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## THE STRONG MAXIMUM PRINCIPLE AND THE HARNACK INEQUALITY FOR A CLASS OF HYPOELLIPTIC NON-HÖRMANDER OPERATORS

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ABSTRACT. — We consider a class of hypoelliptic second-order operators  $\mathcal{L}$  in divergence form, arising from CR geometry and Lie group theory, and we prove the Strong and Weak Maximum Principles and the Harnack Inequality. The operators are not assumed in the Hörmander hypoellipticity class, nor to satisfy subelliptic estimates or Muckenhoupt-type degeneracy conditions; indeed our results hold true in the infinitely-degenerate case and for operators which are not necessarily sums of squares. We use a Control Theory result on hypoellipticity to recover a meaningful geometric information on connectivity and maxima propagation, in the absence of any maximal rank condition. For operators  $\mathcal{L}$  with  $C^\omega$  coefficients, this control-theoretic result also implies a Unique Continuation property for the  $\mathcal{L}$ -harmonic functions. The Harnack theorem is obtained via a weak Harnack inequality by means of a Potential Theory argument and the solvability of the Dirichlet problem for  $\mathcal{L}$ .

RÉSUMÉ. — Nous considérons une classe d'opérateurs du second ordre hypoelliptiques  $\mathcal{L}$  sous la forme de divergence et nous prouvons les principes du maximum fort et faible, et l'inégalité de Harnack.  $\mathcal{L}$  n'est pas assumé dans la classe de hypoellipticité de Hörmander, ni satisfaisant des estimations sous-elliptique ou de dégénérescence à la Muckenhoupt; en effet nos résultats sont valables dans le cas infiniment dégénéré et pour des opérateurs qui ne sont pas des sommes de carrés. Nous utilisons un résultat de la théorie du contrôle sur l'hypoellipticité pour récupérer une information géométrique sur la connectivité et la propagation des maximums, en l'absence de la condition de rang maximal. Quand  $\mathcal{L}$  a coefficients  $C^\omega$ , ce résultat implique également une propriété de prolongement unique pour les fonctions  $\mathcal{L}$ -harmoniques. Le théorème de Harnack est obtenue par une inégalité faible de Harnack au moyen d'un argument de la théorie du potentiel et la solvabilité du problème de Dirichlet.

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## 1. Introduction and main results

Throughout the paper, we shall be concerned with linear second order partial differential operators (PDOs, in the sequel), possibly degenerate-elliptic, in divergence form

$$(1.1) \quad \mathcal{L} := \frac{1}{V(x)} \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left( V(x) a_{i,j}(x) \frac{\partial}{\partial x_j} \right), \quad x \in \mathbb{R}^N,$$

where  $V$  is a  $C^\infty$  strictly positive function on  $\mathbb{R}^N$ , the matrix  $A(x) := (a_{i,j}(x))_{i,j}$  is symmetric and *positive semi-definite* at every point  $x \in \mathbb{R}^N$ , and it has real-valued  $C^\infty$  entries. In particular,  $\mathcal{L}$  is symmetric, when restricted to smooth and compactly supported functions, on  $L^2(\mathbb{R}^N, d\nu)$  with respect to the measure  $d\nu(x) = V(x) dx$ , which clarifies the role of  $V$  (see precisely identity (1.13)). We tacitly understand these structural assumptions on  $\mathcal{L}$  throughout. It is well-known that PDOs as in (1.1) naturally intervene in the study of function theory of several complex variables and CR Geometry (see e.g., [28, 29, 41, 45, 61]); moreover, prototypes for the PDOs (1.1) also arise in the theory of sub-Laplace operators on real Lie groups (e.g., for Carnot groups, [7]), as well as in Riemannian Geometry (e.g., the Laplace-Beltrami operator has the form  $\sqrt{|g|}^{-1} \sum \partial_i (\sqrt{|g|} g^{ij} \partial_j)$ ), or in the study of Brownian motion and Stochastic PDEs. In the present paper we are interested in establishing *Maximum Principles* and a *Harnack Inequality* related to the above  $\mathcal{L}$ .

Starting from the 50's/60's seminal works by De Giorgi [18], Moser [55], Nash [56], Serrin [63], the literature on Harnack inequalities and on regularity issues for divergence-form operators like (1.1) in the uniformly-elliptic case has widely grown through the decades. The same is true of the vast literature on Hörmander operators, starting from the 60's/70's pioneering papers by Bony [10], Folland [27], Folland and Stein [29], Hörmander [38], Rothschild and Stein [61]. It is therefore out of our scopes to try to collect all the numerous references for the uniformly-elliptic or Hörmander cases. Instead, we here restrict to mentioning some investigations pertinent to the topics of our paper in the *degenerate-elliptic* case only, and in particular we compare the assumptions on the PDOs involved.

During the 80's, many important results on degenerate-elliptic operators under the divergence-form (1.1) were established, with a special emphasis to the Harnack Inequality and to the Maximum Principles; see e.g. the results

by: Jerison and Sánchez-Calle [41]; Chanillo and Wheeden [13]; Fabes, Jerison and Kenig [21, 22]; Fabes, Kenig and Serapioni [23]; Franchi and Lanconelli [30, 31]; Gutiérrez [35]. As for the assumptions made on the involved PDOs, in [41] a suitable *subellipticity* hypothesis is assumed, whereas in the other cited papers, operators as in (1.1) are considered (with  $V \equiv 1$ ) with very low regularity assumptions on the coefficients, under the hypothesis that the degeneracy of  $A(x)$  be controlled on both sides by some appropriate weights (for example, by *Muckenhoupt-type* weights, [21, 22, 23, 35]; or by doubling weights, [13]; or by a family of diagonal vector fields, [30]). The Muckenhoupt-type condition on the degeneracy is still one of the most frequently assumed hypotheses in obtaining Harnack theorems (see e.g. recent investigations in [17, 51, 66]; see also [44] for a Harnack inequality in the case of the so-called  $X$ -elliptic weight condition).

Another type of assumption can be made in facing potential-theoretic problems for operators  $\mathcal{L}$  as in (1.1): indeed, very recently a systematic study of the Potential Theory for the harmonic/subharmonic functions related to  $\mathcal{L}$  has been carried out in the series of papers [1, 5, 8, 9], under the assumption that  $\mathcal{L}$  possesses a smooth, global and positive *fundamental solution*. For the use of the fundamental solution in obtaining the Harnack Inequality for Hörmander sums of squares, see: Citti, Garofalo and Lanconelli [15]; Garofalo and Lanconelli [32, 33]; Pascucci and Polidoro [58, 59]; see also the recent survey by Bramanti, Brandolini, Lanconelli and Uguzoni, [11], for the same relevant use of the fundamental solution for heat PDOs structured on Hörmander vector fields.

Nowadays, it is known that the natural framework for Harnack-type theorems is the setting of *doubling metric spaces*: see e.g., Aimar, Forzani and Toledano [2]; Barlow and Bass [4]; Di Fazio, Gutiérrez and Lanconelli [19]; Grigor'yan and Saloff-Coste [34]; Gutiérrez and Lanconelli [36]; Hebisch and Saloff-Coste [37]; Indratno, Maldonado and Silwal [40]; Kinnunen, Marola, Miranda and Paronetto [43]; Mohammed [52]; Saloff-Coste [62]. In this framework it appears that the Harnack Inequality holds true whenever some axiomatic assumptions are satisfied (roughly speaking, a doubling condition and a Poincaré inequality): this fact has a strong theoretical impact, even if it is not always simple to verify whether these axiomatic assumptions are satisfied for a given PDO.

We explicitly remark that in the present paper we do not require  $\mathcal{L}$  to be a Hörmander operator, our results holding true in the infinitely-degenerate case as well, nor we make any assumption of subellipticity or Muckenhoupt-weighted degeneracy (see Example 1.2); furthermore, we do not assume the

existence of any global fundamental solution for  $\mathcal{L}$  nor the verification of any associated metric/Poincaré-type conditions. Hence our results are not contained in any of the aforementioned papers.

As a counterpart of allowing for less assumptions (our hypotheses are, broadly speaking, more qualitative than quantitative), we presently have to renounce to lower the regularity of the coefficients (as in Moser-type techniques) or to obtain an *invariant* Harnack inequality (roughly put, an inequality with a constant independent of the radius of the balls involved). Nonetheless, our results are sufficient to prime a great part of the Potential-Theory machinery needed to investigate further relevant topics concerning our PDOs  $\mathcal{L}$ , such as Wiener-type regularity results as well as the study of the distinguished properties of the harmonic sheaf and of the subharmonic cone related to  $\mathcal{L}$ , which will be part of future investigations, completing the results in [1, 5, 8, 9].

Most importantly, our main novelty is to obtain the Strong Maximum Principle and the Harnack Inequality for hypoelliptic operators with infinitely degenerate coefficients, allowing some eigenvalues of the principal matrix  $A(x)$  of (1.1) to vanish at infinite order, as in Fedii operator, [24],

$$\frac{\partial^2}{\partial x_1^2} + \left( \exp(-1/x_1^2) \frac{\partial}{\partial x_2} \right)^2 \quad \text{in } \mathbb{R}^2$$

(see also Example 1.2 for other models of infinitely-degenerate PDOs to which our theory applies). Note that this operator violates the Hörmander maximal rank condition on  $\{x_1 = 0\}$ , it does not satisfy subelliptic estimates, and its quadratic form does not satisfy Muckenhoupt-type weight conditions.

We now describe the main results of this paper concerning  $\mathcal{L}$ , namely the Strong Maximum Principle and the Harnack Inequality for  $\mathcal{L}$ ; gradually as we need to specify them, we introduce the three assumptions under which our theorems are proved. As we shall see in a moment, the main hypothesis is a *hypoellipticity* assumption.

In obtaining our main results we are much indebted to the ideas in the pioneering paper by Bony, [10], where Hörmander operators are considered. The main novelty of our framework is that we have to renounce to the geometric information encoded in Hörmander's Rank Condition: the latter implies a connectivity/propagation property (leading to the Strong Maximum Principle), as well as it implies hypoellipticity, due to the well-known Hörmander's theorem [38]. In our setting, the approach is somewhat reversed: hypoellipticity is the main assumption, and we need to derive from

it some appropriate connectivity and propagation features, even in the absence of a maximal rank condition. This will be made possible by exploiting a Control Theory result by Amano [3] on hypoelliptic PDOs (long forgotten in the PDE literature), as we shall describe in detail. Once the Strong Maximum Principle is established, the path to the Harnack Inequality is traced in [10]: we pass through the solvability of the Dirichlet problem, the relevant Green kernel and a weak form of the Harnack Inequality. Finally, the gap between the so-called weak and strong Harnack Inequalities is filled by an abstract Potential Theory result, due to Mokobodzki and BreLOT, [12].

In order to describe our results more closely, we first fix some notation and definition: we say that a linear second order PDO on  $\mathbb{R}^N$

$$(1.2) \quad L := \sum_{i,j=1}^N \alpha_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N \beta_i(x) \frac{\partial}{\partial x_i} + \gamma(x)$$

is *non-totally degenerate* at a point  $x \in \mathbb{R}^N$  if the matrix  $(\alpha_{i,j}(x))_{i,j}$  (which will be referred to as the principal matrix of  $L$ ) is non-vanishing. We observe that the principal matrix of an operator  $\mathcal{L}$  of the form (1.1) is precisely  $A(x) = (a_{i,j}(x))_{i,j}$ . We also recall that  $L$  is said to be  $(C^\infty)$ -hypoelliptic in an open set  $O \subseteq \mathbb{R}^N$  if, for every  $u \in \mathcal{D}'(O)$ , every open set  $U \subseteq O$  and every  $f \in C^\infty(U, \mathbb{R})$ , the equation  $Lu = f$  in  $U$  implies that  $u$  is (a function-type distribution associated with) a  $C^\infty$  function on  $U$ .

In the sequel, if  $O \subseteq \mathbb{R}^N$  is open, we say that  $u$  is  $L$ -harmonic (resp.,  $L$ -subharmonic) in  $O$  if  $u \in C^2(O, \mathbb{R})$  and  $Lu = 0$  (resp.,  $Lu \geq 0$ ) in  $O$ . The set of the  $L$ -harmonic functions in  $O$  will be denoted by  $\mathcal{H}_L(O)$ . We observe that, if  $L$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ , then  $\mathcal{H}_L(O) \subset C^\infty(O, \mathbb{R})$ ; under this hypoellipticity assumption,  $\mathcal{H}_L(O)$  has important topological properties, which will be crucially used in the sequel (Remark 4.3).

In order to introduce our first main result we assume the following hypotheses on  $\mathcal{L}$ :

**(NTD):**  $\mathcal{L}$  is *non-totally degenerate at every point of  $\mathbb{R}^N$* , or equivalently (recalling that  $A(x)$  is symmetric and positive semi-definite),

$$(1.3) \quad \text{trace}(A(x)) > 0, \quad \text{for every } x \in \mathbb{R}^N.$$

**(HY):**  $\mathcal{L}$  is  $C^\infty$ -hypoelliptic in every open subset of  $\mathbb{R}^N$ .

Under these two assumptions we shall prove the *Strong Maximum Principle for  $\mathcal{L}$* .

Condition (NTD), if compared to the previously mentioned Muckenhoupt-type weights on the degeneracies of  $A(x)$ , does not allow a *simultaneous* vanishing of the eigenvalues of  $A(x)$ , but it has the advantage of permitting a very fast vanishing of small eigenvalues (see Example 1.2) together with a very fast growing of large eigenvalues (see Example 1.1); both phenomena can happen at an exponential rate (e.g., like  $e^{-1/x^2}$  as  $x \rightarrow 0$  in the first case, and like  $e^x$  as  $x \rightarrow \infty$  in the second case), which is not allowed when Muckenhoupt weights are involved.

Meaningful examples of operators satisfying hypotheses (NTD) and (HY), providing prototype PDOs to which our theory applies and a motivation for our investigation, are now described in the following two examples.

*Example 1.1.* — The following PDOs satisfy the assumptions (NTD) and (HY).

(a). — If  $\mathbb{R}^N$  is equipped with a Lie group structure  $\mathbb{G} = (\mathbb{R}^N, *)$ , and if we fix a set  $X := \{X_1, \dots, X_m\}$  of Lie-generators for the Lie algebra  $\mathfrak{g}$  of  $\mathbb{G}$  (this means that the smallest Lie algebra containing  $X$  is equal to  $\mathfrak{g}$ ), then a direct computation shows that

$$(1.4) \quad \mathcal{L}_X := - \sum_{j=1}^m X_j^* X_j$$

is of the form (1.1), where  $V(x)$  is the density of the Haar measure  $\nu$  on  $\mathbb{G}$ , and  $(a_{i,j})_{i,j}$  is equal to  $SS^T$ , where  $S$  is the  $N \times m$  matrix whose columns are given by the coefficients of the vector fields  $X_1, \dots, X_m$ ; here  $X_j^*$  denotes the (formal) adjoint of  $X_j$  in the Hilbert space  $L^2(\mathbb{R}^N, d\nu)$ . Most importantly,  $\mathcal{L}_X$  in (1.4) satisfies the assumptions (NTD) and (HY) above. Indeed:

- The non-total-degeneracy is a consequence of  $X$  being a set of Lie-generators of  $\mathfrak{g}$ .
- $\mathcal{L}_X$  is a Hörmander operator, of the form  $\sum_{j=1}^m X_j^2 + X_0$ , where  $X_0$  is a linear combination (with smooth coefficients) of  $X_1, \dots, X_m$ . Therefore  $\mathcal{L}_X$  is hypoelliptic due to Hörmander's Hypoellipticity Theorem, [38], jointly with the cited fact that  $X$  is a set of Lie-generators of  $\mathfrak{g}$ .

The density  $V$  need not be identically 1, as for example for the Lie group  $(\mathbb{R}^2, *)$ , where

$$(x_1, x_2) * (y_1, y_2) = (x_1 + y_1 e^{x_2}, x_2 + y_2),$$

since in this case  $V(x) = e^{-x_2}$ . The left-invariant PDO associated with the set of generators  $X = \{e^{x_2} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$  has fast-growing coefficients:

$$\mathcal{L}_X = e^{2x_2} \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial}{\partial x_2}.$$

Note that the eigenvalues of the principal matrix of  $\mathcal{L}_X$  are  $e^{2x_2}$  and 1, so that the largest eigenvalue cannot be controlled (for  $x_2 > 0$ ) by any integrable weight.

(b). — More generally (arguing as above), if  $X = \{X_1, \dots, X_m\}$  is a family of smooth vector fields in  $\mathbb{R}^N$  satisfying Hörmander's Rank Condition, if  $d\nu(x) = V(x) dx$  is the Radon measure associated with any positive smooth density  $V$  on  $\mathbb{R}^N$ , then the operator  $-\sum_{j=1}^m X_j^* X_j$  is of the form (1.1) and it satisfies (NTD) and (HY). Here  $X_j^*$  denotes the formal adjoint of  $X_j$  in  $L^2(\mathbb{R}^N, d\nu)$ . As already observed, PDOs of this form naturally arise in CR Geometry and in the function theory of several complex variables (see [41]).

The above examples show that geometrically meaningful PDOs belonging to the class of our concern actually fall in the hypoellipticity class of the Hörmander operators. Nonetheless, hypotheses (NTD) and (HY) are general enough to comprise *non-Hörmander* and *non-subelliptic* operators, as it is shown in the next example. Applications to this kind of *infinitely-degenerate* PDOs also furnish one of the main motivation for our study.

*Example 1.2.* — Let us consider the class of operators in  $\mathbb{R}^2$  defined by

$$(1.5a) \quad \mathcal{L}_a = \frac{\partial^2}{\partial x_1^2} + \left( a(x_1) \frac{\partial}{\partial x_2} \right)^2,$$

with  $a \in C^\infty(\mathbb{R}, \mathbb{R})$ ,  $a$  even, nonnegative, nondecreasing on  $[0, \infty)$  and vanishing only at 0. Then  $\mathcal{L}_a$  satisfies (NTD) (obviously) and (HY), thanks to a result by Fedii, [24]. Note that  $\mathcal{L}_a$  does not satisfy Hörmander's Rank Condition at  $x_1 = 0$  if all the derivatives of  $a$  vanish at 0, as for  $a(x_1) = \exp(-1/x_1^2)$ . Other examples of operators satisfying our assumptions (NTD) and (HY) but failing to be Hörmander operators can be found, e.g., in the following papers: Bell and Mohammed [6]; Christ [14, Section 1];



Kohn [46]; Kusuoka and Stroock [48, Theorem 8.41]; Morimoto [54]. Explicit examples are, for instance,

$$(1.5b) \quad \frac{\partial^2}{\partial x_1^2} + \left( \exp(-1/|x_1|) \frac{\partial}{\partial x_2} \right)^2 + \left( \exp(-1/|x_1|) \frac{\partial}{\partial x_3} \right)^2 \quad \text{in } \mathbb{R}^3,$$

$$(1.5c) \quad \frac{\partial^2}{\partial x_1^2} + \left( \exp(-1/\sqrt{|x_1|}) \frac{\partial}{\partial x_2} \right)^2 + \frac{\partial^2}{\partial x_3^2} \quad \text{in } \mathbb{R}^3,$$

$$(1.5d) \quad \frac{\partial^2}{\partial x_2^2} + \left( x_2 \frac{\partial}{\partial x_1} \right)^2 + \frac{\partial^2}{\partial x_4^2} + \left( \exp(-1/\sqrt[3]{|x_1|}) \frac{\partial}{\partial x_3} \right)^2 \quad \text{in } \mathbb{R}^4.$$

For the hypoellipticity of (1.5b) see [14]; for (1.5c) see [48]; for (1.5d) see [54]. Later on, in proving the Harnack Inequality, we shall add another hypothesis to (NTD) and (HY) and, as we shall show, the operators from (1.5a) to (1.5d) (and those in Example 1.1) will fulfil this assumption as well. Hence the main results of this paper (except for the Unique Continuation result in Section 3, proved for operators with  $C^\omega$  coefficients) fully apply to these PDOs.

Moreover, since the PDOs (1.5a)-to-(1.5d) are *not subelliptic* (see Remark 1.6), they do not fall in the class considered by Jerison and Sánchez-Calle in [41]. Finally, note that the smallest eigenvalue in all the above examples vanishes very quickly (like  $\exp(-1/|x|^\alpha)$  for  $x \rightarrow 0$ , with positive  $\alpha$ ) and it cannot be bounded from below by any weight  $w(x)$  with locally integrable reciprocal function.

Our first main result under conditions (NTD) and (HY) is the following one.

**THEOREM 1.3 (Strong Maximum Principle for  $\mathcal{L}$ ).** — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j})_{i,j} \geq 0$ , and that it satisfies (NTD) and (HY). Let  $O \subseteq \mathbb{R}^N$  be a connected open set. Then, the following facts hold.*

(1) *Any function  $u \in C^2(O, \mathbb{R})$  satisfying  $\mathcal{L}u \geq 0$  on  $O$  and attaining a maximum in  $O$  is constant throughout  $O$ .*

(2) *If  $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$  is nonnegative on  $\mathbb{R}^N$ , and if we set*

$$(1.6) \quad \mathcal{L}_c := \mathcal{L} - c,$$

*then any function  $u \in C^2(O, \mathbb{R})$  satisfying  $\mathcal{L}_c u \geq 0$  on  $O$  and attaining a nonnegative maximum in  $O$  is constant throughout  $O$ .*

The role of the nonnegativity of the zero-order term  $c$  in the above statement (2) in obtaining Strong Maximum Principles is well-known (see e.g., Pucci and Serrin [60]).

*Remark 1.4.*

(a). — Obviously, the Strong Maximum Principle (SMP, shortly) in Theorem 1.3 will immediately provide the Weak Maximum Principle (WMP, shortly) for operators  $\mathcal{L}$  and  $\mathcal{L} - c$ , for any nonnegative zero-order term  $c$  (and any bounded open set  $O$ ), see Theorem 2.3 for the precise statement.

(b). — We will show that, in order to obtain the SMP and WMP for  $\mathcal{L} - c$ , it is also sufficient to replace the hypothesis on the hypoellipticity of  $\mathcal{L}$  with the (more natural hypothesis of the) hypoellipticity of  $\mathcal{L} - c$ , still under assumption (NTD) and the divergence-form structure of  $\mathcal{L}$ ; see Remark 2.4 for the precise result.

Our proof of the SMP in Theorem 1.3 follows a rather classical scheme, in that it rests on a Hopf Lemma for  $\mathcal{L}$  (see Lemma 2.1). However, the passage from the Hopf Lemma to the SMP is, in general, non-trivial and the same is true in our framework. For example, in the paper [10] by Bony, where Hörmander operators are considered, this passage is accomplished by means of a maximum propagation principle, crucially based on Hörmander's Rank Condition, the latter ensuring a connectivity property (the so-called *Chow's Connectivity Theorem* for Hörmander vector fields). The novelty in our setting is that, since hypotheses (NTD) and (HY) do *not* necessarily imply that  $\mathcal{L}$  is a Hörmander operator (see for instance Example 1.2), we have to supply for a lack of geometric information. Due to this main novelty, we describe more closely our argument in deriving the SMP.

As anticipated, we are able to supply the lack of Hörmander's Rank Condition by using a notable control-theoretic property encoded in the hypoellipticity assumption (HY), proved by Amano in [3]: indeed, thanks to the hypothesis (NTD), we are entitled to use [3, Theorem 2] which states that (HY) ensures the *controllability* of the ODE system

$$\dot{\gamma} = \xi_0 X_0(\gamma) + \sum_{i=1}^N \xi_i X_i(\gamma), \quad (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{R}^{1+N},$$

on every open and connected subset of  $\mathbb{R}^N$ . Here  $X_1, \dots, X_N$  denote the vector fields associated with the rows of the principal matrix of  $\mathcal{L}$ , whereas  $X_0$  is the drift vector field obtained by writing  $\mathcal{L}$  (this being always possible) in the form

$$\mathcal{L}u = \sum_{i=1}^N \frac{\partial}{\partial x_i} (X_i u) + X_0 u.$$

By definition of a controllable system, Amano's controllability result provides another geometric *connectivity property* (a substitute for Chow's

Theorem): any couple of points can be joined by a continuous path which is piece-wise an integral curve of some vector field  $Y$  belonging to  $\text{span}_{\mathbb{R}}\{X_0, X_1, \dots, X_N\}$ . The SMP will then follow if we show that there is a propagation of the maximum of any  $\mathcal{L}$ -subharmonic function  $u$  along all integral curves  $\gamma_Y$  of every  $Y \in \text{span}_{\mathbb{R}}\{X_0, X_1, \dots, X_N\}$ . In other words, we need to show that if the set  $F(u)$  of the maximum points of  $u$  intersects any such  $\gamma_Y$ , then  $\gamma_Y$  is wholly contained in  $F(u)$ : briefly, if this happens we say that  $F(u)$  is  $Y$ -invariant. In its turn, this  $Y$ -invariance property can be characterized (see Bony, [10, §2]) in terms of a tangentiality property of  $Y$  with respect to  $F(u)$  (the reader is referred to Section 2 below for this notion of tangentiality).

Now, the divergence-form structure of our PDO  $\mathcal{L}$  in (1.1) ensures that  $X_0$  is a linear combination with smooth coefficients of  $X_1, \dots, X_N$ . Hence, by the very definition of tangentiality (see e.g., (2.10)), the tangentiality of  $X_0$  w.r.t.  $F(u)$  will be inherited from the tangentiality of  $X_1, \dots, X_N$  w.r.t.  $F(u)$ . By means of the above argument of controllability/propagation, this allows us to reduce the proof of the SMP to showing that any of the vector fields  $X_1, \dots, X_N$  is tangent to  $F(u)$ . Luckily, this tangentiality is a consequence of the choice of  $X_1, \dots, X_N$  as deriving from the rows of the principal matrix of  $\mathcal{L}$ , together with the Hopf-type Lemma 2.1 for  $\mathcal{L}$ . This argument is provided, in all detail, in Section 2.

The use of the above ideas, plus the classical Holmgren's Theorem, will allow us to proving that, when  $\mathcal{L}$  has real-analytic coefficients, a *Unique Continuation* result holds true for  $\mathcal{L}$ : any  $\mathcal{L}$ -harmonic function defined on a connected open set  $U$  which vanishes on some non-void open subset is necessarily null on the whole of  $U$  (see Theorem 3.1). We observe that the  $C^\omega$  assumption is satisfied, for example, if  $\mathcal{L}$  is a left invariant operator on a Lie group (e.g., a sub-Laplacian on a Carnot group, as in [7]), since, as it is well-know, any Lie group can be endowed with a compatible  $C^\omega$  structure.

*Remark 1.5.* — We explicitly remark that, as it is proved by Amano in [3, Theorem 1], the above controllability property ensures the validity of the Hörmander Rank Condition only on an open *dense* subset of  $\mathbb{R}^N$  which may fail to coincide with the whole of  $\mathbb{R}^N$ . This actual possible lack of the Hörmander Rank Condition is clearly exhibited in Example 1.2 (of non-Hörmander operators which nonetheless satisfy our assumptions (NTD) and (HY), and hence the SMP).

To the best of our knowledge, Amano's controllability result for hypoelliptic non-totally-degenerate operators has been long forgotten in the literature; only recently, it has been used by the third-named author and B. Abbondanza [1] in studying the Dirichlet problem for  $\mathcal{L}$ , and in obtaining Potential Theoretic results for the harmonic sheaf related to  $\mathcal{L}$ .

In order to give the second main result of the paper (namely, the Harnack Inequality for  $\mathcal{L}$ ), we shall need a further assumption, very similar to (HY) (and, indeed, equivalent to it in many important cases), together with some technical results on the solvability of the Dirichlet problem related to  $\mathcal{L}$ . Our next assumption is the following one:

**(HY) $_{\varepsilon}$** : *There exists  $\varepsilon > 0$  such that  $\mathcal{L} - \varepsilon$  is  $C^{\infty}$ -hypoelliptic in every open subset of  $\mathbb{R}^N$ .*

For operators  $\mathcal{L}$  satisfying hypotheses (NTD), (HY) and (HY) $_{\varepsilon}$  we are able to prove the Harnack Inequality (see Theorem 1.10).

We postpone the description of the relationship between assumptions (HY) and (HY) $_{\varepsilon}$  (and their actual equivalence for large classes of operators: for subelliptic PDOs, for instance) in Remark 1.6 below. Instead, we anticipate the role of the perturbation  $\mathcal{L} - \varepsilon$  of the operator  $\mathcal{L}$ : this is motivated by a crucial comparison argument (which we generalize to our setting), due to Bony [10, Proposition 7.1, p.298], giving the lower bound

$$(1.7) \quad u(x_0) \geq \varepsilon \int_{\Omega} u(y) k_{\varepsilon}(x_0, y) V(y) dy \quad \forall x_0 \in \Omega,$$

for every nonnegative  $\mathcal{L}$ -harmonic function  $u$  on the open set  $\Omega$  which possesses a Green kernel  $k_{\varepsilon}(x, y)$  relative to the perturbed operator  $\mathcal{L} - \varepsilon$  (see Theorem 1.9 for the notion of a Green kernel, and see Lemma 5.1 for the proof of (1.7)). This lower bound, plus some topological facts on hypoellipticity, is the key ingredient for a weak Harnack Inequality related to  $\mathcal{L}$ , as we shall explain shortly.

Some remarks on assumption (HY) $_{\varepsilon}$  are now in order.

*Remark 1.6.* — Hypothesis (HY) $_{\varepsilon}$  is implicit in hypothesis (HY) for notable classes of operators, whence our assumptions for the validity of the Harnack Inequality for  $\mathcal{L}$  reduce to (NTD) and (HY) solely: namely, (HY) implies (HY) $_{\varepsilon}$  in the following cases:

- for Hörmander operators, and, more generally, for second order subelliptic operators (in the usual sense of fulfilling a subelliptic estimate, see e.g., [41, 46]); indeed, any operator  $L$  in these classes

of PDOs is hypoelliptic (see Hörmander [38], Kohn and Nirenberg [47]), and  $L$  still belongs to these classes after the addition of a smooth zero-order term;

- for operators with *real-analytic coefficients*. Indeed, in the  $C^\omega$  case, one can apply known results by Oleĭnik and Radkevič ensuring that, for a general  $C^\omega$  operator  $L$  as in (1.2), hypoellipticity is equivalent to the verification of Hörmander's Rank Condition for the vector fields  $X_0, X_1, \dots, X_N$  obtained by rewriting  $L$  as  $\sum_{i=1}^N \partial_i(X_i) + X_0 + \gamma$ ; this condition is clearly invariant under any change of the zero-order term  $\gamma$  of  $L$  so that (HY) and (HY) $_\varepsilon$  are indeed equivalent.

The problem of establishing, in general, whether (HY) implies (HY) $_\varepsilon$  seems non-trivial and it is postponed to future investigations.<sup>(1)</sup> In this regard we recall that, for example, in the complex coefficient case the presence of a zero-order term (even a small  $\varepsilon$ ) may drastically alter hypoellipticity (see for instance the example given by Stein in [64]; see also the very recent paper [57] by Parmeggiani, for related topics).

We explicitly remark that the operators (1.5a)-to-(1.5d) are *not* subelliptic (nor  $C^\omega$ ), yet they satisfy hypotheses (NTD), (HY) and (HY) $_\varepsilon$ . The lack of subellipticity is a consequence of the characterization of the subelliptic PDOs due to Fefferman and Phong [26, 25] (see also [46, Prop.1.3] or [41, Th.2.1 and Prop.2.1], jointly with the presence of a coefficient with a zero of infinite order in (1.5a)-to-(1.5d)). The second assertion concerning the verification of (HY) $_\varepsilon$  (the other hypotheses being already discussed) derives from the following result by Kohn, [46]: any operator of the form

$$L_1 + \lambda(x) L_2 \quad \text{in } \mathbb{R}_x^n \times \mathbb{R}_y^m$$

is hypoelliptic, where  $\lambda \in C^\infty(\mathbb{R}_x)$ ,  $\lambda \geq 0$  has a zero of infinite order at 0 (and no other zeroes of infinite order), and  $L_1$  (operating in  $x \in \mathbb{R}^n$ ) and  $L_2$  (operating in  $y \in \mathbb{R}^m$ ) are general second order PDOs (as in (1.2)) with smooth coefficients and they are assumed to be subelliptic. It is straightforward to recognize that by subtracting  $\varepsilon$  to any PDO in (1.5a)-to-(1.5d) we get an operator of the form  $(L_1 - \varepsilon) + \lambda(x) L_2$ , where  $\lambda$  has the required features,  $L_2$  is uniformly elliptic (indeed, a classical Laplacian in all the examples), and  $L_1 - \varepsilon$  is a uniformly elliptic operator (cases (1.5a)-to-(1.5c)) or it is a Hörmander operator (case (1.5d)).

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<sup>(1)</sup>It appears that having some quantitative information on the loss of derivatives may help in facing this question (personal communication by A. Parmeggiani).

Before describing the approach to the Harnack Inequality, inspired by the techniques in [10], we state the main needed technical tools on the solvability of the Dirichlet problem for  $\mathcal{L}$  and for the perturbed operator  $\mathcal{L} - \varepsilon$ .

LEMMA 1.7. — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and that  $\mathcal{L}$  satisfies (NTD). Let  $\varepsilon \geq 0$  be fixed (the case  $\varepsilon = 0$  being admissible). We set  $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$  and we assume that  $\mathcal{L}_\varepsilon$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ .*

*Then, there exists a basis for the Euclidean topology of  $\mathbb{R}^N$ , independent of  $\varepsilon$ , made of open, bounded and connected sets  $\Omega$  (with Lipschitz boundary) with the following properties: for every continuous function  $f$  on  $\overline{\Omega}$  and for every continuous function  $\varphi$  on  $\partial\Omega$ , there exists one and only one solution  $u \in C(\overline{\Omega}, \mathbb{R})$  of the Dirichlet problem*

$$(1.8) \quad \begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad (\text{in the weak sense of distributions}), \\ u = \varphi & \text{on } \partial\Omega \quad (\text{point-wise}). \end{cases}$$

*Furthermore, if  $f, \varphi \geq 0$  then  $u \geq 0$  as well. Finally, if  $f$  belongs to  $C^\infty(\Omega, \mathbb{R}) \cap C(\overline{\Omega}, \mathbb{R})$ , then the same is true of  $u$ , and  $u$  is a classical solution of (1.8).*

This result can be proved via the techniques in [10, Section 5] used for Hörmander operators. These techniques are perfectly suited to our more general case, since they only rely on hypoellipticity and on the Weak Maximum Principle. Therefore we omit the proof. We only recall Bony’s notable construction of the sets  $\{\Omega\}$  (see Lemma 4.1) for the sake of completeness.

With the existence of the weak solution of the Dirichlet problem for  $\mathcal{L}_\varepsilon$  on a bounded open set  $\Omega$ , we can define the associated Green operator as usual:

DEFINITION 1.8 (Green operator and Green measure). — *Let  $\varepsilon \geq 0$  be fixed, and let  $\mathcal{L}_\varepsilon$  and  $\Omega$  satisfy, respectively, the hypothesis and the thesis of Lemma 1.7. We consider the operator (depending on  $\mathcal{L}_\varepsilon$  and  $\Omega$ ; we avoid keeping track of the dependency on  $\Omega$  in the notation)*

$$(1.9) \quad G_\varepsilon : C(\overline{\Omega}, \mathbb{R}) \longrightarrow C(\overline{\Omega}, \mathbb{R})$$

*mapping  $f \in C(\overline{\Omega}, \mathbb{R})$  into the function  $G_\varepsilon(f)$  which is the unique distributional solution  $u$  in  $C(\overline{\Omega}, \mathbb{R})$  of the Dirichlet problem*

$$(1.10) \quad \begin{cases} \mathcal{L}_\varepsilon u = -f & \text{on } \Omega \quad (\text{in the weak sense of distributions}), \\ u = 0 & \text{on } \partial\Omega \quad (\text{point-wise}). \end{cases}$$

We call  $G_\varepsilon$  the Green operator related to  $\mathcal{L}_\varepsilon$  and to the open set  $\Omega$ .

By the Riesz Representation Theorem (which is applicable thanks to the monotonicity properties in Lemma 1.7 with respect to the function  $f$ ), for every  $x \in \overline{\Omega}$  there exists a (nonnegative) Radon measure  $\lambda_{x,\varepsilon}$  on  $\overline{\Omega}$  such that

$$(1.11) \quad G_\varepsilon(f)(x) = \int_{\overline{\Omega}} f(y) d\lambda_{x,\varepsilon}(y), \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}).$$

We call  $\lambda_{x,\varepsilon}$  the Green measure related to  $\mathcal{L}_\varepsilon$  (to the open set  $\Omega$  and to the point  $x$ ).

Let  $\mathcal{L}$  be as in (1.1); in the rest of the paper, we set once and for all

$$(1.12) \quad d\nu(x) := V(x) dx,$$

that is,  $\nu$  is the (Radon) measure on  $\mathbb{R}^N$  associated with the (positive) density  $V$  in (1.1), absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}^N$ . It is clear that the measure  $\nu$  plays the following key role:

$$(1.13) \quad \int \varphi \mathcal{L}\psi d\nu = \int \psi \mathcal{L}\varphi d\nu, \quad \text{for every } \varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}).$$

We observe that (in general) the classical adjoint operator  $\mathcal{L}^*$  of  $\mathcal{L}$  may not coincide with  $\mathcal{L}$  itself; indeed  $\mathcal{L}^*$  is related to  $\mathcal{L}$  by the following identity (a consequence of (1.13))

$$(1.14) \quad \mathcal{L}^*u = V \mathcal{L}(u/V), \quad \text{for every } u \text{ of class } C^2.$$

The possibility of dealing with non-identically 1 densities  $V$  (as in the case of Lie groups, see Example 1.1-(a)) makes it more convenient to decompose the Green measure  $\lambda_{x,\varepsilon}$  with respect to  $\nu$  in (1.12), rather than w.r.t. Lebesgue measure. Hence we prove the following:

**THEOREM 1.9 (Green kernel).** — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and that  $\mathcal{L}$  satisfies (NTD). Let  $\varepsilon \geq 0$  be fixed. We set  $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$  and we assume that  $\mathcal{L}_\varepsilon$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ .*

Let  $\Omega$  be a bounded open set as in Lemma 1.7. If  $G_\varepsilon$  and  $\lambda_{x,\varepsilon}$  are the Green operator and the Green measure related to  $\mathcal{L}_\varepsilon$  (Definition 1.8), there exists a function  $k_\varepsilon : \Omega \times \Omega \rightarrow \mathbb{R}$ , smooth and positive out of the diagonal of  $\Omega \times \Omega$ , such that the following representation holds true:

$$(1.15) \quad G_\varepsilon(f)(x) = \int_{\Omega} f(y) k_\varepsilon(x, y) d\nu(y), \quad \text{for every } x \in \Omega,$$

and for every  $f \in C(\overline{\Omega}, \mathbb{R})$ . We call  $k_\varepsilon$  the Green kernel related to  $\mathcal{L}_\varepsilon$  (and to the open set  $\Omega$ ).

Furthermore, we have the following properties:

(i) *Symmetry of the Green kernel:*

$$(1.16) \quad k_\varepsilon(x, y) = k_\varepsilon(y, x) \quad \text{for every } x, y \in \Omega.$$

(ii) *For every fixed  $x \in \Omega$ , the function  $k_\varepsilon(x, \cdot)$  is  $\mathcal{L}_\varepsilon$ -harmonic in  $\Omega \setminus \{x\}$ ; moreover  $G_\varepsilon(\mathcal{L}_\varepsilon\varphi) = -\varphi = \mathcal{L}_\varepsilon(G_\varepsilon(\varphi))$  for any  $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ , that is*

$$(1.17) \quad \begin{aligned} -\varphi(x) &= \int_\Omega \mathcal{L}_\varepsilon\varphi(y) k_\varepsilon(x, y) \, d\nu(y) \\ &= \mathcal{L}_\varepsilon\left(\int_\Omega \varphi(y) k_\varepsilon(x, y) \, d\nu(y)\right), \quad \text{for every } \varphi \in C_0^\infty(\Omega, \mathbb{R}). \end{aligned}$$

(iii) *For every fixed  $x \in \Omega$ , one has*

$$(1.18) \quad \lim_{y \rightarrow y_0} k_\varepsilon(x, y) = 0 \quad \text{for any } y_0 \in \partial\Omega.$$

(iv) *For every fixed  $x \in \Omega$ , the functions  $k_\varepsilon(x, \cdot) = k_\varepsilon(\cdot, x)$  are in  $L^1(\Omega)$ , and  $k_\varepsilon \in L^1(\Omega \times \Omega)$ .*

The key ingredients in the proof of the above results are the following facts:

- the hypoellipticity of  $\mathcal{L}_\varepsilon$  (as assumed in the hypothesis) which will imply the hypoellipticity of the *classical* adjoint of  $\mathcal{L}_\varepsilon$  (see Remark 4.2);
- the  $C^\infty$ -topology on the space of the  $\mathcal{L}_\varepsilon$ -harmonic functions is the same as the  $L^1_{\text{loc}}$ -topology, another consequence of the hypoellipticity of  $\mathcal{L}_\varepsilon$  (Remark 4.3);
- the fact that  $\mathcal{L}$  is symmetric on  $L^2(\mathbb{R}^N, d\nu)$  when restricted to smooth and compactly supported functions (see (1.13), precisely), so that the same is true of  $\mathcal{L}_\varepsilon$ : this will be crucial in proving the symmetry of the Green kernel;
- the Strong Maximum Principle for the perturbed operator  $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$ , which we obtain as a consequence of our previous Strong Maximum Principle for  $\mathcal{L}$  in Theorem 1.3 (see precisely Remark 2.2, where *nonnegative* maxima are considered): this is a key step for the proof of the *positivity* of  $k_\varepsilon$ ;
- the Schwartz Kernel Theorem (used for the regularity of the Green kernel).

The difference with respect to the analogous result given in the framework of the Hörmander operators in [10, Théorème 6.1] is the introduction of the



relevant measure  $\nu$  in the integral representation (1.15); indeed, the symmetry property (1.16) of the kernel  $k_\varepsilon$  is connected with the identity (1.13), which is not true (in general) if we consider Lebesgue measure instead of  $\nu$ .

We are now ready to give the second main result of the paper. (The naming ‘Weak’ or ‘Strong’ Harnack Inequality is non-standard: for example some authors refer to weak Harnack inequalities when at least one side of (1.19) is replaced by some  $L^p$ -norm of  $u$ ; we follow the naming from Potential Theory used by Loeb and Walsh in [49], with the hope that this does not lead to any ambiguity.)

**THEOREM 1.10 (Strong Harnack Inequality).** — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and suppose it satisfies hypotheses (NTD), (HY) and  $(\text{HY})_\varepsilon$ .*

*Then, for every connected open set  $O \subseteq \mathbb{R}^N$  and every compact subset  $K$  of  $O$ , there exists a constant  $M = M(\mathcal{L}, O, K) \geq 1$  such that*

$$(1.19) \quad \sup_K u \leq M \inf_K u,$$

*for every nonnegative  $\mathcal{L}$ -harmonic function  $u$  in  $O$ .*

*If  $\mathcal{L}$  is subelliptic or if it has  $C^\omega$  coefficients, then assumption  $(\text{HY})_\varepsilon$  can be dropped.*

The last assertion follows from Remark 1.6.

We now describe the spine of the proof of Theorem 1.10.

The main step towards the Strong Harnack Inequality is the following Theorem 1.11 from Potential Theory. A proof of a more general abstract version of this useful result, in the framework of axiomatic harmonic spaces, can be found in the survey notes [12, pp.20–24] by BreLOT, where this theorem is attributed to G. Mokobodzki. (See also a further improvement to harmonic spaces which are not necessarily second-countable, by Loeb and Walsh, [49]). Instead of appealing to an abstract Potential-Theoretic statement, we prefer to formulate the result under the following more specific form (where a harmonic sheaf related to a smooth PDO is considered).

**THEOREM 1.11.** — *Let  $L$  be a second order linear PDO in  $\mathbb{R}^N$  with smooth coefficients. Suppose the following conditions are satisfied.*

**(Regularity):** *There exists a basis  $\mathcal{B}$  for the Euclidean topology of  $\mathbb{R}^N$  (consisting of bounded open sets) such that, for every  $\Omega \in \mathcal{B} \setminus \{\emptyset\}$  and for every  $\varphi \in C(\partial\Omega, \mathbb{R})$ , there exists a unique  $L$ -harmonic*

function  $H_\varphi^\Omega \in C^2(\Omega) \cap C(\overline{\Omega})$  solving the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

and satisfying  $H_\varphi^\Omega \geq 0$  whenever  $\varphi \geq 0$ .

**(Weak Harnack Inequality):** For every connected open set  $O \subseteq \mathbb{R}^N$ , every compact subset  $K$  of  $O$  and every  $y_0 \in O$ , there exists a constant  $C(y_0) = C(L, O, K, y_0) > 0$  such that

$$\sup_K u \leq C(y_0) u(y_0),$$

for every nonnegative  $L$ -harmonic function  $u$  in  $O$ .

Then, the following Strong Harnack Inequality for  $L$  holds: for every connected open set  $O$  and every compact subset  $K$  of  $O$  there exists a constant  $M = M(L, O, K) \geq 1$  such that

$$(1.20) \quad \sup_K u \leq M \inf_K u,$$

for every nonnegative  $L$ -harmonic function  $u$  in  $O$ .

See also Remark 5.2 for some equivalent assumptions that can replace the above (Weak Harnack Inequality) to get the Strong Harnack Inequality. The proof of Theorem 1.11 is given in Section 5, starting from a result by Mokobodzki and BreLOT in [12, Chapter I]: in the latter it is shown that if the axioms (Regularity) and (Weak Harnack Inequality) are fulfilled then, for any connected open set  $O \subseteq \mathbb{R}^N$  and any  $x_0 \in O$ , the set

$$(1.21) \quad \Phi_{x_0} := \left\{ h \in \mathcal{H}_L(O) : h \geq 0, \quad h(x_0) = 1 \right\}$$

is equicontinuous at  $x_0$ . The proof of this fact rests on some deep results of Functional Analysis concerning the family of the so-called harmonic measures  $\{\mu_x^\Omega\}_{x \in \partial\Omega}$  related to  $L$  (and to a regular set  $\Omega$  for the Dirichlet problem), jointly with some distinguished properties of the harmonic sheaf associated with the operator  $L$ .

As observed by Bony in [10, Remarque 7.1, p.300], the Strong Harnack Inequality classically relies on two-sided estimates of the ratios  $h(x_1, \cdot)/h(x_2, \cdot)$ , where  $h(x, y)$  is the relevant Poisson kernel; these estimates were unavailable in the setting considered in [10], as they are (to the best of our knowledge) in our setting too. However, like in [10], the unavailability of these estimates can be overcome by the use of the Green kernel for the perturbed operator  $\mathcal{L} - \varepsilon$  and by the Strong Maximum Principle, as they jointly lead to the Weak Harnack Inequality. It is interesting to

observe that, once the Weak Harnack Inequality is available, the equicontinuity of (1.21) (an equivalent version of the Strong Harnack Inequality) is derived by Mokobodzki and Brelot by the comparison (in the sense of measures)  $\mu_{x_1}^\Omega \leq M \mu_{x_2}^\Omega$  for harmonic measures: this comparison seems to be the core substitute for the mentioned pointwise estimates with Poisson kernels centered at different points  $x_1, x_2$ .

Due to Theorem 1.11, the focus on the Strong Harnack Inequality is now shifted to the Weak Harnack Inequality, which is easier to establish. As already anticipated, the latter is based on the lower bound (1.7) as we now briefly describe. First, we remark that the proof of (1.7) is a two-line comparison argument: it suffices to apply  $\mathcal{L} - \varepsilon$  on both sides of (1.7) to see that they produce the same result, namely  $-\varepsilon u$ ; then one uses the Weak Maximum Principle, since the right-hand side is null on  $\partial\Omega$  whereas the left-hand side is nonnegative. Secondly, with inequality (1.7) at hands and the *strict positivity* of  $k_\varepsilon$  (a consequence of the SMP), it is not difficult to prove that  $u(x_0)$  dominates the  $L_{\text{loc}}^1$ -norm of  $u$ , on suitable compact sets. Then, due to the equivalence of the  $L_{\text{loc}}^1$  and  $C^\infty$  topologies on the space of the  $\mathcal{L}$ -harmonic functions (this fact deriving from (HY)), one can infer the following:

**THEOREM 1.12** (Weak Harnack inequality for derivatives). — *Let  $\mathcal{L}$  satisfy (NTD), (HY) and  $(\text{HY})_\varepsilon$ . Then, for every connected open set  $O \subseteq \mathbb{R}^N$ , every compact subset  $K$  of  $O$ , every  $m \in \mathbb{N} \cup \{0\}$  and every  $y_0 \in O$ , there exists a positive  $C(y_0) = C(\mathcal{L}, \varepsilon, O, K, m, y_0)$  such that*

$$(1.22) \quad \sum_{|\alpha| \leq m} \sup_{x \in K} \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right| \leq C(y_0) u(y_0),$$

for every nonnegative  $\mathcal{L}$ -harmonic function  $u$  in  $O$ .

We remark that topological properties similar to those mentioned above for the space of the  $\mathcal{L}$ -harmonic functions are also valid when  $\mathcal{L}$  in (1.1) is *not necessarily hypoelliptic*, provided that it possesses a global positive fundamental solution (not necessarily smooth): see e.g., [5] by the first and third named authors, where Montel-type results are proved (in the sense of [53]), jointly with the equivalence of the topologies induced on  $\mathcal{H}_\mathcal{L}(\Omega)$  by  $L_{\text{loc}}^1$  and by  $L_{\text{loc}}^\infty$ , under no hypoellipticity assumptions.

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## 2. The Strong Maximum Principle for $\mathcal{L}$

The aim of this section is to prove the Strong Maximum Principle for  $\mathcal{L}$  in Theorem 1.3. Clearly, a fundamental step is played by a suitable Hopf-type lemma, furnished in Lemma 2.1. (For a recent interesting survey on maximum principles and Hopf-type results for uniformly elliptic operators, see López-Gómez [50].)

First the relevant definition and notation: given an open set  $O \subseteq \mathbb{R}^N$  and a relatively closed set  $F$  in  $O$ , we say that  $\eta$  is *externally orthogonal to  $F$  at  $y$* , and we write

$$(2.1) \quad \eta \perp F \text{ at } y,$$

if  $y \in O \cap \partial F$ ;  $\eta \in \mathbb{R}^N \setminus \{0\}$ ;  $\overline{B(y + \eta, |\eta|)}$  is contained in  $O$  and it intersects  $F$  only at  $y$ . Here and throughout  $B(x_0, r)$  is the Euclidean ball in  $\mathbb{R}^N$  of centre  $x_0$  and radius  $r > 0$ ; moreover  $|\cdot|$  will denote the Euclidean norm on  $\mathbb{R}^N$ . The notation (2.1) does not explicitly refer to externality, but this will not create any confusion in the sequel. It is not difficult to recognize that if  $O$  is open and connected, and if  $F$  is a proper (relatively closed) subset of  $O$ , then there always exist couples  $(y, \eta)$  such that  $\eta \perp F$  at  $y$ .

Finally, throughout the paper we write  $\partial_i$  for  $\frac{\partial}{\partial x_i}$ .

LEMMA 2.1 (of Hopf-type for  $\mathcal{L}$ ). — Suppose that  $\mathcal{L}$  is an operator of the form (1.1) with  $C^1$  coefficients  $V > 0$  and  $a_{i,j}$ , and let us set  $A(x) := (a_{i,j}(x))_{i,j}$ . (We recall that  $A(x) \geq 0$  for every  $x \in \mathbb{R}^N$ .) Let  $O \subseteq \mathbb{R}^N$  be a connected open set. Then, the following facts hold.

(1) Let  $u \in C^2(O, \mathbb{R})$  be such that  $\mathcal{L}u \geq 0$  on  $O$ ; let us suppose that

$$(2.2) \quad F(u) := \left\{ x \in O : u(x) = \max_O u \right\}$$

is a proper subset of  $O$ . Then

$$(2.3) \quad \langle A(y)\eta, \eta \rangle = 0 \quad \text{whenever } \eta \perp F(u) \text{ at } y.$$

(2) Suppose  $c \in C(\mathbb{R}^N, \mathbb{R})$  is nonnegative on  $\mathbb{R}^N$ , and let us set  $\mathcal{L}_c := \mathcal{L} - c$ . Let  $u \in C^2(O, \mathbb{R})$  be such that  $\mathcal{L}_c u \geq 0$  on  $O$ ; let us suppose that  $F(u)$  in (2.2) is a proper subset of  $O$  and that  $\max_O u \geq 0$ . Then (2.3) holds true.

*Proof.* — We begin by proving part (1) in the statement of the Lemma, from which we also inherit the notation and hypotheses on  $u$  and  $F(u)$ . Notice that the assumption ensures that  $M := \max_O u \in \mathbb{R}$  (since  $u$  is finite-valued). To this aim, let us assume by contradiction that

$$(2.4) \quad \langle A(y)\eta, \eta \rangle > 0, \quad \text{for some } \eta \perp F(u) \text{ at } y.$$

We define a smooth function  $w : \mathbb{R}^N \rightarrow \mathbb{R}$  as follows

$$w(x) := e^{-\lambda|x-(y+\eta)|^2} - e^{-\lambda|\eta|^2},$$

where  $\lambda$  is a positive real number chosen in a moment. We set  $b_j := \sum_{i=1}^N \partial_i(V a_{i,j})/V$ , so that  $\mathcal{L} = \sum_{i,j} a_{i,j} \partial_{i,j} + \sum_j b_j \partial_j$ . A simple computation shows that

$$(2.5) \quad \mathcal{L}w(y) = \lambda^2 e^{-\lambda|\eta|^2} \left( 4\langle A(y)\eta, \eta \rangle - \frac{2}{\lambda} \sum_{j=1}^N (a_{j,j}(y) - b_j(y)\eta_j) \right),$$

and thus, by (2.4), it is possible to choose  $\lambda > 0$  in such a way that  $\mathcal{L}w(y) > 0$ .

By the continuity of  $\mathcal{L}w$ , we can then find a positive real number  $\delta$  such that  $W := B(y, \delta)$  is compactly contained in  $O$  and  $\mathcal{L}w > 0$  on  $W$ . We now define, for  $\varepsilon > 0$ , a function  $v_\varepsilon : \overline{W} \rightarrow \mathbb{R}$  by setting  $v_\varepsilon := u + \varepsilon w$ . Clearly,  $v_\varepsilon \in C^2(W, \mathbb{R}) \cap C(\overline{W}, \mathbb{R})$ , and we claim that the maximum of  $v_\varepsilon$  on  $\overline{W}$  is attained in  $W$ .

Indeed, let us consider the splitting of  $\partial W$  given by the two sets  $K_1 := \partial W \cap \overline{B}(y + \eta, |\eta|)$  and  $K_2 := \partial W \setminus K_1$ . For every  $x \in K_2$ , one has

$$v_\varepsilon(x) = u(x) + \varepsilon w(x) < u(x) \leq M.$$

On the other hand, for all  $x \in K_1$ , we have

$$v_\varepsilon(x) \leq \max_{K_1} u + \varepsilon \max_{K_1} w,$$

and since  $\max_{K_1} u < M$  (observe that  $u < M$  outside  $F(u)$  and that  $K_1$  is a compact set contained in  $O \setminus F(u)$ ), it is possible to choose  $\varepsilon > 0$  so small that  $v_\varepsilon < M$  on  $K_1$ . By gathering together these facts we see that, for every  $x \in \partial W$  (note that  $y \in F(u)$  and  $w(y) = 0$ )

$$v_\varepsilon(x) < M = u(y) = v_\varepsilon(y) \leq \max_{\overline{W}} v_\varepsilon,$$

and this proves the claim. From  $\mathcal{L}v_\varepsilon = \mathcal{L}u + \varepsilon \mathcal{L}w \geq \varepsilon \mathcal{L}w$  (and the latter is  $> 0$  on  $W$ ) the function  $v_\varepsilon$  is a *strictly*  $\mathcal{L}$ -subharmonic function on  $W$ , that is,  $\mathcal{L}v_\varepsilon > 0$  on  $W$ , admitting a maximum point on the open set  $W$ , say  $p_0$ . Then we have (recall that  $A(p_0) \geq 0$  and notice that  $\nabla v_\varepsilon(p_0) = 0$  and  $H(p_0) := (\partial_{i,j} v_\varepsilon(p_0))_{i,j} \leq 0$ )

$$(2.6) \quad 0 < \mathcal{L}v_\varepsilon(p_0) = \sum_{i,j} a_{i,j}(p_0) \partial_{i,j} v_\varepsilon(p_0) = \text{trace}(A(p_0) \cdot H(p_0)) \leq 0,$$

which is clearly a contradiction.

Part (2) in the statement of the Lemma can be proved in a totally analogous way: we replace  $\mathcal{L}$  with  $\mathcal{L}_c$  and we notice that  $w(y) = 0$  so that

$\mathcal{L}_c w(y) = \mathcal{L}w(y)$ , and (2.5) is left unchanged. Arguing as above, we let again  $p_0 \in W$  be such that  $v_\varepsilon(p_0) = \max_{\overline{W}} v_\varepsilon$ . This gives  $v_\varepsilon(p_0) \geq v_\varepsilon(y) = u(y) = M$ . Hence (2.6) becomes

$$0 < \mathcal{L}_c v_\varepsilon(p_0) = \text{trace}(A(p_0) \cdot H(p_0)) - c(p_0) v_\varepsilon(p_0) \leq -c(p_0) M,$$

where in the last inequality we used the assumption  $c \geq 0$  and the fact that  $v_\varepsilon(p_0) \geq M$ . By the assumption  $M \geq 0$  (and again by the assumption on the sign of  $c$ ), we have  $-c(p_0) M \leq 0$ , and we obtain another contradiction.  $\square$

We are now in a position to provide the

*Proof of Theorem 1.3.* — Let  $\mathcal{L}$  be as in the statement of Theorem 1.3; suppose that  $O \subseteq \mathbb{R}^N$  is a connected open set and that  $u \in C^2(O, \mathbb{R})$  satisfies  $\mathcal{L}u \geq 0$  on  $O$  and  $u$  attains a maximum in  $O$ . We set

$$F(u) := \left\{ x \in O : u(x) = \max_O u \right\}.$$

By assumption  $F(u) \neq \emptyset$ , say  $\xi \in F(u)$ . We show that  $F(u) = O$ .

To this aim, let us rewrite  $\mathcal{L}$  as follows:

$$\begin{aligned} \mathcal{L} &= \frac{1}{V} \sum_{i,j} \partial_i \left( V a_{i,j} \partial_j \right) \\ &= \frac{1}{V} \sum_{i,j} V \partial_i (a_{i,j} \partial_j) + \sum_{i,j} \frac{\partial_i V}{V} a_{i,j} \partial_j \\ &= \sum_{i,j} \partial_i (a_{i,j} \partial_j) + \sum_j b_j \partial_j, \end{aligned}$$

where  $b_j := \frac{1}{V} \sum_{i=1}^N \partial_i V a_{i,j}$  (for  $j = 1, \dots, N$ ). Let us consider the vector fields

$$(2.7) \quad X_i := \sum_{j=1}^N a_{i,j} \partial_j, \quad i = 1, \dots, N, \quad X_0 := \sum_{j=1}^N b_j \partial_j.$$

We explicitly remark the following useful fact:  $X_0$  is a linear combination (with smooth-coefficients) of  $X_1, \dots, X_N$ ; indeed

$$(2.8) \quad \begin{aligned} X_0 &= \sum_{j=1}^N b_j \partial_j = \sum_{j=1}^N \frac{1}{V} \sum_{i=1}^N \partial_i V a_{i,j} \partial_j = \sum_{i=1}^N \frac{\partial_i V}{V} \sum_{j=1}^N a_{i,j} \partial_j \\ &= \sum_{i=1}^N \frac{\partial_i V}{V} X_i. \end{aligned}$$

Summing up, we have written  $\mathcal{L}$  as follows

$$\mathcal{L}u = \sum_{i=1}^N \partial_i(X_i u) + \sum_{i=1}^N \frac{\partial_i V}{V} X_i u, \quad \forall u \in C^2.$$

Thanks to the assumption (NTD) of non-total degeneracy of  $\mathcal{L}$  and due to the smoothness of its coefficients, we are entitled to use a notable result [3, Theorem 2] by Amano, which states that the hypoellipticity assumption (HY) ensures the controllability of the ODE system

$$(2.9) \quad \dot{\gamma} = \xi_0 X_0(\gamma) + \sum_{i=1}^N \xi_i X_i(\gamma), \quad (\xi_0, \xi_1, \dots, \xi_N) \in \mathbb{R}^{1+N},$$

on every open and connected subset of  $\mathbb{R}^N$  (see e.g., [42, Chapter 3] for the notion of controllability). Since  $O$  is open and connected, this implies that any point of  $O$  can be joined to  $\xi$  by a continuous curve  $\gamma$  contained in  $O$  which is piecewise an integral curve of a vector field belonging to  $\mathcal{V} := \text{span}_{\mathbb{R}}\{X_0, X_1, \dots, X_N\}$ . It then suffices to prove that if  $\gamma$  is an integral curve of a vector field  $X \in \mathcal{V}$  starting at a point of  $F(u)$  (which is non-empty), then  $\gamma(t)$  remains in  $F(u)$  for every admissible time  $t$ . In this case we say that  $F(u)$  is  $X$ -invariant.

By a result of Bony, [10, Théorème 2.1], the  $X$ -invariance of  $F(u)$  is equivalent to the tangentiality of  $X$  to  $F(u)$ : this latter condition means that

$$(2.10) \quad \langle X(y), \eta \rangle = 0 \quad \text{whenever } \eta \perp F(u) \text{ at } y.$$

Hence, by all the above arguments, the proof of the SMP is complete if we show that (2.10) is fulfilled by any  $X \in \mathcal{V}$ . Since  $X$  is a linear combination of  $X_0, X_1, \dots, X_N$  and due to (2.8), it suffices to prove this identity when  $X$  is replaced by any element of  $\{X_1, \dots, X_N\}$ . Due to identity (2.3) in the Hopf-type Lemma 2.1, it is therefore sufficient to show that for every  $i \in \{1, \dots, N\}$  and every  $x \in \mathbb{R}^N$ , there exists  $\lambda_i(x) > 0$  such that

$$(2.11) \quad \langle X_i(x), \eta \rangle^2 \leq \lambda_i(x) \langle A(x)\eta, \eta \rangle \quad \text{for every } \eta \in \mathbb{R}^N.$$

Indeed, (2.11) together with (2.3) implies that the left-hand side of (2.11) is null whenever  $\eta \perp F(u)$  at  $y$ , which is precisely (2.10) for  $X \in \{X_1, \dots, X_N\}$ . Due to the very definition of  $X_i$ , inequality (2.11) boils down to proving that, given a real symmetric positive semidefinite matrix  $A = (a_{i,j})$ , for every  $i$  there exists  $\lambda_i > 0$  such that

$$\left( \sum_j a_{i,j} \eta_j \right)^2 \leq \lambda_i \sum_{i,j} a_{i,j} \eta_i \eta_j \quad \text{for every } \eta \in \mathbb{R}^N,$$

which is a consequence of the Cauchy-Schwarz inequality and the characterization (for a symmetric  $A \geq 0$ ) of  $\ker(A)$  as  $\{x \in \mathbb{R}^N : \langle Ax, x \rangle = 0\}$ .

This proves part (1) of Theorem 1.3. As for part (2), let  $c$  be smooth and nonnegative on  $\mathbb{R}^N$ , and let us set  $\mathcal{L}_c := \mathcal{L} - c$ . Suppose  $u \in C^2(O, \mathbb{R})$  satisfies  $\mathcal{L}_c u \geq 0$  on  $O$  and that it attains a nonnegative maximum in  $O$ . For  $F(u) \neq \emptyset$  as above, we show again that  $F(u) = O$ . The hypoellipticity and non-total degeneracy of  $\mathcal{L}$  ensure again (by Amano's cited result for  $\mathcal{L}$ ) the controllability of system (2.9). This again grants a connectivity property of  $O$  by means of continuous curves, piecewise integral curves of elements in the above vector space  $\mathcal{V}$ . By Bony's quoted result on invariance/tangentiality, the needed identity  $F(u) = O$  follows if we show again that (2.10) is fulfilled when  $X$  is replaced by  $X_i$ , for  $i = 1, \dots, N$  (the case  $i = 0$  deriving as above from (2.8)).

Now, by part (2) of Lemma 2.1, it is at our disposal a Hopf-type Lemma for operators of the form  $\mathcal{L}_c$ , and for functions  $u$  such that  $\mathcal{L}_c u \geq 0$  and attaining a *nonnegative* maximum. In other words, we know that (2.3) holds true, again as in the previous case (1). The validity of (2.11) allows us to end the proof, as in the previous part.  $\square$

A close inspection to the above proof shows that we have indeed demonstrated the following result as well (replacing the hypothesis of hypoellipticity of  $\mathcal{L}$  by that of  $\mathcal{L} - c$ ), since Amano's results on hypoellipticity/controllobility are independent of the presence of a zero-order term:

*Remark 2.2.* — Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and that it satisfies (NTD). Let  $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$  be nonnegative and suppose that the operator  $\mathcal{L}_c := \mathcal{L} - c$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ .

If  $O \subseteq \mathbb{R}^N$  is a connected open set, then any function  $u \in C^2(O, \mathbb{R})$  satisfying  $\mathcal{L}_c u \geq 0$  on  $O$  and attaining a nonnegative maximum in  $O$  is constant throughout  $O$ .

As a corollary of Theorem 1.3 we immediately get the following result.

**THEOREM 2.3** (Weak Maximum Principle for  $\mathcal{L}$ ). — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and that it satisfies (NTD) and (HY). Suppose also that  $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$  is nonnegative on  $\mathbb{R}^N$  (the case  $c \equiv 0$  is allowed), and let us set  $\mathcal{L}_c := \mathcal{L} - c$ . Then,  $\mathcal{L}_c$  satisfies the Weak Maximum Principle on every bounded open*



set  $\Omega \subseteq \mathbb{R}^N$ , that is:

$$(2.12) \quad \begin{cases} u \in C^2(\Omega, \mathbb{R}) \\ \mathcal{L}_c u \geq 0 \text{ on } \Omega \\ \limsup_{x \rightarrow x_0} u(x) \leq 0 \text{ for every } x_0 \in \partial\Omega \end{cases} \implies u \leq 0 \text{ on } \Omega.$$

As a consequence, if  $\Omega \subseteq \mathbb{R}^N$  is bounded, and if  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is nonnegative and such that  $\mathcal{L}_c u \geq 0$  on  $\Omega$ , then one has  $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$ .

*Proof.* — Suppose that the open set  $\Omega \subset \mathbb{R}^N$  is bounded and  $u$  is as in the left-hand side of (2.12). Let  $x_0 \in \overline{\Omega}$  be such that

$$(2.13) \quad \limsup_{x \rightarrow x_0} u(x) = \sup_{\Omega} u.$$

If  $x_0 \in \partial\Omega$ , then (2.12) ensures that  $\limsup_{x \rightarrow x_0} u(x) \leq 0$ , so that (due to (2.13))  $\sup_{\Omega} u \leq 0$ , proving the right-hand side of (2.12). If  $x_0 \in \Omega$ , then (2.13) gives  $u(x_0) = \max_{\Omega} u$ . If  $u(x_0) < 0$ , we conclude as above that  $\max_{\Omega} u = u(x_0) < 0$ . If  $u(x_0) \geq 0$ , we consider  $\Omega_0 \subseteq \Omega$  the connected component of  $\Omega$  containing  $x_0$ , and, thanks to part (2) of the Strong Maximum Principle in Theorem 1.3, the existence of an interior maximum point of  $u$  on  $\Omega \supseteq \Omega_0$  (and the fact that  $u(x_0) \geq 0$ ) ensures that  $u \equiv u(x_0)$  on  $\Omega_0$ . Let us take any  $\xi_0 \in \partial\Omega_0$ ; we have

$$\max_{\Omega} u = u(x_0) = \limsup_{\Omega_0 \ni x \rightarrow \xi_0} u(x) \leq \limsup_{\Omega \ni x \rightarrow \xi_0} u(x) \leq 0,$$

where the last inequality follows from  $\partial\Omega_0 \subseteq \partial\Omega$  and from the assumption in (2.12).

We remark that when  $c \equiv 0$  the proof is slightly simpler, as an interior maximum of  $u$  propagates up to the boundary, regardless of the sign of this maximum.

Finally we prove the last assertion of the corollary. Let  $\Omega \subseteq \mathbb{R}^N$  be bounded and let  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  be nonnegative on  $\overline{\Omega}$  and satisfying  $\mathcal{L}_c u \geq 0$  on  $\Omega$ ; then we set  $M := \sup_{\partial\Omega} u$  and we observe that  $M \geq 0$  since this is true of  $u$ . We have (recall that  $c \geq 0$ )

$$\mathcal{L}_c(u - M) = \mathcal{L}_c u - \mathcal{L}_c M \geq -\mathcal{L}_c M = -(\mathcal{L} - c)M = cM \geq 0.$$

Since (by definition of  $M$ ) we have  $u - M \leq 0$  on  $\partial\Omega$  (and  $u - M$  is continuous up to  $\partial\Omega$ ), we can apply (2.12) to get  $u - M \leq 0$ , that is  $u \leq \sup_{\partial\Omega} u$  on  $\Omega$ . This clearly proves the needed  $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$ .  $\square$

Arguing as in the previous proof (and exploiting Remark 2.2 instead of Theorem 1.3-(2)) we also get the following result, where we alternatively replace the hypothesis of hypoellipticity of  $\mathcal{L}$  by that of  $\mathcal{L} - c$ :

*Remark 2.4.* — Suppose that  $\mathcal{L}$  is an operator of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and that it satisfies (NTD). Let  $c \in C^\infty(\mathbb{R}^N, \mathbb{R})$  be nonnegative and suppose that the operator  $\mathcal{L}_c := \mathcal{L} - c$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ .

Then  $\mathcal{L}_c$  satisfies the Weak Maximum Principle on every bounded open set  $\Omega \subseteq \mathbb{R}^N$ .

As a consequence, if  $\Omega \subseteq \mathbb{R}^N$  is bounded, and if  $u \in C^2(\Omega) \cap C(\overline{\Omega})$  is nonnegative and such that  $\mathcal{L}_c u \geq 0$  on  $\Omega$ , then one has  $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$ .

### 3. Analytic coefficients: A Unique Continuation result for $\mathcal{L}$

In this short section, by means of the ideas of controllability/propagation introduced in the previous section, we prove the following result.

**THEOREM 3.1** (Unique Continuation for  $\mathcal{L}$ ). — *Suppose that  $\mathcal{L}$  is an operator of the form (1.1) satisfying assumptions (NTD) and (HY). Suppose that  $\mathcal{L}$  has  $C^\omega$  coefficients  $a_{i,j}$  and  $V$ .*

*Let  $O \subseteq \mathbb{R}^N$  be a connected open set. Then any  $\mathcal{L}$ -harmonic function on  $O$  vanishing on some non-empty open subset of  $O$  is identically zero on  $O$ .*

*Proof.* — Let  $u \in \mathcal{H}_{\mathcal{L}}(O)$  be vanishing on the open set  $U \subseteq O$ ,  $U \neq \emptyset$ . Let  $F \subseteq O$  be the support of  $u$ . We argue by contradiction, by assuming that  $F \neq \emptyset$ . Let us fix any  $y \in \partial F \cap O$  and any  $\eta \in \mathbb{R}^N \setminus \{0\}$  such that  $\eta \perp F$  at  $y$  (see the notion of exterior orthogonality at the beginning of Section 2; the assumption  $F \neq \emptyset$  ensures the existence of such a couple  $(y, \eta)$ ). We consider the Euclidean open ball  $B := B(y + \eta, |\eta|)$  which is completely contained in  $O \setminus F$ , so that  $u \equiv 0$  on  $B$ . We observe that  $B$  is the sub-level set  $\{f(x) < 0\}$ , where

$$f(x) = |x - (y + \eta)|^2 - |\eta|^2.$$

There are only two cases:

(a). — The boundary of  $B$  is *non-characteristic* for  $\mathcal{L}$  at  $y$ , that is,  $\langle A(y) \nabla f(y), \nabla f(y) \rangle \neq 0$ . Due to the  $C^\omega$  assumption we are allowed to use the classical Holmgren's Theorem (see e.g., [39, Theorem 8.6.5]), ensuring that  $u$  vanishes in a neighborhood of  $y$ , so that  $y \in O \setminus F$ . Since  $F$  is relatively closed in  $O$ , this is in contradiction with  $y \in \partial F \cap O$ . Hence it is true that:

(b). — The boundary of  $B$  is *characteristic* for  $\mathcal{L}$  at  $y$ , that is,  $\langle A(y) \nabla f(y), \nabla f(y) \rangle = 0$ . Since  $\nabla f(x) = 2(x - y - \eta)$ , this condition boils down to  $\langle A(y)\eta, \eta \rangle = 0$ . Let  $X_0, X_1, \dots, X_N$  be the vector fields introduced in (2.7). The same Linear Algebra argument leading to (2.11) shows that  $\langle A(y)\eta, \eta \rangle = 0$  implies

$$\langle X_i(y), \eta \rangle = 0 \quad \text{for every } i = 1, \dots, N.$$

Identity (2.8) guarantees that the same holds for  $i = 0$  as well. Therefore, one has  $\langle X(y), \eta \rangle = 0$  for every  $X \in \mathcal{V} := \text{span}\{X_0, X_1, \dots, X_N\}$ . Arguing as in Section 2, by means of the result by Bony [10, Théorème 2.1], this geometric condition (holding true for arbitrary  $\eta \perp F$  at  $y$ ) implies that the closed set  $F$  is  $X$ -invariant for any  $X \in \mathcal{V}$  (that is  $F$  contains the trajectories of the integral curves of  $X$  touching  $F$ ).

On the other hand, the hypoellipticity assumption (HY) on  $\mathcal{L}$  ensures (due to the recalled result by Amano [3, Theorem 2]) that any pair of points of the connected open set  $O$  can be joined by a continuous curve which is piecewise an integral curve of some vector fields  $X$  in  $\mathcal{V}$ . Gathering together all the mentioned results, the fact that  $F \neq \emptyset$  implies that any point of  $O$  belongs to  $F$ , contradicting the assumption that  $U \subseteq O \setminus F$ .  $\square$

#### 4. The Green function and the Green kernel for $\mathcal{L} - \varepsilon$

The aim of this section is to prove Theorem 1.9. In the first part of the proof (Steps I–III) we follow the classical scheme by Bony (see [10, Théorème 6.1]), hence we skip many details; it is instead in Step IV that a slight difference is presented, in that we exploit the measure  $d\nu(x) = V(x)dx$  in order to obtain the symmetry property of the Green kernel even when  $\mathcal{L} \neq \mathcal{L}^*$  (where  $\mathcal{L}^*$  is the classical formal adjoint of  $\mathcal{L}$ ). The problem of the behavior of the Green kernel along the diagonal is more subtle, as it is shown by Fabes, Jerison and Kenig in [21] who proved that, for divergence-form operators as in (1.1) (when  $V \equiv 1$  and, roughly put, when the degeneracy of  $A(x)$  is controlled by a suitable weight) the limit of the Green kernel along the diagonal need not be infinite; we plan to investigate this behavior in a future study, since our assumption (NTD) prevents the existence of any vanishing Muckenhoupt-type weight.

Throughout this section, we fix an operator  $\mathcal{L}$  of the form (1.1), with  $C^\infty$  coefficients  $V > 0$  and  $(a_{i,j}) \geq 0$ , and we assume that  $\mathcal{L}$  satisfies (NTD). Moreover, we also fix  $\varepsilon \geq 0$  (note that the case  $\varepsilon = 0$  is allowed) and we set  $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ ; we assume that  $\mathcal{L}_\varepsilon$  is hypoelliptic on every open subset

of  $\mathbb{R}^N$ . Finally, we understand that  $\Omega$  is a fixed bounded open set as in Lemma 1.7, such that the Dirichlet problem (1.8) is (uniquely) solvable. We recall its construction:

LEMMA 4.1. — *Let  $A(x) = (a_{i,j}(x))$  be a matrix with real-valued continuous entries on  $\mathbb{R}^N$ , which is symmetric, positive semi-definite and non-vanishing at a point  $x_0 \in \mathbb{R}^N$ .*

*Then, there exists a basis  $\mathcal{B}_{x_0}$  of open, bounded and connected neighborhoods of  $x_0$  such that any  $\Omega \in \mathcal{B}_{x_0}$  satisfies the following property: for every  $y \in \partial\Omega$  there exists  $\eta \in \mathbb{R}^N \setminus \{0\}$  such that  $\overline{B(y + \eta, |\eta|)}$  intersects  $\overline{\Omega}$  at  $y$  only, and such that*

$$(4.1) \quad \langle A(y) \eta, \eta \rangle > 0.$$

*Proof.* — By the assumptions on  $A(x_0)$  there exists a unit vector  $h_0$  such that

$$(4.2) \quad \langle A(x_0)h_0, h_0 \rangle > 0.$$

Following the idea of Bony [10], we choose the neighborhood basis  $\mathcal{B}_{x_0} = \{\Omega(\varepsilon)\}$  as follows:

$$\Omega(\varepsilon) := B(x_0 + \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2) \cap B(x_0 - \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2).$$

It suffices to show that there exists  $\bar{\varepsilon} > 0$  such that every  $\Omega(\varepsilon)$  with  $0 < \varepsilon \leq \bar{\varepsilon}$  satisfies the requirement of the lemma. Now, the set  $\Omega(\varepsilon)$  (which is trivially an open neighborhood of  $x_0$ ) shrinks to  $\{x_0\}$  as  $\varepsilon$  shrinks to 0. Moreover, every  $y \in \partial\Omega(\varepsilon)$  belongs to one at least of the spheres  $\partial B(x_0 \pm \varepsilon^{-1} h_0, \varepsilon^{-1} + \varepsilon^2)$ ; accordingly, we choose

$$\eta = \eta_\varepsilon(y) := \frac{y - (x_0 \pm \varepsilon^{-1} h_0)}{\varepsilon^{-1} + \varepsilon^2}$$

to get the geometric condition  $\overline{B(y + \eta, |\eta|)} \cap \overline{\Omega(\varepsilon)} = \{y\}$ . It obviously holds that  $\eta_\varepsilon(y)$  tends to  $h(x_0)$  as  $\varepsilon \rightarrow 0$  (uniformly for bounded  $x_0, y, h_0$ ), so that (4.1) follows from (4.2) by continuity arguments, for any  $0 \leq \varepsilon \leq \bar{\varepsilon}$ , with  $\bar{\varepsilon}$  conveniently small.  $\square$

From Lemma 1.7, we know that there exists a monotone operator  $G_\varepsilon$  (which we called the Green operator related to  $\mathcal{L}_\varepsilon$  and  $\Omega$ ); since  $\varepsilon \geq 0$  is fixed, in all this section we drop the subscript  $\varepsilon$  in  $G_\varepsilon, k_\varepsilon, \lambda_{x,\varepsilon}$  and we simply write  $G, k, \lambda_x$ . Hence, for any bounded open  $\Omega$  as in Lemma 1.7, we are given the monotone operator

$$G : C(\overline{\Omega}, \mathbb{R}) \longrightarrow C(\overline{\Omega}, \mathbb{R})$$

mapping  $f \in C(\overline{\Omega}, \mathbb{R})$  into the unique function  $G(f) \in C(\overline{\Omega}, \mathbb{R})$  satisfying

$$(4.3) \quad \begin{cases} \mathcal{L}_\varepsilon(G(f)) = -f & \text{on } \Omega \quad (\text{in the weak sense of distributions}), \\ G(f) = 0 & \text{on } \partial\Omega \quad (\text{point-wise}). \end{cases}$$

We also know that the (Riesz) representation

$$(4.4) \quad G(f)(x) = \int_{\overline{\Omega}} f(y) \, d\lambda_x(y) \quad \text{for every } f \in C(\overline{\Omega}, \mathbb{R}) \text{ and every } x \in \overline{\Omega}$$

holds true, with a unique Radon measure  $\lambda_x$  defined on  $\overline{\Omega}$  (which we called the Green measure related to  $\mathcal{L}_\varepsilon$ ,  $\Omega$  and  $x$ ).

Finally, we set  $d\nu(x) := V(x) \, dx$  and we observe that (as in (1.13))

$$(4.5) \quad \int \varphi \mathcal{L}_\varepsilon \psi \, d\nu = \int \psi \mathcal{L}_\varepsilon \varphi \, d\nu, \quad \text{for every } \varphi, \psi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}).$$

Under the above assumptions and notation, the proof of Theorem 1.9 is split in four steps.

*Step I.* — We fix  $x \in \Omega$ . We begin by proving that  $\lambda_x$  is absolutely continuous with respect to the Lebesgue measure on  $\overline{\Omega}$ . To this end, let  $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ ; by (4.3) it is clear that  $G(\mathcal{L}_\varepsilon \varphi) = -\varphi$ , so that (see (4.4))

$$-\varphi(x) = \int_{\overline{\Omega}} \mathcal{L}_\varepsilon \varphi(y) \, d\lambda_x(y), \quad \text{for every } \varphi \in C_0^\infty(\overline{\Omega}, \mathbb{R}).$$

If we consider  $\lambda_x$  as a distribution on  $\Omega$  in the standard way, this identity boils down to

$$(4.6) \quad (\mathcal{L}_\varepsilon)^* \lambda_x = -\text{Dir}_x \quad \text{in } \mathcal{D}'(\Omega),$$

where  $\text{Dir}_x$  denotes the Dirac mass at  $x$ , and  $(\mathcal{L}_\varepsilon)^*$  is the classical adjoint operator of  $\mathcal{L}_\varepsilon$ . It is noteworthy to observe that, in general,  $(\mathcal{L}_\varepsilon)^*$  is neither equal to  $\mathcal{L}_\varepsilon$  nor of the form  $\tilde{\mathcal{L}} - \varepsilon$  for any  $\tilde{\mathcal{L}}$  a divergence operator as in (1.1).

However, the following crucial property of  $(\mathcal{L}_\varepsilon)^*$  is fulfilled:

*Remark 4.2.* — *The operator  $(\mathcal{L}_\varepsilon)^*$  is hypoelliptic on every open subset of  $\mathbb{R}^N$ .*

Indeed, let  $U \subseteq W$  be open sets and let  $u \in \mathcal{D}'(W)$  be such that  $(\mathcal{L}_\varepsilon)^* u = h$  in  $\mathcal{D}'(U)$ , where  $h \in C^\infty(U, \mathbb{R})$ . This gives the following chain of identities (here  $\psi \in C_0^\infty(U, \mathbb{R})$  is arbitrary)

$$\begin{aligned} \int h \psi &= \langle u, \mathcal{L}_\varepsilon \psi \rangle = \langle u, \mathcal{L} \psi - \varepsilon \psi \rangle \stackrel{(1.14)}{=} \left\langle u, \frac{\mathcal{L}^*(V\psi)}{V} - \varepsilon \psi \right\rangle \\ &= \left\langle \frac{u}{V}, \mathcal{L}^*(V\psi) - \varepsilon \psi V \right\rangle = \left\langle \frac{u}{V}, (\mathcal{L}_\varepsilon)^*(V\psi) \right\rangle. \end{aligned}$$

If we write  $\int h \psi = \int \frac{h}{V} (\psi V)$ , and if we observe that  $C_0^\infty(U, \mathbb{R}) = \{\psi V : \psi \in C_0^\infty(U, \mathbb{R})\}$ , the above computation shows that  $\mathcal{L}_\varepsilon(u/V) = h/V$  in

$\mathcal{D}'(U)$ . The hypoellipticity of  $\mathcal{L}_\varepsilon$  now gives  $u/V \in C^\infty(U, \mathbb{R})$  whence  $u \in C^\infty(U, \mathbb{R})$ , as  $V$  is smooth and positive.  $\square$

Identity (4.6) gives in particular  $(\mathcal{L}_\varepsilon)^* \lambda_x = 0$  in  $\mathcal{D}'(\Omega \setminus \{x\})$ ; thanks to Remark 4.2, this ensures the existence of  $g_x \in C^\infty(\Omega \setminus \{x\}, \mathbb{R})$  such that the distribution  $\lambda_x$  restricted to  $\Omega \setminus \{x\}$  is the function-type distribution associated with the function  $g_x$ ; equivalently

$$(4.7) \quad \int \varphi(y) d\lambda_x(y) = \int \varphi(y) g_x(y) dy, \quad \text{for every } \varphi \in C_0^\infty(\Omega \setminus \{x\}, \mathbb{R}).$$

Clearly  $g_x \geq 0$  on  $\Omega \setminus \{x\}$  and  $(\mathcal{L}_\varepsilon)^* g_x = 0$  in  $\Omega \setminus \{x\}$ . This temporarily proves that  $\lambda_x$  coincides with  $g_x(y) dy$  on  $\Omega \setminus \{x\}$ . We claim that this is also true throughout  $\Omega$ . This will follow if we show that  $C := \lambda_x(\{x\}) = 0$ . Clearly, by the definition of  $C$ , on  $\Omega$  we have

$$\lambda_x = C \text{Dir}_x + (\lambda_x)|_{\Omega \setminus \{x\}} = C \text{Dir}_x + g_x(y) dy.$$

Treating this as an identity between distributions on  $\Omega$ , we apply the operator  $(\mathcal{L}_\varepsilon)^*$  to get

$$C (\mathcal{L}_\varepsilon)^* \text{Dir}_x = -\text{Dir}_x - (\mathcal{L}_\varepsilon)^*(g_x(y) dy).$$

Here we used (4.6). We now proceed as follows:

- we multiply both sides by a  $C^\infty$  function  $\chi$  compactly supported in  $\Omega$  and  $\chi \equiv 1$  near  $x$ ;
- we compute the Fourier transform of the tempered distributions obtained as above;
- on the left-hand side we obtain a function-type distribution associated with function

$$y \mapsto C e^{-i\langle x, y \rangle} \left( - \sum_{i,j} a_{i,j}(x) y_i y_j + \{\text{polynomial in } y \text{ of degree } \leq 1\} \right),$$

where  $(a_{i,j})$  is the principal matrix of  $\mathcal{L}$ ;

- on the right-hand side we obtain a function-type distribution associated with a function which is the sum of  $y \mapsto -e^{-i\langle x, y \rangle}$  with a function of the form

$$y \mapsto - \sum_{i,j} \alpha_{i,j}(x, y) y_i y_j + \{\text{polynomial in } y \text{ of degree } \leq 1\},$$

where

$$\alpha_{i,j}(x, y) = - \int g_x(\xi) \chi(\xi) a_{i,j}(\xi) e^{-i\langle \xi, y \rangle} d\xi.$$

By the Riemann-Lebesgue Theorem one has  $\alpha_{i,j}(x, y) \rightarrow 0$  as  $|y| \rightarrow \infty$ . This implies that  $C = 0$ , since at least one of the entries of  $(a_{i,j}(x))$  is non-vanishing, due to the (NTD) hypothesis on  $\mathcal{L}$ .

We have therefore proved that, for any  $x \in \Omega$ ,

$$(4.8) \quad d\lambda_x(y) = g_x(y) \, dy \text{ on } \Omega.$$

Since  $\lambda_x$  is a finite measure (recalling that  $\bar{\Omega}$  is compact), from (4.8) we get  $g_x \in L^1(\Omega)$  for every  $x \in \Omega$ .

*Step II.* — We next show that  $\lambda_x(\partial\Omega) = 0$  for any  $x \in \bar{\Omega}$ . For small  $\delta > 0$ , we let  $D_\delta$  denote the closed  $\delta$ -neighborhood of  $\partial\Omega$  of the points in  $\mathbb{R}^N$  having distance from  $\partial\Omega$  less than or equal to  $\delta$ ; we then choose a function  $F \in C(\mathbb{R}^N, [0, 1])$  which is identically 1 on  $\partial\Omega$  and is supported in the interior of  $D_\delta$ . We denote by  $f$  the restriction of  $F$  to  $\bar{\Omega}$ . From (4.4) we have

$$(4.9) \quad 0 \leq G(f)(x) = \int_{\bar{\Omega}} f(y) \, d\lambda_x(y) \leq \int_{\bar{\Omega}} d\lambda_x(y) = G(1)(x),$$

for every  $x \in \bar{\Omega}$ .

For any  $x \in \bar{\Omega}$  we have

$$\begin{aligned} \lambda_x(\partial\Omega) &= \int_{\partial\Omega} d\lambda_x(y) = \int_{\partial\Omega} f(y) \, d\lambda_x(y) \leq \int_{\bar{\Omega}} f(y) \, d\lambda_x(y) = G(f)(x) \\ &\leq \sup_{\bar{\Omega}} G(f) = \max \left\{ \sup_{\bar{\Omega} \cap D_\delta} G(f), \sup_{\bar{\Omega} \setminus D_\delta} G(f) \right\} =: \max\{\text{I}, \text{II}\}. \end{aligned}$$

We claim that I and II in the above right-hand side are bounded from above by  $\sup_{\bar{\Omega} \cap D_\delta} G(1)$ . This is true of I, due to (4.9); as for II we invoke the last assertion in Remark 2.4 applied to:

- the hypoelliptic operator  $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$ ,
- the bounded open set  $\Omega_1 := \bar{\Omega} \setminus D_\delta$ ,
- the nonnegative function  $G(f)$ , which satisfies  $\mathcal{L}_\varepsilon G(f) = -f = 0$  on  $\Omega_1$  both weakly and strongly due to the hypoellipticity of  $\mathcal{L}_\varepsilon$ .

The mentioned Remark 2.4 then ensures that the values of  $G(f)$  on  $\bar{\Omega} \setminus D_\delta$  are bounded from above by the values of  $G(f)$  on the boundary of this set, so that  $\text{II} \leq \text{I}$ . Summing up,

$$\lambda_x(\partial\Omega) \leq \max\{\text{I}, \text{II}\} \leq \sup_{\bar{\Omega} \cap D_\delta} G(1).$$

As  $\delta$  goes to 0, the right-hand side tends to  $\sup_{\partial\Omega} G(1) = 0$  by (4.3). This gives the desired  $\lambda_x(\partial\Omega) = 0$ , for any  $x \in \bar{\Omega}$ . By collecting together (4.8) and  $\lambda_x(\partial\Omega) = 0$ , we infer that (for every  $f \in C(\bar{\Omega}, \mathbb{R})$  and  $x \in \Omega$ )

$$G(f)(x) \stackrel{(4.4)}{=} \int_{\bar{\Omega}} f(y) \, d\lambda_x(y) = \int_{\Omega} f(y) \, d\lambda_x(y) \stackrel{(4.8)}{=} \int_{\Omega} f(y) g_x(y) \, dy.$$

This proves the identity

$$(4.10) \quad G(f)(x) = \int_{\Omega} f(y) g_x(y) \, dy,$$

for every  $f \in C(\overline{\Omega}, \mathbb{R})$  and every  $x \in \Omega$ .

If  $\varphi \in C_0^\infty(\Omega, \mathbb{R})$ , since we know that  $G(\mathcal{L}_\varepsilon \varphi) = -\varphi$ , we get

$$(4.11) \quad -\varphi(x) = \int_{\Omega} \mathcal{L}_\varepsilon \varphi(y) g_x(y) \, dy, \quad \text{for every } x \in \Omega.$$

This is equivalent to

$$(4.12) \quad (\mathcal{L}_\varepsilon)^* g_x = -\text{Dir}_x \quad \text{for every } x \in \Omega.$$

*Step III.* — If  $g_x$  is as in Step I, we are ready to set

$$g : \Omega \times \Omega \longrightarrow [0, \infty], \quad g(x, y) := \begin{cases} g_x(y) & \text{if } x \neq y \\ \infty & \text{if } x = y. \end{cases}$$

Hence the representation (4.10) becomes

$$(4.13) \quad G(f)(x) = \int_{\Omega} f(y) g(x, y) \, dy,$$

for every  $f \in C(\overline{\Omega}, \mathbb{R})$  and every  $x \in \Omega$ .

We aim to prove that  $g$  is smooth outside the diagonal of  $\Omega \times \Omega$ .

*Remark 4.3.* — Let  $O$  be any open subset of  $\mathbb{R}^N$ . The hypoellipticity of a general PDO  $L$  as in (1.2) ensures the equality of the topologies on  $\mathcal{H}_L(O)$  inherited by the Fréchet spaces  $C^\infty(O)$  and  $L^1_{\text{loc}}(O)$ .

Indeed, let  $\mathcal{X}$  and  $\mathcal{Y}$  denote respectively the topological space  $\mathcal{H}_L(O)$  with the topologies inherited by  $C^\infty(O)$  and  $L^1_{\text{loc}}(O)$ . Then  $\mathcal{X}$  and  $\mathcal{Y}$  are Fréchet spaces, since, if a sequence  $u_n \in \mathcal{H}_L(O)$  converges to  $u$  uniformly on the compact sets of  $\Omega$  or, more generally in  $L^1_{\text{loc}}$ ,

$$0 = \int u_n L^* \varphi \xrightarrow{n \rightarrow \infty} \int u L^* \varphi, \quad \forall \varphi \in C_0^\infty(O, \mathbb{R}).$$

Now, the identity map  $\iota : \mathcal{X} \rightarrow \mathcal{Y}$  is trivially linear, bijective and continuous, whence, by the Open Mapping Theorem,  $\iota$  is a homeomorphism, whence the mentioned topologies coincide. □

We next resume our main proof. The set  $\{g_x\}_{x \in \Omega}$  is bounded in  $L^1(\Omega)$ , since

$$0 \leq \int_{\Omega} g_x(y) \, dy = G(1)(x) \leq \max_{\Omega} G(1).$$



A fortiori, the set  $\{g_x\}_{x \in \Omega}$  is also bounded in the topological vector space  $L^1_{\text{loc}}(\Omega)$ . We next fix two disjoint open sets  $U, W$  with closures contained in  $\Omega$ . The family of the restrictions

$$\left\{ (g_x)|_U \right\}_{x \in W}$$

is contained in the space of the  $(\mathcal{L}_\varepsilon)^*$ -harmonic functions on  $U$ . By Remark 4.3, the above set is also bounded in the topological vector space

$$\mathcal{H}(\mathcal{L}_\varepsilon)^*(U), \quad \text{endowed with the } C^\infty\text{-topology.}$$

This means that, for every compact set  $K \subset U$  and for every  $m \in \mathbb{N}$ , there exists a constant  $C(K, m) > 0$  such that

$$(4.14) \quad \sup_{|\alpha| \leq m} \sup_{y \in K} \left| \left( \frac{\partial}{\partial y} \right)^\alpha g(x, y) \right| \leq C(K, m), \quad \text{uniformly for } x \in W.$$

Following Bony [10, Section 6], we introduce the operator  $F$  transforming any distribution  $T$  compactly supported in  $U$  into the function on  $W$  defined by

$$F(T) : W \longrightarrow \mathbb{R}, \quad F(T)(x) := \langle T, g_x \rangle \quad (x \in W).$$

The definition is well-posed since  $g_x \in C^\infty(U, \mathbb{R})$  (and  $T$  is compactly supported in  $U$ ). We claim that  $F(T) \in C^\infty(W, \mathbb{R})$ . Once this is proved, by the Schwartz Kernel Theorem (see e.g., [20, Section 11] or [65, Chapter 50]), we can conclude that  $g(x, y)$  is smooth on  $W \times U$ . By the arbitrariness of the disjoint open sets  $U, W$  this proves that  $g(x, y)$  is smooth out of the diagonal of  $\Omega \times \Omega$ , as desired.

As for the proof of the claimed  $F(T) \in C^\infty(W, \mathbb{R})$ , we can take (say, by some appropriate convolution) a sequence of continuous functions  $f_n$ , supported in  $U$ , converging to  $T$  in the weak sense of distributions; due to the compactness of the supports (of the  $f_n$  and of  $T$ ),

$$\lim_{n \rightarrow \infty} \int_U f_n \varphi = \langle T, \varphi \rangle, \quad \text{for every } \varphi \in C^\infty(U, \mathbb{R}).$$

We are hence entitled to take  $\varphi = g_x$  (for any fixed  $x \in W$ ). From (4.13) we get

$$(4.15) \quad \lim_{n \rightarrow \infty} G(f_n)(x) = \langle T, g_x \rangle = F(T)(x), \quad \text{for any } x \in W.$$

We now prove that  $F(T) \in L^\infty(W)$ ; this follows from the next calculation (here  $C > 0$  and  $m \in \mathbb{N}$  are constants depending on  $T$  and on the compact

set  $\bar{U}$ )

$$\begin{aligned} \|F(T)\|_{L^\infty} &= \sup_{x \in W} |\langle T, g_x \rangle| \leq \sup_{x \in W} C \sum_{|\alpha| \leq m} \sup_{y \in \bar{U}} \left| \left( \frac{\partial}{\partial y} \right)^\alpha g(x, y) \right| \\ &\stackrel{(4.14)}{\leq} \tilde{C}(\bar{U}, m) < \infty. \end{aligned}$$

We finally prove that  $\mathcal{L}_\varepsilon(F(T)) = 0$  in the weak sense of distributions on  $W$ ; by the hypoellipticity of  $\mathcal{L}_\varepsilon$  this will yield the smoothness of  $F(T)$  on  $W$ . We aim to show that,

$$\int_W F(T)(x) (\mathcal{L}_\varepsilon)^* \varphi(x) \, dx = 0 \quad \text{for any } \varphi \in C_0^\infty(W).$$

Now, the left-hand side is (by (4.15))

$$\int \lim_{n \rightarrow \infty} G(f_n)(x) (\mathcal{L}_\varepsilon)^* \varphi(x) \, dx.$$

If a dominated convergence can be applied, this is equal to

$$\lim_{n \rightarrow \infty} \int_W G(f_n)(x) (\mathcal{L}_\varepsilon)^* \varphi(x) \, dx \stackrel{(4.3)}{=} - \lim_{n \rightarrow \infty} \int_W f_n(x) \varphi(x) \, dx = 0,$$

the last equality descending from the fact that the  $f_n$  are supported in  $U$  for every  $n$ . We are then left with showing that the dominated convergence is fulfilled: this is a consequence of (4.14), of the boundedness of  $F(T)$  on  $W$ , and of the fact that the convergence in (4.15) is indeed uniform w.r.t.  $x \in W$  (a general result of distribution theory: the uniform convergence for sequences of distributions on bounded sets).

*Step IV.* — We are finally ready to introduce our kernel

$$(4.16) \quad k : \Omega \times \Omega \longrightarrow [0, \infty), \quad k(x, y) := \frac{g(x, y)}{V(y)}.$$

Clearly, from (4.13) and (1.13) we immediately have

$$(4.17) \quad G(f)(x) = \int_\Omega f(y) k(x, y) \, d\nu(y),$$

for every  $f \in C(\bar{\Omega}, \mathbb{R})$  and every  $x \in \Omega$ .

This gives the representation (1.15) whilst (1.17) follows from (4.11).

The integrability of  $k(x, \cdot)$  in  $\Omega$  is a consequence of  $g_x \in L^1(\Omega)$  (and the positivity of the continuous function  $V$  on  $\mathbb{R}^N$ ). Moreover,  $k$  is smooth on  $\Omega \times \Omega$  deprived of the diagonal by Step III. Also, the nonnegative function

$k$  is integrable on  $\Omega \times \Omega$  as this computation shows:

$$0 \leq \int_{\Omega \times \Omega} k(x, y) \, dx dy = \int_{\Omega} \left( \int_{\Omega} \frac{1}{V(y)} k(x, y) \, d\nu(y) \right) dx \\ \stackrel{(4.17)}{=} \int_{\Omega} G(1/V)(x) \, dx < \infty,$$

the last inequality following from the continuity of  $G(1/V)$  on the compact set  $\bar{\Omega}$ .

For fixed  $x \in \Omega$ , the  $\mathcal{L}_\varepsilon$ -harmonicity of the function  $k(x, \cdot)$  in  $\Omega \setminus \{x\}$  is a consequence of the following computation

$$0 \stackrel{(4.12)}{=} (\mathcal{L}_\varepsilon)^* g_x \stackrel{(1.14)}{=} V \mathcal{L}_\varepsilon \left( \frac{g_x}{V} \right) \stackrel{(4.16)}{=} V \mathcal{L}_\varepsilon(k(x, \cdot)).$$

The fact that  $V$  is positive then gives  $\mathcal{L}_\varepsilon(k(x, \cdot)) = 0$  in  $\Omega \setminus \{x\}$ . From the SMP for  $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$  in Remark 2.2, we deduce that the nonnegative function  $k(x, \cdot)$  (which is  $\mathcal{L}_\varepsilon$ -harmonic in  $\Omega \setminus \{x\}$ ) cannot attain the (minimal) value 0; therefore  $k(x, \cdot) > 0$  on the connected open set  $\Omega \setminus \{x\}$ .

A crucial step consists in proving the symmetry property (1.16). We take any nonnegative  $\varphi \in C_0^\infty(\Omega, \mathbb{R})$  and we set (note the reverse order of  $x$  and  $y$ , if compared to  $G(\varphi)$ )

$$\Phi(x) = \int_{\Omega} \varphi(y) k(y, x) \, d\nu(y), \quad x \in \Omega.$$

We claim that  $\Phi \geq G(\varphi)$  on  $\Omega$ ; once the claim is proved, from (4.17) we infer that

$$\int_{\Omega} \varphi(y) k(x, y) \, d\nu(y) \leq \int_{\Omega} \varphi(y) k(y, x) \, d\nu(y), \quad x \in \Omega.$$

The arbitrariness of  $\varphi$  will then give  $k(x, y) \leq k(y, x)$  (recalling that  $d\nu = V(y) \, dy$  with positive  $V$ ) for every  $y \in \Omega$ ; since  $x, y \in \Omega$  are arbitrary, we get  $k(x, y) = k(y, x)$  on  $\Omega \times \Omega$ . We prove the claim. We observe that  $\Phi$  is continuous on  $\Omega$  and that  $\mathcal{L}_\varepsilon \Phi = -\varphi$  in  $\mathcal{D}'(\Omega)$ , as the following computation

shows ( $\psi \in C_0^\infty(\Omega, \mathbb{R})$  is arbitrary):

$$\begin{aligned} \int_{\Omega} \Phi(x) (\mathcal{L}_\varepsilon)^* \psi(x) \, dx &= \int_{\Omega} \varphi(y) \left( \int_{\Omega} k(y, x) (\mathcal{L}_\varepsilon)^* \psi(x) \, dx \right) d\nu(y) \\ &= \int_{\Omega} \varphi(y) \left( \int_{\Omega} k(y, x) \frac{(\mathcal{L}_\varepsilon)^* \psi(x)}{V(x)} \, d\nu(x) \right) d\nu(y) \\ &\stackrel{(1.14)}{=} \int_{\Omega} \varphi(y) \left( \int_{\Omega} k(y, x) \mathcal{L}_\varepsilon \left( \frac{\psi(x)}{V(x)} \right) \, d\nu(x) \right) d\nu(y) \\ &\stackrel{(1.17)}{=} - \int_{\Omega} \varphi(y) \frac{\psi(y)}{V(y)} \, d\nu(y) \\ &= - \int_{\Omega} \varphi(y) \psi(y) \, dy. \end{aligned}$$

From the hypoellipticity of  $\mathcal{L}_\varepsilon$  we get  $\Phi \in C^\infty(\Omega, \mathbb{R})$  and  $\mathcal{L}_\varepsilon \Phi = -\varphi$  point-wise. We now apply the WMP in Remark 2.4 to the operator  $\mathcal{L}_\varepsilon = \mathcal{L} - \varepsilon$  and to the function  $G(\varphi) - \Phi$ : this function is smooth and  $\mathcal{L}_\varepsilon$ -harmonic on  $\Omega$ , and  $G(\varphi) - \Phi \leq G(\varphi)$  on  $\Omega$  (since  $\Phi$  is nonnegative), so that

$$\limsup_{x \rightarrow x_0} (G(\varphi) - \Phi)(x) \leq \limsup_{x \rightarrow x_0} G(\varphi)(x) = 0 \quad \text{for every } x_0 \in \partial\Omega.$$

Therefore  $G(\varphi) - \Phi \leq 0$  on  $\Omega$  as claimed.

We finally prove (1.18). Due to the symmetry property of  $k$ , (1.18) will follow if we show that, given  $x_0 \in \Omega$  and  $y_0 \in \partial\Omega$ , one has

$$(4.18) \quad \lim_{n \rightarrow \infty} k(y_n, x_0) = 0,$$

for every sequence  $y_n$  in  $\Omega$  converging to  $y_0$ . To this end, we fix an open set  $\Omega'$  containing  $x_0$  and with closure contained in  $\Omega$ , and it is non-restrictive to suppose that  $y_n \notin \Omega'$  for every  $n$ . The functions

$$k_n : \Omega' \rightarrow \mathbb{R}, \quad k_n(x) := k(y_n, x), \quad x \in \Omega'$$

are smooth and  $\mathcal{L}_\varepsilon$ -harmonic in  $\Omega'$ . We also have  $k_n \rightarrow 0$  in  $L^1(\Omega')$ , as it follows from

$$\begin{aligned} 0 &\leq \int_{\Omega'} k_n(x) \, dx \leq \int_{\Omega} k(y_n, x) \, dx = \int_{\Omega} \frac{g(y_n, x)}{V(x)} \, dx \\ &\leq \sup_{\Omega} \frac{1}{V} \int_{\Omega} g(y_n, x) \, dx = \sup_{\Omega} \frac{1}{V} G(1)(y_n) \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

From Remark 4.3 we get that  $k_n \rightarrow 0$  in the Fréchet space  $\mathcal{H}_{\mathcal{L}_\varepsilon}(\Omega')$  with the  $C^\infty$ -topology, so that  $k_n \rightarrow 0$  uniformly on the compact sets of  $\Omega'$  and in particular point-wise on  $\Omega'$ . □

## 5. The Harnack inequality

We begin by proving the next crucial lemma. This is the first time that, broadly speaking, the PDOs  $\mathcal{L}$  and the perturbed  $\mathcal{L} - \varepsilon$  clearly interact.

LEMMA 5.1. — *Let  $\mathcal{L}$  be as in (1.1) and let it satisfy (NTD) and  $(\text{HY})_\varepsilon$ . Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set as in the thesis of Lemma 1.7, and let  $\Omega'$  be an open set containing  $\overline{\Omega}$ . Finally, we denote by  $k_\varepsilon$  the Green kernel related to  $\mathcal{L}_\varepsilon$  and to the set  $\Omega$  (as in Theorem 1.9).*

*Then we have the estimate*

$$(5.1) \quad u(x) \geq \varepsilon \int_{\Omega} u(y) k_\varepsilon(x, y) \, d\nu(y), \quad \forall x \in \Omega,$$

*holding true for every smooth nonnegative  $\mathcal{L}$ -harmonic function  $u$  in  $\Omega'$ .*

*Proof.* — We consider the function  $v(x) = \int_{\Omega} u(y) k_\varepsilon(x, y) \, d\nu(y)$  on  $\Omega$ . From (1.15) (and the definition of Green operator) we know that  $v = G_\varepsilon(u)$ , where  $G_\varepsilon$  is the Green operator related to  $\mathcal{L}_\varepsilon$  (and to the open set  $\Omega$ ); moreover, since  $u$  is smooth (by assumption) on  $\overline{\Omega}$ , we know from Lemma 1.7 (and the hypoellipticity of  $\mathcal{L}_\varepsilon$ ) that  $v \in C^\infty(\Omega) \cap C(\overline{\Omega})$  is the solution of

$$(5.2) \quad \begin{cases} \mathcal{L}_\varepsilon v = -u & \text{on } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

This gives  $\mathcal{L}_\varepsilon(\varepsilon v - u) = -\varepsilon u - (\mathcal{L} - \varepsilon)u = -\varepsilon u + \varepsilon u = 0$  on  $\Omega$ ; moreover, on  $\partial\Omega$ ,  $\varepsilon v - u = -u \leq 0$ , by the nonnegativity of  $u$ . By the WMP in Remark 2.4, we get  $\varepsilon v - u \leq 0$  on  $\Omega$  which is equivalent to (5.1).  $\square$

We are ready for the proof of the Weak Harnack Inequality (for higher order derivatives).

*Proof of the Weak Harnack Inequality for derivatives, Theorem 1.12.* — We distinguish two cases:  $y_0 \notin K$  and  $y_0 \in K$ . The second case can be reduced to the former. Indeed, let us assume we have already proved the theorem in the former case, and let  $y_0 \in K$ . If we take any  $y'_0 \in O \setminus K$ , and we consider the inequality

$$u(y'_0) \leq C' u(y_0),$$

resulting from (1.22) by considering  $m = 0$  and the compact set  $\{y'_0\}$ , we get

$$\sum_{|\alpha| \leq m} \sup_{x \in K} \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right| \stackrel{(1.22)}{\leq} C u(y'_0) \leq C C' u(y_0).$$

We are therefore entitled to assume that  $y_0 \notin K$ . By the aid of a classical argument (with a chain of suitable small open sets  $\{\Omega_n\}_{n=1}^p$  covering a connected compact set containing  $K \cup \{y_0\}$ ), it is not restrictive to assume that  $K \cup \{y_0\} \subset \Omega \subset \bar{\Omega} \subset O$ , where  $\Omega$  is one of the (bounded) basis open sets constructed in Lemma 1.7.

Let  $x_0 \in K$  be arbitrarily fixed. The function  $k_\varepsilon(x_0, \cdot)$  (the Green kernel related to  $\mathcal{L}_\varepsilon$  and  $\Omega$ ) is *strictly positive* in  $\Omega \setminus \{x_0\}$  (this is a consequence of the SMP applied to the  $\mathcal{L}_\varepsilon$ -harmonic function  $k_\varepsilon(x_0, \cdot)$ ; see Theorem 1.9). In particular, since  $y_0 \notin K$ , we infer that  $k_\varepsilon(x_0, y_0) > 0$ . Hence, there exist a neighborhood  $W$  of  $x_0$  (contained in  $\Omega$ ) and a constant  $\mathbf{c} = \mathbf{c}(\varepsilon, y_0, x_0) > 0$  such that

$$(5.3) \quad \inf_{z \in W} k_\varepsilon(z, y_0) \geq \mathbf{c} > 0.$$

Our assumptions allow us to apply Lemma 5.1: hence, for every nonnegative  $u \in \mathcal{H}_\mathcal{L}(O)$ , we have the following chain of inequalities

$$\begin{aligned} u(y_0) &\stackrel{(5.1)}{\geq} \varepsilon \int_\Omega u(z) k_\varepsilon(y_0, z) \, d\nu(z) \\ &\geq \varepsilon \int_W u(z) k_\varepsilon(y_0, z) \, d\nu(z) \stackrel{(1.16)}{=} \varepsilon \int_W u(z) k_\varepsilon(z, y_0) \, d\nu(z) \\ &\stackrel{(5.3)}{\geq} \varepsilon \mathbf{c} \int_W u(z) \, d\nu(z) \\ &\geq \varepsilon \mathbf{c} \inf_W V \int_W u(z) \, dz. \end{aligned}$$

Summing up, for every  $x_0 \in K$  there exist a neighborhood  $W$  of  $x_0$  and a constant  $\mathbf{c}_1 > 0$  (also depending on  $x_0$  but independent of  $u$ ) such that

$$(5.4) \quad u(y_0) \geq \mathbf{c}_1 \int_W u(z) \, dz,$$

for every nonnegative  $u \in \mathcal{H}_\mathcal{L}(O)$ .

Next, from Remark 4.3, we know that the hypothesis (HY) for  $\mathcal{L}$  ensures the equality of the topologies on  $\mathcal{H}_\mathcal{L}(W)$  inherited by the Fréchet spaces  $C^\infty(W)$  and  $L^1_{\text{loc}}(W)$ . In particular, to any chosen open neighborhood  $U$  of  $x_0$  (with  $\bar{U} \subset W$ ) we are given a positive constant  $\mathbf{c}_2 = \mathbf{c}_2(U, W, m)$  such that

$$(5.5) \quad \sum_{|\alpha| \leq m} \sup_{x \in U} \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right| \leq \mathbf{c}_2 \int_W u(z) \, dz,$$

for every nonnegative  $u \in \mathcal{H}_\mathcal{L}(O)$ . Gathering together (5.4) and (5.5), we infer that, for every  $x_0 \in K$  there exist a neighborhood  $U$  of  $x_0$  and a

constant  $\mathbf{c}_3 > 0$  (again depending on  $x_0$  but independent of  $u$ ) such that

$$u(y_0) \geq \mathbf{c}_3 \sum_{|\alpha| \leq m} \sup_{x \in U} \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right|,$$

for every nonnegative  $u \in \mathcal{H}_{\mathcal{L}}(O)$ . The compactness of  $K$  allows us to derive (1.22) from the latter inequality, and a covering argument.  $\square$

We now present a proof of Theorem 1.11, crucially based on [12, Chapter I].

*Proof of Theorem 1.11.* — As anticipated in the Introduction, the proof is based in an essential way on the ideas by Mokobodzki-Brelot in [12, Chapter I], ensuring the equivalence of the Strong Harnack Inequality with a series of properties comprising the Weak Harnack Inequality, provided some assumptions are fulfilled. We furnish some details in order to be oriented through these equivalent properties.

We denote by  $\mathcal{H}_L$  the harmonic sheaf on  $\mathbb{R}^N$  defined by  $O \mapsto \mathcal{H}_L(O)$  (here  $O \subseteq \mathbb{R}^N$  is any open set). Under the assumptions of (Regularity) and (Weak Harnack Inequality), Brelot proves that (see [12, pp.22–24]), for any connected open set  $O \subseteq \mathbb{R}^N$ , and any  $x_0 \in O$ , the set

$$(5.6) \quad \Phi_{x_0} := \left\{ h \in \mathcal{H}_L(O) : h \geq 0, \quad h(x_0) = 1 \right\}$$

is equicontinuous at  $x_0$ . The proof of this fact rests on some results of Functional Analysis related to the family of the so-called harmonic measures  $\{\mu_x^\Omega\}_{x \in \partial\Omega}$  associated with  $L$  (and on basic properties of the harmonic sheaf  $\mathcal{H}_L$ ). Next, we show how to prove (1.20) starting from the equicontinuity of  $\Phi_{x_0}$  at  $x_0$ . Indeed, let  $K \subset O$ , where  $K$  is compact and  $O$  is an open and connected subset of  $\mathbb{R}^N$ . By possibly enlarging  $K$ , we can suppose that  $K$  is connected as well. Let  $u \in \mathcal{H}_L(O)$  be nonnegative. If  $u \equiv 0$  then (1.20) is trivial; if  $u$  is not identically zero then (from the Weak Harnack Inequality) one has  $u > 0$  on  $O$ . For every  $x \in K$ , the equicontinuity of  $\Phi_x$  ensures the existence of  $\delta(x) > 0$  such that (with the choice  $h = u/u(x)$  in (5.6))

$$(5.7) \quad \frac{1}{2} u(x) \leq u(\xi) \leq \frac{3}{2} u(x), \quad \text{for all } \xi \in B_x := B(x, \delta(x)).$$

From the open cover  $\{B_x\}_{x \in K}$  we can extract a finite subcover  $B_{x_1}, \dots, B_{x_p}$  of  $K$ . It is also non-restrictive (since  $K$  is connected) to assume that the elements of this subcover are chosen in such a way that

$$B_{x_1} \cap B_{x_2} \neq \emptyset, \quad (B_{x_1} \cup B_{x_2}) \cap B_{x_3} \neq \emptyset, \quad \dots \quad (B_{x_1} \cup \dots \cup B_{x_{p-1}}) \cap B_{x_p} \neq \emptyset.$$

From (5.7) it follows (1.20) with  $K$  replaced by  $B_{x_1}$  (with  $M = 3$ ); since  $B_{x_1}$  intersects  $B_{x_2}$ , one can use again (5.7) in order to prove (1.20) with  $K$  replaced by  $B_{x_1} \cup B_{x_2}$  (with  $M = 3^2$ ); by proceeding in an inductive way, one can prove (1.20) with  $K$  replaced by  $B_{x_1} \cup \dots \cup B_{x_p}$  (and  $M = 3^p$ ), and this finally proves (1.20), since  $B_{x_1} \cup \dots \cup B_{x_p}$  covers  $K$ .  $\square$

*Remark 5.2.* — Following BreLOT [12, pp.14–17], it being understood that axiom (Regularity) in Theorem 1.11 holds true, the axiom (Weak Harnack Inequality) can be replaced by any of the following equivalent assumptions (see also Constantinescu and Cornea [16]):

**(BreLOT Axiom):** For every connected open set  $O \subseteq \mathbb{R}^N$ , if  $\mathcal{F}$  is an up-directed<sup>(2)</sup> family of  $L$ -harmonic functions in  $O$ , then  $\sup_{u \in \mathcal{F}} u$  is either  $+\infty$  or it is  $L$ -harmonic in  $O$ .

**(Harnack Principle):** For every connected open set  $O \subseteq \mathbb{R}^N$ , if  $\{u_n\}_n$  is a non-decreasing sequence of  $L$ -harmonic functions in  $O$ , then  $\lim_{n \rightarrow \infty} u_n$  is either  $+\infty$  or it is an  $L$ -harmonic function in  $O$ .

We are ready to derive our main result for this section: due to all our preliminary results, the proof is now a few lines argument.

*Proof (of Harnack Inequality, Theorem 1.10).* — Due to Theorem 1.11, it suffices to prove that our operator  $\mathcal{L}$  as in the statement of Theorem 1.10 satisfies the properties named (Regularity) and (Weak Harnack Inequality) in Theorem 1.11: the former is a consequence of Lemma 1.7 (with  $f = 0$ ), whilst the latter follows from Theorem 1.12.  $\square$

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<sup>(2)</sup>  $\mathcal{F}$  is said to be up-directed if for any  $u, v \in \mathcal{F}$  there exists  $w \in \mathcal{F}$  such that  $\max\{u, v\} \leq w$ .



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