phi_k)$

$$\tilde{\phi}_k(x) = A + BQ^y + qx_n + A\varepsilon_k p + Bq\varepsilon_k Q^y + \varepsilon_k^{1/2} d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2}$. Since \tilde{u}_k converges uniformly to \tilde{u} and $\tilde{\phi}$ touches \tilde{u} strictly by below at x_0 , we conclude that there exist a sequence of constants $c_k \rightarrow 0$ and of points $x_k \rightarrow x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k by below at x_k . We thus get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is

$$\begin{cases} \Delta \psi_k > \varepsilon_k^2 \beta_k \geq \|f_k\|_\infty, & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu^2 - (\psi_k^-)_\nu^2 > 1, & \text{on } F(\psi_k). \end{cases}$$

It is easily checked that away from the free boundary

$$\Delta \psi_k \geq \beta_k \varepsilon_k^{3/2} \Delta d_k^2(x + \varepsilon_k c_k e_n)$$

and the first condition is satisfied for k large enough.

Finally, since on the zero level set $|\nabla \Gamma_k| = 1$ and $|\nabla d_k^2| = 0$ the free boundary condition reduces to showing that

$$a_k^2 - b_k^2 > 1.$$

Using the definition of a_k, b_k we need to check that

$$(\alpha_k^2 p^2 - \beta_k^2 q^2) \varepsilon_k + 2(\alpha_k^2 p - \beta_k^2 q) > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha}^2 p - \tilde{\beta}^2 q > 0.$$

Thus \tilde{u} is a solution to the linearized problem.

Step 3 – Contradiction. According to estimate (3.2), since $\tilde{u}(0) = 0$ we obtain that

$$|\tilde{u} - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq Cr^2, \quad x \in B_r,$$

with

$$\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \leq C.$$

Thus, since \tilde{u}_k converges uniformly to \tilde{u} (by slightly enlarging C) we get that

$$(5.9) \quad |\tilde{u}_k - (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq Cr^2, \quad x \in B_r.$$

Now set,

$$\beta'_k = \beta_k(1 + \varepsilon_k \tilde{q}), \quad \nu_k = \frac{1}{\sqrt{1 + \varepsilon_k^2 |\nu'|^2}} (e_n + \varepsilon_k(\nu', 0)).$$

Then,

$$\alpha'_k = \sqrt{1 + \beta_k'^2} = \alpha_k(1 + \varepsilon_k \tilde{p}) + O(\varepsilon_k^2), \quad \nu_k = e_n + \varepsilon_k(\nu', 0) + \varepsilon_k^2 \tau, \quad |\tau| \leq C,$$

where to obtain the first equality we used that $\tilde{\alpha}^2 \tilde{p} - \tilde{\beta}^2 \tilde{q} = 0$ and hence

$$\frac{\beta_k^2}{\alpha_k^2} \tilde{q} = \tilde{p} + O(\varepsilon_k).$$

With these choices we can now show that (for k large and $r \leq r_0$)

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq \tilde{u}_k(x) \leq \tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

where again we are using the notation:

$$\tilde{U}_{\beta'_k}(x) = \begin{cases} \frac{\tilde{U}_{\beta'_k}(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(\tilde{U}_{\beta'_k}) \cup F(\tilde{U}_{\beta'_k}) \\ \frac{\tilde{U}_{\beta'_k}(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(\tilde{U}_{\beta'_k}). \end{cases}$$

This will clearly imply that

$$U_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq u_k(x) \leq U_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

and hence leads to a contradiction.

In view of (5.9) we need to show that in B_r

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2$$

and

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

Let us show the second inequality. In the set where

$$(5.10) \quad x \cdot \nu_k + \varepsilon_k \frac{r}{2} < 0$$

by definition we have that

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) = \frac{1}{\beta_k \varepsilon_k} (\beta'_k(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) - \beta_k x_n)$$

which from the formula for β'_k, ν_k gives

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq x' \cdot \nu' + \tilde{q}x_n + \frac{r}{2} - C_0 \varepsilon_k.$$

Using (5.10) we then obtain

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^- + \frac{r}{2} - C_1 \varepsilon_k.$$

Thus to obtain the desired bound it suffices to fix $r_0 \leq 1/(4C)$ and take k large enough.

The other case can be argued similarly. \square

5.2. Degenerate case. In this case, the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0$). We prove below that in this setting only u^+ enjoys an improvement of flatness.

Lemma 5.2 (Improvement of flatness). *Let u satisfy*

$$(5.11) \quad U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u),$$

with

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4,$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2.$$

If $0 < r \leq r_1$ for r_1 universal, and $0 < \varepsilon \leq \varepsilon_1$ for some ε_1 depending on r , then

$$(5.12) \quad U_0(x \cdot \nu_1 - r \frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + r \frac{\varepsilon}{2}) \quad \text{in } B_r,$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ for a universal constant C .

Proof. We argue similarly as in the non-degenerate case.

Step 1 – Compactness. Fix $r \leq r_1$ with r_1 universal (made precise in Step 3). Assume by contradiction that we can find a sequence $\varepsilon_k \rightarrow 0$ and a sequence u_k of solutions to (1.1) in B_1 with right hand side f_k with L^∞ norm bounded by ε_k^4 , such that

$$(5.13) \quad U_0(x_n - \varepsilon_k) \leq u_k^+(x) \leq U_0(x_n + \varepsilon_k) \quad \text{for } x \in B_1, 0 \in F(u_k),$$

with

$$\|u_k^-\|_\infty \leq \varepsilon_k^2$$

but u_k does not satisfy the conclusion (5.12) of the lemma.

Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$

Then (5.13) gives,

$$(5.14) \quad -1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in B_1^+(u_k) \cup F(u_k).$$

As in the non-degenerate case, it follows from Corollary 4.5 that as $\varepsilon_k \rightarrow 0$ the graphs of the \tilde{u}_k converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \tilde{u} over $B_{1/2} \cap \{x_n \geq 0\}$.

Step 2 – Limiting Solution. We now show that \tilde{u} solves the following Neumann problem

$$(5.15) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

As before, the interior condition follows easily thus we focus on the boundary condition.

Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n + BQ(x - y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in \mathbb{R}, B > 0$$

and

$$p > 0.$$

Then we must show that $\tilde{\phi}$ cannot touch u strictly by below at a point $x_0 = (x'_0, 0) \in B_{1/2}$. Suppose that such a $\tilde{\phi}$ exists and let x_0 be the touching point.

Let Γ_k be as in the proof of the non-degenerate case (see (5.8)). Call

$$\phi_k(x) = a_k \Gamma_k^+(x) + (d_k^+(x))^2 \varepsilon_k^2, \quad a_k = (1 + \varepsilon_k p)$$

where $d_k(x)$ is the signed distance from x to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k))$.

Let

$$\tilde{\phi}_k(x) = \frac{\phi_k(x) - x_n}{\varepsilon_k}.$$

As in the previous case, it follows that in $B_1^+(\phi_k) \cup F(\phi_k)$ ($Q^y(x) = Q(x - y)$)

$$\tilde{\phi}_k(x) = A + BQ^y + px_n + A\varepsilon_k p + Bp\varepsilon_k Q^y + \varepsilon_k d_k^2 + O(\varepsilon_k).$$

Hence, $\tilde{\phi}_k$ converges uniformly to $\tilde{\phi}$ on $B_{1/2} \cap \{x_n \geq 0\}$. Since \tilde{u}_k converges uniformly to \tilde{u} and $\tilde{\phi}$ touches \tilde{u} strictly by below at x_0 , we conclude that there exist a sequence of constants $c_k \rightarrow 0$ and of points $x_k \rightarrow x_0$ such that the function

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k by below at $x_k \in B_1^+(u_k) \cup F(u_k)$. We claim that x_k cannot belong to $B_1^+(u_k)$. Otherwise, in a small neighborhood N of x_k we would have that

$$\Delta \psi_k > \varepsilon_k^4 \geq \|f_k\|_\infty = \Delta u_k, \quad \psi_k < u_k \text{ in } N \setminus \{x_k\}, \psi_k(x_k) = u_k(x_k)$$

a contradiction.

Thus $x_k \in F(u_k) \cap \partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k))$. For simplicity we call

$$\mathcal{B} := B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k)).$$

Let N_ρ be a small neighborhood of x_k of size ρ . Since

$$\|u_k^-\|_\infty \leq \varepsilon_k^2, \quad u_k^+ \geq (x_n - \varepsilon_k)^+$$

as in the proof of Harnack inequality using the fact that $x_k \in F(u_k) \cap \partial \mathcal{B}$ we can conclude by the comparison principle that

$$u_k^- \leq c\varepsilon_k^2 (d(x, \partial \mathcal{B}))^-, \quad \text{in } N_{\frac{3}{4}\rho}$$

where d denotes again the signed distance from x to $\partial \mathcal{B}$.

Let

$$(5.16) \quad \Psi_k(x) = \begin{cases} \psi_k & \text{in } \mathcal{B} \\ c\varepsilon_k^2 (3d(x, \partial \mathcal{B}) + d^2(x, \partial \mathcal{B})) & \text{outside of } \mathcal{B}. \end{cases}$$

Then Ψ_k touches u_k strictly by below at $x_k \in F(u_k) \cap F(\Psi_k)$.
 We reach a contradiction if we show that

$$(\Psi_k^+)_\nu^2 - (\Psi_k^-)_\nu^2 > 1, \quad \text{on } F(\Psi_k).$$

This is equivalent to showing that

$$a_k^2 - c\varepsilon_k^4 > 1$$

or

$$(1 + \varepsilon_k p)^2 - c\varepsilon_k^4 > 1.$$

This holds for k large enough, since $p > 0$. We finally reached a contradiction.

Step 3 – Contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [D]. \square

6. PROOF OF THE MAIN THEOREMS

In this section we exhibit the proofs of our main results, Theorem 1.1 and Theorem 1.2. As already pointed out, Theorem 1.2 will follow via a blow-up analysis from the flatness result. Thus, first we present the proof of Theorem 1.1 based on the improvement of flatness lemmas of the previous section.

6.1. Proof of Theorem 1.1. To complete the analysis of the degenerate case, we need to deal with the situation when u is close to a one-plane solution and however the size of u^- is not negligible. Precisely, we prove the following lemma.

Lemma 6.1. *Let u solve (1.1) in B_2 with*

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon^4$$

and satisfy

$$(6.1) \quad \begin{aligned} U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u), \\ \|u^-\|_{L^\infty(B_2)} \leq \bar{C}\varepsilon^2, \quad \|u^-\|_{L^\infty(B_1)} > \varepsilon^2, \end{aligned}$$

for a universal constant \bar{C} . If $\varepsilon \leq \varepsilon_2$ universal, then the rescaling

$$u_\varepsilon(x) = \varepsilon^{-1/2}u(\varepsilon^{1/2}x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C'\varepsilon^{1/2}) \leq u_\varepsilon(x) \leq U_{\beta'}(x_n + C'\varepsilon^{1/2})$$

with $\beta' \sim \varepsilon^2$ and $C' > 0$ depending on \bar{C} .

Proof. For notational simplicity we set

$$v = \frac{u^-}{\varepsilon^2}.$$

From our assumptions we can deduce that

$$F(v) \subset \{-\varepsilon \leq x_n \leq \varepsilon\}$$

$$(6.2) \quad v \geq 0 \quad \text{in } B_2 \cap \{x_n \leq -\varepsilon\}, \quad v \equiv 0 \quad \text{in } B_2 \cap \{x_n > \varepsilon\}.$$

Also,

$$|\Delta v| \leq \varepsilon^2, \quad \text{in } B_2 \cap \{x_n < -\varepsilon\},$$

and

$$(6.3) \quad 0 \leq v \leq \bar{C} \quad \text{on } \partial B_2,$$

$$(6.4) \quad v(\bar{x}) > 1 \quad \text{at some point } \bar{x} \text{ in } B_1.$$

Thus, using comparison with the function w such that

$$\Delta w = -\varepsilon^2 \quad \text{in } D := B_2 \cap \{x_n < \varepsilon\} \text{ and } w = v \text{ on } \partial D$$

we obtain that for some $k > 0$ universal

$$(6.5) \quad v \leq k|x_n - \varepsilon|, \quad \text{in } B_1.$$

This fact forces the point \bar{x} in (6.4) to belong to $B_1 \cap \{x_n < -\varepsilon\}$ at a fixed distance δ from $x_n = -\varepsilon$.

Now, let w be the harmonic function in $B_1 \cap \{x_n < -\varepsilon\}$ such that

$$w = 0 \quad \text{on } B_1 \cap \{x_n = -\varepsilon\}, \quad w = v \quad \text{on } \partial B_1 \cap \{x_n \leq -\varepsilon\}.$$

By the maximum principle we conclude that

$$w + \varepsilon^2(|x|^2 - 3) \leq v \quad \text{on } B_1 \cap \{x_n < -\varepsilon\}.$$

Also, for ε small, in view of (6.5) we obtain that

$$w - k\varepsilon(|x|^2 - 3) \geq v \quad \text{on } \partial(B_1 \cap \{x_n < -\varepsilon\})$$

and hence also in the interior. Thus we conclude that

$$(6.6) \quad |w - v| \leq c\varepsilon \quad \text{in } B_1 \cap \{x_n < -\varepsilon\}.$$

In particular this is true at \bar{x} which forces

$$(6.7) \quad w(\bar{x}) \geq 1/2.$$

By expanding w around $(0, -\varepsilon)$ we then obtain, say in $B_{1/2} \cap \{x_n \leq -\varepsilon\}$

$$|w - a|x_n + \varepsilon|| \leq C|x|^2 + C\varepsilon.$$

This combined with (6.6) gives that

$$|v - a|x_n + \varepsilon|| \leq C\varepsilon, \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \leq -\varepsilon\}.$$

Moreover, in view of (6.7) and the fact that \bar{x} occurs at a fixed distance from $\{x_n = -\varepsilon\}$ we deduce from Hopf lemma that

$$a \geq c > 0$$

with c universal. In conclusion (see (6.5))

$$|u^- - b\varepsilon^2|x_n + \varepsilon|| \leq C\varepsilon^3, \quad \text{in } B_{\varepsilon^{1/2}} \cap \{x_n \leq -\varepsilon\}, \quad u^- \leq b\varepsilon^2|x_n - \varepsilon|, \quad \text{in } B_1$$

with b comparable to a universal constant.

Combining the two inequalities above and the assumption (6.1) we conclude that in $B_{\varepsilon^{1/2}}$

$$(x_n - \varepsilon)^+ - b\varepsilon^2(x_n - C\varepsilon)^- \leq u(x) \leq (x_n + \varepsilon)^+ - b\varepsilon^2(x_n + C\varepsilon)^-$$

with $C > 0$ universal and b larger than a universal constant. Rescaling, we obtain that in B_1

$$(x_n - \varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq (x_n + \varepsilon^{1/2})^+ - \beta'(x_n + C\varepsilon^{1/2})^-$$

with $\beta' \sim \varepsilon^2$. We finally need to check that this implies the desired conclusion in B_1

$$\alpha'(x_n - C\varepsilon^{1/2})^+ - \beta'(x_n - C\varepsilon^{1/2})^- \leq u_\varepsilon(x) \leq \alpha'(x_n + C\varepsilon^{1/2})^+ - \beta'(x_n + C\varepsilon^{1/2})^-$$

with $\alpha'^2 = 1 + \beta'^2 \sim 1 + \varepsilon^4$. This clearly holds in B_1 for ε small, say by possibly enlarging C so that $C \geq 2$. \square

We are finally ready to exhibit the proof of Theorem 2.7, which as already observed, immediately gives the result of Theorem 1.1.

Proof of Theorem 2.7. Let us fix $\bar{r} > 0$ to be a universal constant such that

$$\bar{r} \leq r_0, r_1, 1/16,$$

with r_0, r_1 the universal constants in the improvement of flatness Lemmas 5.1-5.2. Also, let us fix a universal constant $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon} \leq \varepsilon_0(\bar{r}), \frac{\varepsilon_1(\bar{r})}{2}, \frac{1}{2\tilde{C}}, \frac{\varepsilon_2}{2}$$

with $\varepsilon_0, \varepsilon_1, \varepsilon_2, \tilde{C}$, the constants in the Lemmas 5.1-5.2-6.1. Now, let

$$\bar{\varepsilon} = \tilde{\varepsilon}^3.$$

We distinguish two cases. For notational simplicity we assume that u satisfies our assumptions in the ball B_2 and $0 \in F(u)$.

Case 1. $\beta \geq \bar{\varepsilon}$.

In this case, in view of Lemma 2.8 and our choice of $\tilde{\varepsilon}$, we obtain that u satisfies the assumptions of Lemma 5.1,

$$(6.8) \quad U_\beta(x_n - \tilde{\varepsilon}) \leq u(x) \leq U_\beta(x_n + \tilde{\varepsilon}) \quad \text{in } B_1, \quad 0 \in F(u),$$

with $0 < \beta \leq L$ and

$$\|f\|_{L^\infty(B_1)} \leq \tilde{\varepsilon}^3 \leq \tilde{\varepsilon}^2 \beta.$$

Thus we can conclude that, ($\beta_1 = \beta'$)

$$U_{\beta_1}(x \cdot \nu_1 - \tilde{r} \frac{\tilde{\varepsilon}}{2}) \leq u(x) \leq U_{\beta_1}(x \cdot \nu_1 + \tilde{r} \frac{\tilde{\varepsilon}}{2}) \quad \text{in } B_{\tilde{r}},$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\tilde{\varepsilon}$, and $|\beta - \beta_1| \leq \tilde{C}\beta\tilde{\varepsilon}$. In particular, by our choice of $\tilde{\varepsilon}$ we have

$$\beta_1 \geq \tilde{\varepsilon}/2.$$

We can therefore rescale and iterate the argument above. Precisely, set ($k = 0, 1, 2, \dots$)

$$\rho_k = \tilde{r}^k, \quad \varepsilon_k = 2^{-k}\tilde{\varepsilon}$$

and

$$u_k(x) = \frac{1}{\rho_k} u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x).$$

Also, let β_k be the constants generated at each k -iteration, hence satisfying ($\beta_0 = \beta$)

$$|\beta_k - \beta_{k+1}| \leq \tilde{C}\beta_k \varepsilon_k.$$

Then we obtain by induction that each u_k satisfies

$$(6.9) \quad U_{\beta_k}(x \cdot \nu_k - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1,$$

with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq \tilde{C}\tilde{\varepsilon}_k$ ($\nu_0 = e_n$.)

Case 2. $\beta < \bar{\varepsilon}$.

In view of Lemma 2.8 we conclude that

$$U_0(x_n - \tilde{\varepsilon}) \leq u^+(x) \leq U_0(x_n + \tilde{\varepsilon}) \quad \text{in } B_1.$$

Moreover, from the assumption (2.4) and the fact that $\beta < \tilde{\varepsilon}$ we also obtain that

$$\|u^-\|_{L^\infty(B_1)} < 2\tilde{\varepsilon}.$$

Call $(\varepsilon')^2 = 2\tilde{\varepsilon}$. Then u satisfies the assumptions of the (degenerate) improvement of flatness Lemma 5.2.

$$U_0(x_n - \varepsilon') \leq u^+(x) \leq U_0(x_n + \varepsilon') \quad \text{in } B_1,$$

with

$$\|f\|_{L^\infty(B_1)} \leq (\varepsilon')^4, \quad \|u^-\|_{L^\infty(B_1)} < \varepsilon'^2.$$

We conclude that

$$U_0(x \cdot \nu_1 - \bar{r} \frac{\varepsilon'}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + \bar{r} \frac{\varepsilon'}{2}) \quad \text{in } B_{\bar{r}},$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon'$ for a universal constant C . We now rescale as in the previous case and set ($k = 0, 1, 2, \dots$)

$$\rho_k = \bar{r}^k, \quad \varepsilon_k = 2^{-k} \varepsilon'$$

and

$$u_k(x) = \frac{1}{\rho_k} u(\rho_k x), \quad f_k(x) = \rho_k f(\rho_k x).$$

We can iterate our argument and obtain that (with $|\nu_k| = 1$, $|\nu_k - \nu_{k+1}| \leq C\varepsilon_k$)

$$(6.10) \quad U_0(x \cdot \nu_k - \varepsilon_k) \leq u_k^+(x) \leq U_0(x \cdot \nu_k + \varepsilon_k) \quad \text{in } B_1,$$

as long as we can verify that

$$\|u_k^-\|_{L^\infty(B_1)} < \varepsilon_k^2.$$

Let \bar{k} be the first integer $\bar{k} > 1$ for which this fails, that is

$$\|u_{\bar{k}}^-\|_{L^\infty(B_1)} \geq \varepsilon_{\bar{k}}^2,$$

and

$$\|u_{\bar{k}-1}^-\|_{L^\infty(B_1)} < \varepsilon_{\bar{k}-1}^2.$$

Also,

$$U_0(x \cdot \nu_{\bar{k}-1} - \varepsilon_{\bar{k}-1}) \leq u_{\bar{k}-1}^+(x) \leq U_0(x \cdot \nu_{\bar{k}-1} + \varepsilon_{\bar{k}-1}) \quad \text{in } B_1.$$

As argued several times (see for example (4.14)), we can then conclude from the comparison principle that

$$u_{\bar{k}-1}^- \leq M|x_n - \varepsilon_{\bar{k}-1}| \varepsilon_{\bar{k}-1}^2 \quad \text{in } B_{19/20},$$

for a universal constant $M > 0$. Thus, by rescaling we get that

$$\|u_{\bar{k}}^-\|_{L^\infty(B_2)} < \bar{C} \varepsilon_{\bar{k}}^2$$

with \bar{C} universal (depending on the fixed \bar{r}). We obtain that $u_{\bar{k}}$ satisfies all the assumptions of Lemma 6.1 and hence the rescaling

$$v(x) = \varepsilon_{\bar{k}}^{-1/2} u_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2} x)$$

satisfies in B_1

$$U_{\beta'}(x_n - C' \varepsilon_{\bar{k}}^{1/2}) \leq v(x) \leq U_{\beta'}(x_n + C' \varepsilon_{\bar{k}}^{1/2})$$

with $\beta' \sim \varepsilon_{\bar{k}}^2$. Call $\eta = \bar{C} \varepsilon_{\bar{k}}^{1/2}$. Then v satisfies our free boundary problem in B_1 with right hand side

$$g(x) = \varepsilon_{\bar{k}}^{1/2} f_{\bar{k}}(\varepsilon_{\bar{k}}^{1/2} x)$$

and the flatness assumption

$$U_{\beta'}(x_n - \eta) \leq v(x) \leq U_{\beta'}(x_n + \eta)$$

Since $\beta' \sim \varepsilon_k^2$ with a universal constant,

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon_k^{1/2} \varepsilon_k^4 \leq \eta^2 \beta'$$

as long as $\tilde{\varepsilon} \leq C''$ depending on \bar{C} . In conclusion choosing $\tilde{\varepsilon} \leq \frac{\varepsilon_0(\bar{r})^4}{2C^4}$, v falls under the assumptions of the (non-degenerate) improvement of flatness Lemma 5.1 and we can use an iteration argument as in Case 1.

6.2. Proof of Theorem 1.2. Although not strictly necessary, we use the following Liouville type result for global viscosity solutions to a two-phase homogeneous free boundary problem, that could be of independent interest.

Lemma 6.2. *Let U be a global viscosity solution to*

$$(6.11) \quad \begin{cases} \Delta U = 0, & \text{in } \{U > 0\} \cup \{U \leq 0\}^0, \\ (U_\nu^+)^2 - (U_\nu^-)^2 = 1, & \text{on } F(U) := \partial\{U > 0\}. \end{cases}$$

Assume that $F(U) = \{x_n = g(x'), x' \in \mathbb{R}^{n-1}\}$ with $\text{Lip}(g) \leq M$. Then g is linear and $U(x) = U_\beta(x)$ for some $\beta \geq 0$.

Proof. Assume for simplicity, $0 \in F(U)$. Also, balls (of radius ρ and centered at 0) in \mathbb{R}^{n-1} are denoted by \mathcal{B}_ρ .

By the regularity theory in [C1], since U is a solution in B_2 , the free boundary $F(U)$ is $C^{1,\gamma}$ in B_1 with a bound depending only on n and on M . Thus,

$$|g(x') - g(0) - \nabla g(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1$$

with C depending only on n, M . Moreover, since U is a global solution, the rescaling

$$g_R(x') = \frac{1}{R}g(Rx'), \quad x' \in \mathcal{B}_2$$

which preserves the same Lipschitz constant as g , satisfies the same inequality as above i.e.

$$|g_R(x') - g_R(0) - \nabla g_R(0) \cdot x'| \leq C|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

This reads,

$$|g(Rx') - g(0) - \nabla g(0) \cdot Rx'| \leq CR|x'|^{1+\alpha}, \quad x' \in \mathcal{B}_1.$$

Thus,

$$|g(y') - g(0) - \nabla g(0) \cdot y'| \leq C \frac{1}{R^\alpha} |y'|^{1+\alpha}, \quad y' \in \mathcal{B}_R.$$

Passing to the limit as $R \rightarrow \infty$ we obtain the desired claim.

Proof of Theorem 1.2. Let $\bar{\varepsilon}$ be the universal constant in Theorem 2.7. Consider the blow-up sequence

$$u_k(x) = \frac{u(\delta_k)}{\delta_k}$$

with $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. Each u_k solves (1.1) with right hand side

$$f_k(x) = \delta_k f(\delta_k x)$$

and

$$\|f_k(x)\| \leq \delta_k \|f\|_{L^\infty} \leq \bar{\varepsilon}$$

for k large enough. Standard arguments (see for example [ACF]) using the uniform Lipschitz continuity of the u_k 's and the nondegeneracy of their positive part u_k^+ (see Lemma 2.4) imply that (up to a subsequence)

$$u_k \rightarrow \tilde{u} \quad \text{uniformly on compacts}$$

and

$$\{u_k^+ = 0\} \rightarrow \{\tilde{u} = 0\} \quad \text{in the Hausdorff distance.}$$

The blow-up limit \tilde{u} solves the global homogeneous two-phase free boundary problem

$$(6.12) \quad \begin{cases} \Delta \tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \leq 0\}^0 \\ (\tilde{u}_\nu^+)^2 - (\tilde{u}_\nu^-)^2 = 1, & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}. \end{cases}$$

Since $F(u)$ is a Lipschitz graph in a neighborhood of 0, it follows from Lemma 6.2 that \tilde{u} is a two-plane solutions, $\tilde{u} = U_\beta$ for some $\beta \geq 0$. Thus, for k large enough

$$\|u_k - U_\beta\|_{L^\infty} \leq \bar{\varepsilon}$$

and

$$\{x_n \leq -\bar{\varepsilon}\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\}.$$

Therefore, we can apply our flatness Theorem 2.7 and conclude that $F(u_k)$ and hence $F(u)$ is smooth.

6.3. Flatness and ε -monotonicity. The flatness results which are present in the literature (see, for instance [C2]), are often stated in terms of “ ε -monotonicity” along a large cone of directions $\Gamma(\theta_0, e)$ of axis e and opening θ_0 . Precisely, a function u is said to be ε -monotone ($\varepsilon > 0$ small) along the direction τ in the cone $\Gamma(\theta_0, e)$ if for every $\varepsilon' \geq \varepsilon$,

$$u(x + \varepsilon' \tau) \leq u(x).$$

A variant of Theorem 1.1 states the following.

Theorem 6.3. *Let u be a solution to (1.1) in B_1 , $0 \in F(u)$. Suppose that u^+ is non-degenerate. Then there exist $\theta_0 < \pi/2$ and $\varepsilon_0 > 0$ such that if u^+ is ε -monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then u^+ is fully monotone in $B_{1/2}$ along any direction in $\Gamma(\theta_1, e_n)$ for some θ_1 depending on θ_0, ε_0 . In particular $F(u)$ is the graph of a Lipschitz function.*

Geometrically, the ε -monotonicity of u^+ can be interpreted as ε -closeness of $F(u)$ to the graph of a Lipschitz function. Our flatness assumption requires ε -closeness of $F(u)$ to a hyperplane. While this looks like a somewhat stronger assumption, it is indeed a natural one since it is satisfied for example by rescaling of solutions around a “regular” point of the free boundary. Moreover, if $\|f\|_\infty$ is small enough, depending on ε , it is not hard to check that ε -flatness of $F(u)$ implies $c\varepsilon$ -monotonicity of u^+ along the directions of a flat cone, for a c depending on its opening.

The proof of Theorem 6.3 follows immediately from the following elementary lemma:

Lemma 6.4. *Let u be a solution to (1.1) in B_1 , $0 \in F(u)$. Suppose that u^+ is Lipschitz and non-degenerate. Assume that u^+ is ε -monotone along every direction in $\Gamma(\theta_0, e_n)$ for some $\varepsilon \leq \varepsilon_0$, then there exist a radius $r_0 > 0$ and $\delta_0 > 0$ depending on ε_0, θ_0 such that u^+ is δ_0 -flat in B_{r_0} , that is*

$$\{x_n \leq -\delta_0\} \subset B_{r_0} \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta_0\}.$$

7. MORE GENERAL OPERATORS AND FREE BOUNDARY CONDITIONS

7.1. The set up. In this section we analyze the free boundary problem (1.3), that is

$$(7.1) \quad \begin{cases} \mathcal{L}u = f, & \text{in } \Omega^+(u) \cup \Omega^-(u), \\ u_\nu^+ = G(u_\nu^-, x), & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

where f is continuous in $\Omega^+(u) \cup \Omega^-(u)$ with $\|f\|_{L^\infty(\Omega)} \leq L$, and

$$\mathcal{L} = \sum_{i,j=1}^n a_{ij}(x)D_{ij} + \mathbf{b} \cdot \nabla, \quad a_{ij} \in C^{0,\bar{\gamma}}(\Omega), \mathbf{b} \in C(\Omega) \cap L^\infty(\Omega),$$

is uniformly elliptic with constants $0 < \lambda \leq \Lambda$.

We recall that our assumptions on G are:

- (H1) $G(\eta, \cdot) \in C^{0,\bar{\gamma}}(\Omega)$ uniformly in η ; $G(\cdot, x) \in C^{1,\bar{\gamma}}([0, L])$ for every $x \in \Omega$.
- (H2) $G'(\cdot, x) > 0$ with $G(0, x) \geq \gamma_0 > 0$ uniformly in x .
- (H3) There exists $N > 0$ such that $\eta^{-N}G(\eta, x)$ is strictly decreasing in η , uniformly in x .

We assume that $0 \in F(u)$ and that $a_{ij}(0) = \delta_{ij}$. Also, for notational convenience we set

$$G_0(\beta) = G(\beta, 0).$$

Let U_β be the two-plane solution to (7.1) when $\mathcal{L} = \Delta$, $f \equiv 0$ and $G = G_0$, i.e.

$$U_\beta(x) = \alpha x_n^+ - \beta x_n^-, \quad \beta \geq 0, \quad \alpha = G_0(\beta).$$

The following definitions parallel those in Section 2.

Definition 7.1. We say that $v \in C(\Omega)$ is a C^2 strict (comparison) subsolution (resp. supersolution) to (7.1) in Ω , if $v \in C^2(\overline{\Omega^+(v)}) \cap C^2(\overline{\Omega^-(v)})$ and the following conditions are satisfied:

- (i) $\mathcal{L}v > f$ (resp. $< f$) in $\Omega^+(v) \cup \Omega^-(v)$;
- (ii) If $x_0 \in F(v)$, then
 - $v_\nu^+(x_0) > G(v_\nu^-(x_0), x_0)$ (resp. $v_\nu^+(x_0) < G(v_\nu^-(x_0), x_0)$, $v_\nu^+(x_0) \neq 0$.)

Observe that the free boundary of a strict comparison sub/supersolution is C^2 .

Definition 7.2. Let u be a continuous function in Ω . We say that u is a viscosity solution to (1.3) in Ω , if the following conditions are satisfied:

- (i) $\mathcal{L}u = f$ in $\Omega^+(u) \cup \Omega^-(u)$ in the viscosity sense;
- (ii) Any (strict) comparison subsolution v (resp. supersolution) cannot touch u by below (resp. by above) at a point $x_0 \in F(v)$ (resp. $F(u)$.)

From here after, most of the statements and proofs parallel those in Sections 2 to 6. Thus, we only point out the main differences as much as possible.

7.2. Compactness and localization. As for problem (1.1), we prove some basic lemmas to reduce the statement of the flatness theorem to a proper normalized situation. We start with the compactness Lemma 2.5 which generalizes to operators of the form

$$\mathcal{L}_*^k = \sum a_{ij}^k D_{ij}$$

with $a_{ij}^k \in C^{0,\bar{\gamma}}$ uniformly elliptic with constants λ, Λ and free boundary conditions given by a G_k satisfying the hypotheses (H1)-(H3).

Lemma 7.3. *Let u_k be a sequence of (Lipschitz) viscosity solutions to*

$$(7.2) \quad \begin{cases} |\mathcal{L}_*^k u_k| \leq M, & \text{in } \Omega^+(u_k) \cup \Omega^-(u_k), \\ (u_k^+)_\nu = G_k((u_k^-)_\nu, x), & \text{on } F(u_k). \end{cases}$$

Assume that:

- (i) $a_{ij}^k \rightarrow a_{ij}, u_k \rightarrow u^*$ uniformly on compact sets,
- (ii) $G_k(\eta, \cdot) \rightarrow G(\eta, \cdot)$ on compact sets, uniformly on $0 \leq \eta \leq L = \text{Lip}(u_k)$,
- (iii) $\{u_k^+ = 0\} \rightarrow \{(u^*)^+ = 0\}$ in the Hausdorff distance.

Then

$$|\sum a_{ij} D_{ij} u^*| \leq M, \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

and u^ satisfies the free boundary condition*

$$(u^*)_\nu^+ = G((u^*)_\nu^-, x) \quad \text{on } F(u^*),$$

both in the viscosity sense.

Proof. Call

$$\mathcal{L}_* := \sum a_{ij} D_{ij}.$$

The proof that

$$|\mathcal{L}_* u^*| \leq M, \quad \text{in } \Omega^+(u^*) \cup \Omega^-(u^*)$$

is standard. We show for example that

$$\mathcal{L}_* u^* + M \geq 0, \quad \text{in } \Omega^+(u^*).$$

Let $v \in C^2(\Omega^+(u^*))$ touch u^* by above at $\bar{x} \in \Omega^+(u^*)$ and assume by contradiction that

$$\mathcal{L}_* v(\bar{x}) + M < 0.$$

Without loss of generality we can assume that v touches u^* strictly by above (otherwise we replace v with $v + \frac{\eta}{2n\Lambda}|x - \bar{x}|^2$ and η small.) Then, since $u_k \rightarrow u^*$ uniformly in compact sets and $\{u_k^+ = 0\} \rightarrow \{(u^*)^+ = 0\}$ in the Hausdorff distance, there exists $x_k \rightarrow \bar{x}$ and constants $c_k \rightarrow 0$ such that $v + c_k$ touches by above u_k at $x_k \in \Omega^+(u_k)$, for k large. Then, since $|\mathcal{L}_*^k u_k(x_k)| \leq M$ we must have

$$\mathcal{L}_*^k v(x_k) + M \geq 0.$$

This implies, for $k \rightarrow \infty$,

$$\mathcal{L}_* v(\bar{x}) + M \geq 0$$

which is a contradiction.

We now prove that the free boundary condition holds. Let v be a strict comparison super solution such that,

$$(7.3) \quad \mathcal{L}_* v + M < 0, \quad \text{in } \Omega^+(v) \cup \Omega^-(v),$$

and

$$v_\nu^+ < G(v_\nu^-, x), \quad v_\nu^+(x) \neq 0 \quad \text{on } F(v).$$

Assume v touches u^* strictly by above at a point $\bar{x} \in F(u^*) \cap F(v)$ and for notational simplicity let $\nu(\bar{x}) = e_n$. Also, we can assume that the free boundaries $F(v)$ and $F(u^*)$ touch strictly and that (7.3) holds up to $F(v)$. Otherwise, say $v_n^+(\bar{x}) > 0$, we replace v with $\bar{v}(x) = v(x + \eta|x' - \bar{x}'|^2 e_n) + \eta|dist(x, F(v))| - dist(x, F(v))^2$, (η small). Then, for a suitable $c_k \rightarrow 0$, $v(x + c_k e_n)$ touches by above u_k at x_k with $x_k \rightarrow \bar{x}$. Then, either for every (large) k we have $x_k \in \Omega^+(u_k) \cup \Omega^-(u_k)$ or there exists a subsequence, that we still call x_k , such that $x_k \in F(u_k)$ for every large k .

In the first case, we have

$$\sum a_{ij}^k(x_k) D_{ij} v(x_k + c_k e_n) + M \geq 0.$$

while in the second case,

$$\bar{v}_{\nu_k}^+(x_k + c_k e_n) \geq G_k(v_{\nu_k}^-(x_k + c_k e_n), x_k)$$

and we easily reach a contradiction for k large.

Lemma 2.4 on the non-degeneracy of the positive part δ -away from the free boundary continues to hold unaltered; only choose

$$w(x) = \frac{G_0(0)}{2\gamma}(1 - |x|^{-\gamma}).$$

The analogue of Lemma 2.6 is the following:

Lemma 7.4. *Let u be a Lipschitz solution to (1.3) in B_1 , with $Lip(u) \leq L$, $\|b\|_\infty, \|f\|_\infty \leq L$. For any $\varepsilon > 0$ there exist $\bar{\delta}, \bar{r} > 0$ such that if*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\},$$

with $0 \leq \delta \leq \bar{\delta}$, then

$$(7.4) \quad \|u - U_\beta\|_{L^\infty(B_{\bar{r}})} \leq \varepsilon \bar{r}$$

for some $0 \leq \beta \leq L$.

Proof. Given $\varepsilon > 0$ and \bar{r} depending on ε to be specified later, assume by contradiction that there exist a sequence $\delta_k \rightarrow 0$ and a sequence of solutions u_k to the problem (7.2) with $M = L + L^2$, such that $Lip(u_k) \leq L$ and

$$(7.5) \quad \{x_n \leq -\delta_k\} \subset B_1 \cap \{u_k^+(x) = 0\} \subset \{x_n \leq \delta_k\},$$

but the u_k do not satisfy the conclusion (7.4).

Then, up to a subsequence, the u_k converge uniformly on compact set to a function u^* . In view of (7.5) and the non-degeneracy of u_k^+ , δ_k -away from the free boundary (see remark above), we can apply our compactness Lemma 7.3 and conclude that, for some $\tilde{\mathcal{L}} := \sum \tilde{a}_{ij} D_{ij}$ and \tilde{G} in our class,

$$|\tilde{\mathcal{L}}u^*| \leq M, \quad \text{in } B_{1/2} \cap \{x_n \neq 0\}$$

and

$$(7.6) \quad (u^*)_n^+ = \tilde{G}((u^*)_n^-, x) \quad \text{on } F(u^*) = B_{1/2} \cap \{x_n = 0\},$$

in the viscosity sense, with

$$u^* > 0 \quad \text{in } B_{\rho_0} \cap \{x_n > 0\}.$$

Thus, by L^p Schauder estimates

$$u^* \in C^{1,\tilde{\gamma}}(B_{1/2} \cap \{x_n \geq 0\}) \cap C^{1,\tilde{\gamma}}(B_{1/2} \cap \{x_n \leq 0\})$$

for all $\tilde{\gamma} < 1$ and (for any \bar{r} small)

$$\|u^* - (\alpha x_n^+ - \beta x_n^-)\|_{L^\infty(B_{\bar{r}})} \leq C(n, L)\bar{r}^{1+\tilde{\gamma}}$$

with $\beta = (u^*)_n^-(0)$ and $\alpha = (u^*)_n^+(0) > 0$. Thus, from (7.6), we have $\alpha = \tilde{G}_0(\beta)$.

Then we reach a contradiction as in Lemma 2.6.

In view of the lemma above, after proper rescaling, Theorem 1.3 follows from the following result.

Theorem 7.5. *Let u be a Lipschitz solution to (1.3) in B_1 , with $\text{Lip}(u) \leq L$. There exists a universal constant $\bar{\varepsilon} > 0$ such that, if*

$$(7.7) \quad \begin{aligned} \|u - U_\beta\|_{L^\infty(B_1)} &\leq \bar{\varepsilon} \quad \text{for some } 0 \leq \beta \leq L, \\ \{x_n \leq -\bar{\varepsilon}\} &\subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \bar{\varepsilon}\}, \end{aligned}$$

and

$$\begin{aligned} [a_{ij}]_{C^{0,\tilde{\gamma}}(B_1)} &\leq \bar{\varepsilon}, \quad \|\mathbf{b}\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \\ [G(\eta, \cdot)]_{C^{0,\tilde{\gamma}}(B_1)} &\leq \bar{\varepsilon}, \quad \forall 0 \leq \eta \leq L, \end{aligned}$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$.

7.3. Linearized problem. The linearized problem becomes, ($\tilde{\alpha} > 0$)

$$(7.8) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_\rho \cap \{x_n \neq 0\}, \\ \tilde{\alpha}(\tilde{u})_n^+ - \tilde{\beta}G'_0(\tilde{\beta})(\tilde{u})_n^- = 0 & \text{on } B_\rho \cap \{x_n = 0\}, \end{cases}$$

with $\tilde{\alpha} = G_0(\tilde{\beta})$.

Setting $\zeta^2 = \tilde{\alpha}$ and $\xi^2 = \tilde{\beta}G'_0(\tilde{\beta})$ we can write the free boundary condition as

$$\zeta^2 \tilde{u}_n^+ - \xi^2 \tilde{u}_n^- = 0.$$

As a consequence, all the Definitions and conclusions in Section 3 hold, in particular Theorems 3.2, 3.3 and 3.4.

8. THE NON-DEGENERATE CASE FOR GENERAL FREE BOUNDARY PROBLEMS.

In this section, we recover the improvement of flatness lemma in the non-degenerate case, that is when the solution is trapped between parallel two-plane solutions U_β at ε distance, with $\beta > 0$. First we need the Harnack inequality.

8.1. Harnack inequality. As in Section 4, Harnack inequality follows from the following basic lemma.

Lemma 8.1. *Let u be a viscosity solution to (7.1). There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u(x) \geq U_\beta(x), \quad \text{in } B_1$$

for $0 < \beta \leq L$ and for $0 \leq \varepsilon \leq \bar{\varepsilon}$,

$$(8.1) \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \min\{G_0(\beta), \beta\}, \quad \|\mathbf{b}\|_{L^\infty(B_1)} \leq \varepsilon^2,$$

$$(8.2) \quad \|G(\eta, x) - G_0(\eta)\|_{L^\infty(B_1)} \leq \varepsilon^2, \quad \forall 0 \leq \eta \leq L,$$

then, if at $\bar{x} = \frac{1}{5}e_n$

$$(8.3) \quad u(\bar{x}) \geq U_\beta(\bar{x}_n + \varepsilon),$$

then

$$(8.4) \quad u(x) \geq U_\beta(x_n + c\varepsilon) \quad \text{in } \bar{B}_{1/2},$$

for some $0 < c < 1$ universal. Analogously, if

$$u(x) \leq U_\beta(x) \quad \text{in } B_1$$

and

$$u(\bar{x}) \leq U_\beta(\bar{x}_n - \varepsilon)$$

then

$$u(x) \leq U_\beta(x_n - c\varepsilon) \quad \text{in } \bar{B}_{1/2}.$$

Proof. We argue as in the proof of Lemma 4.3 and we only point out the main differences.

By our assumptions, in $B_{1/10}(\bar{x}) \subset B_1^+(u)$, $u - U_\beta \geq 0$ solves

$$\mathcal{L}(u - U_\beta) = f - \alpha b_n.$$

Recall that $\alpha = G_0(\beta)$. By Harnack inequality, we obtain in $\bar{B}_{1/20}(\bar{x})$

$$\begin{aligned} u(x) - U_\beta(x) &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C\|f - \alpha b_n\|_{L^\infty} \\ &\geq c(u(\bar{x}) - U_\beta(\bar{x})) - C(\|f\|_{L^\infty} + \alpha\|b\|_{L^\infty}). \end{aligned}$$

From (8.1), (8.3) and the inequality above we conclude that for ε small enough,

$$(8.5) \quad u - U_\beta \geq \alpha c \varepsilon - \alpha C \varepsilon^2 \geq c_0 \alpha \varepsilon \quad \text{in } \bar{B}_{1/20}(\bar{x}).$$

From (8.5) and the comparison principle it follows that for c_1 small universal

$$(8.6) \quad u - \alpha x_n \geq \alpha c_1 \varepsilon x_n, \quad x \in \{x_n > 0\} \cap \bar{B}_{19/20}.$$

To prove this claim, let ϕ solve

$$\mathcal{L}\phi = 0 \quad \text{in } R := (B_1 \cap \{x_n > 0\}) \setminus \bar{B}_{1/20}(\bar{x})$$

with boundary data

$$\phi = 0 \quad \text{on } \partial(B_1 \cap \{x_n > 0\}), \quad \phi = 1 \quad \text{on } \partial B_{1/20}(\bar{x}).$$

Then, by boundary Harnack

$$\phi \geq c x_n \quad \text{in } \bar{R} \cap B_{19/20}.$$

We now compare $u - \alpha x_n$ with $\frac{1}{2}\alpha c_0 \phi \varepsilon - 8\alpha \varepsilon^2 x_n + 4\alpha \varepsilon^2 x_n^2$ in the domain R to obtain the desired conclusion.

We now proceed similarly as in Lemma 4.3, with w the function defined in (4.5). We compute

$$\begin{aligned} &\sum a_{ij} D_{ij} w(x) \\ &= \gamma(\gamma + 2) |x - \bar{x}|^{-\gamma-4} \text{Tr}(A(x - \bar{x}) \otimes (x - \bar{x})) - \gamma |x - \bar{x}|^{-\gamma-2} \text{Tr}(A) \\ &\geq \gamma(\gamma + 2) |x - \bar{x}|^{-\gamma-2} n\lambda - \gamma |x - \bar{x}|^{-\gamma-2} n\Lambda \\ &= \gamma |x - \bar{x}|^{-\gamma-2} n((\gamma + 2)\lambda - \Lambda). \end{aligned}$$

Then

$$\begin{aligned}\mathcal{L}w &\geq \gamma |x - \bar{x}|^{-\gamma-2} n((\gamma+2)\lambda - \Lambda) - \gamma \|\mathbf{b}\|_{L^\infty} |x - \bar{x}|^{-\gamma-1} \\ &= \gamma |x - \bar{x}|^{-\gamma-2} (n((\gamma+2)\lambda - \Lambda) - \|\mathbf{b}\|_{L^\infty} |x - \bar{x}|) \\ &\geq \gamma |x - \bar{x}|^{-\gamma-2} (n((\gamma+2)\lambda - \Lambda) - \|\mathbf{b}\|_{L^\infty}) \equiv k_0(\gamma, c_0, n, \lambda, \Lambda) > 0,\end{aligned}$$

as long as γ satisfies

$$n((\gamma+2)\lambda - \Lambda) - \|\mathbf{b}\|_{L^\infty} > 0.$$

Now set $\psi = 1 - w$ and for $x \in \overline{B}_{3/4}(\bar{x})$ define

$$v_t(x) = \alpha(1 + c_1\varepsilon)(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^+ - \beta(x_n - \varepsilon c_0\delta\psi(x) + t\varepsilon)^-,$$

with $\delta > 0$ small to be made precise later, and c_1 the constant in (8.6).

Then, for $t = -c_1$ one can easily verify that

$$v_{-c_1} \leq U_\beta \leq u, \quad x \in \overline{B}_{3/4}(\bar{x}).$$

Let \bar{t} be the largest $t \geq -c_1$ such that

$$v_t(x) \leq u(x) \quad \text{in } \overline{B}_{3/4}(\bar{x}),$$

and let \tilde{x} be the first touching point. To guarantee that \tilde{x} cannot belong to $\partial B_{3/4}$ when $\bar{t} < c_0\delta$ we use (8.6). Indeed if $x \in \partial B_{3/4}$ and $v_{\bar{t}}(x) \geq 0$ then $x_n > 0$ and in view of (8.6)

$$v_{\bar{t}}(x) = \alpha(1 + c_1\varepsilon)(x_n - \varepsilon c_0\delta + \bar{t}\varepsilon) < \alpha(1 + c_1\varepsilon)x_n \leq u(x).$$

If $v_{\bar{t}}(x) < 0$ we use that $u \geq U_\beta$ to reach again the conclusion that $v_{\bar{t}}(x) < u(x)$. To proceed as in Lemma 4.3 we now need to show that for $\bar{t} < c_0\delta$, $v_{\bar{t}}$ is a strict subsolution in the annulus A .

Indeed, in $A^+(v_{\bar{t}})$ in view of the assumption (8.1) and the computation above for $\mathcal{L}w$, we have

$$\mathcal{L}v_{\bar{t}} \geq \alpha(\varepsilon c_0\delta k_0 + b_n) \geq \varepsilon^2 \min\{\alpha, \beta\} \geq \|f\|_\infty.$$

A similar estimate holds in $A^-(v_{\bar{t}})$. Thus

$$\mathcal{L}v_{\bar{t}} \geq f \quad \text{in } A^+(v_{\bar{t}}) \cup A^-(v_{\bar{t}})$$

for ε small enough.

Also, since $\psi_n < -c$ on $F(v_{\bar{t}}) \cap A$, for ε small, we have

$$\kappa \equiv |e_n - \varepsilon c_0 \nabla \psi| = (1 - 2\varepsilon c_0 \delta \psi_n + \varepsilon^2 c_0^2 \delta^2 |\nabla \psi|^2)^{1/2} = 1 + \tilde{k}\delta\varepsilon,$$

with \tilde{k} between two universal constants.

Then, on $F(v_{\bar{t}}) \cap A$, using (8.2), we can write, as long as ε is sufficiently small,

$$\begin{aligned}(v_{\bar{t}}^+)_\nu - G((v_{\bar{t}}^-)_\nu, x) &= \alpha(1 + c_1\varepsilon)\kappa - G(\beta\kappa, x) \geq \alpha(1 + c_1\varepsilon)\kappa - G_0(\beta\kappa) - \varepsilon^2 \\ &> (1 + c_1\varepsilon)G_0(\beta) - G_0(\beta)\kappa^N - \varepsilon^2 \\ &\geq \varepsilon G_0(\beta) \left(\frac{c_1}{2} - N\tilde{k}\delta\right) > 0\end{aligned}$$

if $\delta < c_1/(2N\tilde{k})$. We used that $G_0(\beta) \geq G_0(0) > 0$ and that $G_0(\beta\kappa) < G_0(\beta)\kappa^N$, since $\eta^{-N}G_0(\eta)$ is strictly decreasing.

Thus, $v_{\bar{t}}$ is a strict subsolution to (1.1) in A as desired. Hence $\bar{t} \geq c_0\delta$ and we conclude as in the Laplacian case.

With Lemma 8.1 at hand, Harnack Inequality and its Corollary follow as in Section 4. We only state the Corollary, since it is indeed the tool used in the proof of the improvement of flatness lemma in the next subsection.

Corollary 8.2. *Let u satisfies at some point $x_0 \in B_2$*

$$(8.7) \quad U_\beta(x_n + a_0) \leq u(x) \leq U_\beta(x_n + b_0) \quad \text{in } B_1(x_0) \subset B_2,$$

for some $0 < \beta \leq L$, with

$$b_0 - a_0 \leq \varepsilon,$$

and let (8.1)-(8.2) hold, for $\varepsilon \leq \bar{\varepsilon}$, $\bar{\varepsilon}$ universal. Then in $B_1(x_0)$, ($\alpha = G_0(\beta)$)

$$\tilde{u}_\varepsilon(x) = \begin{cases} \frac{u(x) - \alpha x_n}{\alpha \varepsilon} & \text{in } B_2^+(u) \cup F(u) \\ \frac{u(x) - \beta x_n}{\beta \varepsilon} & \text{in } B_2^-(u) \end{cases}$$

has a Hölder modulus of continuity at x_0 , outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

8.2. Improvement of flatness. We now extend the basic induction step towards $C^{1,\gamma}$ regularity at 0. We argue as in the proof of Lemma 5.1.

Lemma 8.3. *Let u be solution of (1.3) and suppose that*

$$(8.8) \quad U_\beta(x_n - \varepsilon) \leq u(x) \leq U_\beta(x_n + \varepsilon) \quad \text{in } B_1,$$

with $0 < \beta \leq L$,

$$\|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \varepsilon, \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon^2 \min\{G_0(\beta), \beta\}, \quad \|\mathbf{b}\|_{L^\infty(B_1)} \leq \varepsilon^2,$$

and

$$\|G(\eta, \cdot) - G_0(\eta)\|_{L^\infty(B_1)} \leq \varepsilon^2, \quad \forall 0 \leq \eta \leq L.$$

If $0 < r \leq r_0$ for r_0 universal, and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r , then

$$(8.9) \quad U_{\beta'}(x \cdot \nu_1 - r \frac{\varepsilon}{2}) \leq u(x) \leq U_{\beta'}(x \cdot \nu_1 + r \frac{\varepsilon}{2}) \quad \text{in } B_r,$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq \tilde{C}\varepsilon$, and $|\beta - \beta'| \leq \tilde{C}\beta\varepsilon$ for a universal constant \tilde{C} .

Proof. We divide the proof in 3 steps.

Step 1 – Compactness. We keep the same notation of Lemma 5.1. In this case, the sequence u_k is a solution of problem (1.3) for operators

$$\mathcal{L}^k = \sum_{ij} a_{ij}^k D_{ij} + \mathbf{b}^k \cdot \nabla$$

with ($\alpha_k = G_k(\beta_k, 0)$)

$$\|a_{ij}^k - \delta_{ij}\|_{L^\infty} \leq \varepsilon_k, \quad \|f_k\|_{L^\infty} \leq \varepsilon_k^2 \min\{\alpha_k, \beta_k\}, \quad \|\mathbf{b}^k\|_{L^\infty} \leq \varepsilon_k^2,$$

and

$$(8.10) \quad \|G_k(\eta, \cdot) - G_k(\eta, 0)\|_\infty \leq \varepsilon_k^2, \quad \forall 0 \leq \eta \leq L.$$

The normalized functions \tilde{u}_k are defined by the same formula. Up to a subsequence, $G_k(\cdot, 0)$ converges, locally uniformly, to some C^1 -function \tilde{G}_0 , while $\beta_k \rightarrow \tilde{\beta}$

so that $\alpha_k \rightarrow \tilde{\alpha} = \tilde{G}_0(\tilde{\beta})$. Moreover, by Corollary 8.2 the graphs of \tilde{u}_k converge in the Hausdorff distance to a Hölder continuous \tilde{u} .

Step 2 – Limiting Solution. We show that \tilde{u} solves

$$(8.11) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n \neq 0\}, \\ \tilde{\alpha} \tilde{u}_n^+ - \tilde{\beta} \tilde{G}'_0(\tilde{\beta}) \tilde{u}_n^- = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

We can write, say in $\Omega^+(u^k)$, (in $\Omega^-(u^k)$ replace α_k with β_k)

$$\sum a_{ij}^k D_{ij} \tilde{u}_k = \frac{1}{\alpha_k \varepsilon_k} \sum a_{ij}^k D_{ij} u_k = \frac{1}{\alpha_k \varepsilon_k} (-\alpha_k \mathbf{b}^k \cdot \nabla u_k + f^k) \equiv F^k,$$

where $|F^k| \leq C\varepsilon_k$.

Thus

$$\Delta \tilde{u}_k = \sum_{i,j=1}^n (\delta_{ij} - a_{ij}^k) D_{ij} \tilde{u}_k + F^k.$$

Hence recalling that $\|a_{ij}^k - \delta_{ij}\|_\infty \leq \varepsilon_k$, and from interior L^p Schauder estimates for second derivatives, we conclude that, for instance, $\Delta \tilde{u}_k \rightarrow 0$ in L^p on every compact set contained in $\Omega^+(\tilde{u}^k)$ or in $\Omega^-(\tilde{u}^k)$. This shows that \tilde{u} is harmonic in $B_{1/2} \cap \{x_n \neq 0\}$.

Next, we prove that \tilde{u} satisfies the transmission condition in (8.11) in the viscosity sense.

Again we argue by contradiction. Let $\tilde{\phi}$ be a function of the form

$$\tilde{\phi}(x) = A + px_n^+ - qx_n^- + BQ(x - y)$$

with

$$Q(x) = \frac{1}{2}[(n-1)x_n^2 - |x'|^2], \quad y = (y', 0), \quad A \in, B > 0$$

and

$$\tilde{\alpha} p - \tilde{\beta} \tilde{G}'_0(\tilde{\beta}) q > 0,$$

and assume that $\tilde{\phi}$ touches u strictly from below at a point $x_0 = (x'_0, 0) \in B_{1/2}$. As in Lemma 5.1, let

$$\phi_k = a_k \Gamma_k^+(x) - b_k \Gamma_k^-(x) + \alpha_k (d_k^+(x))^2 \varepsilon_k^{3/2} + \beta_k (d_k^-(x))^2 \varepsilon_k^{3/2},$$

where, we recall,

$$a_k = \alpha_k (1 + \varepsilon_k p), \quad b_k = \beta_k (1 + \varepsilon_k q)$$

and $d_k(x)$ is the signed distance from x to $\partial B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k))$. Moreover,

$$\psi_k(x) = \phi_k(x + \varepsilon_k c_k e_n)$$

touches u_k from below at x_k , with $c_k \rightarrow 0, x_k \rightarrow x_0$.

We get a contradiction if we prove that ψ_k is a strict subsolution to our free boundary problem, that is

$$\begin{cases} \mathcal{L}^k \psi_k > f_k, & \text{in } B_1^+(\psi_k) \cup B_1^-(\psi_k), \\ (\psi_k^+)_\nu - G_k((\psi_k^-)_\nu, x) > 0, & \text{on } F(\psi_k). \end{cases}$$

We have

$$|\nabla \Gamma_k| \leq C, |D_{ij} \Gamma_k| \leq C\varepsilon_k$$

and $|a_{ij} - \delta_{ij}| \leq \varepsilon_k$. We can write, k large enough, say, in the positive phase of ψ_k ,

$$\begin{aligned} \mathcal{L}_k \psi_k &= (\mathcal{L}^k - \Delta) \psi_k + \Delta \psi_k \geq -C \alpha_k \varepsilon_k^2 + \alpha_k \varepsilon_k^{3/2} \mathcal{L}^k d_k^2(x + \varepsilon_k e_n) \\ &\geq c \min\{\alpha_k, \beta_k\} \varepsilon_k^{3/2} \geq \|f_k\|_{L^\infty} \end{aligned}$$

and the first condition is satisfied. An analogous estimate holds in the negative phase.

Finally, since on the zero level set $|\nabla \Gamma_k| = 1$ and $|\nabla d_k^2| = 0$ the free boundary condition reduces to showing that

$$a_k - G_k(b_k, x) > 0.$$

Using the definition of a_k, b_k we need to check that

$$\alpha_k(1 + \varepsilon_k p) - G_k(\beta_k(1 + \varepsilon_k q), x) > 0.$$

From (8.10), it suffices to check that

$$\alpha_k(1 + \varepsilon_k p) - G_k(\beta_k(1 + \varepsilon_k q), 0) - \varepsilon_k^2 > 0.$$

This inequality holds for k large in view of the fact that

$$\tilde{\alpha} p - \tilde{\beta} \tilde{G}'_0(\tilde{\beta}) q > 0.$$

Thus \tilde{u} is a viscosity solution to the linearized problem.

Step 3 – Contradiction. According to estimate (3.2), since $\tilde{u}(0) = 0$ we obtain that

$$|\tilde{u} - (x' \cdot \nu' + p x_n^+ - q x_n^-)| \leq C r^2, \quad x \in B_r,$$

with

$$\tilde{\alpha} p - \tilde{\beta} \tilde{G}'_0(\tilde{\beta}) q = 0, \quad |\nu'| = |\nabla_{x'} \tilde{u}(0)| \leq C.$$

Thus, since \tilde{u}_k converges uniformly to \tilde{u} (by slightly enlarging C) we get that

$$|\tilde{u}_k - (x' \cdot \nu' + p x_n^+ - q x_n^-)| \leq C r^2, \quad x \in B_r.$$

Now set,

$$\beta'_k = \beta_k(1 + \varepsilon_k q), \quad \nu_k = \frac{1}{\sqrt{1 + \varepsilon_k^2 |\nu'|^2}} (e_n + \varepsilon_k (\nu', 0)).$$

Then,

$$\begin{aligned} \alpha'_k &= G_k(\beta_k(1 + \varepsilon_k q), 0) = G_k(\beta_k, 0) + \beta_k G'_k(\beta_k, 0) \varepsilon_k q + O(\varepsilon_k^2) \\ &= \alpha_k \left(1 + \beta_k \frac{G'_k(\beta_k, 0)}{\alpha_k} q \varepsilon_k\right) + O(\varepsilon_k^2) = \alpha_k(1 + \varepsilon_k p) + O(\varepsilon_k^2) \end{aligned}$$

since from the identity $\tilde{\alpha} p - \tilde{\beta} \tilde{G}'_0(\tilde{\beta}) q = 0$ we derive that

$$\beta_k \frac{G'_k(\beta_k, 0)}{\alpha_k} q = p + O(\varepsilon_k).$$

Moreover

$$\nu_k = e_n + \varepsilon_k (\nu', 0) + \varepsilon_k^2 \tau, \quad |\tau| \leq C.$$

With these choices it follows as in Lemma 5.1 that (for k large and $r \leq r_0$)

$$\tilde{U}_{\beta'_k}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq \tilde{u}_k(x) \leq \tilde{U}_{\beta'_k}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

which leads to a contradiction.

9. THE DEGENERATE CASE FOR GENERAL FREE BOUNDARY PROBLEMS.

In this section, we recover the improvement of flatness lemma in the degenerate case, that is when the negative part of u is negligible and the positive part is close to a one-plane solution (i.e. $\beta = 0, \alpha = G_0(0)$). First we need the Harnack inequality.

9.1. Harnack inequality. As in Section 4, Harnack inequality in the degenerate case follows from the following basic lemma.

Lemma 9.1. *There exists a universal constant $\bar{\varepsilon} > 0$ such that if u satisfies*

$$u^+(x) \geq U_0(x), \quad \text{in } B_1$$

with

$$(9.1) \quad \|u^-\|_{L^\infty} \leq \varepsilon^2, \quad \|\mathbf{b}\|_{L^\infty} \leq \varepsilon^2 \quad \|f\|_{L^\infty} \leq \varepsilon^4,$$

$$(9.2) \quad \|G(\eta, \cdot) - G_0(\eta)\| \leq \varepsilon^2, \quad 0 \leq \eta \leq C\varepsilon^2$$

then if at $\bar{x} = \frac{1}{5}e_n$

$$(9.3) \quad u^+(\bar{x}) \geq U_0(\bar{x}_n + \varepsilon),$$

then

$$(9.4) \quad u^+(x) \geq U_0(x_n + c\varepsilon) \quad \text{in } \bar{B}_{1/2},$$

for some $0 < c < 1$ universal. Analogously, if

$$u^+(x) \leq U_0(x) \quad \text{in } B_1$$

and

$$u^+(\bar{x}) \leq U_0(\bar{x}_n - \varepsilon)$$

then

$$u^+(x) \leq U_0(x_n - c\varepsilon) \quad \text{in } \bar{B}_{1/2}.$$

Proof. The proof is the same as for the model case in Lemma 4.6. To prove that

$$v_{\bar{\varepsilon}}(x) = G_0(0)(x_n - \varepsilon c_0 \psi + \bar{\varepsilon})^+ - \varepsilon^2 C_1(x_n - \varepsilon c_0 \psi(x) + \bar{\varepsilon})^-, \quad x \in \bar{B}_{3/4}(\bar{x})$$

is a subsolution in the annulus A we use the following computation

$$\mathcal{L}v_{\bar{\varepsilon}} \geq c_0 C_1 \varepsilon^3 \mathcal{L}w - C_1 \varepsilon^2 |b_n| \geq \varepsilon^3 K(n, \lambda, \Lambda) > \varepsilon^4 \geq \|f\|_\infty, \quad \text{in } A^+(v_{\bar{\varepsilon}}) \cup A^-(v_{\bar{\varepsilon}})$$

for ε small enough. Here we have used as in Lemma 8.1 that $\mathcal{L}w \geq k_0 > 0$.

Moreover, on $F(v_{\bar{\varepsilon}}) \cap A$ we have

$$(v_{\bar{\varepsilon}}^+)_\nu - G((v_{\bar{\varepsilon}}^-)_\nu) = G_0(0)|e_n - \varepsilon c_0 \nabla \psi| - G(\varepsilon^2 C_1 |e_n - \varepsilon c_0 \nabla \psi|, x) \geq C\varepsilon |\psi_n| + O(\varepsilon^2) > 0$$

as long as ε is small enough.

We state here the Corollary that can be deduced by the degenerate Harnack Inequality.

Corollary 9.2. *Let u satisfies at some point $x_0 \in B_2$*

$$(9.5) \quad U_0(x_n + a_0) \leq u(x) \leq U_0(x_n + b_0) \quad \text{in } B_1(x_0) \subset B_2,$$

with

$$b_0 - a_0 \leq \varepsilon,$$

and let (9.1)-(9.2) hold for $\varepsilon \leq \bar{\varepsilon}$, $\bar{\varepsilon}$ universal. Then in $B_1(x_0)$

$$\tilde{u}_\varepsilon := \frac{u^+(x) - G_0(0)x_n}{\varepsilon G_0(0)}$$

has a Hölder modulus of continuity at x_0 , outside the ball of radius $\varepsilon/\bar{\varepsilon}$, i.e for all $x \in B_1(x_0)$, with $|x - x_0| \geq \varepsilon/\bar{\varepsilon}$

$$|\tilde{u}_\varepsilon(x) - \tilde{u}_\varepsilon(x_0)| \leq C|x - x_0|^\gamma.$$

9.2. Improvement of flatness. We prove here the improvement of flatness in the degenerate setting. Recall that in this case one improves the flatness of u^+ only.

Lemma 9.3. *Let u satisfy*

$$(9.6) \quad U_0(x_n - \varepsilon) \leq u^+(x) \leq U_0(x_n + \varepsilon) \quad \text{in } B_1, \quad 0 \in F(u),$$

with

$$\begin{aligned} \|a_{ij} - \delta_{ij}\| &\leq \varepsilon, \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon^4, \quad \|\mathbf{b}\|_{L^\infty(B_1)} \leq \varepsilon^2, \\ \|G(\eta, \cdot) - G_0(\eta)\|_{L^\infty} &\leq \varepsilon^2, \quad 0 \leq \eta \leq C\varepsilon^2, \end{aligned}$$

and

$$\|u^-\|_{L^\infty(B_1)} \leq \varepsilon^2.$$

If $0 < r \leq r_1$ for r_1 universal, and $0 < \varepsilon \leq \varepsilon_1$ for some ε_1 depending on r , then

$$(9.7) \quad U_0(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \leq u^+(x) \leq U_0(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad \text{in } B_r,$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \leq C\varepsilon$ for a universal constant C .

Proof. Step 1 – Compactness. As in Lemma 5.2, it follows from Corollary 9.2 that as $\varepsilon_k \rightarrow 0$ the graphs of the

$$\tilde{u}_k(x) = \frac{u_k(x) - G_k(0,0)x_n}{G_k(0,0)\varepsilon_k}, \quad x \in B_1^+(u_k) \cup F(u_k)$$

converge (up to a subsequence) in the Hausdorff distance to the graph of a Hölder continuous function \tilde{u} over $B_{1/2} \cap \{x_n \geq 0\}$. Here the u_k solve our free boundary problem (1.3) with coefficients a_{ij}^k , \mathbf{b}^k , right-hand-side f_k and free boundary condition G_k satisfying the assumptions of the lemma for a sequence of ε_k 's going to 0.

Step 2 – Limiting Solution. One shows that \tilde{u} solves the following Neumann problem

$$(9.8) \quad \begin{cases} \Delta \tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases}$$

We can easily adapt the proof of Lemma 5.2, choosing

$$\phi_k(x) = a_k \Gamma_k^+(x) + (d_k^+(x))^2 \varepsilon_k^{3/2}, \quad a_k = G_k(0,0)(1 + \varepsilon_k p).$$

and

$$(9.9) \quad \Psi_k(x) = \begin{cases} \phi_k(x + c_k \varepsilon_k e_n) & \text{in } \mathcal{B} \\ c_k^2 (3d(x, \partial \mathcal{B}) + d^2(x, \partial \mathcal{B})) & \text{outside of } \mathcal{B}, \end{cases}$$

with

$$\mathcal{B} := B_{\frac{1}{B\varepsilon_k}}(y + e_n(\frac{1}{B\varepsilon_k} - A\varepsilon_k - \varepsilon_k c_k)).$$

To check the subsolution condition at the free boundary for the function $\Psi_k(x)$, we need that

$$(\Psi_k^+)_\nu > G_k((\Psi_k^-)_\nu, x), \quad \text{on } F(\Psi_k).$$

This is equivalent to show that, for k large,

$$G_k(0, 0)(1 + \varepsilon_k p) - G_k(c\varepsilon_k^2, x) > 0.$$

Since $p > 0$, this follows immediately from the assumptions on G_k .

Step 3 – Contradiction. In this step we can argue as in the final step of the proof of Lemma 4.1 in [D]. \square

10. PROOFS OF THE MAIN THEOREMS FOR GENERAL FREE BOUNDARY PROBLEMS

The proof of Theorem 1.3 and Theorem 1.4 follow the same scheme of the model case. In particular, for Theorem 1.3, we take care of choosing $\bar{r}^{\bar{\gamma}} < 1/16$, say, while the other assumptions on \bar{r} remain the same. Also, $\tilde{\varepsilon}$ may have to be smaller, depending on γ_0 . The dichotomy degenerate/nondegenerate is handled through Lemma 6.1 which extends to the variable coefficients case, with minor changes in the proof.

In the proof of Theorem 1.4, the blow-up limit \tilde{u} solves the following global homogeneous two-phase free boundary problem

$$(10.1) \quad \begin{cases} \Delta \tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cup \{\tilde{u} \leq 0\}^0 \\ \tilde{u}_\nu^+ = G_0(\tilde{u}_\nu^-), & \text{on } F(\tilde{u}) := \partial\{\tilde{u} > 0\}, \end{cases}$$

Now, Lemma 6.2 holds with identical proof for the free boundary condition $U_\nu^+ = G_0(U_\nu^-)$, so that the proof of Theorem 1.4 does not present any further difficulty.

REFERENCES

- [ACF] Alt H.W., Caffarelli L.A., Friedman A., *Variational problems with two phases and their free boundaries*. Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461.
- [AF] Argiolas R., Ferrari F., *Flat free boundaries regularity in two-phase problems for a class of fully nonlinear elliptic operators with variable coefficients*, Interfaces Free Bound. **11** (2009), no.2, 177-199.
- [B1] Batchelor G.K., *On steady laminar flow with closed streamlines at large Reynolds number*, J. Fluid.Mech. **1** (1956), 177–190.
- [C1] Caffarelli L.A., *A Harnack inequality approach to the regularity of free boundaries. Part I: Lipschitz free boundaries are $C^{1,\alpha}$* , Rev. Mat. Iberoamericana **3** (1987) no. 2, 139–162.
- [C2] Caffarelli L.A., *A Harnack inequality approach to the regularity of free boundaries. Part II: Flat free boundaries are Lipschitz*, Comm. Pure Appl. Math. **42** (1989), no.1, 55–78.
- [CC] Caffarelli L.A., Cabre X., *Fully Nonlinear Elliptic Equations*, Colloquium Publications 43, American Mathematical Society, Providence, RI, 1995.
- [CJK] Caffarelli L.A., Jerison D., Kenig C.E., *Some new monotonicity theorems with applications to free boundary problems*, Ann. of Math. (2) **155** (2002), no. 2, 369–404.
- [CFS] Cerutti M.C., Ferrari F., Salsa S., *Two phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$* , Archive for Rational Mechanics and Analysis, Vol 171, n.3, pp. 329 - 348 (2004)

- [D] De Silva D., *Free boundary regularity for a problem with right hand side*, Interfaces and free boundaries **13** (2011), 223–238.
- [DR] De Silva D., Roquejoffre J-M., *Regularity in a one-phase free boundary problem for the fractional Laplacian*. Preprint 2010. To appear in Annales de l’Institut Henri Poincare.
- [EM] Elcrat A.R, Miller K.G., *Variational formulas on Lipschitz domains*, Trans. Amer. Math. Soc. **347** (1995), 2669–2678.
- [F1] Feldman M., *Regularity of Lipschitz free boundaries in two-phase problems for fully nonlinear elliptic equations*, Indiana Univ. Math. J. **50** (2001), no.3, 1171–1200
- [F2] Feldman M., *Regularity for nonisotropic two-phase problems with Lipschitz free boundaries*, Differential Integral Equations **10** (1997), no.6, 1171–1179.
- [Fe1] Ferrari F., *Two-phase problems for a class of fully nonlinear elliptic operators, Lipschitz free boundaries are $C^{1,\gamma}$* , Amer. J. Math. **128** (2006), 541–571.
- [FS1] Ferrari F., Salsa S., *Regularity of the free boundary in two-phase problems for elliptic operators*, Adv. Math. **214** (2007), 288–322.
- [FS2] Ferrari F., Salsa S., *Subsolutions of elliptic operators in divergence form and application to two-phase free boundary problems*, Bound. Value Probl. **2007**, art. ID 57049, 21pp.
- [FL] Friedman A., Liu Y. *A free boundary problem arising in magnetohydrodynamic system*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. **22** (1994), 375–448.
- [LW] Lederman C., Wolanski N., *A two phase elliptic singular perturbation problem with a forcing term*, J. Math. Pures Appl. (9) **86** (2006), no. 6, 552–589.
- [LT] Leitao R., Teixeira E., *Regularity and geometric estimates for minima of discontinuous functionals*, arXiv:1111.2625.
- [LN] Lewis J., Nyström K., *Regularity of Lipschitz free boundaries in two phases Problems for the p Laplace operator*, Adv. in Math. **225**, (2010) 2565-2597.
- [MP] Matevosyan N., Petrosyan A., *Almost monotonicity formulas for elliptic and parabolic operators with variable coefficients*, Comm. Pure Appl. Math **44** (12011), 271-311.
- [W1] Wang P.Y., *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. I. Lipschitz free boundaries are $C^{1,\alpha}$* , Comm. Pure Appl. Math. **53** (2000), 799–810.
- [W2] Wang P.Y., *Regularity of free boundaries of two-phase problems for fully nonlinear elliptic equations of second order. II. Flat free boundaries are Lipschitz*, Comm. Partial Differential Equations **27** (2002), 1497–1514.

DEPARTMENT OF MATHEMATICS, BARNARD COLLEGE, COLUMBIA UNIVERSITY, NEW YORK, NY 10027

E-mail address: `desilva@math.columbia.edu`

DIPARTIMENTO DI MATEMATICA DELL’UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA S. DONATO, 5, 40126 BOLOGNA, ITALY.

E-mail address: `fausto.ferrari@unibo.it`

DIPARTIMENTO DI MATEMATICA DEL POLITECNICO DI MILANO, PIAZZA LEONARDO DA VINCI, 32, 20133 MILANO, ITALY.

E-mail address: `sandro.salsa@polimi.it`