THE FUETER PRIMITIVE OF BIAXIALLY MONOGENIC FUNCTIONS

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1. **Introduction.** The Fueter mapping theorem has been proved in the thirties by Fueter in [9] as a tool to generate functions in the kernel of the so called Cauchy-Fueter operator. This theorem has been generalized to monogenic functions, that is functions with values in a Clifford algebra and in the kernel of the Dirac operator (see [1, 5, 8, 12, 10]), by several authors. Without claiming completeness we mention the works [15, 16, 17, 18, 19, 20, 22, 23] and the references therein. This field is still very active, in fact it turned out that the Fueter mapping theorem is a very deep result in Clifford analysis. More recently the authors, using the theory of slice monogenic functions, see e.g. the book [6], have given an integral version of the Fueter mapping theorem which is useful to define the so called F-functional calculus see [2]. The inversion of the Fueter mapping theorem is a brand new field of studies that started in the papers [3, 4]. In order to illustrate the problems that we have tackled, we begin by stating the first problem that we have solved in [3].

Let n be an odd number and let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, i.e. let U be an open set invariant under the action of $SO(n)$. Suppose that $\breve{f}$ is an axially monogenic function and f is a slice monogenic function such that

$$\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x).$$

Our problem is to find an explicit description of the map $\breve{f} \mapsto f$.

In the paper [3] we have proved an integral representation formula for the inverse Fueter mapping theorem for axially monogenic functions defined on axially symmetric open sets $U \subseteq \mathbb{R}^{n+1}$, where n is an odd number. In general, one may address the problem of finding an explicit expression for f such that $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$, given a monogenic function $\breve{f}$. However, in order to solve the problem, we need additional hypothesis on the monogenic function $\breve{f}$.
In [4] we have generalized the result by proving the inverse Fueter mapping theorem for axially monogenic functions of degree k, i.e. functions of type

$$\breve{f}_k(x) := [A(x_0, \rho) + \underline{\omega} B(x_0, \rho)]\mathcal{P}_k(\underline{x}),$$

where $A(x_0, \rho)$ and $B(x_0, \rho)$ satisfy a suitable Vekua-type system and $\mathcal{P}_k(\underline{x})$ is a homogeneous monogenic polynomial of degree k. Given an axially monogenic function of degree k, it is possible to explicitly write a holomorphic function f of a paravector variable defined on U such that

$$\Delta^{k+\frac{n-1}{2}}(f(x)\mathcal{P}_k(\underline{x})) = \breve{f}(x)\mathcal{P}_k(\underline{x}). \tag{1}$$

Axially monogenic functions of degree k are important since every monogenic function $\breve{f}$ defined on an axially symmetric open set can be written as a series $\breve{f} = \sum_k f_k$ where f_k are axially monogenic functions of degree k. For each addendum of degree k we can provide a Fueter primitive as described in (1), and so we have, see [4]

$$\breve{f} = \Delta^{\frac{n-1}{2}} \sum_k \Delta^k \varphi_k.$$

The aim of this paper to solve the problem of inverting the Fueter mapping theorem at a different level of generality, in fact we consider the case of biaxially monogenic functions. A monogenic function f is said to be biaxially monogenic if it is invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$, $p \geq 1$, $q \geq 1$. It can be proved, see [14] and Remark 1, that a biaxially monogenic function f is of the form

$$f(\underline{x}, \underline{y}) = A(|\underline{x}|, |\underline{y}|) + \frac{\underline{x}}{|\underline{x}|} B(|\underline{x}|, |\underline{y}|) + \frac{\underline{y}}{|\underline{y}|} C(|\underline{x}|, |\underline{y}|) + \frac{\underline{x}}{|\underline{x}|} \frac{\underline{y}}{|\underline{y}|} D(|\underline{x}|, |\underline{y}|), \tag{2}$$

where the functions A, B, C, D satisfy a Vekua-type system.

Corollary 1 in [21] shows that the Fueter mapping theorem can be extended to this setting. Indeed, functions W of type

$$W(\underline{x}, \underline{y}) = h_1(|\underline{x}|, |\underline{y}|)\frac{\underline{x}}{|\underline{x}|} + h_2(|\underline{x}|, |\underline{y}|)\frac{\underline{y}}{|\underline{y}|} \tag{3}$$

with h_1, h_2 real valued and such that W is in the kernel of the operator $\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}$, are such that

$$\Delta^{\frac{p+q}{2}-1} W(\underline{x}, \underline{y}) = f(\underline{x}, \underline{y}) \tag{4}$$

with f biaxially monogenic. The aim of this paper is to provide the inverse of this version of the Fueter mapping theorem (4). More precisely:

Problem. Let f be a biaxially monogenic function. Determine W, as in (3), that is a solution to (4).

The solution of such problem is the main result of this paper and it is given in Theorem 4.2 and Theorem 4.5, which treat the case of the odd and even part of a biaxially monogenic, respectively. From these results we can construct the Fueter's primitive for a general biaxially monogenic function.

The plan of the paper. In Section 2 we state some preliminary results on the series expansion of functions $W(\underline{x},\underline{y})$ in the kernel of the operator $\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|}+\frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}$. In Section 3 we introduce and study the kernels $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$, for $p,q\in\mathbb{N}$ and for $\lambda>0$ and $\mu>0$, which are obtained by integrating the monogenic Cauchy kernel on suitable spheres. We explicitly determine two Fueter's primitives $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ of $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$, respectively. In Section 4 we use the kernels $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ to state our main result, namely, the inverse Fueter mapping theorem in integral form. We distinguish between odd and even part of biaxially monogenic functions and we obtain Fueter's primitive for a general biaxially monogenic function.

2. **Preliminary material.** The setting in which we work is the real Clifford algebra $\mathbb{R}_m$ over m imaginary units $e_1,\dots,e_m$ satisfying the relations $e_ie_j+e_je_i=-2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A=i_1\dots i_r$, $i_\ell\in\{1,2,\dots,m\}$, $i_1<\dots<i_r$, is a multi-index, $e_A=e_{i_1}e_{i_2}\dots e_{i_r}$ and $e_\emptyset=1$.

Let p,q be two natural numbers. The basis of $\mathbb{R}_{p+q}$ is then

$$e_1,\dots,e_p,e_{p+1},\dots,e_{p+q}.$$

An element $(x_1,\dots,x_p,y_1,\dots,y_q)$ in the Euclidean space $\mathbb{R}^p\times\mathbb{R}^q$ will be identified with the pair $(\underline{x},\underline{y})$ of the two 1-vectors $\underline{x}=\sum_{i=1}^p x_ie_i$, $\underline{y}=\sum_{i=1}^q y_ie_{p+i}$ or also with the 1-vector $\underline{x}+\underline{y}$. By $\partial_{\underline{x}}$, $\partial_{\underline{y}}$, and $\partial_{\underline{x}+\underline{y}}$ we will denote the corresponding Dirac operators.

Definition 2.1 (Biaxially monogenic function)**.** Let U be an open set in $\mathbb{R}^p\times\mathbb{R}^q$, for $p\geq 1$ and $q\geq 1$, invariant under the action of the group $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. Let f be a $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$-invariant monogenic function on U. Then we say that f is a biaxially monogenic function on U.

Remark 1. It can be shown that biaxially monogenic functions are of the form

$$f(\underline{x},\underline{y})=A(|\underline{x}|,|\underline{y}|)+\frac{\underline{x}}{|\underline{x}|}B(|\underline{x}|,|\underline{y}|)+\frac{\underline{y}}{|\underline{y}|}C(|\underline{x}|,|\underline{y}|)+\frac{\underline{x}}{|\underline{x}|}\frac{\underline{y}}{|\underline{y}|}D(|\underline{x}|,|\underline{y}|)$$

defined on an open set U in $\mathbb{R}^p\times\mathbb{R}^q$ invariant under the action of the group $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. In [14], the authors show that this function is biaxially monogenic if the functions A,B,C,D satisfy the following Vekua-type system, where $r=|\underline{x}|$ and $\rho=|\underline{y}|$:

$$\begin{gathered}
\frac{\partial}{\partial\rho}A(r,\rho)+\left(\frac{\partial}{\partial r}+\frac{q-1}{r}\right)D(r,\rho)=0,\\
\left(\frac{\partial}{\partial\rho}+\frac{p-1}{\rho}\right)D(r,\rho)-\frac{\partial}{\partial r}A(r,\rho)=0,\\
\left(\frac{\partial}{\partial\rho}+\frac{p-1}{\rho}\right)B(r,\rho)+\left(\frac{\partial}{\partial r}+\frac{q-1}{r}\right)C(r,\rho)=0,\\
\frac{\partial}{\partial\rho}C(r,\rho)-\frac{\partial}{\partial r}B(r,\rho)=0.
\end{gathered}$$

Moreover, the condition of monogenicity $\partial_{\underline{x}+\underline{y}}f=0$ decomposes into a pair of equations for A and D and another pair for B and C, precisely it decomposes as

$$\partial_{\underline{x}+\underline{y}}\left(\frac{\underline{x}}{|\underline{x}|}B(|\underline{x}|,|\underline{y}|)+\frac{\underline{y}}{|\underline{y}|}C(|\underline{x}|,|\underline{y}|)\right)=0,\ \partial_{\underline{x}+\underline{y}}\left(A(|\underline{x}|,|\underline{y}|)+\frac{\underline{x}}{|\underline{x}|}\frac{\underline{y}}{|\underline{y}|}D(|\underline{x}|,|\underline{y}|)\right)=0.$$

This means that both

$$\frac{\underline{x}}{|\underline{x}|}B(|\underline{x}|,|\underline{y}|)+\frac{\underline{y}}{|\underline{y}|}C(|\underline{x}|,|\underline{y}|), \tag{5}$$

and

$$A(|\underline{x}|,|\underline{y}|)+\frac{\underline{x}}{|\underline{x}|}\frac{\underline{y}}{|\underline{y}|}D(|\underline{x}|,|\underline{y}|), \tag{6}$$

are biaxially monogenic functions.

We recall the following.

Definition 2.2. Functions of type (5) are the "odd part" of a biaxially monogenic function, while (6) corresponds to the "even part" of a biaxially monogenic function.

Definition 2.3. Let $\mathcal{U} \subseteq (\mathbb{R}^+ \cup \{0\}) \times (\mathbb{R}^+ \cup \{0\})$ and let $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$ be the set induced by $\mathcal{U}$. Then we denote by $\mathcal{H}_B(U)$ the set of functions W of the form

$$W(\underline{x},\underline{y}) = h_1(|\underline{x}|,|\underline{y}|)\frac{\underline{x}}{|\underline{x}|} + h_2(|\underline{x}|,|\underline{y}|)\frac{\underline{y}}{|\underline{y}|}$$

with h_1, h_2 real valued and such that W are nullsolutions of the operator

$$\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}.$$

Lemma 2.4. *Let U be an open set in $\mathbb{R}^p \times \mathbb{R}^q$, for $p \geq 1$ and $q \geq 1$, invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$ and let $\underline{x}/r \in \mathbb{S}^{p-1}$, $\underline{y}/\rho \in \mathbb{S}^{q-1}$, where $r = |\underline{x}|$, $\rho = |\underline{y}|$. Then the function W is in the kernel of the operator $\frac{\underline{x}}{|\underline{x}|}\partial_{|\underline{x}|} + \frac{\underline{y}}{|\underline{y}|}\partial_{|\underline{y}|}$ if and only if its components h_1 and h_2 satisfy the equations*

$$\begin{cases} \partial_r h_1(r,\rho) + \partial_\rho h_2(r,\rho) = 0, \\ \partial_\rho h_1(r,\rho) - \partial_r h_2(r,\rho) = 0. \end{cases} \tag{7}$$

Proof. It follows by a direct computation. $\square$

Notation. Let us consider $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C} = \mathbb{R}_m + i\mathbb{R}_m$ where i is the imaginary unit of the algebra of complex numbers $\mathbb{C}$. With an abuse of notation, for any $a+ib$ in $\mathbb{C}_m$ we set $\mathrm{Re}(a+ib) = a$.

Proposition 1. *Let U in $\mathbb{R}^p \times \mathbb{R}^q$, for $p \geq 1$ and $q \geq 1$, be invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$, and assume that $W \in \mathcal{H}_B(U)$. Then we have*

$$W(\underline{x},\underline{y}) = \mathrm{Re}\Big((h_1(r,\rho) + ih_2(r,\rho))(\underline{\omega} - i\underline{\nu})\Big). \tag{8}$$

Moreover, if we set

$$H(r - i\rho) := h_1(r,\rho) + ih_2(r,\rho), \qquad H^{(\ell)}(r) := \partial_r^\ell H(r), \quad \ell = 0,1,2,... \tag{9}$$

then W can be represented in power series as follows:

$$W(\underline{x},\underline{y}) = \sum_{\ell=0}^{+\infty}\Big(\frac{1}{(2\ell)!}\underline{y}^{2\ell}H^{(2\ell)}(r)\frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!}\underline{y}^{2\ell+1}H^{(2\ell+1)}(r)\Big) \tag{10}$$

or equivalently

$$W(\underline{x},\underline{y}) = \sum_{\ell=0}^{+\infty}\Big(\frac{1}{(2\ell)!}\underline{y}^{2\ell}H^{(2\ell)}(|\underline{x}|)\frac{\underline{x}}{|\underline{x}|} - \frac{1}{(2\ell+1)!}\underline{y}^{2\ell+1}H^{(2\ell+1)}(|\underline{x}|)\Big). \tag{11}$$

Proof. Observe that since we have set $r = |\underline{x}|$, $\rho = |\underline{y}|$, $\underline{x}/r = \underline{\omega}$, $\underline{y}/\rho = \underline{\nu}$ we can write

$$\begin{aligned} W(\underline{x}, \underline{y}) =& h_1(|\underline{x}|, |\underline{y}|)\frac{\underline{x}}{|\underline{x}|} + h_2(|\underline{x}|, |\underline{y}|)\frac{\underline{y}}{|\underline{y}|} \\ =& h_1(r, \rho)\underline{\omega} + h_2(r, \rho)\underline{\nu} = \mathrm{Re}\Big((h_1(r, \rho) + ih_2(r, \rho))(\underline{\omega} - i\underline{\nu})\Big) \end{aligned}$$

so we obtain (8). By Lemma 2.4 it follows that the function H defined in (9) is holomorphic in the variable $z := r - i\rho$ so it admits power series expansion with respect to the variable ρ. We can write

$$\begin{aligned} W(\underline{x}, \underline{y}) =& \mathrm{Re}\Big(H(z)[\underline{\omega} - i\underline{\nu}]\Big) = \mathrm{Re}\Big(\sum_{k\geq 0} \frac{1}{k!}(-i\rho)^k H^{(k)}(r)[\underline{\omega} - i\underline{\nu}]\Big) \\ =& \sum_{\ell\geq 0} \frac{1}{(2\ell)!}(-i\rho)^{2\ell} H^{(2\ell)}(r)\underline{\omega} - \frac{1}{(2\ell+1)!}(-i\rho)^{2\ell}\rho H^{(2\ell+1)}(r)\underline{\nu} \\ =& \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!}\underline{y}^{2\ell} H^{(2\ell)}(r)\frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!}\underline{y}^{2\ell+1} H^{(2\ell+1)}(r)\Big). \end{aligned}$$

This is formula (10). □

In the paper [21], the authors prove a version of the Fueter mapping theorem that we state here in a special case:

Theorem 2.5 (See Corollary 1 in [21]). *Let p, q be two odd numbers. Let $f \in \mathcal{H}_B(U)$ where $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$ is an open set invariant under the action of* $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$. *Let $\Delta := \Delta_{\underline{x}} + \Delta_{\underline{y}}$. Then the function*

$$\Delta^{\frac{p+q}{2}-1} f(\underline{x}, \underline{y})$$

is left monogenic with respect to the operator $\partial_{\underline{x}} + \partial_{\underline{y}}$.

Remark 2. Theorem 2.5 is a generalization of the Fueter theorem that considers a holomorphic function f, $p = 1$, $q = 3$, and states that Δf is in the kernel of the Cauchy-Fueter operator. The Fueter-Sce theorem can be obtained by considering a holomorphic function f, $p = 1$, q any odd number.

Analogously to what we did in [3], [4] one may ask if it is possible to solve the following problem.

Problem. *Let f be a biaxially monogenic function on an open set $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$, invariant under the action of the group* $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$. *Determine a function $W \in \mathcal{H}_B(U)$ such that*

$$\Delta^{\frac{p+q}{2}-1} W(\underline{x}, \underline{y}) = f(\underline{x}, \underline{y}),$$

where p, q are odd positive integers.

From the above problem naturally arises the following definition.

Definition 2.6. Let p, q are odd positive integers and let f be a biaxially monogenic function on an open set $U \subseteq \mathbb{R}^p \times \mathbb{R}^q$, invariant under the action of the group $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$. A function $W \in \mathcal{H}_B(U)$ is a Fueter primitive of f if

$$\Delta^{\frac{(p+q)}{2}-1}(W(\underline{x}, \underline{y})) = f(\underline{x}, \underline{y}).$$

We conclude this section with a proposition that we will use in the sequel.

Proposition 2. *Let p and q odd positive integers, let U be a domain in $\mathbb{R}^p \times \mathbb{R}^q$ invariant under the action of the group* $\mathrm{Spin}(p) \times \mathrm{Spin}(q)$ *and let $W \in \mathcal{H}_B(U)$. Set $d := \frac{p+q}{2} - 1$, so we have*

$$(\Delta_{\underline{x}} + \Delta_{\underline{y}})^d W(\underline{x}, \underline{y})|_{\underline{y}=0} = \sum_{k=0}^{d} \binom{d}{k} \frac{C_k}{(2k)!} \Delta_{\underline{x}}^{d-k} \Big(H^{(2k)}(r) \frac{\underline{x}}{r} \Big)$$

where H is defined in (9) and

$$C_0 := 1, \quad and \quad C_k = (-1)^k \Pi_{i=1}^{k} (2i) \ \ \Pi_{j=0}^{k-1} (q+2j), \quad for \quad k \in \mathbb{N}. \tag{12}$$

Proof. We know that $\Delta_{\underline{y}} = -\partial_{\underline{y}}^2$ where $\partial_{\underline{y}}$ is the Dirac operator in dimension q and that

$$\partial_{\underline{y}}(\underline{y}^{\ell}) = \begin{cases} -\ell \ \underline{y}^{\ell-1} & \text{if } \ell \quad \text{is even,} \\ -(\ell + q - 1) \ \underline{y}^{\ell-1} & \text{if } \ell \quad \text{is odd.} \end{cases}$$

We then obtain

$$\Delta_{\underline{y}}^k \left(\underline{y}^{\ell} \right) = \begin{cases} 0 & \text{if } k > 2\ell, \\ C_k & \text{if } k = 2\ell, \\ \mathcal{E}(\underline{y}) & \text{if } k < 2\ell, \end{cases}$$

where C_k are constants depending on k and $\mathcal{E}$ is a continuous function such that $\mathcal{E}(\underline{y}) \to 0$ for $\underline{y} \to 0$. Thus we only have to compute the constants in the case $k = 2\ell$. To this purpose note that $\Delta_{\underline{y}}^0 (\underline{y}^0) = 1$ and

$$\Delta_{\underline{y}}(\underline{y}^2) = -2q, \quad \Delta_{\underline{y}}^2(\underline{y}^4) = 2 \cdot 4 \cdot q(q+2), \quad \Delta_{\underline{y}}^3(\underline{y}^6) = -2 \cdot 4 \cdot 6 \cdot q(q+2)(q+4);$$

and, by recurrence, one can easily verify that

$$C_k = \Delta_{\underline{y}}^k (\underline{y}^{2k}) = (-1)^k \Pi_{i=1}^{k} (2i) \ \Pi_{j=0}^{k-1} (q+2j). \tag{13}$$

We set for simplicity $d = m/2 - 1$ and we compute $(\Delta_{\underline{x}} + \Delta_{\underline{y}})^d W(\underline{x}, \underline{y})$ keeping in mind that we have to consider the restriction to $\underline{y} = \underline{0}$. We have:

$$\begin{aligned}
&(\Delta_{\underline{x}} + \Delta_{\underline{y}})^d W(\underline{x}, \underline{y})\Big|_{\underline{y}=\underline{0}} \\
&= \Big[\sum_{k=0}^{d} \binom{d}{k} \Delta_{\underline{x}}^{d-k} \Delta_{\underline{y}}^{k} \Big] \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!} \underline{y}^{2\ell} H^{(2\ell)}(r) \frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!} \underline{y}^{2\ell+1} H^{(2\ell+1)}(r) \Big) \Big|_{\underline{y}=\underline{0}} \\
&= \Big[\sum_{k=0}^{d} \binom{d}{k} \Delta_{\underline{x}}^{d-k} \Big] \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!} \Delta_{\underline{y}}^{k} (\underline{y}^{2\ell}) H^{(2\ell)}(r) \frac{\underline{x}}{r} - \frac{1}{(2\ell+1)!} \Delta_{\underline{y}}^{k} (\underline{y}^{2\ell+1}) H^{(2\ell+1)}(r) \Big) \Big|_{\underline{y}=\underline{0}} \\
&= \Big[\sum_{k=0}^{d} \binom{d}{k} \Delta_{\underline{x}}^{d-k} \Big] \sum_{\ell=0}^{+\infty} \Big(\frac{1}{(2\ell)!} \Delta_{\underline{y}}^{k} (\underline{y}^{2\ell}) H^{(2\ell)}(r) \frac{\underline{x}}{r} \Big) \Big|_{\underline{y}=\underline{0}} \\
&= \sum_{k=0}^{d} \binom{d}{k} \frac{C_k}{k!} \Delta_{\underline{x}}^{d-k} \Big(H^{(2k)}(r) \frac{\underline{x}}{r} \Big),
\end{aligned}$$

and this concludes the proof.

As an example we explicitly compute $(\Delta_{\underline{x}} + \Delta_{\underline{y}})^d W(\underline{x}, \underline{y})|_{\underline{y}=0}$ for $p = q = 3$.

Proposition 3. *Let U be a domain in $\mathbb{R}^3 \times \mathbb{R}^3$ invariant under the action of the group* $\mathrm{Spin}(3) \times \mathrm{Spin}(3)$ *and let $W \in \mathcal{H}_B(U)$. Then we have*

$$(\Delta_{\underline{x}} + \Delta_{\underline{y}})^2 W(\underline{x}, \underline{y})|_{\underline{y}=0} = -8 \partial_r \Big(\frac{1}{r} \partial_r^2 H(r) \Big) \underline{\omega},$$

where $\underline{x}/r = \underline{\omega}$.

Proof. Using the notations of the previous proposition we have $p = q = 3$ and $d = 2$. Recall that if $\underline{x} \in \mathbb{R}^p$, $p \in \mathbb{N}$, $\underline{x} = r\underline{\omega}$ then

$$\Delta_{\underline{x}} = \partial_r^2 + \frac{p-1}{r}\partial_r + \frac{1}{r^2}\Gamma(p-2-\Gamma)$$

where the Γ operator is such that $\Gamma(\underline{\omega}) = (p-1)\underline{\omega}$. For every function $F(r,\omega)$, with suitable regularity, and such that $F(r,\omega) = f(r)\underline{\omega}$ and for $p = 3$, we have

$$\Delta_{\underline{x}} = \partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}$$

since $\Gamma(1-\Gamma)\underline{\omega} = -2\underline{\omega}$. Let us set the positions:

$$\gamma_k(r,\underline{\omega}) := \binom{2}{k}\frac{C_k}{(2k)!}\Delta_{\underline{x}}^{2-k}\Big(H^{(2k)}(r)\frac{\underline{x}}{r}\Big), \qquad k = 0,1,2,$$

where C_k is given by (12), then

$$(\Delta_{\underline{x}} + \Delta_{\underline{y}})^2 W(\underline{x},\underline{y})|_{\underline{y}=0} = \sum_{k=0}^{2}\gamma_k(r,\underline{\omega}),$$

where, more explicitly,

$$\begin{aligned}
\gamma_0(r,\underline{\omega}) =& \Big(\partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}\Big)^2 (H(r)\underline{\omega}),\\
\gamma_1(r,\underline{\omega}) =& -6\Big(\partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}\Big)(H^{(2)}(r)\underline{\omega}),\\
\gamma_2(r,\underline{\omega}) =& 5H^{(4)}(r)\underline{\omega}.
\end{aligned}$$

With some computations we finally get

$$\sum_{k=0}^{2}\gamma_k(r,\underline{\omega}) = \Big(-\frac{8}{r}\partial_r^3 + \frac{8}{r^2}\partial_r^2\Big)(H(r)\underline{\omega}) = -8\partial_r\Big(\frac{1}{r}\partial_r^2 H(r)\Big)\underline{\omega}.$$

3. **The biaxially Cauchy kernel and Fueter's primitive.** To prove some of our main results we recall Funk-Hecke's theorem. We denote by $P_m(t)$ the Legendre polynomials and by A_{n-1} the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, i.e.

$$A_{n-1} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}.$$

Note that, by the Rodriguez formula, $P_m(t)$ can be expressed by

$$P_m(t) = \Big(-\frac{1}{2}\Big)^m \frac{\Gamma((n-1)/2)}{\Gamma(m+(n-1)/2)}(1-t^2)^{(3-n)/2} D^m (1-t^2)^{m+(n-3)/2}. \qquad (14)$$

Theorem 3.1 (Funk-Hecke (see [13]))**.** *Denote by* $\mathbb{S}^{n-1}$ *the unit sphere in* $\mathbb{R}^n$ *and by* A_{n-1} *its area. Let* $\underline{\xi}$ *and* $\underline{\eta}$ *be two unit vectors in* $\mathbb{R}^n$. *Let* ψ *be a real-valued function whose domain contains* $[-1,\ 1]$ *and let* $\mathcal{S}_m(\xi)$ *be spherical harmonics, of degree* m. *Then we have*

$$\int_{\mathbb{S}^{n-1}} \psi(\langle\underline{\xi},\underline{\eta}\rangle)\mathcal{S}_m(\underline{\eta})\, dS(\underline{\eta}) = A_{n-1}\mathcal{S}_m(\underline{\xi})\int_{-1}^{1}\psi(t)P_m(t)(1-t^2)^{(n-3)/2}\, dt,$$

where $dS(\underline{\eta})$ *is the scalar element of surface area on* $\mathbb{S}^{n-1}$, $\langle\underline{\xi},\underline{\eta}\rangle$ *denotes the scalar product of* $\underline{\xi}$, $\underline{\eta}$ *and* $P_m(t)$ *is defined in (14).*

Definition 3.2 (The monogenic Cauchy kernel)**.** We denote by $\mathcal{G}$ the monogenic Cauchy kernel on $\mathbb{R}^m := \mathbb{R}^p \oplus \mathbb{R}^q$

$$\mathcal{G}(x) = -\frac{\underline{x}+\underline{y}}{|\underline{x}+\underline{y}|^{p+q}}, \qquad \underline{x}+\underline{y} \in \mathbb{R}^{p+q} \setminus \{0\}. \tag{15}$$

The area of the unit sphere in $\mathbb{R}^{p+q}$ will be denoted by

$$A_{p+q} = \frac{2\pi^{(p+q)/2}}{\Gamma(\frac{p+q}{2})}.$$

Definition 3.3 (The kernels $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})$)**.** Let $p,q \in \mathbb{N}$ and let $\mathcal{G}(\underline{x}+\underline{y}-\underline{X}-\underline{Y})$ be the monogenic Cauchy kernel defined in (15) with $\underline{x} \in \mathbb{R}^p$, $\underline{y} \in \mathbb{R}^q$, and assume $\underline{\omega} \in \mathbb{S}^{p-1}$, $\underline{\eta} \in \mathbb{S}^{q-1}$ and for $\lambda > 0$ and $\mu > 0$, we define the kernels

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y}) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\, dS(\underline{\xi})\, dS(\underline{\eta}), \tag{16}$$

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y}) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\, \underline{\xi}\ \underline{\eta}\ dS(\underline{\xi}) dS(\underline{\eta}), \tag{17}$$

where $dS(\underline{\xi})$ and $dS(\underline{\eta})$ are the scalar element of surface area of $\mathbb{S}^{p-1}$ and of $\mathbb{S}^{q-1}$, respectively.

Theorem 3.4 (The restrictions of the kernels $\mathcal{N}^+_{p,q}(x,\underline{y})$ and $\mathcal{N}^-_{p,q}(x,\underline{y})$ to $\underline{y}=0$)**.** *Let p, q be odd numbers. Let $\mathcal{N}^+_{p,q}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q}(\underline{x},\underline{y})$ be the kernels defined in (16) and (17), respectively. Then their restrictions to $\underline{y}=0$ are given by*

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0) = \frac{A_q A_{p-1}}{A_{p+q}} [J_{2,\lambda,\mu}(r;p,q) - r\lambda J_{1,\lambda,\mu}(r;p,q)] \frac{\underline{x}}{r}, \tag{18}$$

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},0) = \frac{A_q}{A_{p+q}} J_{2,\lambda,\mu}(r,p,q)\mu \frac{\underline{x}}{r}, \tag{19}$$

where the functions $J_{j,\lambda,\mu}(r;p,q)$, $j=1,2$ are given by

$$J_{j,\lambda,\mu}(r;p,q) := \int_{-1}^{1} \frac{t^{j-1}(1-t^2)^{(p-3)/2}}{(r^2-2r\lambda t+\lambda^2+\mu^2)^{(p+q)/2}}\, dt, \quad j=1,2. \tag{20}$$

Remark 3. Since p and q are odd numbers we have that $(p-3)/2$ and $(p+q)/2$ are integers and so the integrals (20) can be explicitly computed as integrals of rational functions.

Proof of Theorem 3.4. First we compute the restriction to $\underline{y}=0$ of $\mathcal{N}^+_{p,q}$. We have:

$$\begin{aligned}
\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y}) =& \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\, dS(\underline{\xi}) dS(\underline{\eta}) \\
=& \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{-\underline{x}-\underline{y}+\lambda\underline{\xi}+\mu\underline{\eta}}{(|\underline{x}-\lambda\underline{\xi}|^2+|\underline{y}-\mu\underline{\eta}|^2)^{(p+q)/2}}\, dS(\underline{\xi}) dS(\underline{\eta}).
\end{aligned}$$

Let us now restrict to $\underline{y}=\underline{0}$ and get

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{-\underline{x}+\lambda\underline{\xi}+\mu\underline{\eta}}{(|\underline{x}-\lambda\underline{\xi}|^2+\mu^2)^{(p+q)/2}}\, dS(\underline{\xi}) dS(\underline{\eta}).$$

We now set $\underline{x} = r\underline{\omega}$, where $r = |\underline{x}|$ and $\underline{\omega} \in \mathbb{S}^{p-1}$, and we have

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0) = -\frac{A_q}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \frac{\underline{x}-\lambda\underline{\xi}}{(r^2-2r\lambda\langle\underline{\omega},\underline{\xi}\rangle+\lambda^2+\mu^2)^{(p+q)/2}}\, dS(\underline{\xi})$$

$$= -\frac{A_q}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \frac{\underline{x}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi})$$

$$+ \frac{A_q}{A_{p+q}} \lambda \int_{\mathbb{S}^{p-1}} \frac{\underline{\xi}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}).$$

Using again Funk-Hecke's theorem, we obtain

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x}, 0) = -\frac{A_q A_{p-1}}{A_{p+q}} \underline{x} \int_{-1}^{1} \frac{(1-t^2)^{(p-3)/2}}{(r^2 - 2r\lambda t + \lambda^2 + \mu^2)^{(p+q)/2}} \, dt$$

$$+ \frac{A_q A_{p-1}}{A_{p+q}} \lambda I \int_{-1}^{1} \frac{(1-t^2)^{(p-3)/2} t}{(r^2 - 2r\lambda t + \lambda^2 + \mu^2)^{(p+q)/2}} \, dt.$$

We recall the integrals defined in (20)

$$J_{j,\lambda,\mu}(r; p, q) := \int_{-1}^{1} \frac{t^{j-1}(1-t^2)^{(p-3)/2}}{(r^2 - 2r\lambda t + \lambda^2 + \mu^2)^{(p+q)/2}} \, dt, \quad j = 1, 2$$

and we obtain

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x}, 0) = -\frac{A_q A_{p-1}}{A_{p+q}} J_{1,\lambda,\mu}(r; p, q) \underline{x} + \frac{A_q A_{p-1}}{A_{p+q}} J_{2,\lambda,\mu}(r; p, q) \lambda \frac{\underline{x}}{r}$$

which is formula (18).
With similar computations we treat $\mathcal{N}^-_{p,q}(\underline{x}, 0)$. In fact

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y}) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{-\underline{x} - \underline{y} + \lambda\underline{\xi} + \mu\underline{\eta}}{(|\underline{x} - \lambda\underline{\xi}|^2 + |\underline{y} - \mu\underline{\eta}|^2)^{(p+q)/2}} \, \underline{\xi} \, \underline{\eta} \, dS(\underline{\xi}) dS(\underline{\eta})$$

and taking the restriction to $\underline{y} = 0$, we obtain

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, 0) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{-\underline{x}\,\underline{\xi}\,\underline{\eta} - \lambda\underline{\eta} + \mu\underline{\eta}\,\underline{\xi}\,\underline{\eta}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}) \, dS(\underline{\eta})$$

and so we split in the three integrals

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, 0) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{-\underline{x}\,\underline{\xi}\,\underline{\eta}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}) \, dS(\underline{\eta})$$

$$- \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{\lambda\underline{\eta}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}) \, dS(\underline{\eta})$$

$$+ \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{\mu\underline{\eta}\,\underline{\xi}\,\underline{\eta}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}) \, dS(\underline{\eta}),$$

where the first and the second integral above are zero since

$$\int_{\mathbb{S}^{q-1}} \underline{\eta} \, dS(\underline{\eta}) = 0.$$

To compute the last integral we recall the identity

$$\underline{\xi}\underline{\eta} = -\underline{\eta}\underline{\xi},$$

from which

$$\underline{\eta}\,\underline{\xi}\,\underline{\eta} = \underline{\eta}(-\underline{\eta}\underline{\xi}) = \underline{\xi}$$

so we have

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, 0) = \frac{1}{A_{p+q}} \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \frac{\mu\underline{\xi}}{(r^2 - 2r\lambda\langle \underline{\omega}, \underline{\xi}\rangle + \lambda^2 + \mu^2)^{(p+q)/2}} \, dS(\underline{\xi}) \, dS(\underline{\eta})$$

by Funk-Hecke's theorem we get

$$\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x},0)=\frac{A_q}{A_{p+q}}\mu\underline{\omega}J_2(r)$$

from which we finally have (19).

We explicitly compute the integrals in (20) for $p=q=3$.

Corollary 1 (The restrictions of the kernels $\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ to $\underline{y}=0$)**.** *Let $\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ be the kernels defined in (16) and (17), respectively. Then their restrictions to $\underline{y}=0$ are given by*

$$\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},0)=\frac{A_3A_2}{A_6}\ \frac{2\lambda r(2-(r^2+\lambda^2+\mu^2))}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}\ \frac{\underline{x}}{r},\tag{21}$$

$$\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},0)=\frac{A_3}{A_6}\ \frac{4\lambda\mu r}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}\ \frac{\underline{x}}{r}.\tag{22}$$

Proof. Let us set $p=q=3$ in the integrals $J_{j,\lambda,\mu}$ in (20):

$$J_{j,\lambda,\mu}(r;3,3):=\int_{-1}^{1}\frac{t^{j-1}}{(r^2-2r\lambda t+\lambda^2+\mu^2)^3}\,dt,\quad j=1,2$$

so for $j=1$ we have

$$J_{1,\lambda,\mu}(r;3,3)=2\frac{r^2+\lambda^2+\mu^2}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}.$$

For $j=2$, with some computations, we obtain

$$J_{2,\lambda,\mu}(r;3,3)=\frac{r^2+\lambda^2+\mu^2}{2\lambda r}J_{1,\lambda,\mu}(r;3,3)-\frac{1}{2\lambda r}\ \frac{1}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}$$

from which we obtain

$$J_{2,\lambda,\mu}(r;3,3)=\frac{4\lambda r}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2},$$

and

$$J_{2,\lambda,\mu}(r;3,3)-r\lambda J_{1,\lambda,\mu}(r;3,3)=\frac{2\lambda r(2-(r^2+\lambda^2+\mu^2))}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}.$$

Using (18) and (19) we get the statement.

Definition 3.5. Let p and q be an odd numbers and let $\lambda>0$ and $\mu>0$. We say that $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ are Fueter's primitives of $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$, respectively, if they satisfy

$$\Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y}))=\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y}),\ \ \Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y}))=\mathcal{N}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y}).$$

Since a Fueter primitive $W\in\mathcal{H}_B(U)$ of a given biaxially monogenic function admits the power series expansion given in (10), the function W is known when its coefficients $H^{(\ell)}$, $\ell\in\mathbb{N}\cup\{0\}$ are determined.

Definition 3.6. Let p and q be an odd numbers and let $\lambda>0$ and $\mu>0$. Let $\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ be Fueter primitives of $\mathcal{N}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and

$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})$. We denote by $H^{(\ell)}_{p,q,\lambda,\mu,\pm}(r)$, for $\ell\in\mathbb{N}\cup\{0\}$, the coefficients that appear in the series expansions:

$$\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x},\underline{y})=\sum_{\ell=0}^{+\infty}\Big(\frac{1}{(2\ell)!}\underline{y}^{2\ell}H^{(2\ell)}_{p,q,\lambda,\mu,\pm}(r)\frac{\underline{x}}{r}-\frac{1}{(2\ell+1)!}\underline{y}^{2\ell+1}H^{(2\ell+1)}_{p,q,\lambda,\mu,\pm}(r)\Big).$$

Remark 4. In this paper we assume that functions $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}$ are defined on U, which is a open set invariant under the action of the group $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$. On this set U the functions $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}$ are represented by a convergent series expansion so we have to determine the coefficients $H^{(\ell)}_{p,q,\lambda,\mu,\pm}$.

Theorem 3.7 (The differential equations for coefficients of the restrictions of $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}$)**.** *Let p and q be an odd numbers, $d=(p+q)/2-1$, $\lambda>0$ and $\mu>0$ and let U be a domain in $\mathbb{R}^p\times\mathbb{R}^q$ invariant under the action of the group $\mathrm{Spin}(p)\times\mathrm{Spin}(q)$ and let $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}\in\mathcal{H}_B(U)$ be Fueter's primitives of $\mathcal{N}^{\pm}_{p,q,\lambda,\mu}$. Then the coefficients $H^{(\ell)}_{p,q,\lambda,\mu,\pm}(r)$, for $\ell\in\mathbb{N}\cup\{0\}$, of the series expansions of $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ satisfy the differential equations*

$$\sum_{k=0}^{d}\binom{d}{k}\frac{C_k}{(2k)!}\Delta^{d-k}_{\underline{x}}\Big(H^{(2k)}_{p,q,\lambda,\mu,+}(r)\frac{\underline{x}}{r}\Big)=\frac{A_qA_{p-1}}{A_{p+q}}[J_{2,\lambda,\mu}(r;p,q)-rJ_{1,\lambda,\mu}(r;p,q)]\frac{\underline{x}}{r},\tag{23}$$

$$\sum_{k=0}^{d}\binom{d}{k}\frac{C_k}{(2k)!}\Delta^{d-k}_{\underline{x}}\Big(H^{(2k)}_{p,q,\lambda,\mu,-}(r)\frac{\underline{x}}{r}\Big)=\frac{A_q}{A_{p+q}}J_{2,\lambda,\mu}(r;p,q)\frac{\underline{x}}{r}.\tag{24}$$

Proof. From Proposition 2, for $d=(p+q)/2-1$, the restrictions of

$$(\Delta_{\underline{x}}+\Delta_{\underline{y}})^d\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$$

to $\underline{y}=0$ can be written as

$$(\Delta_{\underline{x}}+\Delta_{\underline{y}})^d\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x},\underline{y})|_{\underline{y}=0}=\sum_{k=0}^{d}\binom{d}{k}\frac{C_k}{(2k)!}\Delta^{d-k}_{\underline{x}}\Big(H^{(2k)}_{p,q,\lambda,\mu,\pm}(r)\frac{\underline{x}}{r}\Big).$$

Let us observe that the kernels $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ defined in (16) and (17), respectively are monogenic, thus they satisfy an elliptic system so $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ are determined by their restrictions $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0)$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},0)$, respectively. From Theorem 3.4 we have that

$$\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x},0)=\frac{A_qA_{p-1}}{A_{p+q}}[J_{2,\lambda,\mu}(r;p,q)-rJ_{1,\lambda,\mu}(r;p,q)]\frac{\underline{x}}{r}\tag{25}$$

$$\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x},0)=\frac{A_q}{A_{p+q}}J_{2,\lambda,\mu}(r;p,q)\frac{\underline{x}}{r}\tag{26}$$

so by the definition of Fueter primitive we get the statement. □

Let us write explicitly the differential equations for the case $p=q=3$.

Corollary 2 (The differential equations for the coefficients of the restrictions of $\mathcal{W}^{\pm}_{3,3,\lambda,\mu}$)**.** *The coefficients $H^{(\ell)}_{3,3,\lambda,\mu,\pm}(r)$, for $\ell\in\mathbb{N}\cup\{0\}$, in the series expansions:*

$$\mathcal{W}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})=\sum_{\ell=0}^{+\infty}\Big(\frac{1}{(2\ell)!}\underline{y}^{2\ell}H^{(2\ell)}_{3,3,\lambda,\mu,\pm}(r)\frac{\underline{x}}{r}-\frac{1}{(2\ell+1)!}\underline{y}^{2\ell+1}H^{(2\ell+1)}_{3,3,\lambda,\mu,\pm}(r)\Big)$$

of Fueter's primitives of $\mathcal{N}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ *are given by the differential equations*

$$-8\partial_r\Big(\frac{1}{r}\partial_r^2 H_{3,3,\lambda,\mu,+}(r)\Big)=\frac{A_3A_2}{A_6}\ \frac{2\lambda r(2-(r^2+\lambda^2+\mu^2))}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2},$$

$$-8\partial_r\Big(\frac{1}{r}\partial_r^2 H_{3,3,\lambda,\mu,-}(r)\Big)=\frac{A_3}{A_6}\ \frac{4\lambda\mu r}{[(r^2+\lambda^2+\mu^2)^2-4\lambda^2r^2]^2}.$$

Proof. Recall that $\mathcal{W}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ are the Fueter primitives of $\mathcal{N}^{+}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{-}_{3,3,\lambda,\mu}(\underline{x},\underline{y})$, respectively. From Proposition 3 we have

$$(\Delta_{\underline{x}}+\Delta_{\underline{y}})^2\mathcal{W}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})|_{\underline{y}=0}=-8\partial_r\Big(\frac{1}{r}\partial_r^2 H_{p,q,\lambda,\mu,\pm}(r)\Big)\underline{\omega},$$

where $\underline{x}/r=\underline{\omega}$. So using the explicit formulas (21) and (22) for $\mathcal{N}^{\pm}\quad_{3,3,\lambda,\mu}(\underline{x},0)$ and setting

$$(\Delta_{\underline{x}}+\Delta_{\underline{y}})^2\mathcal{W}^{\pm}_{3,3,\lambda,\mu}(\underline{x},\underline{y})|_{\underline{y}=0}=\mathcal{N}^{\pm}_{3,3,\lambda,\mu}(\underline{x},0),$$

we get the statement. □

4. **The inverse Fueter mapping theorem for biaxially monogenic functions.** We now recall the Cauchy's integral formula for monogenic functions that with the results of the previous section is the main tool to prove our main result.

Theorem 4.1 (Cauchy's integral representation theorem for monogenic functions)**.** *Let* $\breve{f}$ *be a left monogenic function in* $U\subseteq\mathbb{R}^n$. *Then, for every* $M\subset U$ *and for* $x\in M$, *we have*

$$\breve{f}(x)=\frac{1}{A_n}\int_{\partial M}\mathcal{G}(y-x)d\sigma(y)\breve{f}(y),\tag{27}$$

where ∂M *is an* n*-dimensional compact smooth manifold in* U, *the differential form* $d\sigma(y)$ *is given by* $d\sigma(y)=\eta(y)dS(y)$ *where* $\eta(y)$ *is the outer unit normal to* ∂M *at point* y *and* $dS(y)$ *is the scalar element of surface area on* ∂M.

We are now in position to state and prove the main result of this paper.

Theorem 4.2 (The inverse Fueter mapping theorem for the odd part of a biaxially monogenic function)**.** *Let* $f(x)$ *be a biaxially monogenic function of the form*

$$f(\underline{x},\underline{y})=\underline{\omega}B(\rho,r)+\underline{\nu}C(\rho,r)$$

(where $\underline{\omega}=\underline{x}/r$, $r=|\underline{x}|$, $\underline{\nu}=\underline{y}/\rho$, $\rho=|\underline{y}|$*) defined on an axially symmetric domain* $U\subseteq\mathbb{R}^{p+q}$, *where* p *and* q *are odd numbers. Let* Γ *be the boundary of an open bounded subset* $\mathcal{V}$ *of the half plane*

$$\underline{\xi}\mathbb{R}^+ + \underline{\eta}\mathbb{R}^+$$

and let $V=\{\underline{\xi}u+\underline{\eta}v,\ (u,v)\in\mathcal{V},\ \underline{\xi}\in\mathbb{S}^{p-1},\ \ \underline{\eta}\in\mathbb{S}^{q-1}\}\subset U$. *Moreover suppose, that* Γ *is a regular curve whose parametric equations* $\lambda=\lambda(s)$, $\mu=\mu(s)$ *are expressed in terms of the arc-length* $s\in[0,L]$, $L>0$. *Then the function*

$$\begin{aligned}W(\underline{x},\underline{y}):=&\int_\Gamma\mathcal{W}^{+}_{p,q,\lambda,\mu}(\underline{x},\underline{y})\mu^{p-1}\lambda^{q-1}\left[C(\lambda,\mu)d\lambda-B(\lambda,\mu)d\mu\right]\\&+\int_\Gamma\mathcal{W}^{-}_{p,q,\lambda,\mu}(\underline{x},\underline{y})\,\mu^{p-1}\lambda^{q-1}\left[B(\lambda,\mu)d\lambda+C(\lambda,\mu)d\mu\right],\end{aligned}\tag{28}$$

is a Fueter's primitive of $f(\underline{x},\underline{y})$ *on* U, *where* $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}$ *are as in Theorem 2.*

Proof. We represent a biaxially monogenic functions f by the Cauchy formula using the following manifold

$$\Sigma := \{\underline{\xi}\lambda + \underline{\eta}\mu \mid \ (\mu, \lambda) \in \Gamma, \quad \underline{\xi} \in \mathbb{S}^{p-1}, \underline{\eta} \in \mathbb{S}^{q-1}\},$$

where

$$\Gamma \in \mathbb{R}^+ \times \mathbb{R}^+$$

is a smooth curve. We specify the notations

- ds is the infinitesimal arc-length and $dS(\underline{\xi})dS(\underline{\eta})$ is the infinitesimal element of surface area on $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$
- The the unite tangent vector is easily obtained by $\mathbf{t} = \frac{d}{ds}[\mu\underline{\eta} + \lambda\underline{\xi}] = \mu'(s)\underline{\eta} + \lambda'(s)\underline{\xi}$, so that the unit normal vector to to Γ is given by
$$\mathbf{n} = \lambda'(s)\underline{\eta} - \mu'(s)\underline{\xi}.$$
- The scalar infinitesimal element of the manifold Σ, expressed in terms of ds and $dS(\underline{\xi})dS(\underline{\eta})$ is given by
$$d\Sigma(s, \underline{\xi}, \underline{\eta}) = \mu^{p-1}\lambda^{q-1}\, ds\, dS(\underline{\xi})dS(\underline{\eta}).$$
Finally, the oriented infinitesimal element of manifold $d\sigma(s, \underline{\xi})$ is given by
$$d\sigma(s, \underline{\xi}, \underline{\eta}) = \mathbf{n}d\Sigma = [\lambda'(s)\underline{\eta} - \mu'(s)\underline{\xi}]\mu^{p-1}\lambda^{q-1}\, ds\, dS(\underline{\xi})dS(\underline{\eta})$$
so finally we get
$$d\sigma(s, \underline{\xi}, \underline{\eta}) = [\underline{\eta}d\lambda(s) - \underline{\xi}d\mu(s)]\mu^{p-1}\lambda^{q-1}\, dS(\underline{\xi})dS(\underline{\eta}).$$

Thanks to the above considerations we have:

$$f(\underline{x} + \underline{y}) = -\frac{1}{A_{p+q}} \int_\Gamma \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x} + \underline{y} - \lambda\underline{\xi} - \mu\underline{\eta})\, d\sigma(s, \underline{\xi}, \underline{\eta})\, f(\lambda\underline{\xi} + \mu\underline{\eta}),$$

so we get

$$\begin{aligned} f(\underline{x} + \underline{y}) = -\frac{1}{A_{p+q}} \int_\Gamma \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x} + \underline{y} - \lambda\underline{\xi} - \mu\underline{\eta})\, [\underline{\eta}d\lambda(s) - \underline{\xi}d\mu(s)] \\ \times[\underline{\xi}B(\lambda, \mu) + \underline{\eta}C(\lambda, \mu)]\mu^{p-1}\lambda^{q-1}\, dS(\underline{\xi})dS(\underline{\eta}) \end{aligned}$$

We can now split the integral in the following way

$$\begin{aligned} &f(\underline{x} + \underline{y}) = \\ &\frac{1}{A_{p+q}} \int_\Gamma \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x} + \underline{y} - \lambda\underline{\xi} - \mu\underline{\eta})\, dS(\underline{\xi})dS(\underline{\eta})\mu^{p-1}\lambda^{q-1}\, [C(\lambda, \mu)d\lambda - B(\lambda, \mu)d\mu] \\ &+ \frac{1}{A_{p+q}} \int_\Gamma \int_{\mathbb{S}^{p-1}} \int_{\mathbb{S}^{q-1}} \mathcal{G}(\underline{x} + \underline{y} - \lambda\underline{\xi} - \mu\underline{\eta})\, \underline{\xi}\, \underline{\eta}\, dS(\underline{\xi})\, dS(\underline{\eta})\, [B(\lambda, \mu)d\lambda + C(\lambda, \mu)d\mu] \end{aligned}$$

Taking into account the kernels $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ in Definition 3.3 we have

$$\begin{aligned} f(\underline{x} + \underline{y}) = &\int_\Gamma \mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y})\mu^{p-1}\lambda^{q-1}\, [C(\lambda, \mu)d\lambda - B(\lambda, \mu)d\mu] \\ &+ \int_\Gamma \mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y})\, [B(\lambda, \mu)d\lambda + C(\lambda, \mu)d\mu]. \end{aligned}$$

Now we recall the functions $\mathcal{W}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{W}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ that are the Fueter primitives of $\mathcal{N}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$ and $\mathcal{N}^-_{p,q,\lambda,\mu}(\underline{x}, \underline{y})$, respectively, introduced in Definition 3.5 and we get

$$f(\underline{x} + \underline{y}) = \int_\Gamma \Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^+_{p,q,\lambda,\mu}(\underline{x}, \underline{y}))\mu^{p-1}\lambda^{q-1}\, [C(\lambda, \mu)d\lambda - B(\lambda, \mu)d\mu]$$

$$+\int_\Gamma \Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y}))\,[B(\lambda,\mu)d\lambda+C(\lambda,\mu)d\mu],$$

that we write as

$$f(\underline{x}+\underline{y})=\Delta^{\frac{(p+q)}{2}-1}\int_\Gamma \mathcal{W}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})\mu^{p-1}\lambda^{q-1}\,[C(\lambda,\mu)d\lambda-B(\lambda,\mu)d\mu]$$

$$+\Delta^{\frac{(p+q)}{2}-1}\int_\Gamma \mathcal{W}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})\,[B(\lambda,\mu)d\lambda+C(\lambda,\mu)d\mu].$$

So the Fueter primitive of f is given by

$$W(\underline{x},\underline{y}):=\int_\Gamma \mathcal{W}^+_{p,q,\lambda,\mu}(\underline{x},\underline{y})\mu^{p-1}\lambda^{q-1}\,[C(\lambda,\mu)d\lambda-B(\lambda,\mu)d\mu]$$

$$+\int_\Gamma \mathcal{W}^-_{p,q,\lambda,\mu}(\underline{x},\underline{y})\,\mu^{p-1}\lambda^{q-1}\,[B(\lambda,\mu)d\lambda+C(\lambda,\mu)d\mu].$$

We conclude this section by pointing out what is the form of a Fueter's primitives for even biaxially monogenic functions. Since that computations are very similar the the case of Fueter's primitives for odd type we omit the proof of the main theorem.

4.1. **Fueter's primitives for even biaxially monogenic functions.** When consider biaxially monogenic functions of the form:

$$f(\underline{x},\underline{y})=A(r,\rho)+\underline{\omega}\,\underline{\nu}D(r,\rho)$$

where $\underline{\omega}=\underline{x}/r$, $r=|\underline{x}|$, $\underline{\nu}=\underline{y}/\rho$, $\rho=|\underline{y}|$, we have to replace the kernels $\mathcal{N}^+_{p,q}(\underline{x},\underline{y})$ and $\mathcal{N}^-_{p,q}(\underline{x},\underline{y})$ in Definition 3.3 by the kernels described in next definition.

Definition 4.3 (The kernels $\mathcal{N}^{\oplus}_{p,q}(\underline{x},\underline{y})$ and $\mathcal{N}^{\ominus}_{p,q}(\underline{x},\underline{y})$)**.** Let p, $q\in\mathbb{N}$ and let $\mathcal{G}(\underline{x}+\underline{y}-\underline{X}-\underline{Y})$ be the monogenic Cauchy kernel defined in (15) with $\underline{x}\in\mathbb{R}^p$, $\underline{y}\in\mathbb{R}^q$, and assume $\underline{\omega}\in\mathbb{S}^{p-1}$, $\underline{\eta}\in\mathbb{S}^{q-1}$ and for $\lambda>0$ and $\mu>0$, we define the kernels

$$\mathcal{N}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})=\frac{1}{A_{p+q}}\int_{\mathbb{S}^{p-1}}\int_{\mathbb{S}^{q-1}}\mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\,\underline{\xi}\;dS(\underline{\xi})\,dS(\underline{\eta}),\tag{29}$$

$$\mathcal{N}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})=\frac{1}{A_{p+q}}\int_{\mathbb{S}^{p-1}}\int_{\mathbb{S}^{q-1}}\mathcal{G}(\underline{x}+\underline{y}-\lambda\underline{\xi}-\mu\underline{\eta})\,\underline{\eta}\;dS(\underline{\xi})dS(\underline{\eta}),\tag{30}$$

where $dS(\underline{\xi})$ and $dS(\underline{\eta})$ are the scalar element of surface area of $\mathbb{S}^{p-1}$ and of $\mathbb{S}^{q-1}$, respectively.

Definition 4.4. Let p and q be an odd numbers and let $\lambda>0$ and $\mu>0$. We say that $\mathcal{W}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ are Fueter's primitives of $\mathcal{N}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{N}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$, respectively, if they satisfy

$$\Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y}))=\mathcal{N}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y}),\;\;\Delta^{\frac{(p+q)}{2}-1}(\mathcal{W}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y}))=\mathcal{N}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y}).$$

Computations similar to those done in the proof of Theorem 3.4 for $\mathcal{W}^{\pm}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ can be repeated to obtain the representation formula for the Fueter's primitives $\mathcal{W}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$ and $\mathcal{W}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})$. Thus we can state the inverse Fueter mapping for the even case.

Theorem 4.5 (The inverse Fueter mapping theorem for the even part of a biaxially monogenic function)**.** *Let $f(x)$ be a biaxially monogenic function of the form*

$$f(\underline{x},\underline{y}) = A(\rho,r) + \underline{\omega}\underline{\nu}D(\rho,r)$$

(where $\underline{\omega} = \underline{x}/r$, $r = |\underline{x}|$, $\underline{\nu} = \underline{y}/\rho$, $\rho = |\underline{y}|$) defined on an axially symmetric domain $U \subseteq \mathbb{R}^{p+q}$, where p and q are odd numbers. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half plane

$$\underline{\xi}\mathbb{R}^+ + \underline{\eta}\mathbb{R}^+$$

and let $V = \{\underline{\xi}u + \underline{\eta}v,\ (u,v) \in \mathcal{V},\ \underline{\xi} \in \mathbb{S}^{p-1},\ \ \underline{\eta} \in \mathbb{S}^{q-1}\} \subset U$. Moreover suppose, that Γ is a regular curve whose parametric equations $\lambda = \lambda(s)$, $\mu = \mu(s)$ are expressed in terms of the arc-length $s \in [0,L]$, $L > 0$. Then the function

$$W(\underline{x},\underline{y}) := \int_\Gamma \mathcal{W}^{\oplus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})\mu^{p-1}\lambda^{q-1}\left[D(\lambda,\mu)d\lambda - A(\lambda,\mu)d\mu\right] \tag{31}$$

$$+ \int_\Gamma \mathcal{W}^{\ominus}_{p,q,\lambda,\mu}(\underline{x},\underline{y})\,\mu^{p-1}\lambda^{q-1}\left[A(\lambda,\mu)d\lambda + D(\lambda,\mu)d\mu\right],$$

is a Fueter's primitive of $f(\underline{x},\underline{y})$ on U, where $\mathcal{W}^{\oplus}_{p,q,\lambda,\mu}$ and $\mathcal{W}^{\ominus}_{p,q,\lambda,\mu}$ are determined with analogous computations as in Theorem 2.

Remark 5. Theorems 4.2 and 4.5 allows to determine the structure of a Fueter's primitive for a general biaxially monogenic function.

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