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ON THE REGULARITY OF VECTOR FIELDS UNDERLYING A DEGENERATE-ELLIPTIC PDE

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ABSTRACT. In this note we discuss the relationship, under an algebraic constant rank condition, between the regularity of the characteristic form's coefficients of a degenerate elliptic linear PDO in $\mathbb{R}^N$ and the regularity of vector fields controlling its degeneracy. We consider both the cases where the number of vector fields is N and it is equal to the rank.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^N$, and let $\mathcal{L}$ be a second-order linear partial differential operator of the form

$$(1)\qquad \mathcal{L} = \sum_{i,j=1}^{N} a_{ij}(x)\,\partial_{x_i}\partial_{x_j} + \sum_{k=1}^{N} b_k(x)\,\partial_{x_k}, \qquad \text{for } x \in \Omega.$$

Let us just assume (for now) that, for all x, the matrix $A(x) = \big(a_{ij}(x)\big)_{i,j=1}^{N}$ is symmetric and nonnegative definite; that is, $\mathcal{L}$ is degenerate-elliptic. It is well-known (see e.g. [6,22,38]) that the existence of vector fields with suitable properties allowing us to write $\mathcal{L}$ as a sum of squares (possibly up to first order terms) can be crucial for studying qualitative and quantitative properties for the solutions or subsolutions to $\mathcal{L}u = 0$: the properties of the metric space related to such vector fields have been widely investigated and successfully exploited. Moreover, motivated by the studies on certain nonlinear degenerate-elliptic equations of sub-Riemannian type and on linear subelliptic equations with nonsmooth coefficients (see [1,11,12,14,42,45] and the monographs [7,8,43], with the references therein), there have been recent investigations concerning the minimal regularity assumptions for having vector fields with some Hörmander-type properties [9,10,24,31–36,40]. For these reasons we think it is worthwhile to focus on conditions under which we can guarantee the existence in Ω of vector fields with the desired regularity just looking at the quadratic form $A(x)$.

Before stating our results, let us be more precise about the PDE setting we are dealing with. In [39] Phillips and Sarason showed that if $x \mapsto A(x)$ is a C^2 map, then its (symmetric, nonnegative definite) square root $S(x)$ has locally Lipschitz-continuous entries. Then we can identify the N columns of S with N locally

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Lipschitz-continuous vector fields $X_1, \dots, X_N$. Up to lower order terms we can write $\mathcal{L}$ as the sum of their squares. The fact they are locally Lipschitz allows us to consider their integral curves and to look at the Carnot-Carathéodory distance d_{CC}. Essential tools in order to trigger some PDE methods are the X-connectivity (which makes d_{CC} an actual distance), the doubling condition for the metric balls, and Poincaré inequalities. These properties provide Sobolev embeddings (see [17, 18, 20]) and the possibility to develop the De Giorgi-Moser technique for the Hölder regularity of the solutions. In this perspective there is the study of operators, even with possibly very rough coefficients, as the X-elliptic operators in the sense of [19, 27], where the existence of Lipschitz vector fields with doubling and Poincaré properties controlling the degeneracy are required axiomatically [19, 26, 27, 44, 46] (see the subsection below; see also [25, 27, 28] for a more general condition about an X-controllable almost exponential map). On the other hand, one can ask for which families of vector fields the connectivity/doubling/Poincaré properties are satisfied. It is very well-known that such properties hold true for smooth vector fields satisfying the Hörmander condition thanks to the milestones [23, 37]. In the case of nonsmooth vector fields, it is known for vector fields in specific diagonal forms since the pioneering works [15, 16]; see also the results and the references in [43]. For nonsmooth vector fields in general form there has been a renewed interest, as we mentioned, in step-r Hörmander-type conditions with minimal regularity assumptions (such as $C^{r-1,1}$ or peculiar intrinsic regularities) in order to have doubling conditions [10, 33, 36] and Poincaré inequalities [4, 10, 13, 28, 30, 33, 36].

Let us denote, for any $x \in \Omega$, by $X_1(x), \dots, X_N(x)$ the vector fields identified with the columns of the square root $S(x)$ of $A(x)$. In general it is not possible to improve the Phillips-Sarason result either for the Lipschitz outcome or for the C^2-assumption, as we can trivially show respectively with the examples

$$A(x) = \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \qquad \text{or} \qquad A(x) = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{2-\epsilon} \end{pmatrix}.$$

Nevertheless, in the first section of his celebrated paper [6] Bony quickly stated that it is possible to have smooth vector fields as soon as we have a smooth $A(x)$ with constant rank. This is our starting point. Thus, throughout the paper we assume that

$$\begin{gathered} \text{there exists } m \in \{1, \dots, N\} \text{ such that} \\ \mathrm{rk}\,(A(x)) \equiv m \qquad \text{for all } x \in \Omega. \end{gathered} \tag{2}$$

Under this constant rank condition we can prove the following:

Theorem 1.1. *The vector fields $X_1, \dots, X_N$ have the same regularity as A; that is, if A satisfies* (2) *and $A \in C^{k,\alpha}(\Omega)$ for some $\alpha \in [0,1]$ and $k \in \mathbb{N} \cup \{0, \infty, \omega\}$, then every $X_j \in C^{k,\alpha}(\Omega)$.*

In our hypotheses, the N vectors $X_1(x), \dots, X_N(x)$ span at any x an m-dimensional subspace of $\mathbb{R}^N$, and they are thus linearly dependent (in the meaningful case $m < N$). In the applications it might be useful to have m linearly independent vector fields and to have information about their regularity. We can consider, at any fixed x, the decomposition

$$A(x) = R(x)R^t(x),$$

for some $N \times m$ matrix $R(x)$ with maximal rank. We may denote by $\tilde{X}_1(x), \ldots, \tilde{X}_m(x)$ the linearly independent vector fields identified with the m columns of $R(x)$. The decomposition (and hence the possible choice for the vector fields at any x) is, of course, not unique. We can prove the following:

Theorem 1.2. *Let Ω be a contractible open subset of $\mathbb{R}^N$. Suppose A satisfies* (2) *and $A \in C^k(\Omega)$ for some $k \in \mathbb{N} \cup \{0, \infty\}$. Then, there exist m linearly independent vector fields $\tilde{X}_1, \ldots, \tilde{X}_m \in C^k(\Omega)$ such that $A(x) = R(x)R^t(x)$ for all $x \in \Omega$.*

Therefore, in the case $k \geq 1$, we can write $\mathcal{L}$ as the sum of $\tilde{X}_j^2$ up to adjusting the first order terms. Let us explicitly remark that the reason for the C^k-regularity and not for the $C^{k,\alpha}$-regularity in Theorem 1.2 is technical: in our arguments we exploit a differential topology tool for which we have found a clear reference just in the C^k-smoothness category; as long as it holds true also with Lipschitz/Hölder/C^ω regularity, Theorem 1.2 applies even in those situations.

The plan of the paper is as follows. In Section 2 we prove Theorem 1.1, which will readily follow from a simple differential geometry lemma. In Section 3 we finally prove Theorem 1.2. Before doing this, let us add here some further comments on our results.

1.1. **Comments and consequences.** In this subsection we denote, with abuse of notation, by $X_1, \ldots, X_r$ the vector fields whose existence and regularity in Ω is ensured by Theorem 1.1 ($r = N$) or by Theorem 1.2 ($r = m$, with Ω contractible) under the constant rank assumption (2). Let us write them as $X_j = \sum_{i=1}^N \sigma_{ij}(x)\partial_{x_i}$ (the $N \times r$ matrix $\sigma(x)$ is $S(x)$ or $R(x)$, respectively).

We would like at first to give the details of the statement we already mentioned by Bony in [6, p. 279], since it was the initial motivation for the present study. Suppose we have a second order operator $\mathcal{L}$ as in (1), where the nonnegative matrix $A(x)$ and the first order terms $(b_h(x))_{h=1}^N$ are C^∞-smooth functions. Then our vector fields $X_1, \ldots, X_r$ are C^∞-smooth. We have

$$\mathcal{L} = \sum_{j=1}^r X_j^2 + X_0 \quad \text{in } \Omega, \tag{3}$$

where the C^∞-smooth vector field $X_0 = \sum_{h=1}^N \Big(b_h(x) - \sum_{j=1}^r X_j(\sigma_{hj})(x)\Big)\partial_{x_h}$. Thus, under our assumptions, it is completely equivalent to test the Hörmander condition on $X_1, \ldots, X_r$ or on the vector fields $A_1, \ldots, A_N$ identified with the columns of A. In fact, at any $x \in \Omega$, we have

$$\operatorname{Lie}\{A_1, \ldots, A_N\}(x) = \operatorname{Lie}\{X_1, \ldots, X_r\}(x) \qquad \text{and} \tag{4}$$

$$\operatorname{Lie}\{A_1, \ldots, A_N, B\}(x) = \operatorname{Lie}\{X_1, \ldots, X_r, X_0\}(x), \tag{5}$$

where $B(x) = \sum_{h=1}^N \big(b_h(x) - \sum_{j=1}^N \partial_{x_j}(a_{jh})(x)\big)\partial_{x_h}$. As is well-known, the fact that such vector spaces coincide for every x with the whole $\mathbb{R}^N$ is a sufficient condition for the hypoellipticity of $\mathcal{L}$ in Ω. In case of C^ω-coefficients, the condition coming from (5) is even necessary for the hypoellipticity [38] (see also [3, 5] for discussions and applications concerning connectivity properties for C^∞-hypoellipticity).

Since we have been careful with the regularity properties, we can be more precise. Suppose the nonnegative matrix A to be C^k, and the first order terms $(b_h)_{h=1}^N \in C^{k-1}$, for some $\mathbb{N} \ni k \geq 1$. Then we have (3) with $X_1, \ldots, X_r \in C^k$, and $X_0 \in C^{k-1}$. Moreover, if we define for $s \in \mathbb{N}$ the vector space $\mathrm{Lie}_s\{Y_j\} = \mathrm{span}\{Y_j, [Y_{j_1}, Y_{j_2}], \ldots, [Y_{j_1}, [Y_{j_2}, [\ldots, [Y_{j_{s-1}}, Y_{j_s}]], \ldots]]\}$, for any $x \in \Omega$ we have

$$\mathrm{Lie}_s\{A_1, \ldots, A_N\}(x) = \mathrm{Lie}_s\{X_1, \ldots, X_r\}(x) \quad \text{for all } s \leq k+1 \quad \text{and} \tag{6}$$

$$\mathrm{Lie}_s\{A_1, \ldots, A_N, B\}(x) = \mathrm{Lie}_s\{X_1, \ldots, X_r, X_0\}(x) \quad \text{for all } s \leq k. \tag{7}$$

Remark 1.3. For the reader's convenience we provide here an outline of the proof of (6) and (7). This will give a fortiori a proof of (4) and (5).

In the notation we are using, for any $j \in \{1, \ldots, N\}$ and $l \in \{1, \ldots, r\}$, we have $A_j(x) = \sum_{i=1}^N a_{ij}(x)\partial_{x_i}$, $X_l(x) = \sum_{i=1}^N \sigma_{il}(x)\partial_{x_i}$. We recall that the existence of the C^k-matrix valued function $\sigma(\cdot)$ globally defined in Ω with rank m comes from our results. Since we have $A(x) = \sigma(x)\sigma^t(x)$, we can write $A_j(x) = \sum_{l=1}^r \sigma_{jl}(x)X_l(x)$ for every j. This yields in particular (6) with $s=1$, which reads as

$$\mathrm{span}\{A_1(x), \ldots, A_N(x)\} = \mathrm{span}\{X_1(x), \ldots, X_r(x)\} \quad \text{for all } x,$$

since both vector spaces are m-dimensional. Let us now prove (6) with $s=2$, which means we have to prove for all $x \in \Omega$ that

$$\mathrm{span}\{A_j(x), [A_j, A_h](x), j, h=1, \ldots, N\} = \mathrm{span}\{X_l(x), [X_l, X_q](x), l, q=1, \ldots, r\}.$$

The inclusion $\subseteq$ is trivial since

$$\begin{aligned} [A_j, A_h](x) &= \sum_{l,q=1}^r \sigma_{jl}(x)\sigma_{hq}(x)[X_l, X_q](x) + \sum_{l,q=1}^r \sigma_{jl}(x)X_l(\sigma_{hq})(x)X_q(x) \\ &- \sum_{l,q=1}^r \sigma_{hq}(x)X_q(\sigma_{jl})(x)X_l(x) \qquad \text{for all } x \in \Omega. \end{aligned}$$

To deal with the opposite inclusion, we can fix any $x_0 \in \Omega$ and find an open neighborhood of x_0 where $\mathrm{span}\{A_1(x), \ldots, A_N(x)\} = \mathrm{span}\{A_{j_1}(x), \ldots, A_{j_m}(x)\}$ and $\mathrm{span}\{X_1(x), \ldots, X_r(x)\} = \mathrm{span}\{X_{l_1}(x), \ldots, X_{l_m}(x)\}$ (recall that r can be either m or N). Thus, in such a neighborhood, we can write $A_{j_p}(x) = \sum_{i=1}^m \hat{\sigma}_{pi}(x)X_{l_i}(x)$ for all $p \in \{1, \ldots, m\}$ for some $\hat{\sigma}(\cdot) \in C^k$. By construction $\hat{\sigma}(x)$ is invertible and the inverse has to be C^k. Hence, we have an open neighborhood $U_{x_0} \subset \Omega$ of x_0 such that $X_{l_i}(x) = \sum_{p=1}^m c_{ip}(x)A_{j_p}(x)$ for all $i \in \{1, \ldots, m\}$ and $x \in U_{x_0}$, where $c(\cdot) \in C^k(U_{x_0})$. This is enough to prove that $\mathrm{Lie}_2\{X_l\}(x) = \mathrm{Lie}_2\{X_{l_i}\}(x) \subseteq \mathrm{Lie}_2\{A_{j_p}\}(x) \subseteq \mathrm{Lie}_2\{A_j\}(x)$ for all $x \in U_{x_0}$. The arbitrariness of x_0 completes the proof of the $(s=2)$-case in (6). We can proceed in the same way for higher s, until the coefficients are differentiable (that is, $s \leq k+1$). The proof of (7) is completely analogous since $X_0(x) = B(x) + \sum_{j=1}^N \sum_{l=1}^r \partial_{x_j}(\sigma_{jl})(x)X_l(x)$ for all $x \in \Omega$.

The identifications in (6) and (7) might be useful (in the operative sense) in order to check quantitative properties of the distances related to the vector fields under the assumptions of Hörmander condition or involutive properties. The quantitative properties we are referring to are related to the analysis of the exponential map and almost-exponential maps carried out in [10, 33–35, 37].

Finally we would like to mention that, under our constant rank assumption, every operator $\mathcal{L}$ in (1) can be seen as an X-elliptic operator as long as we can control the first order term given by X_0. Let us clarify this point while recalling the definition in [19]. Suppose A to be C^1 and the first order terms b_k's to be measurable. We can write, besides (3), the operator as $\mathcal{L} = \operatorname{div}(A\nabla) + B$. By the definition of our vector fields, we get

$$\langle A(x)\xi, \xi\rangle = \left\|\sigma^t(x)\xi\right\|^2 = \sum_{j=1}^r \langle X_j(x), \xi\rangle^2 \quad \forall\, \xi \in \mathbb{R}^N \text{ and } \forall\, x \in \Omega,$$

which gives at one time the conditions (1.2) and (1.5) in [19]. If we want $\mathcal{L}$ to be X-elliptic in Ω, we are thus left to satisfy the condition (1.4) in [19]; i.e. there has to exist a measurable function γ such that $\langle B(x), \xi\rangle^2 \le \gamma^2(x) \sum_{j=1}^r \langle X_j(x), \xi\rangle^2$. Comparing $B(x)$ and $X_0(x)$ we have

$$\langle B(x) - X_0(x), \xi\rangle^2 \le rN^2 \max_{jl} \left|\partial_{x_j}(\sigma_{jl})(x)\right|^2 \sum_{j=1}^r \langle X_j(x), \xi\rangle^2 .$$

Hence, according to the definition in [19], $\mathcal{L}$ is uniformly X-elliptic in Ω if there exists a function γ_0 such that

$$\langle X_0(x), \xi\rangle^2 \le \gamma_0^2(x) \sum_{j=1}^r \langle X_j(x), \xi\rangle^2 \quad \forall\, \xi \in \mathbb{R}^N \text{ and } \forall\, x \in \Omega.$$

The results proved in [19, 46] for such operators require the metric, doubling, and Poincaré properties we mentioned in the Introduction, as well as some integrability conditions for the function γ. In the case of the vector fields considered in Theorem 1.1, we can assume A to be locally Lipschitz in Ω and the conditions we have just seen regarding the X-ellipticity are meant to be satisfied almost everywhere.

2. The squared case

Theorem 1.1 concerns the regularity of the square root $S(x)$ of $A(x)$. The square root of a nonnegative definite matrix is easily defined via diagonalization. One could think, in a naive way, to look separately at the regularity of the eigenvalues and of the eigenvectors as functions of x. Unfortunately, there are well-known examples of smooth matrices for which it is not possible to find an even continuous choice of eigenvectors (see e.g. [41]). In this direction, let us give here the example (often used in the PDE-setting we deal with) of the Heisenberg Laplacian in $\mathbb{R}^3$ for which

$$A(x_1, x_2, x_3) = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \\ x_2 & -x_1 & x_1^2 + x_2^2 \end{pmatrix}. \tag{8}$$

We note that, for such A, the condition (2) is satisfied with $m = 2$: in fact the eigenvalues of $A(x_1, x_2, x_3)$ are $1+x_1^2+x_2^2, 1, 0$. For $x_1^2+x_2^2 \neq 0$, the unit eigenvectors (up to a sign choice) are respectively

$$v_{1+r}(x) = \frac{(x_2, -x_1, x_1^2 + x_2^2)}{\sqrt{x_1^2 + x_2^2}\sqrt{1 + x_1^2 + x_2^2}},\ v_1(x) = \frac{(x_1, x_2, 0)}{\sqrt{x_1^2 + x_2^2}},\ \text{and } v_0(x) = \frac{(-x_2, x_1, 1)}{\sqrt{1 + x_1^2 + x_2^2}}.$$

Thus, for any choice of an orthonormal basis of eigenvectors at a point $(0, 0, x_3)$, v_{1+r} and v_1 cannot be continuous at that point. Nonetheless, denoting by Λ and V the eigenvalue and eigenvector matrices, one can check that the square root $S(x) = V(x)\sqrt{\Lambda(x)}V^t(x)$ is real-analytic in $\mathbb{R}^3$ (see (10)), as it is also a consequence of our result. In order to give a proof of Theorem 1.1, let us fix some notation.

Let us fix the natural numbers $N \geq m \geq 1$. Denoting by $\mathrm{Sym_N}(\mathbb{R})$ the real vector space of all $N \times N$ symmetric matrices with real coefficients, let us put

$$\mathrm{S}_N^+(m) = \{A \in \mathrm{Sym_N}(\mathbb{R}) : A \geq 0 \text{ and } \mathrm{rk}(A) = m\}.$$

Let us consider the smooth (real-analytic) map

$$q : \mathrm{S}_N^+(m) \to \mathrm{S}_N^+(m), \quad q(M) = M^2.$$

The square root function is, of course, the global inverse of q. In our notation, for any $x \in \Omega$, the vector fields $X_1(x), \ldots, X_N(x)$ are identified with the columns of $q^{-1}(A(x))$. The key assumption (2) ensures the well-position of this composition, for all $x \in \Omega$. The proof of Theorem 1.1 will thus be a straight consequence of the following:

Lemma 2.1. *The function q is a C^ω-diffeomorphism.*

Let us first recall that $\mathrm{S}_N^+(m)$ is an embedded smooth submanifold of $\mathrm{Sym_N}(\mathbb{R}) \equiv \mathbb{R}^{N(N+1)/2}$, with dimension $d_N(m) = Nm - \frac{m(m-1)}{2}$. As a matter of fact, the C^ω-differentiable structure of $\mathrm{S}_N^+(m)$ locally around a matrix M_0 (with a nonvanishing $m \times m$ minor) is basically given by being the zero set of the minors of order $(m+1)$ (in the case $m < N$; if $m = N$, it is just an open set of $\mathrm{Sym_N}(\mathbb{R})$).

Proof of Lemma 2.1. By implicit function theorem, it is enough to show that the differential of q at any point $M \in \mathrm{S}_N^+(m)$ is an isomorphism between $T_M(\mathrm{S}_N^+(m))$ and $T_{M^2}(\mathrm{S}_N^+(m))$. Without loss of generality we can consider M in the diagonal form $\mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$, for positive $\lambda_1, \ldots, \lambda_m$. For such diagonal matrices it is easy to describe the tangent space. The map

$$\Psi : \mathrm{Sym_N}(\mathbb{R}) \to \mathbb{R}^{\frac{N(N+1)}{2} - d_N(m)}, \quad \Psi(A) = \left(\det \left(a_{ij}\right)_{\substack{i \in \{1,\ldots,m\} \cup \{p\} \\ j \in \{1,\ldots,m\} \cup \{q\}}} \right)_{\substack{p,q = m+1,\ldots,N \\ p \leq q}}$$

can be used as a local defining function for $\mathrm{S}_N^+(m)$ around M. Hence, one has

$$T_M(\mathrm{S}_N^+(m)) = \ker(d_M\Psi) = \{H \in \mathrm{Sym_N}(\mathbb{R}) : h_{ij} = 0 \text{ for all } i, j = m+1, \ldots, N\}. \tag{9}$$

For all $H \in T_M(\mathrm{S}_N^+(m))$, we have

$$d_M q(H) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} (M + tH)^2 = MH + HM.$$

In order to show that $d_M q$ is an isomorphism, we may just prove the injectivity. If $d_M q(H) = 0$ for some H, then for every $j = 1, \ldots, N$ we have

$$0 = d_M q(H)\, e_j = M(He_j) + H(Me_j).$$

When $1 \leq j \leq m$, from the above identity we infer that He_j is an eigenvector for M with corresponding eigenvalue $(-\lambda_j)$: since M is nonnegative definite, this implies that $He_j = 0$ for all $j = 1, \ldots, m$. On the other hand, if $j \in \{m+1, \ldots, N\}$, we get that He_j is in the kernel of M which is spanned by $e_{m+1}, \ldots, e_N$. Since $H \in T_M(\mathrm{S}_N^+(m))$ and recalling (9), we immediately deduce that $He_j = 0$ even for all $j = m+1, \ldots, N$. This proves that $H = 0$ and the lemma. ☐

After having proved Theorem 1.1, let us give here a slight generalization of the last lemma. As a matter of fact, all the pth-powers of a matrix give a C^ω-diffeomorphism if restricted to $\mathrm{S}_N^+(m)$. Even for the sake of more generalization, we could consider a real-analytic function $f : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

$f(0) = 0$, $f \not\equiv 0$, and the derivatives of f of every order are nonnegative.

Under these assumptions, Pringsheim's theorem ensures that the Maclaurin series of f converges to f itself for all $x \in \mathbb{R}$, that is,

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n, \quad \text{for all } x \in \mathbb{R}.$$

Moreover, we have that f is (strictly) monotonically increasing on $I := [0, +\infty[$, and it restricts to a bijection from I to itself. Let us now define the real-analytic map

$$q_f : \mathrm{S}_N^+(m) \to \mathrm{S}_N^+(m), \quad q_f(M) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\, M^n.$$

If $M \in \mathrm{S}_N^+(m)$ and P is an orthogonal $N \times N$ matrix such that $P^t M P$ is the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_m, 0, \dots, 0)$, one can write $q_f(A)$ as

$$q_f(M) = P q_f(\Lambda) P^t = P \mathrm{diag}(f(\lambda_1), \dots, f(\lambda_m), 0, \dots, 0) P^t.$$

It is then easy to recognize that q_f is globally invertible (for all $m = 1, \dots, N$).

Lemma 2.2. *The map q_f is a real-analytic diffeomorphism.*

Proof. As in the proof of Lemma 2.1, we have to show that the differential of q_f at any point $M \in \mathrm{S}_N^+(m)$ is an isomorphism between $T_M(\mathrm{S}_N^+(m))$ and $T_{q_f(M)}(\mathrm{S}_N^+(m))$: without loss of generality we can assume that $M = \mathrm{diag}(\lambda_1, \dots, \lambda_m, 0, \dots, 0)$, for positive $\lambda_1, \dots, \lambda_m$. For $H \in T_M(\mathrm{S}_N^+(m))$ we have

$$\begin{aligned} d_M(q_f)(H) &= \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\, (M + tH)^n \\ &= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\, (M^{n-1} H + M^{n-2} H M + \cdots + H M^{n-1}), \end{aligned}$$

by the uniform convergence of the power series. Let us prove the injectivity of $d_M(q_f)$. To this aim, let $d_M(q_f)(H) = 0$ for some $H \in T_M(\mathrm{S}_N^+(m))$. For $j \in \{1, \dots, m\}$, we have

$$0 = d_M(q_f)(H) e_j = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} \left[\left(\sum_{k=0}^{n-2} \lambda_j^k M^{n-1-k} \right) \cdot (H e_j) + \lambda_j^{n-1} \cdot (H e_j) \right].$$

If we now define, for all $n \in \mathbb{N}$, $B_n := \sum_{k=0}^{n-2} \lambda_j^k M^{n-1-k} \in \mathrm{S}_N^+(m)$, then it is immediate to see that the series $\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} B_n$ is absolutely convergent on the Banach space $\mathrm{Sym_N}(\mathbb{R})$, and thus it converges to a matrix B_0 which is actually (symmetric and) nonnegative definite. We can then write the above equality in the following way:

$$0 = B_0(H e_j) + \left(\sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}\, \lambda_j^{n-1} \right) (H e_j) = B_0(H e_j) + \frac{f(\lambda_j)}{\lambda_j} (H e_j).$$

It follows that $H e_j$ is an eigenvector of $B_0 \geq 0$ with eigenvalue $(-f(\lambda_j)/\lambda_j) < 0$, and thus $H e_j = 0$ for all $j = 1, \dots, m$. On the other hand, for $j \in \{m + 1, \dots, N\}$, the vector e_j is in the kernel of M. By setting

$$M_0 := \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} M^{n-1} = \mathrm{diag}(f(\lambda_1)/\lambda_1, \dots, f(\lambda_m)/\lambda_m, f'(0), \dots, f'(0)),$$

we get that $0 = d_M(q_f)(H)e_j = M_0(He_j)$; i.e. the vector He_j is in the kernel of M_0. There are two cases. If $f'(0) \neq 0$, this gives immediately $He_j = 0$ for all $j = m+1, \dots, N$. If $f'(0) = 0$, then the vectors $He_{m+1}, \dots, He_N$ must be included in the span$\{e_{m+1}, \dots, e_N\}$: since $H \in T_M(\mathrm{S}_N^+(m))$, we obtain again $He_j = 0$ for all $j = m+1, \dots, N$. In both cases we have $H = 0$, and the proof is complete. □

3. The rectangular case

Let us go back to the example of the quadratic form $A(x)$ in (8) for the Heisenberg Laplacian. In the previous section we dealt with its square root
(10)

$$S(x) = \begin{pmatrix} \frac{1}{x_1^2+x_2^2}\left(x_1^2 + \frac{x_2^2}{\sqrt{1+x_1^2+x_2^2}}\right) & \frac{x_1x_2}{x_1^2+x_2^2}\left(1 - \frac{1}{\sqrt{1+x_1^2+x_2^2}}\right) & \frac{x_2}{\sqrt{1+x_1^2+x_2^2}} \\ \frac{x_1x_2}{x_1^2+x_2^2}\left(1 - \frac{1}{\sqrt{1+x_1^2+x_2^2}}\right) & \frac{1}{x_1^2+x_2^2}\left(x_2^2 + \frac{x_1^2}{\sqrt{1+x_1^2+x_2^2}}\right) & -\frac{x_1}{\sqrt{1+x_1^2+x_2^2}} \\ \frac{x_2}{\sqrt{1+x_1^2+x_2^2}} & -\frac{x_1}{\sqrt{1+x_1^2+x_2^2}} & \frac{x_1^2+x_2^2}{\sqrt{1+x_1^2+x_2^2}} \end{pmatrix}.$$

The entries of $S(x)$ have nothing to do with the usual Heisenberg vector fields $\partial_{x_1} + x_2\partial_{x_3}$, $\partial_{x_2} - x_1\partial_{x_3}$, which are a couple of smooth and linearly independent vector fields defined globally in $\mathbb{R}^3$. In this section we thus want to deal with the existence and the regularity of a *global* decomposition $A = RR^t$, which in the Heisenberg example can be seen, for example, as

$$R(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x_2 & -x_1 \end{pmatrix}.$$

Le us fix some notation. Fix two natural numbers $N \geq m \geq 1$. Denoting by $\mathrm{M}_{N\times m}(\mathbb{R})$ the vector space of the $N \times m$ matrices with real coefficients, let us define

$$\Omega_N(m) = \{M \in \mathrm{M}_{N\times m}(\mathbb{R}) : \mathrm{rk}(M) = m\},$$

which is an open subset of $\mathrm{M}_{N\times m}(\mathbb{R}) \equiv \mathbb{R}^{Nm}$. Let us restate Theorem 1.2 in this notation.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a contractible open set and let $A : \Omega \to \mathrm{S}_N^+(m)$ be a map of class C^k (for some $k \in \mathbb{N} \cup \{0, \infty\}$). Then there exists a map $R : \Omega \to \Omega_N(m)$ of class C^k such that*

(11) $$A(x) = R(x)R(x)^t \quad \textit{for all } x \in \Omega.$$

In order to prove this theorem, we are going to exploit a smooth identification between $\mathrm{S}_N^+(m)$ and the quotient of $\mathrm{M}_{N\times m}(\mathbb{R})$ by the action of the orthogonal group. This is probably a known fact, but, since we want to use some specific properties of this isomorphism and of the quotient manifold, we will briefly show the construction for the sake of completeness. After that, we are going to invoke a homotopy lifting property for the related principal bundle, which will allow us to prove the desired theorem.

Let us denote by $\mathrm{O}(m)$ the group of the $m \times m$ orthogonal matrices with real coefficients, which is a Lie group of dimension $m(m-1)/2$. There is a standard real-analytic (right-)action of $\mathrm{O}(m)$ on the manifold $\Omega_N(m)$, that is,

$$\rho : \Omega_N(m) \times \mathrm{O}(m) \to \Omega_N(m), \quad \rho(M, O) = MO.$$

It is not difficult to see that this action ρ is free and proper; i.e. (see also [29])

- if there exist $M \in \Omega_N(m)$ and $O \in \mathrm{O}(m)$ such that $MO = M$, then $O = \mathrm{Id}_m$;
- the map $\Omega_N(m) \times \mathrm{O}(m) \ni (M, O) \mapsto (MO, M) \in \Omega_N(m) \times \Omega_N(m)$ is proper (by compactness of $\mathrm{O}(m)$).

Therefore the orbit space

$$V_N(m) = \Omega_N(m)/\mathrm{O}(m)$$

is a topological manifold of dimension $\dim(\Omega_N(m)) - \dim(\mathrm{O}(m)) = d_N(m) = \dim(\mathrm{S}^+_N(m))$. Moreover it has a unique smoothly differentiable structure with the property that the quotient map $\pi : \Omega_N(m) \to V_N(m)$ is a smooth submersion (see e.g. [29, Theorem 21.10]). Let us now consider the real-analytic map Θ defined as follows:

$$\Theta : \Omega_N(m) \to \mathrm{S}^+_N(m), \qquad \Theta(M) = MM^t.$$

Since Θ is constant on each orbit of the action ρ, it defines a smooth map $\overline{\Theta}$ on the quotient $V_N(m)$:

$$\overline{\Theta} : V_N(m) \to \mathrm{S}^+_N(m), \qquad \overline{\Theta}\big([M]\big) = \Theta(M) = MM^t.$$

In other words, $\overline{\Theta}$ is the unique (smooth) map such that the following diagram commutes:

$$\begin{tikzcd} \Omega_N(m) \arrow[r, "\Theta"'] \arrow[d, "\pi"'] & \mathrm{S}^+_N(m) \\ V_N(m) \arrow[ur, dashed, "\overline{\Theta}"'] & \end{tikzcd}$$

Since it is easy to see (e.g. via diagonalization) that Θ is surjective, then $\overline{\Theta}$ is surjective as well. Concerning the injectivity of $\overline{\Theta}$, it is a straightforward consequence of the following lemma.

Lemma 3.1. *Let $M, N \in \Omega_N(m)$ with $MM^t = NN^t$. Then there exists $O \in \mathrm{O}(m)$ such that $M = NO$.*

Proof. Considering $A = MM^t = NN^t \in \mathrm{S}^+_N(m)$, let $\lambda_1, \dots, \lambda_m$ be its strictly positive eigenvalues. Let us take an orthonormal basis $\{x_1, \dots, x_N\}$ of $\mathbb{R}^N$ of eigenvectors of A such that $Ax_i = \lambda_i x_i$ for $i \in \{1, \dots, m\}$. Let us define two more orthonormal bases $\mathcal{B}_1 = \{u_1, \dots, u_m\}$ and $\mathcal{B}_2 = \{v_1, \dots, v_m\}$ of $\mathbb{R}^m$ as follows:

$$u_i = \frac{1}{\sqrt{\lambda_i}}(M^t x_i), \qquad v_i = \frac{1}{\sqrt{\lambda_i}}(N^t x_i) \qquad \text{for all } i = 1, \dots, m.$$

Let us make O the orthogonal $m \times m$ matrix bringing $\mathcal{B}_1$ into $\mathcal{B}_2$. For any $i \in \{1, \dots, m\}$, we get

$$Mu_i = \frac{1}{\sqrt{\lambda_i}}(MM^t)x_i = \frac{1}{\sqrt{\lambda_i}}(NN^t)x_i = Nv_i = N(Ou_i).$$

This ends the proof. $\square$

Hence $\overline{\Theta}$ is a smooth bijection from $V_N(m)$ to $\mathrm{S}_N^+(m)$. We want to see that it is actually a smooth diffeomorphism.

Lemma 3.2. *$V_N(m)$ and $\mathrm{S}_N^+(m)$ are smoothly diffeomorphic via $\overline{\Theta}$.*

Proof. We have to show that the differential $d_{[M]}\overline{\Theta}$ at any point $[M] \in V_N(m)$ is nonsingular. Since the $V_N(m)$ and $\mathrm{S}_N^+(m)$ have the same dimension, it is enough to show the surjectivity of $d_{[M]}\overline{\Theta}$ which is equivalent (π being a submersion) to the surjectivity of $d_M\Theta : T_M(\Omega_N(m)) \to T_{\Theta(M)}(\mathrm{S}_N^+(m))$ for any $M \in \Omega_N(m)$.

We claim that, without loss of generality, we can just consider $M \in \Omega_N(m)$ with $\Theta(M) = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$. Let us convince ourselves of this claim. Fix $M_0 \in \Omega_N(m)$ and $A = \Theta(M_0)$ and take an orthogonal $N \times N$ matrix U such that

$$U^tAU = (U^tM_0)\cdot(U^tM_0)^t = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0),$$

for some positive $\lambda_1, \ldots, \lambda_m$. Denote by L_U the automorphism of $\mathrm{M}_{N\times m}(\mathbb{R})$ defined by $L_U(M) = U^tM$, and by C_U the automorphism of $\mathrm{Sym_N}(\mathbb{R})$ defined by $C_U(A) = UAU^t$. For all $M \in \Omega_N(m)$ we have by construction that $(C_U \circ \Theta \circ L_U)(M) = \Theta(M)$. We thus get

$$d_{M_0}\Theta = C_U \circ d_{(U^tM_0)}\Theta \circ L_U$$

on $T_{M_0}(\Omega_N(m)) = \mathrm{M}_{N\times m}(\mathbb{R})$. By our choice of U, the matrix $U^tM \in \Omega_N(m)$ is such that $\Theta(U^tM) = \Lambda$, and the claim is proved.

We can now fix $M \in \Omega_N(m)$ with $\Theta(M) = \mathrm{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$, and look for the surjectivity of $d_M\Theta$. This assumption implies that $M_{ij} = 0$ for all $i = m+1, \ldots, N$ and $j = 1, \ldots, m$. For every $H \in \mathrm{M}_{N\times m}(\mathbb{R}) \equiv T_M(\Omega_N(m))$, we get

$$d_M\Theta(H) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Theta(M + tH) = M \cdot H^t + H \cdot M^t.$$

If we denote, for fixed $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$, by $E^{(ij)}$ the standard (i,j)-elementary matrix of the canonical basis of $\mathrm{M}_{N\times m}(\mathbb{R})$, we have $(M(E^{(ij)})^t)_{hk} = M_{hj}\delta_{ki}$ and $(E^{(ij)}M^t)_{hk} = M_{kj}\delta_{hi}$. We can thus verify that, for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, m\}$,

$$d_M\Theta\big(E^{(ij)}\big) = \big(M(E^{(ij)})^t\big) + \big(E^{(ij)}M^t\big) = \sum_{l=1}^{m} M_{lj}\big(\tilde{E}^{(li)} + \tilde{E}^{(il)}\big)$$

where, for $r, s \in \{1, \ldots, N\}$, $\tilde{E}^{(rs)}$ denotes the standard (r,s)-elementary matrix of the canonical basis of $\mathrm{M}_N(\mathbb{R})$. It is now quite easy to see that the image of $d_M\Theta$, which is generated by the Nm vectors $d_M\Theta(E^{(11)}), \ldots, d_M\Theta(E^{(Nm)})$, spans the whole $T_{\Theta(M)}(\mathrm{S}_N^+(m))$ (by recalling (9)). The proof is complete. $\square$

Therefore, we can think of the C^k map A in Theorem 1.2 as a map which takes value in the quotient manifold $V_N(m)$. Our aim is somehow to extract a map taking value in $\Omega_N(m)$. We can do that thanks to the structure of a principal $\mathrm{O}(m)$-bundle. Let us recall (we also refer to e.g. [21, Chapter IV] and [47, Chapter II]) that a principal G-bundle is a (locally trivial) smooth fiber bundle with typical fiber a Lie group G, for which the action preserves the fibers and there is an equivariance property between the action and the bundle charts. Moreover, if M is a smooth manifold and G is a Lie group acting smoothly, freely, and properly on M, then the quadruple $(M, M \to M/G, M/G, G)$ is a principal G-bundle (see, for example, [29, Exercise 21-6]). That is why the quadruple $\mathcal{F}_m = (\Omega_N(m), \pi, V_N(m), \mathrm{O}(m))$ is a principal $\mathrm{O}(m)$-bundle.

The main ingredient for our purposes is the following differential topology tool for principal G-bundles:

> let $\mathcal{F} = (E, \mathfrak{p}, M, G)$ be a principal G-bundle and let X be a manifold of class C^k; then $\mathcal{F}$ satisfies the following *homotopy lifting property* with respect to X:
>
> > if $f \in C^k(X; E)$ and $H : X \times [0,1] \to M$ is a map of class $C^k(X \times [0,1])$ such that $H(\cdot, 0) = \mathfrak{p} \circ f$, then there exists a map $\tilde{H} : X \times [0,1] \to E$ of class $C^k(X \times [0,1])$ satisfying $H = \mathfrak{p} \circ \tilde{H}$.

For a proof we refer the reader to [21, Chapter III, Theorem 2.4] (for a particular class of fiber bundles), [2, Chapter III], and [47, Chapter II, Corollary 6.1]. Roughly speaking, if we have a C^k-homotopy $H : X \times [0,1] \to M$ for which $H(\cdot, 0)$ admits a C^k-lift to E, then the whole homotopy H admits a C^k-lift to E. We can now complete the proof.

Proof of Theorem 1.2. Let us set

$$\overline{A} : \Omega \to V_N(m), \qquad \overline{A}(x) = \overline{\Theta}^{\,-1}\big(A(x)\big).$$

Since Ω is contractible, we can find a point $x_0 \in \Omega$ which is a deformation retract of Ω; i.e. there exists a smooth map $r : \Omega \times [0,1] \to \Omega$ such that $r(x,0) = x_0$ and $r(x,1) = x$ for all $x \in \Omega$. Let us now consider the homotopy

$$H : \Omega \times [0,1] \to V_N(m), \qquad H(x,t) = \overline{A}\big(r(x,t)\big).$$

Since r is smooth and $\overline{A}$ is C^k in Ω, we have $H \in C^k(\Omega \times [0,1]; V_N(m))$. Moreover, being $H(\cdot, 0)$ the constant map $\overline{A}(x_0)$, it trivially admits a lift of class C^k from Ω to $\Omega_N(m)$ (namely, the constant map $\Omega \ni x \mapsto R_0 \in \Omega_N(m)$, where R_0 is some matrix in $\Omega_N(m)$ such that $[R] = \overline{A}(x_0)$). By the homotopy lifting property there exists $\overline{H} : \Omega \times [0,1] \to \Omega_N(m)$ of class C^k such that $H = \pi \circ \overline{H}$. We can then define a map R from Ω to $\Omega_N(m)$ in the following way:

$$R : \Omega \to \Omega_N(m), \qquad R(x) = \overline{H}(x, 1).$$

By construction the map R is of class C^k on Ω, that is, the same regularity as A. For every $x \in \Omega$ we also have

$$R(x)R(x)^t = \Theta\big(R(x)\big) = \overline{\Theta}\big(\pi \circ \overline{H}(x,1)\big) = \overline{\Theta}\big(H(x,1)\big) = \overline{\Theta}\left(\overline{A}(x)\right) = A(x),$$

which is exactly the desired identity (11). ☐

Let us remark explicitly that the map R (and therefore the vector fields) we have constructed depends on A in a much *stronger* way with respect to the square-root construction of Section 2. More precisely, we cannot hope to find a smooth map $\sigma : V_N(m) \to \Omega_N(m)$ (independent of A) in order to define $R = \sigma\left(\overline{\Theta}^{\,-1}(A)\right)$. As a matter of fact, if such a σ exists, then it would be a smooth global section of our bundle $\mathcal{F}_m$. Since $\mathcal{F}_m$ is a principal $\mathrm{O}(m)$-bundle, it should be globally trivial (see e.g. [47, Theorem 4.2]). This means, in particular, that $\Omega_N(m)$ should be diffeomorphic to the product manifold $V_N(m) \times \mathrm{O}(m)$, which is a contradiction (if $m < N$ the space $\Omega_N(m)$ is indeed connected, whereas $V_N(m) \times \mathrm{O}(m)$ has at least two connected components). The case $m = N$ is slightly different. In fact, the polar decomposition provides a global trivialization of the principal bundle $\mathcal{F}_N = (\mathrm{GL}_N(\mathbb{R}), \pi, V_N(N), \mathrm{O}(N))$, and the map $\sqrt{\cdot} : V_N(N) \equiv \mathrm{S}_N^+(N) \to \mathrm{GL}_N(\mathbb{R})$ defines a smooth global section.

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